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# Generalization of the Regularization Method to Singularly Perturbed Integro-Differential Systems of Equations with Rapidly Oscillating Inhomogeneity

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**Abstract:** In this paper, we consider systems of singularly perturbed integro-differential equations with a rapidly oscillating right-hand side, including an integral operator with a slowly varying kernel. Differential equations of this type and integro-differential equations with slowly varying inhomogeneity and with a rapidly oscillating coefficient at an unknown function are studied. The main goal of this work is to generalize the Lomov’s regularization method and to reveal the influence of the rapidly oscillating right-hand side on the asymptotics of the solution to the original problem.

**Keywords:** singular perturbation; integro-differential equation; regularization; rapidly oscillating inhomogeneity; asymptotic convergence; resonant exhibitors

**MSC:** 35R09; 45K05



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## 1. Introduction

When studying various applied problems related to the properties of media with a period structure, it is necessary to study differential equations with rapidly oscillating irregularities. Equations of this type are often found, for example, in electrical tric systems under the influence of high-frequency external forces. The presence of such forces creates serious problems for the numerical integration of the corresponding differential equations. Therefore, asymptotic methods are usually applied to such equations, the most famous of which are the Feshchenko–Shkil–Nikolenko splitting method [1,2] and the Lomov’s regularization method [3–14]. The splitting method is especially effective when applied to equations with a rapidly oscillating inhomogeneity, and in the case of an inhomogeneity containing both rapidly and slow components, the Lomov’s regularization method turned out to be the most effective. However, both of these methods were developed mainly for singularly perturbed equations that do not contain an integral operator. The transition from differential equations to integro-differential equations requires a significant restructuring of the algorithm of the regularization method. The integral term generates new types of singularities in solutions that differ from the previously known ones, which complicates the development of the algorithm for the regularization method. As far as we know, the splitting method has not been applied to integro-differential equations.

In the present study, the Lomov’s regularization method is generalized to previously not studied classes of problems such as

$$L_\varepsilon y(t, \varepsilon) \equiv \varepsilon \frac{dy}{dt} - A(t)y - \int_{t_0}^t K(t, s)y(s, \varepsilon)ds = h_1(t) + h_2(t)e^{\frac{i\beta(t)}{\varepsilon}}, \quad y(t_0, \varepsilon) = y^0, \quad t \in [t_0, T] \quad (1)$$

where  $y = \{y_1(t, \varepsilon), \dots, y_n(t, \varepsilon)\}$  is an unknown vector function,  $A(t)$  is  $(n \times n)$ -matrix,  $h_j(t) = (h_{1j}(t), \dots, h_{nj}(t))$  are known vector functions,  $\beta'(t) > 0$  is the frequency of a rapidly

oscillating inhomogeneity,  $\varepsilon > 0$  is a small parameter. Problem (1) will be considered under the following conditions:

- (1)  $A(t) \in C^\infty([t_0, T], \mathbb{C}^{n^2}), h_1(t), h_2(t) \in C^\infty([t_0, T], \mathbb{C}^n), K(t, s) \in C^\infty(\{t_0 \leq s \leq t \leq T\}, \mathbb{C}^{n^2});$
- (2) The spectrum  $\{\lambda_1(t), \dots, \lambda_n(t)\}$  of the matrix  $A(t)$  and the frequency  $\beta'(t)$  satisfy for all  $t \in [t_0, T]$  the requirements:
  - (a)  $\beta'(t) > 0, \lambda_i(t) \neq \lambda_j(t), i \neq j, \lambda_j(t) \neq 0, \lambda_j(t) \neq i\beta'(t), i, j = \overline{1, n};$
  - (b)  $Re\lambda_j(t) \leq 0, j = \overline{1, n}.$

Thus, problem (1) is considered (in the terminology of [3]) in the nonresonant case  $\lambda_j(t) \neq i\beta'(t), i, j = \overline{1, n}, \forall t \in [t_0, T]$ . The resonant case  $\lambda_j(t) \equiv i\beta'(t), i, j = \overline{1, n}, \forall t \in [t_0, T]$  it is supposed to be considered in our subsequent work.

Note that in [15], the scalar case of a singularly perturbed integro-differential equation with a rapidly oscillating factor for an unknown function was considered. In this paper, in contrast to [15], we investigate a more general case of a system of integro-differential equations, and the rapidly oscillating factor is a coefficient on the right-hand side, for one of its inhomogeneities.

### 2. Regularization of the Problem (1)

We denote  $\lambda_{n+1}(t) \equiv i\beta'(t)$  and introduce the regularizing variables

$$\tau_j = \frac{1}{\varepsilon} \int_{t_0}^t \lambda_j(\theta) d\theta \equiv \frac{\psi_j(t)}{\varepsilon}, j = \overline{1, n+1}$$

and instead of problem (1) consider the problem

$$\begin{aligned} L_\varepsilon \tilde{y}(t, \tau, \varepsilon) &\equiv \varepsilon \frac{\partial \tilde{y}}{\partial t} + \sum_{j=1}^{n+1} \lambda_j(t) \frac{\partial \tilde{y}}{\partial \tau_j} - A(t) \tilde{y} - \\ &- \int_{t_0}^t K(t, s) \tilde{y} \left( s, \frac{\psi(s)}{\varepsilon}, \varepsilon \right) ds = h_1(t) + h_2(t) e^{\tau_{n+1}} \sigma, \end{aligned} \tag{2}$$

$$\tilde{y}(t, \tau, \varepsilon)|_{t=t_0, \tau=0} = y^0, t \in [t_0, T]$$

for the function  $\tilde{y} = \tilde{y}(t, \tau, \varepsilon)$ , which is denoted  $\tau = (\tau_1, \dots, \tau_{n+1}), \psi = (\psi_1, \dots, \psi_{n+1}), \sigma = e^{i\beta(t_0)}$ .

It is clear that, if  $\tilde{y} = \tilde{y}(t, \tau, \varepsilon)$  is the solution of the problem (2), then the vector function  $y = \tilde{y} \left( t, \frac{\psi(t)}{\varepsilon}, \varepsilon \right)$  is an exact solution of the problem (1); therefore, the problem (2) is extended with respect to the problem (1). However, it cannot be considered completely regularized, since the integral term

$$J\tilde{y} \equiv J \left( \tilde{y}(t, \tau, \varepsilon) |_{t=s, \tau=\psi(s)/\varepsilon} \right) = \int_{t_0}^t K(t, s) \tilde{y} \left( s, \frac{\psi(s)}{\varepsilon}, \varepsilon \right) ds$$

has not been regularized in it. To regularize it, we introduce a class  $M_\varepsilon$  that is asymptotically invariant with respect to the operator  $J\tilde{y}$  (see [3]; p. 62).

Let us first consider the space  $U$  of vector functions  $y(t, \tau, \sigma)$ , represented by the sums

$$y(t, \tau, \sigma) = y_0(t, \sigma) + \sum_{j=1}^{n+1} y_j(t, \sigma) e^{\tau_j}, y_j(t, \sigma) \in C^\infty([t_0, T], \mathbb{C}^n), j = \overline{0, n+1}. \tag{3}$$

Note that in (3) elements of the space  $U$  depend on bounded in  $\varepsilon > 0$  the constant  $\sigma = \sigma(\varepsilon)$ , which does not affect the development of the algorithm described below; therefore, henceforth, in the record of the element (3) of this space, for the sake of brevity, we omit the dependence on  $\sigma$ .

Let us show that the class  $M_\varepsilon = U|_{\tau=\psi(t)/\varepsilon}$  is asymptotically invariant with respect to the operator  $J$ . The image of the operator  $J$  on the element (3) of the space  $U$  has the form:

$$Jy(t, \tau) = \int_{t_0}^t K(t, s)y_0(s)ds + \sum_{j=1}^{n+1} \int_{t_0}^t K(t, s)y_j(s)e^{\frac{1}{\varepsilon} \int_{t_0}^s \lambda_j(\theta)d\theta} ds.$$

Integrating by parts, we will have

$$\begin{aligned} J_j(t, \varepsilon) &= \int_{t_0}^t K(t, s)y_j(s)e^{\frac{1}{\varepsilon} \int_{t_0}^s \lambda_j(\theta)d\theta} ds = \\ &= \varepsilon \int_{t_0}^t \frac{K(t, s)y_j(s)}{\lambda_j(s)} de^{\frac{1}{\varepsilon} \int_{t_0}^s \lambda_j(\theta)d\theta} = \varepsilon \left[ \frac{K(t, s)y_j(s)}{\lambda_j(s)} e^{\frac{1}{\varepsilon} \int_{t_0}^s \lambda_j(\theta)d\theta} \Big|_{s=t_0}^{s=t} - \right. \\ &- \left. \int_{t_0}^t \left( \frac{\partial}{\partial s} \frac{K(t, s)y_j(s)}{\lambda_j(s)} \right) e^{\frac{1}{\varepsilon} \int_{t_0}^s \lambda_j(\theta)d\theta} ds \right] = \varepsilon \left[ \frac{K(t, t)y_j(t)}{\lambda_j(t)} e^{\frac{1}{\varepsilon} \int_{t_0}^t \lambda_j(\theta)d\theta} - \frac{K(t, t_0)y_j(t_0)}{\lambda_j(t_0)} \right] - \\ &- \varepsilon \int_{t_0}^t \left( \frac{\partial}{\partial s} \frac{K(t, s)y_j(s)}{\lambda_j(s)} \right) e^{\frac{1}{\varepsilon} \int_{t_0}^s \lambda_j(\theta)d\theta} ds, j = \overline{1, n+1}. \end{aligned}$$

Continuing this process further, we obtain the expansion

$$J_j(t, \varepsilon) = \sum_{j=1}^{n+1} \sum_{\nu=0}^{\infty} (-1)^\nu \varepsilon^{\nu+1} \left[ \left( I_j^\nu (K(t, s)y_j(s)) \right)_{s=t} e^{\frac{1}{\varepsilon} \int_{t_0}^t \lambda_j(\theta)d\theta} - \left( I_j^\nu (K(t, s)y_j(s)) \right)_{s=t_0} \right]$$

where the operators are introduced:

$$I_j^0 = \frac{1}{\lambda_j(s)}, I_j^\nu = \frac{1}{\lambda_j(s)} \frac{\partial}{\partial s} I_j^{\nu-1}, j = \overline{1, n+1}, (\nu \geq 1).$$

Hence, the image of the operator  $J$  on an element (3) of the space  $U$  can be represented as a series

$$\begin{aligned} Jy(t, \varepsilon) &= \int_{t_0}^t K(t, s)y_0(s)ds + \\ &+ \sum_{j=1}^{n+1} \sum_{\nu=0}^{\infty} (-1)^\nu \varepsilon^{\nu+1} \left[ \left( I_j^\nu (K(t, s)y_j(s)) \right)_{s=t} e^{\frac{1}{\varepsilon} \int_{t_0}^t \lambda_j(\theta)d\theta} - \left( I_j^\nu (K(t, s)y_j(s)) \right)_{s=t_0} \right]. \end{aligned}$$

It is easy to show (see, for example, [16], pp. 291–294) that this series converges asymptotically for  $\varepsilon \rightarrow +0$  (uniformly in  $t \in [t_0, T]$ ). This means that the class  $M_\varepsilon$  is asymptotically invariant (for  $\varepsilon \rightarrow +0$ ) with respect to the operator  $J$ .

Let us introduce operators  $R_\nu : U \rightarrow U$ , acting on each element  $y(t, \tau) \in U$  of the form (3) according to the law:

$$R_0y(t, \tau) = \int_{t_0}^t K(t, s)y_0(s)ds, \tag{4a}$$

$$R_1y(t, \tau) = \sum_{j=1}^{n+1} \left[ \left( I_j^0 (K(t, s)y_j(s)) \right)_{s=t} e^{\tau_j} - \left( I_j^0 (K(t, s)y_j(s)) \right)_{s=t_0} \right], \tag{4b}$$

$$R_{\nu+1}y(t, \tau) = \sum_{j=1}^{n+1} \sum_{\nu=0}^{\infty} (-1)^\nu \left[ \left( I_j^\nu (K(t, s)y_j(s)) \right)_{s=t} e^{\tau_j} - \left( I_j^\nu (K(t, s)y_j(s)) \right)_{s=t_0} \right], \nu \geq 1. \tag{4c}$$

Now let  $\tilde{y}(t, \tau, \varepsilon)$  be an arbitrary continuous for  $(t, \tau) \in [t_0, T] \times \{\tau : \operatorname{Re}\tau_j \leq 0, j = \overline{1, n+1}\}$  function, having an asymptotic expansion

$$\tilde{y}(t, \tau, \varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k y_k(t, \tau), \quad y_k(t, \tau) \in U \tag{5}$$

converging when  $\varepsilon \rightarrow +0$  (uniformly in  $(t, \tau) \in [t_0, T] \times \{\tau : \operatorname{Re}\tau_j \leq 0, j = \overline{1, n+1}\}$ ). Then the image  $J\tilde{y}(t, \tau, \varepsilon)$  of this function expands into an asymptotic series

$$J\tilde{y}(t, \tau, \varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k Jy_k(t, \tau) = \sum_{r=0}^{\infty} \varepsilon^r \sum_{s=0}^r R_{r-s} y_s(t, \tau)|_{\tau=\psi(t)/\varepsilon}.$$

This equality is the basis for introducing an extension of the operator  $J$  on series of the form (5):

$$\tilde{J}\tilde{y}(t, \tau, \varepsilon) \equiv \tilde{J}\left(\sum_{k=0}^{\infty} \varepsilon^k y_k(t, \tau)\right) \stackrel{\text{def}}{=} \sum_{r=0}^{\infty} \varepsilon^r \sum_{s=0}^r R_{r-s} y_s(t, \tau).$$

Although the operator  $\tilde{J}$  is defined formally, its usefulness is obvious, since in practice the  $N$ -th approximation of the asymptotic solution of problem (2) is usually constructed, in which only  $N$ -th partial sums of the series (5) will participate, which have not formal but true meaning. Now we can write down the problem that is completely regularized with respect to the original problem (1):

$$\begin{aligned} L_\varepsilon \tilde{y}(t, \tau, \varepsilon) &\equiv \frac{\partial \tilde{y}}{\partial t} + \sum_{j=1}^{n+1} \lambda_j(t) \frac{\partial \tilde{y}}{\partial \tau_j} - A(t) \tilde{y} - \tilde{J}\tilde{y} = h_1(t) + \\ &+ h_2(t) e^{\tau_{n+1} \sigma}, \quad \tilde{y}(t, \tau, \varepsilon)|_{t=t_0, \tau=0} = y^0, \quad t \in [t_0, T]. \end{aligned} \tag{6}$$

### 3. Iterative Problems and Their Solvability in the Space $U$

Substituting the series (5) into (6) and equating the coefficients at the same degrees of  $\varepsilon$ , we obtain the following iterative problems:

$$Ly_0(t, \tau) \equiv \sum_{j=1}^{n+1} \lambda_j(t) \frac{\partial y_0}{\partial \tau_j} - A(t)y_0 - R_0 y_0 = h_1(t) + h_2(t) e^{\tau_{n+1} \sigma}, \quad y_0(t_0, 0) = y^0; \tag{7a}$$

$$Ly_1(t, \tau) = -\frac{\partial y_0}{\partial t} + R_1 y_0, \quad y_1(t_0, 0) = 0; \tag{7b}$$

$$Ly_2(t, \tau) = -\frac{\partial y_1}{\partial t} + R_1 y_1 + R_2 y_0, \quad y_2(t_0, 0) = 0; \tag{7c}$$

...

$$Ly_k(t, \tau) = -\frac{\partial y_{k-1}}{\partial t} + R_k y_0 + \dots + R_1 y_{k-1}, \quad y_k(t_0, 0) = 0, \quad k \geq 1. \tag{7d}$$

Each of the iterative problems (7d) can be written as

$$Ly(t, \tau) \equiv \sum_{j=1}^{n+1} \lambda_j(t) \frac{\partial y}{\partial \tau_j} - A(t)y - R_0 y = H(t, \tau), \quad y(t_0, 0) = y_* \tag{8}$$

where  $H(t, \tau) = H_0(t) + \sum_{j=1}^{n+1} H_j(t) e^{\tau_j}$  is the known function of the space  $U$ , and the operator  $R_0$  has the form (see (4a)):

$$R_0 y \equiv R_0 \left( y_0(t) + \sum_{j=1}^{n+1} y_j(t) e^{\tau_j} \right) = \int_{t_0}^t K(t, s) y_0(s) ds.$$

We introduce the scalar (for each  $t \in [t_0, T]$ ) product in the space  $U$  :

$$\begin{aligned} \langle z, w \rangle &\equiv \langle z_0(t) + \sum_{j=1}^{n+1} z_j(t)e^{\tau_j}, w_0(t) + \sum_{j=1}^{n+1} w_j(t)e^{\tau_j} \rangle \stackrel{def}{=} \\ &\stackrel{def}{=} (z_0(t), w_0(t)) + \sum_{j=1}^{n+1} (z_j(t), w_j(t)) \end{aligned}$$

where  $(*, *)$  denotes the usual scalar product in a complex space  $\mathbb{C}^n$ . We prove the following statement.

**Theorem 1.** *Let conditions (1), (2a) hold and the right-hand side  $H(t, \tau) = H_0(t) + \sum_{j=1}^{n+1} H_j(t)e^{\tau_j}$  of the system (8) belongs to the space  $U$ . Then, for the solvability of system (8) in  $U$  it is necessary and sufficient that the identities*

$$\langle H(t, \tau), \chi_k(t)e^{\tau_k} \rangle \equiv 0, k = \overline{1, n}, \forall t \in [t_0, T] \tag{9}$$

are carried out (here  $\chi_k(t)$  are the eigenvectors of the conjugate matrix  $A^*(t)$ , corresponding to the eigenvalues  $\bar{\lambda}_j(t), j = \overline{1, n}$ .)

**Proof.** We will define the solution of the system (8) as an element (3) of the space  $U$ :

$$y(t, \tau) = y_0(t) + \sum_{j=1}^{n+1} y_j(t)e^{\tau_j}. \tag{10}$$

Substituting (10) into the system (8), we will have

$$\sum_{j=1}^{n+1} [\lambda_j(t)I - A(t)]y_j(t)e^{\tau_j} - A(t)y_0(t) - \int_{t_0}^t K(t, s)y_0(s)ds = H_0(t) + \sum_{j=1}^{n+1} H_j(t)e^{\tau_j}.$$

Equating here separately the free term and coefficients at the same exponents, we obtain the following equations:

$$-A(t)y_0(t) - \int_{t_0}^t K(t, s)y_0(s)ds = H_0(t), \tag{11a}$$

$$[\lambda_j(t)I - A(t)]y_j(t) = H_j(t), j = \overline{1, n+1}. \tag{11b}$$

Due to the smoothness of the kernel  $-A^{-1}(t)K(t, s)$  and heterogeneity  $-A^{-1}(t)H_0(t)$ , the integral Equation (11a) has a unique solution  $y_0(t) \in C^\infty([t_0, T], \mathbb{C}^n)$ . As  $\lambda_{n+1}(t) = i\beta'(t)$  is not a point in the spectrum of the matrix  $A(t)$  (see condition 2a)), then the Equation (11b) has a unique solution  $y_{n+1}(t) = [\lambda_{n+1}(t)I - A(t)]^{-1}H_{n+1}(t)$  in the space  $C^\infty([t_0, T], \mathbb{C}^n)$ .

Systems (11b) for  $j = 1, 2, \dots, n$ , are solvable in the space  $C^\infty([t_0, T], \mathbb{C}^n)$  if and only if the identities  $(H_j(t), \chi_j(t)) \equiv 0, j = \overline{1, n}, \forall t \in [t_0, T]$  hold. It is easy to see that these identities coincide with the identities (9). Thus, the conditions (9) are necessary and sufficient for the solvability of the system (8) in the space  $U$ . The theorem is proved.  $\square$

**Remark 1.** *If the identities (9) hold, then under the conditions (1) and (2a) the system (8) has the following solution in the space  $U$  :*

$$\begin{aligned} y(t, \tau) &= y_0(t) + \sum_{j=1}^n \left[ \alpha_j(t)\varphi_j(t) + \sum_{s=1, s \neq j}^n \frac{(H_j(t), \chi_s(t))}{\lambda_j(t) - \lambda_s(t)} \varphi_s(t) \right] e^{\tau_j} + \\ &+ (\lambda_{n+1}(t)I - A(t))^{-1}H_{n+1}(t)e^{\tau_{n+1}} \end{aligned} \tag{12}$$

where  $\varphi_k(t)$  are eigenvectors of the matrix  $A(t)$ , corresponding to the eigenvalues  $\lambda_j(t), j = \overline{1, n}$ ,  $(\varphi_k(t), \chi_j(t)) \equiv \delta_{kj}$ ,  $\alpha_j(t) \in C^\infty([t_0, T], \mathbb{C}^n)$  are arbitrary scalar functions  $j = \overline{1, n}$ ,  $y_0(t)$  is the solution of the integral system (11a).

**4. Unique Solvability of the General Iterative Problem in the Space  $U$**   
**Remainder Theorem**

As seen from (12), the solution of the system (8) is determined ambiguously. However, if its solution satisfies to the additional conditions

$$y(t_0, 0) = y_*,$$

$$\langle -\frac{\partial y}{\partial t} + R_1 y + Q(t, \tau), \chi_k(t) e^{\tau k} \rangle \equiv 0, k = \overline{1, n}, \forall t \in [t_0, T]$$
(13a)

where  $Q(t, \tau) = Q_0(t) + \sum_{j=1}^{n+1} Q_j(t) e^{\tau j}$  is a known function of the space  $U$ ,  $y_*$  is a constant number of the complex space  $\mathbb{C}^n$ , then system (8) will be uniquely solvable in the space  $U$ . More precisely, the following result holds.

**Theorem 2.** *Let conditions (1), (2a) be satisfied, the right-hand side  $H(t, \tau)$  of the system (8) belongs to the space  $U$  and satisfies the orthogonality condition (9). Then system (8) under additional conditions (13a) is uniquely solvable in  $U$ .*

**Proof.** Under condition (9), system (8) has a solution (12) in the space  $U$ , where the functions  $\alpha_j(t) \in C^\infty([t_0, T], \mathbb{C}^n), j = \overline{1, n}$ , are still arbitrary. Subordinating (12) to the first condition (13a), i.e.,  $y(t_0, 0) = y_*$ , we obtain the equation

$$\sum_{j=1}^n \alpha_j(t_0) \varphi_j(t_0) = y^*$$
(13b)

where

$$y^* = y_* + A^{-1}(t_0)H_0(t_0) - \sum_{j=1}^n \sum_{s=1, s \neq j}^n \frac{(H_j(t_0), \chi_s(t_0))}{\lambda_j(t_0) - \lambda_s(t_0)} \varphi_s(t_0) - (\lambda_{n+1}(t_0)I - A(t_0))^{-1}H_{n+1}(t_0).$$

Multiplying scalarly the equality (13b) by  $\chi_j(t_0)$  and taking into account the biorthogonality of the systems  $\{\varphi_j(t)\}$  and  $\{\chi_j(t)\}$ , we find the values

$$\alpha_j(t_0) = (y^*, \chi_j(t_0)), j = \overline{1, n}.$$
(14)

Let us now subordinate solution (12) to the second condition (13a). The right-hand side of this equation has the form

$$-\frac{\partial y_0}{\partial t} + R_1 y_0 + Q(t, \tau) = -\dot{y}_0(t) - \sum_{j=1}^n \left[ (\alpha_j(t) \varphi_j(t))^\bullet + \sum_{s=1, s \neq j}^n \left( \frac{(H_j(t), \chi_s(t))}{\lambda_j(t) - \lambda_s(t)} \varphi_s(t) \right)^\bullet \right] - \left( (\lambda_{n+1}(t)I - A(t))^{-1} H_{n+1}(t) \right)^\bullet e^{\tau_{n+1}} + \sum_{j=1}^{n+1} \left[ \frac{K(t, t) y_j(t)}{\lambda_j(t)} e^{\tau_j} - \frac{K(t, t_0) y_j(t_0)}{\lambda_j(t_0)} \right] + Q(t, \tau).$$
(15)

Now multiplying (15) scalarly by  $\chi_j(t) e^{\tau_j}, j = \overline{1, n}$ , we obtain equations

$$\dot{\alpha}_j(t) + \left( \dot{\varphi}_j(t) - \frac{K(t, t)}{\lambda_j(t)} \varphi_j(t), \chi_j(t) \right) \alpha_j(t) + \sum_{s=1, s \neq j}^n \frac{(H_j(t), \chi_s(t))}{\lambda_j(t) - \lambda_s(t)} (\dot{\varphi}_s(t), \chi_j(t)) + (Q_j(t), \chi_j(t)) = 0, j = \overline{1, n}.$$

Adding the initial conditions (14) to them, we can uniquely find the functions  $\alpha_j(t)$ ,  $j = \overline{1, n}$  :

$$\alpha_j(t) = e^{-\int_{t_0}^t \left[ \left( \dot{\varphi}_j(x) - \frac{K(x,x)\varphi_j(x)}{\lambda_j(x)} \right) \chi_j(x) \right] dx} [\alpha_j(t_0) + \int_{t_0}^t e^{\int_{t_0}^s \left[ \left( \dot{\varphi}_j(x) - \frac{K(x,x)\varphi_j(x)}{\lambda_j(x)} \right) \chi_j(x) \right] dx} l_j(s) ds], j = \overline{1, n}$$

where  $l_j(t) = -(Q_j(t), \chi_j(t)) - \sum_{s=1, s \neq j}^n \frac{(H_j(t), \chi_s(t))}{\lambda_j(t) - \lambda_s(t)} (\dot{\varphi}_s(t), \chi_j(t))$ , and hence, we define the solution (12) of the system (8) in the space in a unique way. The theorem is proved.

Applying Theorems 1 and 2 to iterative problems (7d), we uniquely find their solutions in the space  $U$  and construct series (5). Let  $y_{\varepsilon N}(t) = \sum_{k=0}^N \varepsilon^k y_k \left( t, \frac{\varphi(t)}{\varepsilon} \right)$  is the restriction of the  $N$ -th partial sum of series (5) for  $\tau = \frac{\varphi(t)}{\varepsilon}$ . The same as in [3], it is easy to prove the following statement.  $\square$

**Lemma 1.** *Let conditions (1) and (2) be satisfied. Then the partial sum  $y_{\varepsilon N}(t)$  satisfies problem (1) up to  $O(\varepsilon^{N+1})(\varepsilon \rightarrow +0)$ , i.e.,*

$$\varepsilon \frac{dy_{\varepsilon N}(t)}{dt} \equiv A(t)y_{\varepsilon N}(t) + \int_{t_0}^t K(t,s)y_{\varepsilon N}(s)ds + h_1(t) + h_2(t)e^{\frac{i\beta(t)}{\varepsilon}}\sigma + \varepsilon^{N+1}R_N(t, \varepsilon), \quad (16)$$

$$y_{\varepsilon N}(t_0) = y^0, \forall t \in [t_0, T]$$

where  $\|R_N(t, \varepsilon)\|_{C[t_0, T]} \leq \bar{R}_N$  for all  $t \in [t_0, T]$  and for all  $\varepsilon \in (0, \varepsilon_N]$ .

Consider now the following problem:

$$\varepsilon \frac{dz(t, \varepsilon)}{dt} = A(t)z(t, \varepsilon) + \int_{t_0}^t K(t,s)z(s, \varepsilon)ds + \Phi(t, \varepsilon), z(t_0, \varepsilon) = 0, t \in [t_0, T]. \quad (17)$$

Let us show that this problem is solvable in the space  $C^1([t_0, T], \mathbb{C}^n)$  (i.e., it has a solution for any right-hand side  $\Phi(t, \varepsilon) \in C([t_0, T], \mathbb{C}^n)$ ) and that in this case there is an estimate

$$\|z(t, \varepsilon)\|_{C[t_0, T]} \leq \frac{v_0}{\varepsilon} \|\Phi(t, \varepsilon)\|_{C[t_0, T]}. \quad (18)$$

**Theorem 3.** *Let conditions (1) and (2) be satisfied. Then, for sufficiently small  $\varepsilon \in (0, \varepsilon_0]$ , problem (17) for any right-hand side  $\Phi(t, \varepsilon) \in C[t_0, T]$  has a unique solution  $z(t, \varepsilon)$  in the space  $C^1([t_0, T], \mathbb{C}^n)$  and estimate (18) holds, where  $v_0$  is a constant independent of  $\varepsilon > 0$ .*

**Proof.** Introduce an additional unknown function

$$u(t, \varepsilon) = \int_{t_0}^t K(t,s)z(s, \varepsilon)ds.$$

Differentiating it with respect to  $t$ , we will have

$$\frac{du(t, \varepsilon)}{dt} = K(t,t)z(t, \varepsilon) + \int_{t_0}^t \frac{\partial K(t,s)}{\partial t} z(s, \varepsilon)ds.$$

From this and (17) it follows that the vector function  $w = \{z, u\}$  satisfies the following system:

$$\varepsilon \frac{dw(t, \varepsilon)}{dt} = \begin{pmatrix} A(t) & I_n \\ 0 & 0 \end{pmatrix} w(t, \varepsilon) + \varepsilon \begin{pmatrix} 0 \\ K(t,t)z + \int_{t_0}^t \frac{\partial K(t,s)}{\partial t} z(s, \varepsilon)ds \end{pmatrix} + \begin{pmatrix} \Phi(t, \varepsilon) \\ 0 \end{pmatrix}, w(t_0, \varepsilon) = 0. \quad (19a)$$

Denote by  $Y(t, s, \varepsilon)$  the normal fundamental matrix of the homogeneous system  $\varepsilon \frac{dw}{dt} = \begin{pmatrix} A(t) & I_n \\ 0 & 0 \end{pmatrix} w$ , i.e., the matrix satisfying the equation

$$\varepsilon \frac{dY(t, s, \varepsilon)}{dt} = \begin{pmatrix} A(t) & I_n \\ 0 & 0 \end{pmatrix} Y(t, s, \varepsilon), \quad Y(s, s, \varepsilon) = I, \quad t_0 \leq s \leq t \leq T.$$

Since the matrix  $\begin{pmatrix} A(t) & I_n \\ 0 & 0 \end{pmatrix}$  is a matrix of simple structure and its spectrum  $\{\lambda_1(t), \dots, \lambda_n(t), 0\}$  lies in the half-plane  $\text{Re } \lambda(t) \leq 0$ , then the Cauchy matrix  $Y(t, s, \varepsilon)$  is uniformly bounded, i.e.,

$$\|Y(t, s, \varepsilon)\| \leq c_0 \quad \forall (t, s, \varepsilon) : t_0 \leq s \leq t \leq T, \varepsilon > 0$$

where the constant  $c_0 > 0$  does not depend on  $\varepsilon > 0$  (see, for example, [3], pp. 119–120). We now write down an integral system equivalent to system (19a):

$$\begin{aligned} w(t, \varepsilon) = & \int_{t_0}^t Y(t, \zeta, \varepsilon) \left( K(\zeta, \zeta)z(\zeta, \varepsilon) + \int_{t_0}^{\zeta} \frac{\partial K(\zeta, s)}{\partial t} z(s, \varepsilon) ds \right) d\zeta + \\ & + \frac{1}{\varepsilon} \int_{t_0}^t Y(t, \zeta, \varepsilon) \begin{pmatrix} \Phi(\zeta, \varepsilon) \\ 0 \end{pmatrix} d\zeta. \end{aligned} \tag{19b}$$

Since for each  $\varepsilon > 0$  there exists the solution  $w(t, \varepsilon)$  of the system (19a) in the space  $C^1([t_0, T], \mathbb{C}^n)$ , then substituting it into (19b), we obtain the identity. Let us move on to the norms:

$$\begin{aligned} \|w(t, \varepsilon)\| \leq & \int_{t_0}^t \|Y(t, \zeta, \varepsilon)\| \cdot \|K(\zeta, \zeta)\| \cdot \|z(\zeta, \varepsilon)\| d\zeta + \int_{t_0}^t \|Y(t, \zeta, \varepsilon)\| \times \\ & \times \int_{t_0}^{\zeta} \left\| \frac{\partial K(\zeta, s)}{\partial t} \right\| \cdot \|z(s, \varepsilon)\| ds d\zeta + \frac{1}{\varepsilon} \int_{t_0}^t \|Y(t, \zeta, \varepsilon)\| \cdot \|\Phi(\zeta, \varepsilon)\| d\zeta \leq \\ & \leq c_0 k_0 \int_{t_0}^t \|w(\zeta, \varepsilon)\| d\zeta + c_0 k_1 \int_{t_0}^t \int_{t_0}^{\zeta} \|w(s, \varepsilon)\| ds d\zeta + \\ & + \frac{T_0}{\varepsilon} c_0 \|\Phi(t, \varepsilon)\|_{C[t_0, T]} \leq c_0 k_0 \int_{t_0}^t \|w(s, \varepsilon)\| ds + \\ & + c_0 k_1 \int_{t_0}^t \int_{t_0}^{\zeta} \|w(s, \varepsilon)\| ds d\zeta + \frac{c_0 T_0}{\varepsilon} \|\Phi(t, \varepsilon)\|_{C[t_0, T]} \leq \\ & \leq c_0 k_0 \int_{t_0}^t \|w(s, \varepsilon)\| ds + c_0 k_1 \int_{t_0}^T \int_{t_0}^t \|w(s, \varepsilon)\| ds d\zeta + \\ & + \frac{c_0 T_0}{\varepsilon} \|\Phi(t, \varepsilon)\|_{C[t_0, T_0]} \leq (c_0 k_0 + c_0 k_1 T_0) \int_{t_0}^t \|w(s, \varepsilon)\| ds + \\ & + \frac{c_0 T_0}{\varepsilon} \|\Phi(t, \varepsilon)\|_{C[t_0, T_0]} \end{aligned}$$

where  $T_0 = T - t_0$ ,  $\|K(t, s)\|_{C([t_0, T] \times [t_0, T])} = k_0$ ,  $\|\partial K(t, s)/\partial t\|_{C([t_0, T] \times [t_0, T])} = k_1$ . We get the inequality

$$\|w(t, \varepsilon)\| \leq \frac{c_0 T_0}{\varepsilon} \|\Phi(t, \varepsilon)\|_{C[t_0, T_0]} + (c_0 k_0 + c_0 k_1 T_0) \int_{t_0}^t \|w(s, \varepsilon)\| ds.$$



Applying the Gronwall–Bellman lemma to this inequality, we have

$$\begin{aligned} \|w(t, \varepsilon)\| &\leq \frac{c_0 T_0}{\varepsilon} \|\Phi(t, \varepsilon)\|_{C[t_0, T]} e^{(c_0 k_0 + c_0 k_1 T_0) \int_{t_0}^t ds} = \\ &= \frac{c_0 T_0}{\varepsilon} \|\Phi(t, \varepsilon)\|_{C[t_0, T]} e^{(c_0 k_0 + c_0 k_1 T_0)(t-t_0)} \leq \frac{\nu_0}{\varepsilon} \|\Phi(t, \varepsilon)\|_{C[t_0, T]} \Rightarrow \\ &\Rightarrow \|z(t, \varepsilon)\|_{C[t_0, T]} \leq \frac{\nu_0}{\varepsilon} \|\Phi(t, \varepsilon)\|_{C[t_0, T]} \end{aligned}$$

where  $\nu_0 = c_0 T_0 \cdot \max_{t \in [t_0, T]} e^{(c_0 k_0 + c_0 k_1 T_0)(t-t_0)}$ . The theorem is proved.  $\square$

**Theorem 4.** Let conditions (1) and (2) be satisfied. Then for any  $\varepsilon \in (0, \varepsilon_0]$ , where  $\varepsilon_0 > 0$  is small enough, the problem (1) has a unique solution  $y(t, \varepsilon) \in C^1([t_0, T], \mathbb{C}^n)$ ; in this case, the estimate

$$\|y(t, \varepsilon) - y_{\varepsilon N}(t)\|_{C[t_0, T]} \leq C_N \varepsilon^{N+1} \quad (N = 0, 1, 2, \dots)$$

holds true, where the constant  $C_N > 0$  does not depend on  $\varepsilon \in (0, \varepsilon_0]$ .

**Proof.** By the lemma, the partial sum  $y_{\varepsilon N}(t)$  satisfies the problem (16), so the remainder  $r_N(t, \varepsilon) \equiv y(t, \varepsilon) - y_{\varepsilon N}(t)$  satisfies the following problem:

$$\varepsilon \frac{r_N(t, \varepsilon)}{dt} = A(t)r_N(t, \varepsilon) + \int_{t_0}^t K(t, s)r_N(s, \varepsilon)ds + \varepsilon^{N+1}R_N(t, \varepsilon), \quad r_N(t_0, \varepsilon) = 0$$

where  $\Phi(t, \varepsilon) = -\varepsilon^{N+1} \int_{t_0}^t R_N(s, \varepsilon)ds$ . By Theorem 3, we have the estimate

$$\|r_N(t, \varepsilon)\|_{C[t_0, T]} \leq \varepsilon^N \bar{R}_N$$

for all  $N = 0, 1, 2, \dots$  and all  $\varepsilon \in (0, \varepsilon_N]$ , which means that the partial sum  $y_{\varepsilon, N+1}(t) = y_{\varepsilon N}(t) + \varepsilon^{N+1}y_{N+1}(t, \frac{\psi(t)}{\varepsilon})$  satisfies the inequality

$$\|y(t, \varepsilon) - y_{\varepsilon, N+1}(t)\|_{C[t_0, T]} \equiv \|(y(t, \varepsilon) - y(t)) - \varepsilon^{N+1}y_{N+1}(t, \frac{\psi(t)}{\varepsilon})\|_{C[t_0, T]} \leq \bar{C}_{N+1}\varepsilon^{N+1}.$$

Using the inequality  $\|a - b\| \geq \| \|a\| - \|b\| \|$ , valid for any numbers  $a$  and  $b$ , we will have

$$\|y(t, \varepsilon) - y_{\varepsilon N}(t)\|_{C[t_0, T]} \leq \left( \bar{C}_N + \left\| y_{N+1}(t, \frac{\psi(t)}{\varepsilon}) \right\|_{C[t_0, T]} \right) \varepsilon^{N+1}$$

whence we derive the estimate

$$\|y(t, \varepsilon) - y_{\varepsilon N}(t)\|_{C[t_0, T]} \leq C_N \varepsilon^{N+1}$$

where the constant  $C_N > 0$  does not depend on  $\varepsilon \in (0, \varepsilon_N]$ .  $\square$

### 5. Construction of the Solution of the First Iteration Problem in the Space $U$

Using Theorem 1, we will try to find a solution to the first iterative problem (7d). Since the right-hand side  $h_1(t) + h_2(t)e^{\tau_{n+1}}$  of the system (7a), satisfies the condition (9), this system has (according to (12)) a solution in the space  $U$  in the form

$$y_0(t, \tau) = y_0^{(0)}(t) + \sum_{k=1}^n \alpha_k^{(0)}(t) \varphi_k(t) e^{\tau_k} + (\lambda_{n+1}(t)I - A(t))^{-1} h_2(t) e^{\tau_{n+1}} \sigma \tag{20}$$

where  $\alpha_k^{(0)}(t) \in C^\infty([t_0, T], \mathbb{C}^n)$  are arbitrary functions,  $k = \overline{1, n}$ ,  $y_0^{(0)}(t)$  is the solution of the integral system  $-A(t)y_0(t) - \int_{t_0}^t K(t, s)y_0(s)ds = h_1(t)$ .

Subordinating (20) to the initial condition  $y_0(t_0, 0) = y^0$ , we will have

$$y_0^{(0)}(t_0) + \sum_{k=1}^n \alpha_k^{(0)}(t_0)\varphi_k(t_0) + (\lambda_{n+1}(t_0)I - A(t_0))^{-1}h_2(t_0)\sigma = y^0 \Leftrightarrow$$

$$\Leftrightarrow \sum_{k=1}^n \alpha_k^{(0)}(t_0)\varphi_k(t_0) = y^0 + A^{-1}(t_0)h_1(t_0) - (\lambda_{n+1}(t_0)I - A(t_0))^{-1}h_2(t_0)\sigma.$$

Now, scalarly multiplying this equality by  $\chi_j(t_0), j = \overline{1, n}$ , we obtain values

$$\alpha_k^{(0)}(t_0) = (y^0 + A^{-1}(t_0)h_1(t_0) - (\lambda_{n+1}(t_0)I - A(t_0))^{-1}h_2(t_0)\sigma, \chi_k(t_0)), k = \overline{1, n}. \tag{21}$$

For a complete calculation of the function  $\alpha_k^{(0)}(t)$ , we pass to the next iterative problem (7b). Substituting the solution (20) of the system (7a) into it, we arrive at the following equation:

$$Ly_1(t, \tau) = -\frac{d}{dt}y_0^{(0)}(t) - \sum_{k=1}^n \frac{d}{dt}(\alpha_k^{(0)}(t)\varphi_k(t))e^{\tau_k} -$$

$$-\frac{d}{dt}((\lambda_{n+1}(t)I - A(t))^{-1}h_2(t))e^{\tau_{n+1}}\sigma + \tag{22a}$$

$$+ \sum_{j=1}^n \left[ \frac{(K(t, t)\alpha_j^{(0)}(t)\varphi_j(t))}{\lambda_j(t)}e^{\tau_j} - \frac{(K(t, t_0)\alpha_j^{(0)}(t_0)\varphi_j(t_0))}{\lambda_j(t_0)} \right].$$

Subordinating the right-hand side of this system to the solvability conditions (9), we obtain the system of ordinary differential equations

$$-\frac{d\alpha_k^{(0)}(t)}{dt} + \left( \frac{K(t, t)\varphi_k(t)}{\lambda_k(t)} - \dot{\varphi}_k(t), \chi_k(t) \right) \alpha_k^{(0)}(t) = 0, k = \overline{1, n}.$$

Adding the initial condition (21) to them, we find

$$\alpha_k^{(0)}(t) = \alpha_k^{(0)}(t_0) \exp \left\{ \int_{t_0}^t \left( \frac{K(\theta, \theta)\varphi_k(\theta)}{\lambda_k(\theta)} - \dot{\varphi}_k(\theta), \chi_k(\theta) \right) d\theta \right\}, k = \overline{1, n}$$

and, therefore, the solution (15) of the problem will be found uniquely in the space  $U$ . In this case, the leading term of the asymptotics has the following form:

$$y_{\varepsilon 0}(t) = y_0^{(0)}(t) + \sum_{k=1}^n \alpha_k^{(0)}(t_0) e^{\int_{t_0}^t \left( \frac{K(\theta, \theta)\varphi_k(\theta)}{\lambda_k(\theta)} - \dot{\varphi}_k(\theta), \chi_k(\theta) \right) d\theta + \frac{1}{\varepsilon} \int_{t_0}^t \lambda_k(\theta) d\theta} \varphi_k(t) +$$

$$+ (\lambda_{n+1}(t)I - A(t))^{-1} h_2(t) e^{\frac{1}{\varepsilon} \int_{t_0}^t \lambda_{n+1}(\theta) d\theta} \sigma \tag{22b}$$

where  $y_0^{(0)}(t)$  is the solution of the integral system

$$-A(t)y_0(t) - \int_{t_0}^t K(t, s)y_0(s)ds = h_1(t).$$

**Example 1.** Let us consider the following problem:

$$\begin{cases} \varepsilon \frac{dy}{dt} = -y + z + \int_0^t ts \cdot y(s, \varepsilon) ds + 2t + te^{\frac{i(t+t^3)}{\varepsilon}}, & y(0, \varepsilon) = y^0, \\ \varepsilon \frac{dz}{dt} = -z + \int_0^t (t-s) \cdot z(s, \varepsilon) ds + t + t^2 e^{\frac{i(t+t^3)}{\varepsilon}}, & z(0, \varepsilon) = z^0, t \in [0, T] \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \varepsilon \frac{dw}{dt} = Aw + \int_0^t K(t,s)w(s, \varepsilon) ds + H_1(t) + H_2(t)e^{\frac{i(t+t^3)}{\varepsilon}}, w(0, \varepsilon) = w^0 \quad (23)$$

where

$$A = \begin{pmatrix} -1 & 1 \\ 0 & -2 \end{pmatrix}, K(t,s) = \begin{pmatrix} ts & 0 \\ 0 & t-s \end{pmatrix}, H_1(t) = \begin{pmatrix} 2t \\ t \end{pmatrix},$$

$$H_2(t) = \begin{pmatrix} t \\ t^2 \end{pmatrix}, w = \begin{pmatrix} y(t, \varepsilon) \\ z(t, \varepsilon) \end{pmatrix}, w^0 = \begin{pmatrix} y^0 \\ z^0 \end{pmatrix}, t \in [0, T].$$

Let's try to construct its main term of the asymptotic solution. In this system, the matrix is constant and has the following spectrum:  $\{\lambda_1(t), \lambda_2(t)\} = \{-1, -2\}$ , and a rapidly oscillating multiplier with the inhomogeneity  $H_2(t)$  corresponds to the spectral value  $\lambda_3(t) = i(1 + 3t^2)$ . In this case, the eigenvectors of constant matrices  $A$  and  $A^*$  will have the form

$$\varphi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \varphi_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \chi_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \chi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Regularizing problem (23) using the functions

$$\tau_j = \frac{1}{\varepsilon} \int_0^t \lambda_j(\theta) d\theta, j = \overline{1, 3} \Leftrightarrow \tau_1 = -\frac{t}{\varepsilon}, \tau_2 = -\frac{2t}{\varepsilon},$$

$$\tau_3 = \frac{1}{\varepsilon} \int_0^t (i(1 + 3\theta^2)) d\theta = \frac{i}{\varepsilon} (t + t^3)$$

we get the following extended problem:

$$L_\varepsilon \tilde{w}(t, \tau, \varepsilon) \equiv \varepsilon \frac{\partial \tilde{w}}{\partial t} + \lambda_1 \frac{\partial \tilde{w}}{\partial \tau_1} + \lambda_2 \frac{\partial \tilde{w}}{\partial \tau_2} + \lambda_3(t) \frac{\partial \tilde{w}}{\partial \tau_3} -$$

$$-A\tilde{w} - \tilde{J}\tilde{w} = H_1(t) + H_2(t)e^{\tau_3}, \tilde{w}(t, \tau, \varepsilon)|_{t=0, \tau=(0,0,0)} = w^0 \quad (24)$$

where  $\tau = (\tau_1, \tau_2, \tau_3)$ ,  $\tilde{J}$  is extension of the integral operator  $J$  on series of the form

$$\tilde{w}(t, \tau, \varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k w_k(t, \tau) \quad (25)$$

with coefficients  $w_k(t, \tau)$  from the space  $U$  of vector functions

$$w(t, \tau) = w_0(t) + \sum_{j=1}^3 w_j(t) e^{\tau_j}, w_k(t) \in C^\infty([0, T], \mathbb{C}^2), k = \overline{0, 3}.$$

This extension has the form  $\tilde{J}\tilde{w}(t, \tau, \varepsilon) = \sum_{r=0}^{\infty} \varepsilon^r \sum_{s=0}^r R_{r-s} y_s(t, \tau)$ , where the operators  $R_\nu : U \rightarrow U$  are calculated by the formulas:

$$R_0 w(t, \tau) = \int_0^t K(t,s) w_0(s) ds,$$

$$R_1 w(t, \tau) = \sum_{j=1}^3 \left[ \frac{K(t,t) w_j(t)}{\lambda_j(t)} e^{\tau_j} - \frac{K(t,0) w_j(0)}{\lambda_j(0)} \right], \dots$$

(operators  $R_\nu$  at  $\nu \geq 2$  we do not write out, because we do not need them when constructing the leading term of the asymptotics). Defining the solution of the problem (24) in the form of series (25), we obtain the following iterative problems:

$$Lw_0 \equiv \lambda_1 \frac{\partial w_0}{\partial \tau_1} + \lambda_2 \frac{\partial w_0}{\partial \tau_2} + \lambda_3(t) \frac{\partial w_0}{\partial \tau_3} - Aw_0 - Rw_0 = H_1(t) + H_2(t)e^{\tau_3}, w_0(t, \tau)|_{t=0, \tau=0} = w^0; \tag{26}$$

$$Lw_1 = -\frac{\partial w_1}{\partial t} + R_1w_1, w_1(0, 0) = 0; \tag{27}$$

...

The solution of the first iterative problem (26) will be as follows:

$$w_0(t, \tau) = w_0(t) + \sum_{k=1}^2 \alpha_k(t) \varphi_k e^{\tau_k} + (\lambda_3(t)I - A)^{-1} H_2(t) e^{\tau_3} \tag{28}$$

where  $\alpha_k(t) \in C^\infty([t_0, T], \mathbb{C}^2)$  are arbitrary functions,  $k = 1, 2$ ,  $y_0^{(0)}(t)$  is the solution of the integral system

$$-Aw_0(t) - \int_0^t K(t, s)w_0(s)ds = H_1(t) \Leftrightarrow \Leftrightarrow w_0(t) = \int_0^t \begin{pmatrix} ts & \frac{t-s}{2} \\ 0 & \frac{t-s}{2} \end{pmatrix} w_0(s)ds + \begin{pmatrix} \frac{5}{2}t \\ \frac{1}{2}t \end{pmatrix}. \tag{29}$$

Subordinate (28) to the initial condition  $w_0(t, \tau)|_{t=0, \tau=0} = w^0$ . Taking into account the form of vector functions (29) and  $H_2(t)$ , we obtain the equation  $\sum_{k=1}^2 \alpha_k(0) \varphi_k = w^0$ . Multiplying this equality scalarly by  $\chi_1(0)$  and  $\chi_2(0)$ , we find the values  $\alpha_1(0) = y^0 + z^0, \alpha_2(0) = z^0$ . For a complete calculation of the functions  $\alpha_k(t)$ , we pass to the next iterative problem (27). Taking into account that, under the conditions of solvability (9) of problem (27) only exponentials  $e^{\tau_1}$  and  $e^{\tau_2}$  are involved, we keep in its right-hand side only terms depending on these exponentials:

$$-\frac{\partial}{\partial t} \sum_{k=1}^2 \alpha_k(t) \varphi_k e^{\tau_k} + \sum_{j=1}^2 \frac{K(t, t) \varphi_k}{\lambda_j(t)} \alpha_k(t) e^{\tau_j}$$

and subordinate the resulting sum to the orthogonality conditions (9). We will have

$$-\frac{d\alpha_k(t)}{dt} + \left( \frac{K(t, t) \varphi_k}{\lambda_k(t)}, \chi_k \right) \alpha_k(t) = 0, k = 1, 2$$

or (in more detail, taking into account the eigenvectors  $\varphi_j, \chi_j$  and the kernel  $K(t, t)$ ):

$$\begin{aligned} -\dot{\alpha}_1(t) - \left( \begin{pmatrix} t^2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) \alpha_1(t) &= 0, \\ -\dot{\alpha}_2(t) - \frac{1}{2} \left( \begin{pmatrix} t^2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \alpha_2(t) &= 0 \end{aligned} \Leftrightarrow \dot{\alpha}_1(t) = -t^2 \alpha_1(t), \dot{\alpha}_2(t) = 0.$$

Adding to these equations the initial conditions  $\alpha_1(0) = y^0 + z^0, \alpha_2(0) = z^0$ , found earlier, we uniquely find the functions  $\alpha_1(t) = e^{-\frac{t^3}{3}}(y^0 + z^0), \alpha_2(t) = z^0$ , and hence, we will uniquely construct solution (28) of the first iterative problem (26). Making a narrowing in it at  $\tau_1 = -\frac{t}{\varepsilon}, \tau_2 = -\frac{2t}{\varepsilon}, \tau_3 = \frac{i}{\varepsilon}(t + t^3)$ , we obtain the leading term of the asymptotic solution of the problem (23):

$$w_{\varepsilon 0}(t) = w_0(t) + e^{-\frac{t^3}{\varepsilon}} (y^0 + z^0) \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-\frac{t}{\varepsilon}} + z^0 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-\frac{2t}{\varepsilon}} + \left( \frac{t(3it^2+2+i+t)}{(3it^2+1+i)(3it^2+2+i)} \right) \frac{t^2}{3it^2+2+i} e^{\frac{i}{\varepsilon}(t+t^3)} \tag{30}$$

where  $w_0(t)$  is the solution of the integral system (29). It is seen from (30) that, at the exact solution  $w(t, \varepsilon)$  of the problem (23) does not tend to the solution  $w_0(t)$  of the integral system (29) at  $\varepsilon \rightarrow +0$ , but performs quick oscillations near it.

## 6. Conclusions

From the expression (22b) for  $y_{\varepsilon 0}(t)$ , it can be seen that the construction of the leading term of the asymptotics of the solution to problem (1) is significantly influenced by both the rapidly oscillating inhomogeneity and the kernel of the integral operator.

Note that the application of other asymptotic methods (for example, the method of boundary functions [17–19]) to problems of type (1) with rapidly oscillating inhomogeneities is problematic, since many of them rely heavily on the fact that all points of the spectrum (including the spectral value of the inhomogeneity) lie in the open half-plane  $\operatorname{Re} \lambda < 0$ .

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