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Investigation of Proximal Spaces Using Relators

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Abstract: In this paper, we define uniformities and proximities as relators and show the equivalences of these definitions with classical ones. For this, we summarize the essential properties of relators, using their theory from earlier works of Á. Száz. Moreover, we prove implications between important topological properties of relators and disprove others. Finally, we add an analogous definition for uniformly and proximally filtered properties.

Keywords: (generalized) uniformities; (generalized) proximities; relators

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1. Introduction

At the beginning of the 20. century some mathematicians tried to define abstract topological structures. The most relevant results are Poincaré 1895, Fréchet 1906, Hausdorff 1914, and Kuratowski 1922.

Uniform spaces in terms of relations were introduced by Weil in 1937 [1].

Proximities were first investigated by Riesz in 1909 [2], Effremovič and Smirnov in 1952 [3,4].

After the works of Davis, Pervin, and Nakano [5–7] in 1987, Száz [8] introduces the notion of relator and relator space in the following way.

Definition 1. *A nonvoid family $\mathcal{R}$ of relations on a nonvoid set X is called a relator on X, and the ordered pair $(X, \mathcal{R})$ is called a relator space.*

In the last decades, a few authors investigated the interpretation of well-known topological properties in terms of relators. In 2016, relators defined in terms of proximity spaces were introduced in [9].

For more details, see, for instance, ref. [10], but for the readers' convenience, we summarize the necessary notions and notations.

Remark 1. *With the usual notations, $\mathcal{R}$ is a relator on X means that*

$$X \neq \emptyset, \qquad \emptyset \neq \mathcal{R} \subset \operatorname{Exp}(X^2),$$

where $\operatorname{Exp}(X)$ is the power set of X, and $X^2 = X \times X$.

If R is a relation on X, $x \in X$, and $A \subset X$, then the sets

$$R(x) = \{y \in X : (x,y) \in R\}, \qquad \text{and} \qquad R[A] = \bigcup_{x \in A} R(x)$$

are called the images of x and A under R, respectively.

2. Preliminary Concepts

Definition 2. *If R and S are relations on X, then the composition of R and S can be defined, such that $(R \circ S)(x) = R[S(x)]$ for all $x \in X$.*

Moreover, let $R^{-1} = \{(y,x) : (x,y) \in R\}$, $R^0 = \Delta_X = \{(x,x) : x \in X\}$ and $R^n = R \circ R^{n-1}$, for all $n = 1, 2, \ldots$. Finally, we say that R is

- *reflexive if* $R^0 \subset R$,
- *symmetric if* $R^{-1} \subset R$,
- *transitive if* $R^2 \subset R$.

Lemma 1. *If R is a relation on X, and $A, B \subset X$, then*

$$R[A] \subset B \iff R^{-1}[X \setminus B] \subset X \setminus A.$$

Proof.

$$R[A] \subset B \iff (\forall y \in A : x \in R(y) \implies x \in B) \iff$$
$$\iff (\forall y \in A : x \in X \setminus B \implies x \notin R(y)) \iff \forall (x,y) \in (X \setminus B) \times A : (x,y) \notin R,$$

therefore

$$R[A] \subset B \iff ((X \setminus B) \times A) \cap R = \emptyset.$$

Using this, with required objects, we have that

$$R^{-1}[X \setminus B] \subset X \setminus A \iff ((X \setminus (X \setminus A)) \times (X \setminus B) \cap R^{-1} = \emptyset.$$

□

Definition 3. *If $\mathcal{R}$ is a relator on X, then the relators*

$$\mathcal{R}^* = \{S \subset X^2 : \exists R \in \mathcal{R} : R \subset S\},$$

and

$$\mathcal{R}^\# = \{S \subset X^2 : \forall A \subset X : \exists R \in \mathcal{R} : R[A] \subset S[A]\},$$

are called the uniform and the proximal refinements of $\mathcal{R}$, respectively. For more details, see [10].

Moreover, for all $n = -1, 0, 1, 2, \ldots$, we define

$$\mathcal{R}^n = \{R^n : R \in \mathcal{R}\}.$$

Remark 2. *$*$ and $\#$ are really refinements as we defined in [10], that is they are self-increasing in the sense that*

$$\mathcal{R} \subset \mathcal{S}^* \iff \mathcal{R}^* \subset \mathcal{S}^* \quad \text{and} \quad \mathcal{R} \subset \mathcal{S}^\# \iff \mathcal{R}^\# \subset \mathcal{S}^\#,$$

or equivalently they are expansive, increasing and idempotent, in the sense that

$$\mathcal{R} \subset \mathcal{R}^*, \ \mathcal{R} \subset \mathcal{S} \implies \mathcal{R}^* \subset \mathcal{S}^*, \ \mathcal{R}^{**} = \mathcal{R}^*$$

and

$$\mathcal{R} \subset \mathcal{R}^\#, \ \mathcal{R} \subset \mathcal{S} \implies \mathcal{R}^\# \subset \mathcal{S}^\#, \ \mathcal{R}^{\#\#} = \mathcal{R}^\#,$$

for all $\mathcal{R}$ and $\mathcal{S}$ relators on X.

Moreover, $\#$ is $*$-dominating, $*$-invariant, $*$-absorbing and $*$-compatible, that is

$$\mathcal{R}^* \subset \mathcal{R}^\#, \qquad \mathcal{R}^\# = \mathcal{R}^{\#*}, \qquad \mathcal{R}^\# = \mathcal{R}^{*\#}, \qquad \mathcal{R}^{\#*} = \mathcal{R}^{*\#}.$$

For all $n = -1, 0, 1, 2, \ldots$ the mapping $\mathcal{R} \mapsto \mathcal{R}^n$ of relators on X is increasing.

Finally, $*$ *and* # *are inversion-compatible, that is for all* $\mathcal{R}$ *relators on* X

$$\mathcal{R}^{*-1} = \mathcal{R}^{-1*} \qquad \textit{and} \qquad \mathcal{R}^{\#-1} = \mathcal{R}^{-1\#}.$$

And we have that for all relators on X

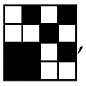
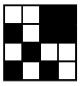
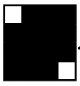
$$\mathcal{R}^{2*} = \mathcal{R}^{*2*}.$$

The following example shows, that the analog assertion is not true for #.

Example 1. *Let* $X = \{1, 2, 3, 4\}$*, and*

$$\mathcal{R} = \{\Delta_X \cup \{(1,2),(4,2),(2,1),(2,4)\}, \Delta_X \cup \{(1,3),(4,3),(3,1),(3,4)\}\}$$

is an elementwise reflexive and symmetric relator on X*. Now,* $\mathcal{R}^{\#2} \not\subset \mathcal{R}^{2\#}$*, since* $R = X^2 \setminus \{(1,4),(4,1)\} \in \mathcal{R}^{\#2}$ *however* $R \notin \mathcal{R}^{2\#}$*.*

Note, that $\mathcal{R} = \Big\{$  ,  $\Big\}$*, and* $R =$  .

Riesz or Efremovič proximity and strong inclusion are interdefinable. We will use the inverse relation of strong inclusion. For details, see, for instance, refs. [10,11].

Definition 4. *If* $\mathcal{R}$ *is a relator on* X*, then for any* $A, B \subset X$ *we write:*

$$B \in \mathrm{Int}_{\mathcal{R}}(A) \text{ if } R[B] \subset A \text{ for some } R \in \mathcal{R}.$$

The relation $\mathrm{Int}_{\mathcal{R}}$ *is called the proximal interiors induced by* $\mathcal{R}$ *on* X*.*

Theorem 1.

$$\mathrm{Int} : \mathrm{Exp}(\mathrm{Exp}(X^2)) \setminus \{\emptyset\} \to \mathrm{Exp}(\mathrm{Exp}(X)^2)$$

is a #–increasing set-valued function for relators on X *in the sense that*

$$\mathcal{S} \subset \mathcal{R}^{\#} \iff \mathrm{Int}_{\mathcal{S}} \subset \mathrm{Int}_{\mathcal{R}}$$

for any two relators $\mathcal{R}$ *and* $\mathcal{S}$ *on* X*.*

Moreover, it follows that Int *is increasing and if* $\mathcal{R}$ *is a relator on* X*, then* $\mathcal{R}^{\#}$ *is the largest relator on* X *such that*

$$\mathrm{Int}_{\mathcal{R}} = \mathrm{Int}_{\mathcal{R}^{\#}}.$$

3. A New Form of Generalized Proximities

Definition 5. *Let* $\mathcal{R}$ *is a relator on* X*, and* $\square \in \{*, \#\}$ *is a refinement for relators on* X*. We define the followings.*

- $\mathcal{R}$ *is* $\square$*-reflexive, if* $\mathcal{R} \subset \mathcal{R}^{0\square}$*;*
- $\mathcal{R}$ *is* $\square$*-symmetric, if* $\mathcal{R} \subset \mathcal{R}^{-1\square}$*;*
- $\mathcal{R}$ *is* $\square$*-transitive, if* $\mathcal{R} \subset \mathcal{R}^{2\square}$*;*
- $\mathcal{R}$ *is* $\square$*-fine, if* $\mathcal{R} = \mathcal{R}^{\square}$*.*

For instance, we say that $\mathcal{R}$ *is uniformly-symmetric or proximally-transitive instead of* $*$*-symmetric or #-transitive, respectively.*

Following Weil, we say that the $\mathcal{R}$ relator on X, is a generalized uniformity / generalized proximity on X, and the ordered pair $(X, \mathcal{R})$ is a generalized uniform space / generalized proximal space if it is

- uniformly/proximally reflexive;

- uniformly/proximally symmetric;
- uniformly/proximally transitive;
- uniformly/proximally fine.

Definition 6. *If the binary relation* Int *on the powerset* $\text{Exp}(X)$ *of a non-empty set* X *satisfies the following axioms for arbitrary* $A, B, C, D \subset X$*, then we say that it is a generalized set-proximity on* X*.*

P1 $A \in \text{Int}(B) \implies A \subset B$;

P2 $A \in \text{Int}(B) \implies X \setminus B \in \text{Int}(X \setminus A)$;

P3 $A \in \text{Int}(B) \implies \exists C \subset X : A \in \text{Int}(C)$ *and* $C \in \text{Int}(B)$;

P4 $A \subset B$ *and* $C \subset D : B \in \text{Int}(C) \implies A \in \text{Int}(D)$;

P5 $X \in \text{Int}(X)$ *and* $\emptyset \in \text{Int}(\emptyset)$.

Moreover, if it also satisfies

P6 $\text{Int}(A) \cap \text{Int}(B) = \text{Int}(A \cap B)$;

then we say that Int *is a set-proximity on* X*.*

P4 is missing from axioms of strong inclusions in [11]. The following example shows that all of the other axioms do not imply P4.

Example 2. *Assume that* X *has at least 3 elements, and let*

$$\text{Int} = \bigcup_{A \subset X} \{(\emptyset, A), (A, A), (A, X)\}.$$

In contrast, an Int generated by an additive δ relation has P4 property. The δ relation is named additive, if $A\delta(B \cup C) \iff A\delta B$ or $A\delta C$. Moreover, a δ relation generates Int by rule $A \notin \text{Int}(X \setminus B) \iff A\delta B$.

Proposition 1. *If* $\mathcal{R}$ *is a relator on* X*, and*

$$\psi : \text{Exp}(\text{Exp}(X)^2)) \to \text{Exp}(\text{Exp}(X^2)),\ \psi(\text{Int}) = \{A \times B \cup (X \setminus A) \times X : (B, A) \in \text{Int}\},$$

then

1. $\psi(\text{Int}_{\mathcal{R}})^{\#} = \mathcal{R}^{\#}$;
2. $\text{Int}_{\mathcal{R}}$ *satisfies* P4 *and* P5;
3. *If* P4 *and* P5 *hold for* $\text{Int} \subset \text{Exp}(X)^2$*, then* $\text{Int} = \text{Int}_{\psi(\text{Int})}$;
4. $\mathcal{R}$ *is proximally reflexive iff* $\text{Int}_{\mathcal{R}}$ *is* P1;
5. $\mathcal{R}$ *is proximally symmetric iff* $\text{Int}_{\mathcal{R}}$ *is* P2;
6. *if* $\mathcal{R}$ *is proximally transitive, then* $\text{Int}_{\mathcal{R}}$ *is* P3;

Proof. 1. At first, let $S \in \psi(\text{Int}_{\mathcal{R}})$ and $H \subset X$. If $S[H] = X$ or $H = \emptyset$, then for an arbitrary $R \in \mathcal{R}$ obviously $R[H] \subset S[H]$.

If $S[H] \neq X$ and $H \neq \emptyset$, then there exists an $A \supset H$ such that $A \times S[H] \cup (X \setminus A \times X) = S$, that is $(S[H], A) \in \text{Int}_{\mathcal{R}}$. By using the definition of $\text{Int}_{\mathcal{R}}$, we have that, $R[H] \subset R[A] \subset S[H]$ for some $R \in \mathcal{R}$.

Now, we have $\psi(\text{Int}_{\mathcal{R}}) \subset \mathcal{R}^{\#}$, which implies

$$\psi(\text{Int}_{\mathcal{R}})^{\#} \subset \mathcal{R}^{\#}.$$

For the converse inclusion let $R \in \mathcal{R}$. For any $A \subset X$, $(R[A], A) \in \mathrm{Int}_{\mathcal{R}}$ and then $S = A \times R[A] \cup (X \setminus A) \times X \in \psi(\mathrm{Int}_{\mathcal{R}})$. Since $S[A] = R[A]$ we have $R \in \psi(\mathrm{Int}_{\mathcal{R}})^{\#}$, that is $\mathcal{R} \subset \psi(\mathrm{Int}_{\mathcal{R}})^{\#}$, which implies

$$\mathcal{R}^{\#} \subset \psi(\mathrm{Int}_{\mathcal{R}})^{\#}.$$

2. $R[X] \subset X$ and $R[\emptyset] = \emptyset$ for all R relations on X, therefore P5 is true for $\mathrm{Int}_{\mathcal{R}}$.

   Moreover, if $\mathcal{R}$ is a relator on X, $A \subset B$, $C \subset D$ and $B \in \mathrm{Int}_{\mathcal{R}}(C)$, then there exists an $R \in \mathcal{R}$ such that $R[A] \subset R[B] \subset C \subset D$. In this case, $A \in \mathrm{Int}_{\mathcal{R}}(D)$, that is P4 is also true for $\mathrm{Int}_{\mathcal{R}}$.
3. At first, note that if P5 holds, then $\mathrm{Int} \neq \emptyset$, therefore $\psi(\mathrm{Int}) \neq \emptyset$, that is $\psi(\mathrm{Int})$ is a relator on X.

   If $A \in \mathrm{Int}(D)$, then $R = A \times D \cup (X \setminus A) \times X \in \psi(\mathrm{Int})$. Now, $R[A] \subset D$ follows $A \in \mathrm{Int}_{\psi(\mathrm{Int})}(D)$. Note, that $R[A] = D$ holds only if $A \neq \emptyset$ or $D = \emptyset$.

   Conversely, if $A \in \mathrm{Int}_{\psi(\mathrm{Int})}(D)$, then there exists $R \in \psi(\mathrm{Int})$ such that $R[A] \subset D$. Such an R is equal to $B \times C \cup (X \setminus B) \times X$ for some $(C, B) \in \mathrm{Int}$. In the case of $D \neq X$ and $A \neq \emptyset$, we have that $C = R[A] \subset D$, and $A \subset B$, and then P4 follows that $A \in \mathrm{Int}(D)$.

   In the case of $D = X$ or $A = \emptyset$, it is quite obvious that if P4 and P5 hold for $\mathrm{Int} \subset \mathrm{Exp}(X)^2$, then $A \in \mathrm{Int}(X)$ and $\emptyset \in \mathrm{Int}(D)$ for all $A, D \subset X$.
4. We note that $\mathcal{R}$ is proximally reflexive iff R is a reflexive relation for all $R \in \mathcal{R}$.

   If $A \in \mathrm{Int}_{\mathcal{R}}(B)$, then by using the reflexivity of $\mathcal{R}$ and the definition of $\mathrm{Int}_{\mathcal{R}}$, we have that $A \subset R[A] \subset B$ for some $R \in \mathcal{R}$.

   Conversely, for all $R \in \mathcal{R}$ and $x \in X$, $R[\{x\}] \subset R(x)$ implies $\{x\} \in \mathrm{Int}_{\mathcal{R}}(R(x))$. P1 follows $x \in R(x)$.
5. $A \in \mathrm{Int}_{\mathcal{R}}(B) \implies \exists R \in \mathcal{R} : R[A] \subset B$. If $\mathcal{R}$ is proximally symmetric, then there exists $Q \in \mathcal{R}$ such that $Q^{-1}[A] \subset R[A]$. Lemma 1 follows $X \setminus B \in \mathrm{Int}_{\mathcal{R}}(X \setminus A)$.

   Conversely, if $R \in \mathcal{R}$ and $A \subset X$, then $A \in \mathrm{Int}_{\mathcal{R}}(R[A])$, therefore since P2 $X \setminus R[A] \in \mathrm{Int}_{\mathcal{R}}(X \setminus A)$ and there exists $S \in \mathcal{R}$ such that $S[X \setminus R[A]] \subset X \setminus A$. By using Lemma 1 there exists $Q = S^{-1} \in \mathcal{R}^{-1}$ such that $Q[A] \subset R[A]$, proving that $R \in \mathcal{R}^{-1\#}$.
6. $A \in \mathrm{Int}_{\mathcal{R}}(B) \implies \exists R \in \mathcal{R} : R[A] \subset B$. If $\mathcal{R}$ is proximally transitive, then there exists $S \in \mathcal{R}$ such that $S^2[A] \subset R[A]$. With $C = S[A]$, we have that $A \in \mathrm{Int}_{\mathcal{R}}(C)$. Moreover $S[C] \subset R[A] \subset B$, and it follows that $C \in \mathrm{Int}_{\mathcal{R}}(B)$. □

Some papers write $\{R \circ S : R, S \subset \mathcal{R}\}$ instead of our $\mathcal{R}^2$. With this definition, the converse implication of 6 follows, but because of the definition of uniformities, we need this one. Otherwise, in the proof of the following Theorem, we will see that with some other assumption, the converse implication of 6 is true with our notions.

Remark 3. *Note, that with the above notations*

$$\psi(\mathrm{Int}_{\mathcal{R}}) = \Big\{R \in \mathcal{R}^{\#} : \exists B \subset X : \forall x \in X : R(x) \in \{B, X\}\Big\}.$$

Theorem 2. *Let $\mathcal{R}$ be a relator on X.*

1. *$\mathcal{R} \mapsto \mathrm{Int}_{\mathcal{R}}$ is a bijection of the set of all proximally fine relators on X onto the set of P4 and P5 relations on the powerset of X.*
2. *$\mathcal{R} \mapsto \mathrm{Int}_{\mathcal{R}}$ is a bijection of the set of generalized proximities on X onto the set of generalized set-proximities on X.*

Proof. Since Proposition 1, we need to prove only that if $\mathrm{Int}_{\mathcal{R}}$ is a generalized set-proximity, and $\mathcal{R}$ is proximally fine, then $\mathcal{R}$ is proximally transitive. For this, let $R \in \mathcal{R}$ be arbitrary.

Then for any $A \subset X$ we have that $A \in \mathrm{Int}_{\mathcal{R}}(R[A])$, therefore since P3 there exists $C \subset X$ such that $A \in \mathrm{Int}_{\mathcal{R}}(C)$ and $C \in \mathrm{Int}_{\mathcal{R}}(R[A])$. That is, there exist $P, Q \in \mathcal{R}$ such that $Q[A] \subset C$ and $P[C] \subset R[A]$. Now

$$S = \Big(A \times Q[A]\Big) \cup \Big((Q[A] \setminus A) \times P[Q[A]]\Big) \cup \Big((X \setminus Q[A]) \times X\Big) \in \mathcal{R}$$

and $S^2[A] \subset R[A]$, that is $R \in \mathcal{R}^{2\#}$.

Really, since P1 $\mathcal{R}$ is proximally reflexive, therefore we have that $A \subset Q[A] \subset P[Q[A]] \subset P[C] \subset R[A]$. Now, for an arbitrary $B \subset X$, $Q[B] \subset S[B]$, if $B \subset A$ and $P[B] \subset S[B]$ if $B \not\subset A$, that is $S \in \mathcal{R}^\# = \mathcal{R}$ because of $\mathcal{R}$ is proximally fine.

Moreover, it is easy to see that if $A \neq Q[A]$, then

$$S^2[A] = S[S[A]] = S[Q[A]] = S[Q[A] \setminus A] \cup S[A] = P[Q[A]] \cup Q[A] = P[Q[A]] \subset R[A],$$

otherwise if $A = Q[A]$, then

$$S^2[A] = S[S[A]] = S[Q[A]] = S[A] = Q[A] \subset R[A]. \quad \square$$

4. A New Form of Proximities

Definition 7. *Let $\mathcal{A}$ be a family of sets, or equivalently $\mathcal{A} \subset \mathrm{Exp}(X)$ for some set X. We call*

$$\Phi(\mathcal{A}) = \Big\{\bigcap \mathcal{B} : \emptyset \neq \mathcal{B} \subset \mathcal{A}, \text{ and } \mathcal{B} \text{ is finite}\Big\}$$

the filtered family of sets generated by $\mathcal{A}$.

Moreover, we say that $\mathcal{A}$ is filtered if $\Phi(\mathcal{A}) = \mathcal{A}$.

Finally, if R is a relation on $\mathrm{Exp}(X)$ for some set X, then we say that R is a filtered relation if

$$\Big\{\Big(\bigcap \mathrm{Dom}_I, \bigcap \mathrm{Ran}_I\Big) : \emptyset \neq I \subset R \text{ is finite}\Big\} = R.$$

Remark 4. *Since Φ is a refinement for relators on X, we write $\mathcal{R}^\Phi$ instead of $\Phi(\mathcal{R})$, if $\mathcal{R}$ is a relator on X. Note also that $\mathcal{R}$ is filtered iff $\mathcal{R}^\Phi \subset \mathcal{R}$.*

Moreover, note that Φ is an inversion compatible refinement for relators on X, that is, if $\mathcal{R}$ is a relator on X, then $\mathcal{R}^{-1\Phi} = \mathcal{R}^{\Phi-1}$.

Finally, if $\mathcal{R}$ is finite, then $\mathcal{R}^{\Phi*} = \{\bigcap \mathcal{R}\}^*$.

Lemma 2. *If $\mathcal{R}$ is a relator on X, then $\mathcal{R}^{*\Phi} = \mathcal{R}^{\Phi*}$, $\mathcal{R}^{2\Phi} \subset \mathcal{R}^{\Phi2*}$.*

Proof. It is easy to see, that

$$\exists \emptyset \neq \mathcal{S} \subset \mathcal{R}^* \text{ finite } : \bigcap \mathcal{S} = R \iff \exists \emptyset \neq \mathcal{Q} \subset \mathcal{R} \text{ finite } : \bigcap \mathcal{Q} \subset R.$$

Moreover, if $R \in \mathcal{R}^{2\Phi}$, then there exists an $\emptyset \neq \mathcal{S} \subset \mathcal{R}$ finite such that $R = \bigcap \mathcal{S}^2$. $\bigcap \mathcal{S} \in \mathcal{R}^\Phi$ and $(\bigcap \mathcal{S})^2 \subset \bigcap \mathcal{S}^2$ implies that $R \in \mathcal{R}^{\Phi2*}$. $\square$

Proposition 2. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent.*

1. $\mathcal{R}^\Phi \subset \mathcal{R}^*$;
2. *there exists an $\mathcal{S}$ relator on X, such that $\mathcal{S}^\Phi \subset \mathcal{S}^* = \mathcal{R}^*$.*

Proof. We need only to prove the (2) $\Longrightarrow$ (1) implication. For this, we note that if (2) is true then

$$\mathcal{R}^\Phi \subset \mathcal{R}^{\Phi*} = \mathcal{R}^{*\Phi} = \mathcal{S}^{*\Phi} = \mathcal{S}^{\Phi*} \subset \mathcal{S}^* = \mathcal{R}^*.$$

$\square$

In contrast to Example 1, the following lemma shows that if $\mathcal{R}$ is a filtered relator on X, then $\mathcal{R}^{\#2\#} = \mathcal{R}^{2\#}$.

Lemma 3. *If $\mathcal{R}$ is a relator on X, then $\mathcal{R}^{\#2\#} \subset \mathcal{R}^{\Phi 2\#}$.*

Proof. By definition of # if $R \in \mathcal{R}^{\#2\#}$, then for all $A \subset X$, there exists an $S \in \mathcal{R}^{\#}$ such that $S^2[A] \subset R[A]$. For such an $S \in \mathcal{R}^{\#}$, there exists $P, Q \in \mathcal{R}$ with $Q[A] \subset S[A]$, and $P[S[A]] \subset S[S[A]]$. Now, for $T = P \cap Q \in \mathcal{R}^{\Phi}$ we have that

$$T^2[A] = T[T[A]] \subset P[Q[A]] \subset P[S[A]] \subset S^2[A] \subset R[A].$$

We get that for all $A \subset X$ there exists a $T \in \mathcal{R}^{\Phi}$ such that $T^2[A] \subset R[A]$, that is $R \in \mathcal{R}^{\Phi 2\#}$. □

Lemma 4. *If R is a filtered relation on X with P2 property, and $\emptyset \neq I \subset R$ is finite, then*

$$\left(\bigcup \mathrm{Dom}_I, \bigcup \mathrm{Ran}_I\right) \in R.$$

Proof. Since P2 property of R we have that

$$J = \{(X \setminus A, X \setminus B) : (B, A) \in I\} \subset R.$$

Of course, J is nonempty and finite, therefore $(\bigcap \mathrm{Dom}_J, \bigcap \mathrm{Ran}_J) \in R$. Moreover, it is also quite obvious, that

$$\left(\bigcap \mathrm{Dom}_J, \bigcap \mathrm{Ran}_J\right) = \left(\bigcap_{(B,A)\in I} X \setminus A, \bigcap_{(B,A)\in I} X \setminus B\right) = \left(X \setminus \bigcup \mathrm{Ran}_I, X \setminus \bigcup \mathrm{Dom}_I\right).$$

Now, P2 property of R proves the Lemma. □

Proposition 3. *If $\mathcal{R}$ is a proximally symmetric relator on X, and*

$$\psi : \mathrm{Exp}(\mathrm{Exp}(X)^2) \to \mathrm{Exp}(\mathrm{Exp}(X^2)),\ \psi(\mathrm{Int}) = \{A \times B \cup (X \setminus A) \times X : (B, A) \in \mathrm{Int}\},$$

then the following assertions are equivalent.

1. $\mathrm{Int}_{\mathcal{R}}(A) \cap \mathrm{Int}_{\mathcal{R}}(B) \subset \mathrm{Int}_{\mathcal{R}}(A \cap B)$;

P6 $\mathrm{Int}_{\mathcal{R}}(A) \cap \mathrm{Int}_{\mathcal{R}}(B) = \mathrm{Int}_{\mathcal{R}}(A \cap B)$;

2. $\mathrm{Int}_{\mathcal{R}}$ *is a filtered relation;*
3. $(\psi(\mathrm{Int}_{\mathcal{R}}))^{\Phi} \subset (\psi(\mathrm{Int}_{\mathcal{R}}))^{\#}$
4. *there exists an $\mathcal{S}$ relator on X, such that $\mathcal{S}^{\Phi} \subset \mathcal{S}^{\#} = \mathcal{R}^{\#}$.*

Proof. (1) $\Longrightarrow$ P6: Since $\mathrm{Int}_{\mathcal{R}}$ has the P4 property, hence $\mathrm{Int}_{\mathcal{R}}(A \cap B) \subset \mathrm{Int}_{\mathcal{R}}(A) \cap \mathrm{Int}_{\mathcal{R}}(B)$.

P6 $\Longrightarrow$ (2): If P6 holds for $\mathrm{Int}_{\mathcal{R}}$ and $(B, A), (D, C) \in \mathrm{Int}_{\mathcal{R}}$, then by using again the P4 property of $\mathrm{Int}_{\mathcal{R}}$ we have that $A \cap C \in \mathrm{Int}_{\mathcal{R}}(B) \cap \mathrm{Int}_{\mathcal{R}}(D) = \mathrm{Int}_{\mathcal{R}}(B \cap D)$. Now, we can prove the implication by induction.

(2) $\Longrightarrow$ (3): Let $\mathcal{A}$ be an arbitrary nonempty and finite subset of $\psi(\mathrm{Int}_{\mathcal{R}})$, and let $C \subset X$ be arbitrary. We need to prove that there exists an $R \in \psi(\mathrm{Int}_{\mathcal{R}})$ such that $R[C] \subset (\bigcap \mathcal{A})[C]$.

For this, let $I \subset \mathrm{Int}_{\mathcal{R}}$ be nonempty and finite such that $\mathcal{A} = \psi(I)$.

If there exists $x \in C \setminus \bigcup \mathrm{Ran}_I$, then such an x we have that $x \notin A$ for all $(B, A) \in I$, and hence $S(x) = X$ for all $S \in \psi(I)$, therefore $R[C] \subset X = (\bigcap \psi(I))(x) \subset (\bigcap \psi(I))[C]$ for all $R \in \psi(\mathrm{Int}_{\mathcal{R}})$.

If $C = \emptyset$, then $R[C] = R[\emptyset] = \emptyset = (\bigcap \psi(I))[\emptyset] = (\bigcap \psi(I))[C]$ for all $R \in \mathcal{R}$. Otherwise, if $\emptyset \neq C \subset \bigcup \mathrm{Ran}_I$, then with

$$\mathbb{E} = \left\{ J \subset I : \left(\bigcap \mathrm{Ran}_J \setminus \bigcup \mathrm{Ran}_{I \setminus J} \right) \cap C \neq \emptyset \right\}$$

we have that J is nonempty finite for all $J \in \mathbb{E}$ and $\mathbb{E}$ is nonempty and finite and

$$C \subset \bigcup_{J \in \mathbb{E}} \left(\bigcap \mathrm{Ran}_J \setminus \bigcup \mathrm{Ran}_{I \setminus J} \right) \subset \bigcup_{J \in \mathbb{E}} \bigcap \mathrm{Ran}_J .$$

Since $\mathrm{Int}_{\mathcal{R}}$ is a filtered relation, and all $J \in \mathbb{E}$ is a nonempty and finite subset of $\mathrm{Int}_{\mathcal{R}}$, it is clear that $\left(\bigcap \mathrm{Dom}_J, \bigcap \mathrm{Ran}_J \right) \in \mathrm{Int}_{\mathcal{R}}$ for all $J \in \mathbb{E}$. By using Lemma 4 we have that

$$\bigcup_{J \in \mathbb{E}} \bigcap \mathrm{Ran}_J \in \mathrm{Int}_{\mathcal{R}} \left(\bigcup_{J \in \mathbb{E}} \bigcap \mathrm{Dom}_J \right).$$

$\bigcap \mathrm{Dom}_J = (\bigcap \psi(I)) \left[\left(\bigcap \mathrm{Ran}_J \setminus \bigcup \mathrm{Ran}_{I \setminus J} \right) \cap C \right] \subset (\bigcap \psi(I))[C]$ for all $J \in \mathbb{E}$. Therefore,

$$\bigcup_{J \in \mathbb{E}} \bigcap \mathrm{Dom}_J \subset \left(\bigcap \psi(I) \right)[C].$$

P4 property of $\mathrm{Int}_{\mathcal{R}}$ follows that $C \in \mathrm{Int}_{\mathcal{R}}((\bigcap \psi(I))[C])$

Figure 1 shows the case, when

$$I = \{(B_1, A_1), (B_2, A_2), (B_3, A_3)\},$$

and

$$\mathbb{E} = \left\{ \begin{matrix} \{(B_1, A_1)\}, \{(B_2, A_2)\}, \\ \{(B_1, A_1), (B_2, A_2)\}, I \end{matrix} \right\}.$$

The striped, dotted, and filled partitions are $\bigcap \mathrm{Ran}_J \setminus \bigcup \mathrm{Ran}_{I \setminus J}$ for $J \in \mathbb{E}$.

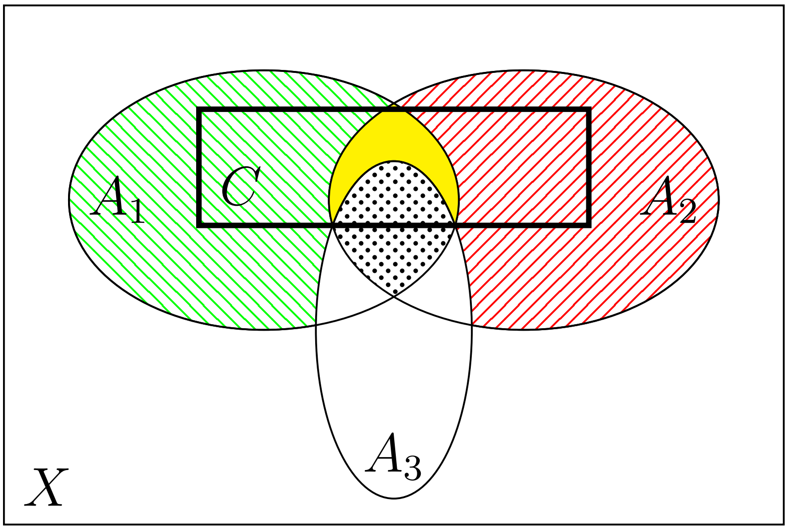


Figure 1. A Venn diagram.

(3)$\Longrightarrow$ (4): It is quite obvious since assertion (1) of Proposition 1.

(4)$\Longrightarrow$ (1): If $C \in \mathrm{Int}_{\mathcal{R}}(A) \cap \mathrm{Int}_{\mathcal{R}}(B)$, then $C \in \mathrm{Int}_{\mathcal{R}}(A) = \mathrm{Int}_{\mathcal{R}^\#}(A) = \mathrm{Int}_{\mathcal{S}^\#}(A) = \mathrm{Int}_{\mathcal{S}}(A)$ and $C \in \mathrm{Int}_{\mathcal{R}}(B) = \mathrm{Int}_{\mathcal{R}^\#}(B) = \mathrm{Int}_{\mathcal{S}^\#}(B) = \mathrm{Int}_{\mathcal{S}}(B)$, therefore $R[C] \subset A$ and

$S[C] \subset B$ for some $R, S \in \mathcal{S}$. $R \cap S \in \mathcal{S}^{\Phi} \subset \mathcal{S}^{\#}$ and $(R \cap S)[C] \subset R[C] \cap S[C] \subset A \cap B$, that is $C \in \mathrm{Int}_{\mathcal{S}^{\#}}(A \cap B) = \mathrm{Int}_{\mathcal{R}^{\#}}(A \cap B) = \mathrm{Int}_{\mathcal{R}}(A \cap B)$. □

Int is the inverse of strong inclusion of Riesz and Efremovič proximities if it satisfies P1–P6.

Unfortunately, Int is a filtered relation on X, is neither sufficient nor necessary for P6 property of Int without P4 is assumed. For instance, if X is a nonvoid set, then $\mathrm{Int} = \{(\emptyset, \emptyset)\}$ is a filtered relation, but not P6. On the other hand, $\{\emptyset, X\}^2 \setminus \mathrm{Int}$ is not a filtered relation, but P6.

Because of the above Proposition, we define the uniformly and proximally filtered properties of a relator by the following.

Definition 8. *If $\square$ is a refinement for relators on X, then we say that the $\mathcal{R}$ relator on X is $\square$-filtered if there exists an $\mathcal{S}$ relator on X such that $\mathcal{S}^{\Phi} \subset \mathcal{S}^{\square} = \mathcal{R}^{\square}$.*

We use the uniformly filtered and proximally filtered notions instead of $*$-filtered and #-filtered.

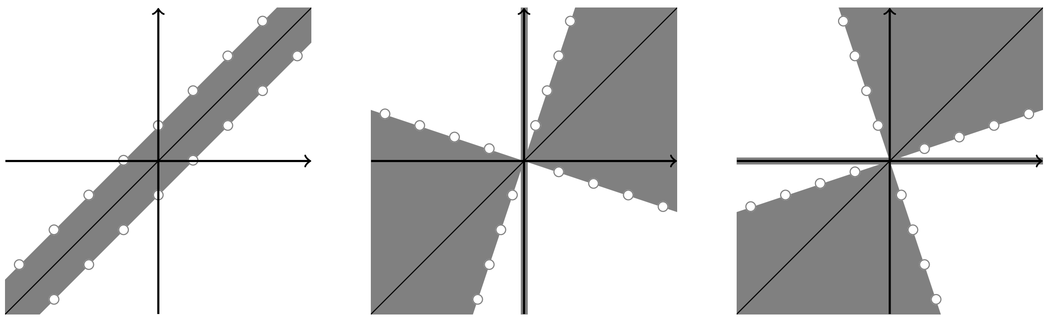
Note, that by Proposition 2 we have that $\mathcal{R}$ is a uniformly filtered relator on X iff $\mathcal{R}^{\Phi} \subset \mathcal{R}^{*}$ which can be found in definition of uniformities by Weil.

Example 3. *In general, $\mathcal{R}^{\Phi} \subset \mathcal{R}^{\#}$ is only a sufficient but not necessary condition of proximally filtered property of $\mathcal{R}$. Namely, let $\mathbb{R}$ be the set of reals, and $\mathcal{M} = \{B_\varepsilon : \varepsilon > 0\}$ where $B_\varepsilon = \{(x, y) \in \mathbb{R}^2 : |x - y| < \varepsilon\}$ for all $\varepsilon > 0$ (see Figure 2a). Then $\mathcal{M}$ is elementwise reflexive, elementwise symmetric, (proximally) filtered, and $\mathcal{M} = \mathcal{M}^2$.*

Moreover, let $R(x) = \begin{cases}]-\frac{1}{3}x, 3x[, & \text{if } x > 0, \\ \mathbb{R}, & \text{if } x = 0, \\ ]3x, -\frac{1}{3}x[, & \text{if } x < 0 \end{cases}$ be a relation on $\mathbb{R}$. Now, $R \in \mathcal{M}^{\#}$, therefore $R^{-1} \in \mathcal{M}^{\#-1} = \mathcal{M}^{-1\#} = \mathcal{M}^{\#}$ (see Figure 2b,c).

Really, if $\emptyset \neq A \subset \mathbb{R}$ is bounded and $0 \notin A$, then $R[A] \supset B_\varepsilon[A]$, with $\varepsilon = \frac{1}{3}\max\{|\inf A|, |\sup A|\}$. In all other cases $B_1[A] \subset R[A]$.

We have that $R \cap R^{-1} \in \mathcal{M}^{\#\Phi}$, but $R \cap R^{-1} \notin \mathcal{M}^{\#} = \mathcal{M}^{\#\#}$ because if $I =]0, 1[$, then $-\frac{\varepsilon}{2} \in B_\varepsilon[I] \setminus (R \cap R^{-1})[I]$ for all $\varepsilon > 0$. Therefore $(\mathcal{M}^{\#})^{\Phi} \not\subset (\mathcal{M}^{\#})^{\#}$, however $\mathcal{M}^{\#}$ is proximally filtered because $\mathcal{M}^{\Phi} = \mathcal{M} \subset \mathcal{M}^{\#} = (\mathcal{M}^{\#})^{\#}$.



(**a**) $B_\varepsilon \in \mathcal{M}$ (**b**) R (**c**) R^{-1}

Figure 2. Relations on $\mathbb{R}$.

Definition 9. *If the generalized uniformity/generalized proximity $\mathcal{R}$ on X is also*

- *uniformly/proximally filtered,*

then we say that $\mathcal{R}$ is a uniformity/proximity on X, and $(X, \mathcal{R})$ is a uniform space/proximal space.

Propositions 1 and 3 give the following.

Theorem 3. *If $\mathcal{R}$ is a relator on X, then $\mathcal{R} \mapsto \mathrm{Int}_{\mathcal{R}}$ is a bijection of the set of proximities on X onto the set of set-proximities on X.*

Proof. Since Proposition 3, the range of the restriction of the bijection in Theorem 2 (2) to the set of proximities on X is the set of set-proximities on X. □

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References

1. Weil, A. *Sur les Espaces a Structure Uniforme at sur la Topologie Générale*; Actualités Sci. Ind. 551; Herman and Cie: Paris, France, 1937.
2. Riesz, F. Stetigkeit und abstrakte Mengenlehre. *Rom. Math. Kongr.* **1909**, *4*, 18–24.
3. Efremovič, V.A. The geometry of proximity. *Mat. Sb.* **1952**, *73*, 189–200.
4. Smirnov, Y.M. On proximity spaces. *Mat. Sb.* **1952**, *73*, 543–574.
5. Davis, A.S. Indexed systems of neighborhoods for general topological spaces. *Amer. Math. Mon.* **1961**, *68*, 886–893. [CrossRef]
6. Nakano, H.; Nakano, K. Connector theory. *Pac. J. Math.* **1975**, *56*, 195–213. [CrossRef]
7. Pervin, W.J. Quasi-uniformization of topological spaces. *Math. Ann.* **1962**, *147*, 316–317. [CrossRef]
8. Száz, Á. Basic tools and mild continuities in relator spaces. *Acta Math. Hungar.* **1987**, *50*, 177–201. [CrossRef]
9. Peters, J.F. Proximal relator spaces. *Filomat* **2016**, *30*, 469–472. [CrossRef]
10. Pataki, G. and Száz, Á. A unified treatment of well-chainedness and connectedness properties. *Acta Math. Acad. Paedagog. Nyházi. (N.S.)* **2003**, *19*, 101–166.
11. Di Concilio, A. Proximity: A powerful tool in extension theory, function spaces, hyperspaces, boolean algebras and point-free geometry. *Beyond Topol.* **2009**, *486*, 89–112.