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Article

A Weak Turnpike Property for Perturbed Dynamical Systems with a Lyapunov Function

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Abstract: In this work, we obtain a weak version of the turnpike property of trajectories of perturbed discrete disperse dynamical systems, which have a prototype in mathematical economics.

Keywords: compact metric space; global attractor; Lyapunov function; set-valued mapping; turnpike

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1. Introduction

In [1,2], A. M. Rubinov introduced a discrete dispersive dynamical system, which was investigated in [1–7]. This dynamical system is determined by a set-valued mapping and has a prototype in the economic growth theory [1,8,9]. Our dynamical system is described by a compact metric space of states and a transition operator, which is set-valued. Usually in the dynamical systems theory a transition operator is single-valued. In [1–7] and in the present paper we study dynamical systems with a set-valued transition operator. Such dynamical systems correspond to certain models of economic dynamics [1,8,9].

Let (X, ρ) be a compact metric space and let $a : X \to 2^X \setminus \{\emptyset\}$ be a set-valued mapping of which the graph

$$\text{graph}(a) = \{(x, y) \in X \times X : \ y \in a(x)\}$$

is a closed subset of $X \times X$. For each nonempty subset $E \subset X$ set

$$a(E) = \cup\{a(x) : \ x \in E\} \text{ and } a^0(E) = E.$$

By induction we define $a^n(E)$ for any positive integer n and any nonempty subset $E \subset X$ as follows:

$$a^n(E) = a(a^{n-1}(E)).$$

In the present work we investigate convergence and structure of trajectories of the perturbed dynamical system determined by the set-valued mapping a. Following [1,2] this system is called a discrete dispersive dynamical system.

A sequence $\{x_t\}_{t=0}^{\infty} \subset X$ is called a trajectory of a (or just a trajectory if the mapping a is understood) if $x_{t+1} \in a(x_t)$ for all nonnegative integers t.

Let $T_2 > T_1$ be integers. A sequence $\{x_t\}_{t=T_1}^{T_2} \subset X$ is called a finite trajectory of a (or just a trajectory if the mapping a is understood) if $x_{t+1} \in a(x_t)$ for all integers $t \in \{T_1, \dots, T_2 - 1\}$.

Define

$$\Omega(a) = \{z \in X : \ \text{for very positive number } \epsilon \text{ there exists a trajectory} \{x_t\}_{t=0}^{\infty}$$

$$\text{for which } \liminf_{t\to\infty} \rho(z, x_t) \le \epsilon\}.$$

By the compactness of X, $\Omega(a)$ is a nonempty closed subset of the metric space (X, ρ). In the dynamical systems theory the set $\Omega(a)$ is called a global attractor of a. In [1,2] $\Omega(a)$ is called a turnpike set of a. This terminology is motivated by economic growth theory [1,8,9].

For each $x \in X$ and each nonempty closed subset $E \subset X$ put

$$\rho(x, E) = \inf\{\rho(x, y) : \ y \in E\}.$$

Evidently, for every trajectory $\{x_t\}_{t=0}^{\infty}$,

$$\lim_{t \to \infty} \rho(x_t, \Omega(a)) = 0.$$

Let $\phi : X \to R^1$ be a continuous function such that

$$\phi(z) \geq 0 \text{ for all } z \in X,$$

$$\phi(y) \leq \phi(x) \text{ for all } x \in X \text{ and all } y \in a(x).$$

Evidently, the function ϕ is a Lyapunov function for the dynamical system generated by the mapping a. In economic growth theory usually X is a subset of the finite-dimensional Euclidean space and ϕ is a linear functional on this space [1,8,9]. Our goal in [7] was to study approximate solutions of the problem

$$\phi(x_T) \to \max,$$

$$\{x_t\}_{t=0}^{T} \text{ is a trajectory satisfying } x_0 = x,$$

where $x \in X$ and a natural number T are given.

The following theorem was obtained in [7].

Theorem 1. *The following properties are equivalent:*

(1) If a sequence $\{x_t\}_{t=-\infty}^{\infty} \subset X$ satisfies $x_{t+1} \in a(x_t)$ and $\phi(x_{t+1}) = \phi(x_t)$ for all integers t, then

$$\{x_t\}_{t=-\infty}^{\infty} \subset \Omega(a).$$

(2) For every positive number ϵ there exists a positive integer $T(\epsilon)$ such that for every trajectory $\{x_t\}_{t=0}^{\infty} \subset X$ satisfying $\phi(x_t) = \phi(x_{t+1})$ for all integers $t \geq 0$ the relation $\rho(x_t, \Omega(a)) \leq \epsilon$ is valid for all integers $t \geq T(\epsilon)$.

Our results are obtained under the assumption that property (1) holds. This property indeed holds for models of economic dynamics, which are prototypes of our dynamical system [1,8,9].

For each bounded function $\psi : X \to R^1$ set

$$\|\psi\| = \sup\{|\psi(z)| : \ z \in X\}.$$

We denote by $\mathrm{Card}(A)$ the cardinality of a set A and suppose that the sum over empty set is zero.

For every point $(x_1, x_1), \ (y_1, y_2) \in X \times X$ put

$$\rho_1((x_1, x_2), (y_1, y_2)) = \rho(x_1, y_1) + \rho(x_2, y_2).$$

For every point $(x_1, x_2) \in X \times X$ and every nonempty closed subset $E \subset X \times X$ set

$$\rho_1((x_1, x_2), E) = \inf\{\rho_1((x_1, x_2), (y_1, y_2)) : \ (y_1, y_2) \in E\}.$$

In [7] we established the turnpike properties for approximate solutions of the problem

$$\phi(x_T) \to \max,$$

$$\{x_t\}_{t=0}^{T} \text{ is a trajectory satisfying } x_0 = x,$$

where $x \in X$ and a positive integer T are given. In [10] we established a weak version of the turnpike property that holds for all finite trajectories of our dynamical system, which are of a sufficient length and which are not necessarily approximate solutions of the problem above. This turnpike result usually holds for models of economic dynamics [1,8,9].

More precisely, in [10] we prove the following result.

Theorem 2. *Assume that property (1) of Theorem 1 holds and that $\epsilon > 0$. Then there exists a natural number L such that for each integer $T > L$ and each finite trajectory $\{x_t\}_{t=0}^T$ the following inequality holds:*

$$Card(\{t \in \{0, \ldots, T\} : \rho(x_t, \Omega(a)) > \epsilon\}) \le L.$$

In this paper we show that a weak version of the turnpike property established in Theorem 2 is stable under small perturbations.

Turnpike properties are well known in mathematical economics. See, for example, references [2,8,9,11] and the references mentioned there. Recently it was shown that the turnpike phenomenon holds for many important classes of problems arising in various areas of research [12–20]. For related infinite horizon problems see [9,21–28].

2. The Main Results

For every pair of nonempty sets $A, B \subset X$ set

$$H(A, B) = \max\{\sup\{\rho(x, B) : x \in A\}, \sup\{\rho(y, A) : y \in B\}\}.$$

We assume that the following assumption is true:

(A) for every positive number ϵ there is a positive number δ such that for every pair of points $x, y \in X$ satisfying $\rho(x, y) \le \delta$,

$$H(a(x), a(y)) \le \epsilon.$$

We also assume that property (1) of Theorem 1 is true and obtain the following two theorems.

Theorem 3. *Let ϵ be a positive number. Then there is a positive integer L_0 such that for every natural number $L > L_0$ there is a positive number δ such that for each sequence $\{x_t\}_{t=0}^L \subset X$ satisfying*

$$\rho_1((x_t, x_{t+1}), graph(a)) \le \delta,\ t = 0, \ldots, T-1$$

the following relation holds:

$$Card(\{t \in \{0, \ldots, L\} : \rho(x_t, \Omega(a)) > \epsilon\}) \le L_0.$$

Theorem 4. *Let ϵ be a positive number. Then there is a positive integer L_1 and a positive number δ such that for every integer $T > L_1$ and every sequence $\{x_t\}_{t=0}^T \subset X$ satisfying*

$$\rho_1((x_t, x_{t+1}), graph(a)) \le \delta,\ t = 0, \ldots, T-1$$

the following relation holds:

$$T^{-1} Card(\{t \in \{0, \ldots, T\} : \rho(x_t, \Omega(a)) > \epsilon\}) \le \epsilon.$$

In Theorems 3 and 4 we deal with the structure of inexact trajectories of our dynamical system. They are important because in the real world applications computational errors and errors of measurements always take place.

3. An Auxiliary Result

The following lemma shows that our dynamical system has the so-called shadowing property [29,30].

Lemma 1. *Let $\epsilon > 0$ and L be a positive integer. Then there is a positive number δ such that for every sequence $\{x_t\}_{t=0}^{L} \subset X$ satisfying for all integers $t = 0, \ldots, L-1$,*

$$\rho_1((x_t, x_{t+1}), graph(a)) \le \delta$$

there is a finite trajectory $\{y_t\}_{t=0}^{L} \subset X$ for which

$$y_0 = x_0,$$

$$\rho(x_t, y_t) \le \epsilon, \ t = 0, \ldots, L.$$

Proof. Let

$$\delta_L = \epsilon/4. \tag{1}$$

By induction and assumption (A), we define positive numbers $\delta_i > 0$, $i = 0, \ldots, L-1$ such that for every integer $i \in \{1, \ldots, L\}$

$$\delta_{i-1} < \delta_i/8 \tag{2}$$

and for every pair of points $x, y \in X$ for which $\rho(x, y) \le \delta_{i-1}$ we have

$$H(a(x), a(y)) < \delta_i/8. \tag{3}$$

Set

$$\delta = \delta_0. \tag{4}$$

Assume that $\{x_t\}_{t=0}^{L} \subset X$ satisfies

$$\rho_1((x_t, x_{t+1}), \text{graph}(a)) \le \delta, \ t = 0, \ldots, L-1. \tag{5}$$

Set

$$x_0 = y_0. \tag{6}$$

In view of (5) and (6),

$$\rho_1((y_0, x_1), \text{graph}(a)) \le \delta. \tag{7}$$

By (7), there exists

$$(z_0, z_1) \in \text{graph}(a) \tag{8}$$

such that

$$\rho(y_0, z_0) \le \delta, \ \rho(x_1, z_1) \le \delta. \tag{9}$$

It follows from (3), (4) and (9) that

$$H(a(y_0), a(z_0)) \le \delta_1/8. \tag{10}$$

Equations (8) and (10) imply that

$$\rho(z_1, a(y_0)) \le \delta_1/8. \tag{11}$$

Equation (11) implies that there is

$$y_1 \in a(y_0) \tag{12}$$

for which

$$\rho(y_1, z_1) \le \delta_1/4. \tag{13}$$

It follows from (2), (4), (9) and (13) that

$$\rho(y_1, x_1) \le \delta + \delta_1/4 < \delta_1/2. \quad 14)$$

Suppose that an integer $k \in \{1, \dots, L\} \setminus \{L\}$ and that we have already defined a trajectory $\{y_i\}_{i=1}^{k}$ such that

$$y_0 = x_0$$

and that for integers $i = 1, \dots, k$,

$$\rho(x_i, y_i) < \delta_i/2. \quad (15)$$

(In view of (6), (12) and (14), our assumption is valid for $k = 1$). Equation (5) implies that there is a point

$$(\xi_k, \xi_{k+1}) \in \mathrm{graph}(a) \quad (16)$$

for which

$$\rho_1((x_k, x_{k+1}), (\xi_k, \xi_{k+1})) \le \delta.$$

This implies that

$$\rho(x_k, \xi_k) \le \delta,\ \rho(x_{k+1}, \xi_{k+1}) \le \delta. \quad (17)$$

Equations (2), (4), (15) and (17) imply that

$$\rho(\xi_k, y_k) \le \rho(\xi_k, x_k) + \rho(x_k, y_k) \le \delta + \delta_k/2 \le \delta_k/8 + \delta_k/2. \quad (18)$$

By (3) and (18),

$$H(a(\xi_k), a(y_k)) \le \delta_{k+1}/8. \quad (19)$$

Equations (16) and (19) imply that

$$\rho(\xi_{k+1}, a(y_k)) \le \delta_{k+1}/8. \quad (20)$$

In view of (20), there exists

$$y_{k+1} \in a(y_k) \quad (21)$$

such that

$$\rho(\xi_{k+1}, y_{k+1}) \le \delta_{k+1}/4. \quad (22)$$

It follows from (2), (4), (17) and (22) that

$$\rho(y_{k+1}, x_{k+1}) \le \rho(y_{k+1}, \xi_{k+1}) + \rho(\xi_{k+1}, x_{k+1}) \le \delta + \delta_{k+1}/4 < \delta_{k+1}/2.$$

Thus the assumption made for k is also true for $k+1$. Therefore by induction we constructed the trajectory $\{y_t\}_{t=0}^{L} \subset X$ such that

$$y_0 = x_0,$$

$$\rho(x_t, y_t) \le \epsilon,\ t = 0, \dots, L.$$

This completes the proof of Lemma. □

Proof of Theorem 3. Theorem 2 implies that there is a positive integer L_0 for which the following property is valid:

(i) for every integer $T > L_0$ and every finite trajectory $\{x_t\}_{t=0}^{T}$,

$$\mathrm{Card}(\{t \in \{0, \dots, T\} :\ \rho(x_t, \Omega(a)) > \epsilon/2\}) \le L_0.$$

Let $L > L_0$ be an integer. By Lemma 1, there is a positive number δ such that the following property is valid:

(ii) for every sequence $\{x_t\}_{t=0}^{L} \subset X$, which satisfies

$$\rho_1((x_t, x_{t+1}), \mathrm{graph}(a)) \leq \delta, \ t = 0, \ldots, L-1 \tag{23}$$

there is a finite trajectory $\{y_t\}_{t=0}^{L} \subset X$ for which

$$y_0 = x_0,$$

$$\rho(x_t, y_t) \leq \epsilon/4, \ t = 0, \ldots, L. \tag{24}$$

Assume that $\{x_t\}_{t=0}^{L} \subset X$ satisfies (23). Property (ii) and (23) imply that there exists a finite trajectory $\{y_t\}_{t=0}^{L}$ satisfying (24). Property (i) implies that

$$\mathrm{Card}(\{t \in \{0, \ldots, L\} : \ \rho(y_t, \Omega(a)) > \epsilon/2\}) \leq L_0. \tag{25}$$

In view of (24), for integers $t = 0, \ldots, L$,

$$\rho(x_t, \Omega(a)) \leq \rho(x_t, y_t) + \rho(y_t, \Omega(a)) \leq \epsilon/4 + \rho(y_t, \Omega(a))$$

and if

$$\rho(x_t, \Omega(a)) > \epsilon,$$

then

$$\rho(y_t, \Omega(a)) > \epsilon/2.$$

Together with (25) this implies that

$$\mathrm{Card}(\{t \in \{0, \ldots, L\} : \ \rho(x_t, \Omega(a)) > \epsilon\})$$

$$\leq \mathrm{Card}(\{t \in \{0, \ldots, L\} : \ \rho(y_t, \Omega(a)) > \epsilon/2\}) \leq L_0.$$

This completes the proof of Theorem 3. □

Proof of Theorem 4. We may assume without loss of generality that $\epsilon < 1$. Theorem 3 implies that there is a positive integer L_0 for which the following property is valid:

(i) for every integer $L > L_0$ there is a positive number δ such that for every sequence $\{x_t\}_{t=0}^{L}$ satisfying

$$\rho_1((x_t, x_{t+1}), \mathrm{graph}(a)) \leq \delta, \ t = 0, \ldots, T-1$$

the relation

$$\mathrm{Card}(\{t \in \{0, \ldots, L\} : \ \rho(x_t, \Omega(a)) > \epsilon\}) \leq L_0$$

is true.

Fix a natural number

$$L > 4L_0\epsilon^{-1}. \tag{26}$$

Let a positive number δ be as guaranteed by property (i). Choose a natural number

$$k_0 > 4\epsilon^{-1}. \tag{27}$$

Set

$$L_1 = k_0 L. \tag{28}$$

Assume that $T > L_1$ is an integer and that a sequence $\{x_t\}_{t=0}^{T}$ satisfies

$$\rho_1((x_t, x_{t+1}), \mathrm{graph}(a)) \leq \delta, \ t = 0, \ldots, T-1. \tag{29}$$

There is an integer $k_1 \geq 1$ for which

$$k_1 L \leq T < (k_1 + 1)L. \tag{30}$$

By (28) and (30),

$$k_1 > k_0. \tag{31}$$

Property (i), (29) and (30) imply that for integers $i = 0, \dots, k_1 - 1$,

$$\mathrm{Card}(\{t \in \{iL, \dots, (i+1)L\} : \rho(x_t, \Omega(a)) > \epsilon\}) \le L_0.$$

Combined with (30) the equation above implies that

$$\mathrm{Card}(\{t \in \{0, \dots, T\} : \rho(x_t, \Omega(a)) > \epsilon\}) \le k_1 L_0 + L.$$

Combined with (26), (27), (30) and (31) the equation above implies that

$$T^{-1}\mathrm{Card}(\{t \in \{0, \dots, T\} : \rho(x_t, \Omega(a)) > \epsilon\})$$
$$\le k_1 L_0 T^{-1} + L T^{-1} \le L_0 L^{-1} + k_0^{-1} < \epsilon.$$

Theorem 4 is proved. □

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