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On Smoothness of the Solution to the Abel Equation in Terms of the Jacobi Series Coefficients

Maksim V. Kukushkin [1,2]

[1] Moscow State University of Civil Engineering, 129337 Moscow, Russia; kukushkinmv@rambler.ru
[2] Kabardino-Balkarian Scientific Center, RAS, 360051 Nalchik, Russia

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Abstract: In this paper, we continue our study of the Abel equation with the right-hand side belonging to the Lebesgue weighted space. We have improved the previously known result— the existence and uniqueness theorem formulated in terms of the Jacoby series coefficients that gives us an opportunity to find and classify a solution by virtue of an asymptotic of some relation containing the Jacobi series coefficients of the right-hand side. The main results are the following—the conditions imposed on the parameters, under which the Abel equation has a unique solution represented by the series, are formulated; the relationship between the values of the parameters and the solution smoothness is established. The independence between one of the parameters and the smoothness of the solution is proved.

Keywords: Riemann-Liouville operator; Abel equation; Jacobi polinomials; weighted Lebesgue spaces

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1. Introduction

In the beginning, let us be reminded that the so called mapping theorems for the Riemann-Liouville operator were first studied by G.H. Hardy and J.E. Littlewood; it was proved that $I^{\alpha}_{a+} : L_p \to L_q$, $1 < p < 1/\alpha$, $q < p/(1-\alpha p)$, $\alpha \in (0,1)$. This proposition was afterwards clarified [1] and nowadays is known as the Hardy-Littlewood theorem with limit index $I^{\alpha}_{a+} : L_p \to L_q$, $q = p/(1-\alpha p)$. However, there was an attempt to extend this theorem on some class of weighted Lebesgue spaces defined as functional spaces endowed with the following norm

$$\|f\|_{L_p(I,\omega)} := \left\{ \int\limits_a^b |f(x)|^p \, \omega(x) dx \right\}^{1/p},$$

$$\omega(x) := (x-a)^{\beta}(b-x)^{\gamma},\ \beta,\gamma \in \mathbb{R},\ I := (a,b).$$

In this direction, mathematicians such as Rubin B.S. [2–4], Karapetyants N.K. [5,6], Vakulov B.G. [7], and Samko S.G. [8,9] (the results of [2,3,5] are also presented in Reference [10]) had great success. Analogues of the Hardy-Littlewood theorem were formulated for a class of weighted Lebesgue spaces. The main disadvantage of the results presented in Reference [10] consists of some gaps of the parameter values in the conditions; moreover, the notorious problem related to $p = 1/\alpha$ (the detailed information can be found in paragraph 3.3 [10], p. 91) remained completely unsolvable. All these create the prerequisite to invent another approach for studying the Riemann-Liouville operator action that was successfully investigated in Reference [11], and below we write out some of its highlights. Despite the fact that the idea of using the Jacobi polynomials is not novel and many papers have been devoted to this topic [12–18], we confirm the main advantage of the method, used in Reference [11] and based on the results [19–23], which are still relevant and allow us to obtain some interesting results. We stress that the method was not considered in the well-known monographs [10,24] devoted to the topic, Reference [25] may be mentioned also.

The main challenge of this paper is to improve and clarify the results of Reference [11]. In particular we need to find a simple condition, imposed on the right-hand side of the Abel equation, under which Theorem 2 [11] is applicable. For this purpose we make an attempt to solve this problem by using absolute convergence of a series. The main relevance of the improvement is based on the fact that the previously used methods were determined by the relation between order α of the fractional integral and index p of a Lebesgue space (for instance the case $p = 1/\alpha$ has not been studied in the monograph [10]). All these create a strong motivation for research in this direction, but the highlight is in the following—the relationship between the values of the parameters and order α, by virtue of which we can provide a description of the solution smoothness, will be established and the conditions providing the existence and uniqueness of the solution, formulated in terms of Jacobi series coefficients, will be obtained and the independence between one of the parameters and the solution smoothness will be proved.

2. Preliminaries

Let C be a positive real constant, we assume that the values of C can be different in various formulas and expressions. The orthonormal system of the Jacobi polynomials is denoted by

$$p_n^{(\beta,\gamma)}(x) = \delta_n(\beta,\gamma)\, y_n^{(\beta,\gamma)}(x),\ n \in \mathbb{N}_0,$$

where the normalized multiplier $\delta_n(\beta,\gamma)$ is defined by the formula

$$\delta_n(\beta,\gamma) = \frac{(-1)^n}{(b-a)^{n+(\beta+\gamma+1)/2}} \cdot \sqrt{\frac{(\beta+\gamma+2n+1)\Gamma(\beta+\gamma+n+1)}{n!\Gamma(\beta+n+1)\Gamma(\gamma+n+1)}}\,,$$

$$\delta_0(\beta,\gamma) = \frac{1}{\sqrt{\Gamma(\beta+1)\Gamma(\gamma+1)}}\,,\ \beta+\gamma+1=0,$$

the orthogonal polynomials $y_n^{(\beta,\gamma)}$ are defined by the formula

$$y_n^{(\beta,\gamma)}(x) = (x-a)^{-\beta}(b-x)^{-\gamma}\frac{d^n}{dx^n}\left[(x-a)^{\beta+n}(b-x)^{\gamma+n}\right],\ \beta,\gamma > -1.$$

Consider the orthonormal Jacobi polynomials

$$p_n^{(\beta,\gamma)}(x) = \delta_n y_n(x),\ \beta,\gamma > -1/2,\ n \in \mathbb{N}_0.$$

If it is necessary, we also use the shorthand notations $p_n(x) := p_n^{(\beta,\gamma)}(x)$. It is clear that

$$p_n^{(k)}(a) = (-1)^{n+k}(b-a)^{-k-(\beta+\gamma+1)/2}\delta_n'\tilde{C}_n^k(\beta,\gamma),\ k \le n,$$

where

$$\delta_n' := \sqrt{\frac{(\beta+\gamma+2n+1)\Gamma(\beta+\gamma+n+1)}{n!\Gamma(\beta+n+1)\Gamma(\gamma+n+1)}},$$

$$\tilde{C}_n^k(\beta,\gamma) := \sum_{i=0}^{k} C_n^i \binom{n+\beta}{n-i}\binom{n+\gamma}{i} C_k^i \binom{n-i}{k-i} i!,$$

$$\binom{\eta}{\mu} := \Gamma(\eta+1)/\Gamma(\eta-\mu+1),\ \eta,\mu \in \mathbb{R},\ \mu \neq -1,-2,\ldots.$$

In the same way, we get

$$p_n^{(k)}(b) = (b-a)^{k+(\beta+\gamma+1)/2}\delta'_n \tilde{C}_n^k(\gamma,\beta),\ k \le n.$$

Using the Taylor series expansion for the Jacobi polynomials, we get

$$p_n^{(\beta,\gamma)}(x) = \sum_{k=0}^{n}(-1)^{n+k}\frac{\delta'_n \tilde{C}_n^k(\beta,\gamma)}{k!(b-a)^{k+(\beta+\gamma+1)/2}}(x-a)^k =$$

$$= \sum_{k=0}^{n}(-1)^{k}\frac{\delta'_n \tilde{C}_n^k(\gamma,\beta)}{k!(b-a)^{k+(\beta+\gamma+1)/2}}(b-x)^k.$$

Applying the formulas (2.44), (2.45) of the fractional integral and derivative of a power function [10], p. 40, we obtain

$$(I^\alpha_{a+}p_n)(x) = \sum_{k=0}^{n}(-1)^{n+k}\frac{\delta'_n \tilde{C}_n^k(\beta,\gamma)}{(b-a)^{k+(\beta+\gamma+1)/2}\Gamma(k+1+\alpha)}(x-a)^{k+\alpha},$$

$$(I^\alpha_{b-}p_n)(x) = \sum_{k=0}^{n}(-1)^{k}\frac{\delta'_n \tilde{C}_n^k(\gamma,\beta)}{(b-a)^{k+(\beta+\gamma+1)/2}\Gamma(k+1+\alpha)}(b-x)^{k+\alpha},$$

$$\alpha \in (-1,1),$$

here we used the formal notation $I^{-\alpha}_{a+} := D^\alpha_{a+}$. Thus, using integration by parts, we get

$$\int_a^b p_m(x)(I^\alpha_{a+}p_n)(x)\omega(x)dx =$$

$$= (-1)^n\delta'_m\delta'_n\sum_{k=0}^{n}(-1)^k\frac{\tilde{C}_n^k(\beta,\gamma)B(\alpha+\beta+k+1,\gamma+m+1)}{\Gamma(k+\alpha-m+1)}.$$

In the same way, we get

$$(p_m, I^\alpha_{b-}p_n)_{L_2(I,\omega)} =$$

$$= (-1)^m\delta'_m\delta'_n\sum_{k=0}^{n}(-1)^k\frac{\tilde{C}_n^k(\gamma,\beta)B(\alpha+\gamma+k+1,\beta+m+1)}{\Gamma(k+\alpha-m+1)}.$$

Using the notation

$$A^{\alpha,\beta,\gamma}_{mn} := \delta'_m\delta'_n\sum_{k=0}^{n}(-1)^k\frac{\tilde{C}_n^k(\beta,\gamma)B(\alpha+\beta+k+1,\gamma+m+1)}{\Gamma(k+\alpha-m+1)},$$

we have

$$(p_m, I^\alpha_{a+}p_n)_{L_2(I,\omega)} = (-1)^n A^{\alpha,\beta,\gamma}_{mn},$$

$$(p_m, I^\alpha_{b-}p_n)_{L_2(I,\omega)} = (-1)^m A^{\alpha,\gamma,\beta}_{mn}. \tag{1}$$

Further, we use the ordinary notation

$$S_k f := \sum_{i=0}^{k} p_i f_i,\ f_i := \int_a^b p_i(x)f(x)\omega(x)dx.$$

Consider the Abel equation with the most general assumptions regarding the right-hand side

$$I^{-\alpha}_{b-}\varphi = f \in L_p(I,\omega),\ 1 \le p < \infty,\ \alpha \in (-1,0). \tag{2}$$

The following theorem is the very mapping theorem (see Reference [11]) formulated in terms of the Jacobi series coefficients. Here we give the modified form corresponding to the right-hand side case.

Theorem 1. *(Theorem 2 in [11]) Suppose $\omega(x) = (x-a)^{\beta}(b-x)^{\gamma}$, $\beta, \gamma \in [-1/2, 1/2]$, the Pollard condition holds*

$$4\max\left\{\frac{\beta+1}{2\beta+3}, \frac{\gamma+1}{2\gamma+3}\right\} < p < 4\min\left\{\frac{\beta+1}{2\beta+1}, \frac{\gamma+1}{2\gamma+1}\right\},$$

the right side of the Abel Equation (2) is such that

$$\left\| D_{b-}^{-\alpha} S_k f \right\|_{L_p(I,\omega)} \leq C,\ k \in \mathbb{N}_0,\ \left| \sum_{n=0}^{\infty} f_n A_{mn}^{\alpha,\gamma,\beta} \right| \leq C(m+1)^{-\lambda},\ m \in \mathbb{N}_0,$$

$$\lambda \in [0, \infty); \tag{3}$$

then there exists a unique solution of the Abel Equation (2) in $L_p(I,\omega)$, the solution belongs to $L_q(I,\omega)$, where: $q = p$, if $0 \leq \lambda \leq 1/2$; $q = \max\{p, t\}$, $t < (2s-1)/(s-\lambda)$, if $1/2 < \lambda < s$ $(s = 3/2 + \max\{\beta, \gamma\})$; q is arbitrary large, if $\lambda \geq s$. Moreover if $\lambda > 1/2$, then the solution is represented by the convergent in $L_q(I,\omega)$ series

$$\psi(x) = \sum_{m=0}^{\infty} p_m(x)(-1)^m \sum_{n=0}^{\infty} f_n A_{mn}^{\alpha,\gamma,\beta}. \tag{4}$$

We also need an adopted version (see [11]) of the Zigmund-Marczinkevich theorem (see Reference [26]), which establishes the following.

Theorem 2. *If $q \geq 2$ and we have*

$$\Omega_q(c) = \left(\sum_{n=0}^{\infty} |c_n|^q n^{(\max\{\beta,\gamma\}+3/2)(q-2)} \right)^{1/q} < \infty,\ \max\{\beta, \gamma\} \geq -1/2, \tag{5}$$

then the series

$$\sum_{n=0}^{\infty} c_n p_n^{(\beta,\gamma)}(x)$$

converges in $L_q(I,\omega)$, $\omega(x) = (x-a)^{\beta}(b-x)^{\gamma}$ to some function $f \in L_q(I,\omega)$, $f_n = c_n$ and $\|f\|_{L_q(I,\omega)} \leq C\Omega_q(c)$.

3. Main Results

From now on, contrary to Reference [11], we consider the right-hand side case, assuming that $\alpha \in (-1, 0)$, but the reasonings corresponding to the right-hand side case are absolutely analogous.

Lemma 1. *Suppose $k < m$, $m \in \mathbb{N}$,*

$$I_{mk} := \delta'_m \frac{\Gamma(\beta+m+1) \prod_{i=1}^{m-k} (m-k-\alpha-i)}{\Gamma(\alpha+\beta+k+\gamma+m+2)};$$

then the following estimates hold

$$I_{mk} \leq C m^{-2\alpha-\gamma-3/2},\ I_{mk} \leq C e^{2k} m^{2\xi-2\alpha-\gamma-5/2-2k},\ k = 0, 1, \ldots, m-1,$$

where $\xi = 0.577215\ldots$ is the Mascheroni constant.

Proof. Consider

$$I_{mk} = \sqrt{\frac{(\beta+\gamma+2m+1)\Gamma(\beta+\gamma+m+1)\Gamma(\beta+m+1)}{m!\Gamma(\gamma+m+1)}} \times$$

$$\times \frac{\Gamma(m-k-\alpha)}{\Gamma(-\alpha)\Gamma(\alpha+\beta+k+\gamma+m+2)} <$$

$$< \sqrt{\frac{(\beta+\gamma+2m+1)\Gamma(\beta+\gamma+m+1)\Gamma(\beta+m+1)}{m!\Gamma(\gamma+m+1)}} \times$$

$$\times \frac{\Gamma(m-\alpha)}{\Gamma(-\alpha)\Gamma(\alpha+\beta+\gamma+m+2)}.$$

Now we should take into account the following relation (1.66) [10], p. 17

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} = z^{a-b} + \sum_{k=0}^{N} \frac{c_k}{z_k} + z^{a-b} O(z^{-N-1}),$$

$$a, b, z \in \mathbb{C},\ N \in \mathbb{N},\ c_0 = 1,\ |\arg(z+a)| < \pi,\ |z| \to \infty. \tag{6}$$

Having applied formula (3), we obtain

$$\frac{\Gamma(\beta+\gamma+m+1)}{\Gamma(\beta+m+1)} \sim m^{\gamma}. \tag{7}$$

In an analogous way, we have

$$\frac{\Gamma(m-\alpha)}{\Gamma(\alpha+\beta+\gamma+m+2)} \sim m^{-2\alpha-\beta-\gamma-2};$$

$$\sqrt{\frac{(\beta+\gamma+2m+1)\Gamma(\beta+\gamma+m+1)\Gamma(\beta+m+1)}{m!\Gamma(\gamma+m+1)}} \sim m^{\beta+1/2}. \tag{8}$$

Combining these two relations, we obtain the first estimate this theorem has claimed. However, this estimate can be improved for sufficiently large values m and k. To manage such a result, we should take into account the following relation (see Reference [27])

$$\frac{x^{x-\xi}}{e^{x-1}} < \Gamma(x) < \frac{x^{x-1/2}}{e^{x-1}},\ x > 1,\ \xi = 0.577215\ldots.$$

Having taken into account this formula, we can estimate

$$\frac{\Gamma(x+\delta)}{\Gamma(x)} < e^{-\delta} \frac{(x+\delta)^{x+\delta-1/2}}{x^{x-\xi}} =$$

$$= e^{-\delta} \frac{(x+\delta)^{x+\delta-1/2}}{x^{x+\delta-1/2}} x^{\xi+\delta-1/2} = e^{-\delta} \left(1+\frac{\delta}{x}\right)^{x+\delta-1/2} x^{\xi+\delta-1/2} =$$

$$= e^{-\delta} \left(1+\frac{\delta}{x}\right)^{\frac{x}{\delta} \cdot \frac{x+\delta-1/2}{x} \cdot \delta} x^{\xi+\delta-1/2} \sim x^{\xi+\delta-1/2},\ x \to \infty.$$

In an analogous way, it is not hard to prove the following estimate

$$\frac{\Gamma(x+\delta)}{\Gamma(x)} > e^{-\delta} \frac{(x+\delta)^{x+\delta-\xi}}{x^{x-1/2}} = e^{-\delta} \frac{(x+\delta)^{x+\delta-\xi}}{x^{x+\delta-\xi}} x^{1/2+\delta-\xi} =$$

$$= e^{-\delta}\left(1+\frac{\delta}{x}\right)^{x+\delta-\xi} x^{1/2+\delta-\xi} =$$

$$= e^{-\delta}\left(1+\frac{\delta}{x}\right)^{\frac{x}{\delta}\cdot\frac{x+\delta-\xi}{x}\cdot\delta} x^{1/2+\delta-\xi} \sim x^{1/2+\delta-\xi},\ x\to\infty.$$

Using these formulas we have

$$J := \frac{\Gamma(-\alpha)\prod_{i=1}^{m-k}(m-k-\alpha-i)}{\Gamma(\alpha+\beta+k+\gamma+m+2)} = \frac{\Gamma(m-k-\alpha)}{\Gamma(\alpha+\beta+k+\gamma+m+2)} <$$

$$< \frac{e^{-\delta_1}\left(1+\frac{\delta_1}{m}\right)^{m+\delta_1-1/2} m^{\xi+\delta_1-1/2}}{e^{-\delta_2}\left(1+\frac{\delta_2}{m}\right)^{m+\delta_2-\xi} m^{1/2+\delta_2-\xi}} =$$

$$= e^{-\delta_1+\delta_2}\left(1+\frac{\delta_1}{m}\right)^{m+\delta_1-1/2}\left(1+\frac{\delta_2}{m}\right)^{-m-\delta_2+\xi} m^{2\xi-1+\delta_1-\delta_2},$$

where $\delta_1=-k-\alpha,\ \delta_2=k+\alpha+\beta+\gamma+2$. Note that for concrete δ_1,δ_2 we have the following tending

$$\left(1+\frac{\delta_1}{m}\right)^{m+\delta_1-1/2} \to e^{\delta_1},\ \left(1+\frac{\delta_2}{m}\right)^{-m-\delta_2+\xi} \to e^{-\delta_2},\ m\to\infty.$$

Having taken into account this reasoning and these lemma conditions, we obtain

$$J < Ce^{-\delta_1+\delta_2}m^{2\xi-1+\delta_1-\delta_2} = Ce^{2k+2\alpha+\beta+\gamma+2}m^{2\xi-1-(2k+2\alpha+\beta+\gamma+2)},$$

using formula (8), we have

$$\delta'_m\Gamma(\beta+m+1) \le Cm^{\beta+1/2}.$$

Combining these results, we obtain

$$I_{mk} < Ce^{2k+2\alpha+\beta+\gamma+2}m^{2\xi-1-(2k+2\alpha+\gamma+3/2)},\ k=0,1,\ldots,m-1.$$

Thus, the claimed result has been proved. □

Lemma 2. *Suppose*

$$d_k(\eta) := \eta^k\sum_{i=0}^{k}\frac{2^i}{i!(k-i)!\Gamma(\gamma+i+1)},\ \eta\in\mathbb{N};$$

then

$$\forall\eta,\ \exists N : d_{k+1}(\eta) < d_k(\eta),\ k>N.$$

Proof. Assume that $k>\eta$ and consider the following relation

$$\eta^{k+1}\sum_{i=0}^{k+1}\frac{2^i}{i!(k+1-i)!\Gamma(\gamma+i+1)} - \eta^k\sum_{i=0}^{k}\frac{2^i}{i!(k-i)!\Gamma(\gamma+i+1)} =$$

$$= \eta^k\sum_{i=0}^{k}\frac{2^i(\eta-1+i-k)}{i!(k+1-i)!\Gamma(\gamma+i+1)} + \frac{\eta^{k+1}2^{k+1}}{(k+1)!\Gamma(\gamma+k+2)} =$$

$$= \eta^k\left\{\sum_{i=0}^{k-\eta}\frac{2^i(\eta-1+i-k)}{i!(k+1-i)!\Gamma(\gamma+i+1)} + \sum_{i=k-\eta+1}^{k}\frac{2^i(\eta-1+i-k)}{i!(k+1-i)!\Gamma(\gamma+i+1)}\right\} +$$

$$+\eta^k \frac{\eta 2^{k+1}}{(k+1)!\Gamma(\gamma+k+2)} =$$

$$= \eta^k \sum_{i=\eta}^{k-\eta} \frac{2^i(\eta-1+i-k)}{i!(k+1-i)!\Gamma(\gamma+i+1)} + \eta^k \sum_{i=k-\eta+1}^{k} \frac{2^i(\eta-1+i-k)}{i!(k+1-i)!\Gamma(\gamma+i+1)} +$$

$$+\eta^k \sum_{i=0}^{\eta-1} \frac{2^i(\eta-1+i-k)}{i!(k+1-i)!\Gamma(\gamma+i+1)} + \eta^k \frac{\eta 2^{k+1}}{(k+1)!\Gamma(\gamma+k+2)} =$$

$$= \eta^k \sum_{i=\eta}^{k-\eta} \frac{2^i(\eta-1+i-k)}{i!(k+1-i)!\Gamma(\gamma+i+1)} +$$

$$+\eta^k \left\{ \sum_{i=k-\eta+2}^{k} \frac{2^i(\eta-1+i-k)}{i!(k+1-i)!\Gamma(\gamma+i+1)} + \sum_{i=1}^{\eta-1} \frac{2^i(\eta-1+i-k)}{i!(k+1-i)!\Gamma(\gamma+i+1)} \right\} +$$

$$+\eta^k \left\{ \frac{\eta 2^{k+1}}{(k+1)!\Gamma(\gamma+k+2)} + \frac{(\eta-1-k)}{(k+1)!\Gamma(\gamma+1)} \right\} = S_1 + S_2 + S_3.$$

Note that S_1, S_3 are negative for a sufficiently large value k. Consider separately the expression

$$S_2 = \sum_{i=k-\eta+2}^{k} \frac{2^i(\eta-1+i-k)}{i!(k+1-i)!\Gamma(\gamma+i+1)} + \sum_{i=1}^{\eta-1} \frac{2^i(\eta-1+i-k)}{i!(k+1-i)!\Gamma(\gamma+i+1)} =$$

$$= \sum_{i=k-\eta+2}^{k-1} \frac{2^i(\eta-1+i-k)}{i!(k+1-i)!\Gamma(\gamma+i+1)} + \sum_{i=2}^{\eta-1} \frac{2^i(\eta-1+i-k)}{i!(k+1-i)!\Gamma(\gamma+i+1)} +$$

$$+ \frac{2^k(\eta-1)}{k!\Gamma(\gamma+k+1)} + \frac{2(\eta-k)}{k!\Gamma(\gamma+2)} = \sum_{i=k-\eta+2}^{k-2} \frac{2^i(\eta-1+i-k)}{i!(k+1-i)!\Gamma(\gamma+i+1)} +$$

$$+ \sum_{i=3}^{\eta-1} \frac{2^i(\eta-1+i-k)}{i!(k+1-i)!\Gamma(\gamma+i+1)} +$$

$$+ \frac{2^k(\eta-1)}{k!\Gamma(\gamma+k+1)} + \frac{2(\eta-k)}{k!\Gamma(\gamma+2)} + \frac{2^{k-1}(\eta-2)}{(k-1)!2!\Gamma(\gamma+k)} + \frac{2^2(\eta+1-k)}{2!(k-1)!\Gamma(\gamma+3)} =$$

$$= \sum_{i=0}^{\eta-2} \left\{ \frac{2^{k-i}(\eta-1-i)}{(k-i)!(i+1)!\Gamma(\gamma+k+1-i)} + \frac{2^{i+1}(\eta-k+i)}{(i+1)!(k-i)!\Gamma(\gamma+2+i)} \right\} \leq$$

$$\leq \sum_{i=0}^{\eta-2} \left\{ \frac{2^{k-i}(\eta-1-i)}{(k-i)!(i+1)!\Gamma(\gamma+3)2^{k-2-i}} + \frac{2(2\eta-k-2)}{(i+1)!(k-i)!\Gamma(\gamma+\eta)} \right\} <$$

$$< \sum_{i=0}^{\eta-2} \left\{ \frac{4(\eta-1)}{(k-i)!(i+1)!\Gamma(\gamma+3)} + \frac{2(2\eta-k-2)}{(i+1)!(k-i)!\Gamma(\gamma+\eta)} \right\}.$$

Now, it is clear that $S_2 < 0$ for a sufficiently large value k. Combining this fact with the previously established fact regarding S_1, S_3, we obtain the desired result. □

Lemma 3. *Assume that the following series is absolutely convergent*

$$\sum_{n=0}^{\infty} f_n c_n, \tag{9}$$

where $f_n \in \mathbb{R}$,

$$c_n = \frac{\delta'_n n!\Gamma(n+\beta+1)\Gamma(n+\gamma+1)}{4^n},$$

then

$$\sum_{n=0}^{\infty} |f_n A_{mn}| \leq C(m+1)^{-2\alpha-\gamma-3/2},\ m \in \mathbb{N}_0.$$

Proof. It is easy to see that

$$A_{mn}^{\alpha,\gamma,\beta} := \delta'_m\delta'_n \sum_{k=0}^{n}(-1)^k \frac{\tilde{C}_n^k(\gamma,\beta)B(\alpha+\gamma+k+1,\beta+m+1)}{\Gamma(k+\alpha-m+1)} =$$

$$= \delta'_m\delta'_n \sum_{k=0}^{\min\{m-1,n\}}(-1)^k \frac{\tilde{C}_n^k(\beta,\gamma)B(\alpha+\gamma+k+1,\beta+m+1)}{\Gamma(k+\alpha-m+1)} +$$

$$+\delta'_m\delta'_n \sum_{k=m}^{n}(-1)^k \frac{\tilde{C}_n^k(\beta,\gamma)B(\alpha+\gamma+k+1,\beta+m+1)}{\Gamma(k+\alpha-m+1)} = I_1 + I_2,$$

$$m, n \in \mathbb{N}_0,$$

here and further, using notations we mean that $\sum_{k=l}^{n} a_k = 0$, if $n < l$. Consider I_1, we need the following formula (see (1.56) [10])

$$\Gamma(z) = \frac{\Gamma(z+n)}{z(z+1)\dots(z+n-1)},\ \mathrm{Re} z > -n,\ n = 1,2,\dots,\ z \neq 0,-1,-2,\dots,.$$

Denote $\alpha+1-[m-k] =: z$, then for the case $m > k$, we have $z > -[m-k]$. Hence we obtain

$$I_1 = \delta'_m\delta'_n \sum_{k=0}^{\min\{m-1,n\}}(-1)^k \frac{\tilde{C}_n^k(\gamma,\beta)B(\alpha+\gamma+k+1,\beta+m+1)}{\Gamma(\alpha+1-[m-k])}$$

$$= (-1)^m\delta'_m\delta'_n \sum_{k=0}^{\min\{m-1,n\}} \frac{\tilde{C}_n^k(\gamma,\beta)(m-k-\alpha-1)(m-k-\alpha-2)\dots(-\alpha)}{B^{-1}(\alpha+\gamma+k+1,\beta+m+1)\Gamma(1+\alpha)}$$

$$= (-1)^m\delta'_m\delta'_n \sum_{k=0}^{\min\{m-1,n\}} \frac{\tilde{C}_n^k(\gamma,\beta)\prod_{i=1}^{m-k}(m-k-\alpha-i)}{B^{-1}(\alpha+\gamma+k+1,\beta+m+1)\Gamma(1+\alpha)}$$

$$= (-1)^m\delta'_n \sum_{k=0}^{\min\{m-1,n\}} \frac{\tilde{C}_n^k(\gamma,\beta)\Gamma(\alpha+\gamma+k+1)}{\Gamma(\alpha+1)} I_{mk},\ m \in \mathbb{N}. \tag{10}$$

Besides, having noticed that $\Gamma(\beta+k) \geq \Gamma(\beta+3)2^{k-3}, k = 3,4,\dots,$ we have the following reasonings:

$$\tilde{C}_n^k(\gamma,\beta) = \sum_{i=0}^{k} C_n^i \binom{n+\beta}{i}\binom{n+\gamma}{n-i} C_k^i \binom{n-i}{k-i} i! =$$

$$= \Gamma(n+\beta+1)\Gamma(n+\gamma+1)\sum_{i=0}^{k} \frac{C_n^i C_k^i \Gamma(n-i+1)i!}{\Gamma(n+\beta-i+1)\Gamma(\gamma+i+1)\Gamma(n-k+1)} =$$

$$= \frac{n!k!}{\Gamma(n-k+1)}\sum_{i=0}^{k} \frac{\Gamma(n+\beta+1)\Gamma(n+\gamma+1)}{i!(k-i)!\Gamma(n+\beta-i+1)\Gamma(\gamma+i+1)} \leq$$

$$\leq \frac{n!k!\Gamma(n+\beta+1)\Gamma(n+\gamma+1)}{\Gamma(\beta+3)2^{n-k}} \sum_{i=0}^{k} \frac{2^3}{i!(k-i)!2^{n-i}\Gamma(\gamma+i+1)} =$$

$$= \frac{n!k!\Gamma(n+\beta+1)\Gamma(n+\gamma+1)}{\Gamma(\beta+3)2^{2n-3}} \sum_{i=0}^{k} \frac{2^{i+k}}{i!(k-i)!\Gamma(\gamma+i+1)}.$$

Using Lemma 2, we conclude that there exists such a constant $C > 0$ so that

$$\sum_{i=0}^{k} \frac{2^{i+k}}{i!(k-i)!\Gamma(\gamma+i+1)} \leq Ce^{-2k}.$$

Hence

$$\tilde{C}_n^k(\gamma,\beta) \leq C\frac{n!k!\Gamma(n+\beta+1)\Gamma(n+\gamma+1)}{\Gamma(\beta+3)4^n e^{2k}}.$$

Therefore, applying Lemma 1 to (3), we have

$$|I_1| \leq C_1\delta_n' \frac{n!\Gamma(n+\beta+1)\Gamma(n+\gamma+1)}{4^n} \times$$

$$\times \sum_{k=1}^{\min\{m-1,n\}} \frac{k!\Gamma(\alpha+\beta+k+1)}{m^{2k-1}} m^{2\xi-2-2\alpha-\gamma-3/2} +$$

$$+C_2\delta_n' \frac{n!\Gamma(n+\beta+1)\Gamma(n+\gamma+1)}{4^n}\Gamma(\alpha+\beta+1)m^{-2\alpha-\gamma-3/2}.$$

Consider

$$\Xi_{m-1} := \sum_{k=1}^{m-1} \frac{k!\Gamma(\alpha+\beta+k+1)}{m^{2k-1}},$$

we have

$$\Xi_m - \Xi_{m-1} = \sum_{k=1}^{m-1} \frac{k!\Gamma(\alpha+\beta+k+1)[m^{2k-1}-(m+1)^{2k-1}]}{m^{2k-1}(m+1)^{2k-1}} +$$

$$+\frac{m!\Gamma(\alpha+\beta+m+1)}{(m+1)^{2m-1}}.$$

Having applied the asymptotic Stirling formula (1.63) [10], p. 16, we obtain

$$\frac{m!\Gamma(\alpha+\beta+m+1)}{(m+1)^{2m-1}} \to 0,\ m \to \infty.$$

Hence $\Xi_m - \Xi_{m-1} < 0$ for sufficiently large m. It implies that $\Xi_m < C$, $m = 1, 2, \ldots$. Therefore, we have

$$|I_1| \leq Cm^{-2\alpha-\gamma-3/2}\delta_n' \frac{n!\Gamma(n+\beta+1)\Gamma(n+\gamma+1)}{4^n} \times \left\{ \sum_{k=1}^{\min\{m-1,n\}} \frac{k!\Gamma(\alpha+\beta+k+1)}{m^{2k-1}} m^{2\xi-2} + 1 \right\} <$$

$$< C\delta_n' \frac{n!\Gamma(n+\beta+1)\Gamma(n+\gamma+1)}{4^n} m^{-2\alpha-\gamma-3/2},\ m \in \mathbb{N},\ n \in \mathbb{N}_0.$$

Consider I_2, we have the following reasonings

$$|I_2| \leq \delta_m'\delta_n' \sum_{k=m}^{n} \frac{\tilde{C}_n^k(\beta,\gamma)B(\alpha+\gamma+k+1,\beta+m+1)}{\Gamma(k+\alpha-m+1)} \leq$$

$$\leq \frac{\delta_m'\delta_n' n!\Gamma(n+\beta+1)\Gamma(n+\gamma+1)}{\Gamma(\beta+3)2^{2n-3}} \times$$

$$\times \sum_{k=m}^{n} \frac{k! d_k B(\alpha+\gamma+k+1, \beta+m+1)}{2^k \Gamma(k+\alpha-m+1)} =$$

$$= \frac{\delta'_m \delta'_n n! \Gamma(n+\beta+1)\Gamma(n+\gamma+1)}{\Gamma(\beta+3) 2^{2n-3}} \times$$

$$\times \sum_{k=m}^{n} \frac{d_k k! \Gamma(\alpha+\gamma+k+1)\Gamma(m+\beta+1)}{4^k \Gamma(\alpha+\gamma+k+\beta+m+2)\Gamma(k+\alpha-m+1)},$$

where

$$d_k := 8^k \sum_{i=0}^{k} \frac{2^i}{i!(k-i)!\Gamma(\gamma+i+1)},$$

and we know, as it was proved, that $d_{k+1} < d_k$ for sufficiently large k. Consider

$$J_{mn} = \sum_{k=m}^{n} \frac{d_k k! \Gamma(\alpha+\gamma+k+1)}{4^k \Gamma(\alpha+\gamma+k+\beta+m+2)\Gamma(k+\alpha-m+1)}, \ m, k \in \mathbb{N}_0.$$

Note that

$$\frac{1}{\Gamma(\alpha+\gamma+k+\beta+m+2)\Gamma(k+\alpha-m+1)} >$$

$$> \frac{1}{\Gamma(\alpha+\gamma+k+\beta+m+3)\Gamma(k+\alpha-m)}, \ k \geq m.$$

On the other hand, due to relation (8), it is clear that

$$\frac{k! \Gamma(\alpha+\gamma+k+1)}{\Gamma(\alpha+\gamma+k+\beta+2)\Gamma(k+\alpha+1)} \leq C(k+1)^{-\beta-\alpha-1}.$$

Combining these estimates, we obtain

$$J_{mn} \leq C \sum_{k=m}^{n} \frac{1}{4^k (k+1)^{\beta+\alpha+1}} \leq C \sum_{k=m}^{n} \frac{1}{2^k} \leq \frac{C}{2^{m-1}}.$$

Hence, applying relation (8), we obtain

$$|I_2| \leq C \frac{\delta'_m \delta'_n n! \Gamma(n+\beta+1)\Gamma(n+\gamma+1)\Gamma(m+\beta+1)}{2^m \Gamma(\beta+3) 4^n} \leq$$

$$\leq C\delta'_n \frac{n! \Gamma(n+\beta+1)\Gamma(n+\gamma+1)}{4^n} \cdot \frac{(m+1)^{1/2+\beta}}{2^m} \leq$$

$$\leq C\delta'_n \frac{n! \Gamma(n+\beta+1)\Gamma(n+\gamma+1)}{4^n} (m+1)^{-2\alpha-\gamma-3/2}, \ m \in \mathbb{N}_0.$$

Finally, combining the obtained results regarding to I_1, I_2, we have

$$|A_{mn}^{\alpha,\gamma,\beta}| \leq C\delta'_n \frac{n! \Gamma(n+\beta+1)\Gamma(n+\gamma+1)}{4^n} m^{-2\alpha-\gamma-3/2} = C\delta'_n c_n m^{-2\alpha-\gamma-3/2}, \ m \in \mathbb{N}.$$

Having noticed that due to the absolute convergence of the series (9) we can extract a multiplier in each term of the series, we get

$$\sum_{n=0}^{\infty} \left| f_n A_{mn}^{\alpha,\gamma,\beta} \right| \leq C(m+1)^{-2\alpha-\gamma-3/2} \sum_{n=0}^{\infty} |f_n| c_n \leq C(m+1)^{-2\alpha-\gamma-3/2}, \ m \in \mathbb{N}_0.$$

The proof is complete. □

Theorem 3. *Assume that the Jacobi coefficients f_n of the right-hand side of the Abel Equation (2) such that corresponding series (9) is absolutely convergent, the condition $2\alpha+\gamma+1>0$ holds, then there exists a unique solution of the Abel Equation (2), the solution is represented by series (4), and in accordance with the notations of Theorem 2: $\lambda=2\alpha+\gamma+3/2$.*

Proof. Due to Lemma 3, we have

$$\left|\sum_{n=0}^{\infty} f_n A_{mn}^{\alpha,\gamma,\beta}\right| \leq C[m+1]^{-2\alpha-\gamma-3/2},\ m\in\mathbb{N}_0.$$

and it is clear that $\lambda>1/2$, since $2\alpha+\gamma+1>0$. Thus, to fulfill the conditions of Theorem 1, we must show that

$$\|D_{b-}^{-\alpha}S_k f\|_{L_p}\leq C,\ k=0,1,\dots,$$

where p is such an index value that the Pollard condition holds. Consider

$$\left(D_{b-}^{-\alpha}S_k f, p_m\right)=\sum_{n=0}^{k} f_n\left(D_{b-}^{-\alpha}p_n, p_m\right)=$$

$$=(-1)^m\sum_{n=0}^{k} f_n A_{mn}^{\alpha,\gamma,\beta}=:c_{mk},\ m\in\mathbb{N}_0.$$

Let us impose the conditions on α,γ under which the following estimate holds

$$\|D_{b-}^{-\alpha}S_k f\|_{L_q}\leq C\left(\sum_{m=0}^{\infty}|c_{mk}|^q m^{(\max\{\beta,\gamma\}+3/2)(q-2)}\right)^{1/q}<\infty,\ q\geq 2. \tag{11}$$

Using Lemma 1 and calculating powers, we have the following sufficient conditions of series (11) convergence in terms of Theorem 1 : $q<(2s-1)/(s-\lambda)$. Regarding to the fulfilment of the Pollard conditions, we should notice that

$$4\max\left\{\frac{\beta+1}{2\beta+3},\frac{\gamma+1}{2\gamma+3}\right\}=\frac{2s-1}{s}$$

(this is the consequence of the fact that the function $z(x):=(x+1)/(2x+3)$ is an increasing function), hence we can choose p so that

$$4\max\left\{\frac{\beta+1}{2\beta+3},\frac{\gamma+1}{2\gamma+3}\right\}<p<q.$$

Combining these facts and using the Hölder inequality, we come to the conclusion

$$\|D_{b-}^{-\alpha}S_k f\|_{L_p}\leq C\left(\sum_{m=0}^{\infty}|c_{mk}|^q m^{(\max\{\beta,\gamma\}+3/2)(q-2)}\right)^{1/q}<C,\ k=0,1,\dots,$$

where index p satisfies to the Pollard conditions. Hence the conditions of Theorem 1 are fulfilled. This gives us the desired result. □

Remark 1. *Observe that Theorem 3 claims that there exists a relationship between the values of the parameters and smoothness of the solution φ of Equation (2), that is, $\varphi\in L_q(I,\omega)$, where q is an arbitrary large number satisfying $q<(2s-1)/(s-\lambda)$, $s:=\max\{\beta,\gamma\}+3/2$, $\lambda:=2\alpha+\gamma+3/2$, $\lambda>1/2$, $\alpha\in(-1,0)$. Thus the given above relation establishes the independence between the parameter β and the solution smoothness, if $\beta\leq\gamma$. In this case, we have that q is an arbitrary large value satisfying $q<-(\gamma+1)/\alpha$.*

4. Conclusions

The main aim of the paper is to improve and clarify the results [11]. Generally, the basic result is the successful achievement of conditions imposed on the right-hand side of the Abel equation, under which Theorem 2 [11] is applicable. It has been done by using structural properties of the Riemann-Liouville operator, in contrast to Reference [11], where a novel method has been claimed for solving the problem in general. We recognize that a principled attempt was made to achieve the goal, which does not exclude the significance of a quality result, which creates the prerequisites for improvement of any kind and for achieving conditions imposed on the right-hand side that could be as weak as possible. More precisely, a new type of condition imposed on the right-hand side has been obtained; even if we considered the results in the framework of the classical Abel equation, they would be undoubtedly relevant and we see that some problems take on a new aspect under such a point of view. For instance, the problem $p = 1/\alpha$ would be understood by new tools and methods. A characteristic feature of the paper is the absence of the fractional nature of formulated conditions, contrary to the approach corresponding to the classical Abel equation. This advantage (as the above-mentioned feature can be treated) has been achieved due to special properties of the Legendre polynomials (if we consider a non-weighted case). At the same time, the results corresponding to a weighted case are completely novel—the relationship between the values of the parameters and order α—by virtue of which we can provide a description of the solution smoothness—has been established, the conditions providing the existence and uniqueness of the solution, formulated in terms of the Jacobi series coefficients, have been obtained. The quality result—the independence between one of the parameters and the solution smoothness has been proved.

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