



# General Properties on Differential Sets of a Graph

Ludwin A. Basilio<sup>1</sup>, Sergio Bermudo<sup>2</sup>, Juan C. Hernández-Gómez<sup>1</sup>  and José M. Sigarreta<sup>1,\*</sup> 

<sup>1</sup> Faculty of Mathematics, Autonomous University of Guerrero, Carlos E. Adame 5, Col. La Garita, Acapulco 39650, Mexico; ludwin.ali@gmail.com (L.A.B.); jcarloshg@gmail.com (J.C.H.-G.)

<sup>2</sup> Department of Economy, Quantitative Methods and Economic History, Pablo de Olavide University, Carretera de Utrera Km. 1, 41013 Sevilla, Spain; sbernav@upo.es

\* Correspondence: josemariasigarretaalmira@hotmail.com

**Abstract:** Let  $G = (V, E)$  be a graph, and let  $\beta \in \mathbb{R}$ . Motivated by a service coverage maximization problem with limited resources, we study the  $\beta$ -differential of  $G$ . The  $\beta$ -differential of  $G$ , denoted by  $\partial_\beta(G)$ , is defined as  $\partial_\beta(G) := \max\{|B(S)| - \beta|S| \text{ such that } S \subseteq V\}$ . The case in which  $\beta = 1$  is known as the differential of  $G$ , and hence  $\partial_\beta(G)$  can be considered as a generalization of the differential  $\partial(G)$  of  $G$ . In this paper, upper and lower bounds for  $\partial_\beta(G)$  are given in terms of its order  $|G|$ , minimum degree  $\delta(G)$ , maximum degree  $\Delta(G)$ , among other invariants of  $G$ . Likewise, the  $\beta$ -differential for graphs with heavy vertices is studied, extending the set of applications that this concept can have.

**Keywords:** differential; domination number; independence number



**Citation:** Basilio, L.A.; Bermudo, S.; Hernández-Gómez, J.C.; Sigarreta, J.M. General Properties on Differential Sets of a Graph. *Axioms* **2021**, *10*, 265. <https://doi.org/10.3390/axioms10040265>

Academic Editor: Elena Guardo

Received: 4 August 2021

Accepted: 11 October 2021

Published: 20 October 2021

**Publisher's Note:** MDPI stays neutral with regard to jurisdictional claims in published maps and institutional affiliations.



**Copyright:** © 2021 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

## 1. Introduction

The boom that graph theory has had as an object of study has boosted its application in different contexts and its use as a form of mathematical modeling has become common. The increasing practice of modeling with networks or graphs when it is desired to represent the interactions between elements of a discrete system has made the study of graphs more and more attractive in different mathematical contexts. From the first studies carried out in this area of mathematics to the present day, there has been a long road that has led to the development of new theories, concepts, and methods that allow the study of increasingly complex structures but immersed in today's world. If we think about the type of applications that networks have today, we can see their importance, for example, the much mentioned social networks, supply chains, comorbidities that converge in some disease, forms of contagion or spread of communicable diseases, among others. In the field of optimization, the study of supply networks, for example, has led researchers to ask questions not only about the minimum size that a set with certain properties should have, but also about which elements should be contained in that set. When one wishes to meet the needs of a region by minimizing the number of supply centers but guaranteeing access to each locality in the region, there is an inherent cost-benefit function that one wishes to optimize. However, the problem of finding such a set is related to the differential set, which in [1] is proved to be an NP-complete problem. Let us start with a review of the most important concepts of this topic.

As usual, for the graph  $G = (V, E)$  we use  $n := |V|$ ,  $m := |E|$ ,  $\delta(G)$ , and  $\Delta(G)$  to denote its order, size, minimum degree, and maximum degree, respectively. For  $u, v \in V$ , the distance  $d(u, v)$ , is the length of shortest path between  $u$  and  $v$  in  $G$ . The diameter of a graph  $G$  denoted by  $D(G)$ , is the greatest distance between two vertices of  $G$ . We denote two adjacent vertices  $u$  and  $v$  by  $u \sim v$ . If  $v \in V$ , then  $N(v) = \{u \in V : u \sim v\}$ , also if  $S \subseteq V$ ,  $N_S(v) = \{u \in S : u \sim v\}$  is the set of neighbors of  $v$  in  $S$ ,  $N(S) = \cup_{v \in S} N(v)$ ,  $N[S] = N(S) \cup S$ , furthermore  $\deg(v) = |N(v)|$  is the degree of  $v$  in  $G$  and  $\deg_S(v) = |N_S(v)|$  is the degree of  $v$  in  $S$ . A dominating set  $D$  of a graph  $G$  is a set of

vertices of  $G$  such that every vertex of  $V \setminus D$  is adjacent to some vertex of  $D$ . The domination number of a graph  $G$ , denoted by  $\gamma(G)$  is the minimum cardinality of a dominating set of  $G$ . For  $S \subseteq V$ , let  $B(S)$  be the set of vertices in  $V \setminus S$  that have a neighbor in  $S$ , and let  $C(S) := V \setminus (S \cup B(S))$ . Then  $\{S, B(S), C(S)\}$  is a partition of  $V$ . Finally, an open interval will be denoted by  $(a, b)$  and a closed interval by  $[a, b]$ .

Suppose that  $G(V, E)$  represents the hospital network of a country, consisting of  $|V|$  hospitals interconnected by  $|E|$  roads. It is desired to provide specialized medical equipment to a certain number of these hospitals, with the condition that each hospital ( $h$ ) in the country serves to the population of the place where  $h$  is located, and serves also to those hospitals that are connected by a direct road to  $h$ , in addition to the fact that each hospital serves the same number of patients and that not necessarily all the hospitals in the country will be benefited. If the cost of equipping each hospital with the medical equipment is  $\alpha > 0$  times the benefit of choosing it, it follows that the total benefit of equipping  $|S|$  hospitals with such medical equipment will be  $|B(S)| + |S| - \alpha|S|$ , i.e.,  $|B(S)| - \beta|S|$  with  $\beta = \alpha - 1 > -1$ . This value is known as  $\beta$ -differential of the set  $S$  and is denoted by  $\partial_\beta(S)$ . We want to maximize the profit expression, it is, we want to find the following value:

$$\partial_\beta(G) := \max\{\partial_\beta(S) : S \subseteq V\} = \max\{|B(S)| - \beta|S| : S \subseteq V\}. \tag{1}$$

Note that if we take  $v \in V$  such that  $\deg(v) = \Delta(G)$ , then  $\partial_\beta(G) \geq \partial_\beta(\{v\}) = \Delta(G) - \beta$ , that is, if  $\beta < \Delta(G)$  there will always be at least one way to choose  $S$  such that  $|B(S)| - \beta|S| > 0$ , i.e., we will have benefits. On the other hand, if  $\beta \geq \Delta(G)$  then we have  $|B(S)| - \beta|S| \leq \Delta(G)|S| - \beta|S| = (\Delta(G) - \beta)|S| \leq 0$ , then there is no  $S \subseteq V$  that yields profit.

Then, we can restrict the value of  $\beta$  to the interval  $(-1, \Delta(G))$ . We recall that if we substitute  $\beta = 1$  in Equation (1), then we have the differential of  $G$ , which is usually written as  $\partial(G)$ . The differential of a graph and other kinds of differentials of a set were introduced in [2], where dimensions are given for the differential of a graph. Additionally, this parameter has been studied in [1,3–9]. In addition, the differential of the Cartesian product of graphs was studied in [10]. In [11], the differential of a set  $S$  was also considered, in that paper it was denoted by  $\eta(S)$ , on the other hand, the minimal differential of an independent set was studied in [12]. Moreover, in [2,13] the  $B$ -differential of a graph, also called enclavess number, was studied, which is defined as  $\psi(G) = \max\{|B(S)| : S \subseteq V\}$ .

Notice that, for every graph  $G$  with connected components  $G_1, \dots, G_k$ , we have  $\partial_\beta(G) = \partial_\beta(G_1) + \dots + \partial_\beta(G_k)$ . Therefore, we will only consider connected graphs.

## 2. Some Bounds for the Cardinality of $S$

The authors in [1,2,4–8,10] gave a lot of bounds on  $\partial(G)$ , but they did not obtain many on the cardinality of the differential sets. Now, considering the possible applications mentioned in the introduction, it seems convenient to find them.

We say that  $S \subseteq V$  is a *minimum dominating* set if it is a set of minimum cardinality among all dominating sets (see [14,15]). A subset  $S$  of  $V$  is a  $\beta$ -differential set of  $G$  if  $\partial_\beta(S) = \partial_\beta(G)$  and a *maximum (minimum)  $\beta$ -differential* set is a set of maximum (minimum) cardinality among all  $\beta$ -differential sets. A graph  $G$  is said to be *differential dominating* if  $G$  contains a differential set which is also a dominating set. Thus, we have that a graph  $G$  is differential dominating if and only if  $\partial(G) + 2\gamma(G) = n$ .

The following result is shown in [16].

**Lemma 1.** *Let  $G = (V, E)$  be a graph and let  $A$  be a dominating set in  $G$ . If  $S \subseteq V$  with  $|S| > |A|$ , then  $\partial_\beta(S) < \partial_\beta(A)$ . In particular,*

$$\partial_\beta(G) = \max\{\partial_\beta(S) : S \subseteq V, |S| \leq \gamma(G)\}.$$

Given a vertex  $v \in S$  we will say that  $u \in V \setminus S$  is an  $S$ -private neighbor of  $v$  if  $N(u) \cap S = \{v\}$ . We denote by  $PN_S(v)$  the set of all private neighbors of  $v$  in  $S$ .

**Proposition 1.** *If  $G$  is a differential dominating graph, we have the following properties:*

- (a)  $S$  is a maximum differential set if and only if it is a minimum dominating set.
- (b) If  $S$  is a minimum dominating set, then  $PN_S(v) \neq \emptyset$  for every  $v \in S$ .
- (c) If  $S$  is a minimum dominating set and  $v \in S$  satisfies that  $|PN_S(v)| = 1$ , then  $\deg_S(v) = 0$ .

**Proof.** (a) If  $S$  is a differential set, then, Lemma 1 implies that  $|S| \leq \gamma(G)$ , and so every minimum dominating set is a maximum differential set. If  $D$  is a minimum dominating set, and  $S$  is a maximum differential set, then,  $\gamma(G) = |D| \leq |S| \leq \gamma(G)$ . Using now that  $\partial(G) = |B(S)| - |S| = n - 2\gamma(G)$ , we have that  $|B(S)| = n - \gamma(G)$ , then  $S$  is a minimum dominating set.

(b) Let  $S$  be a minimum dominating set, and let  $v \in S$ . If  $PN_S(v) = \emptyset$ , then  $\partial(S \setminus \{v\}) \geq \partial(S) + 1 > \partial(G)$ , a contradiction.

(c) Let  $S$  be a minimum dominating set, and let  $v \in S$  be a vertex such that  $|PN_S(v)| = 1$ . If  $\delta_S(v) \geq 1$ , then  $\partial(S \setminus \{v\}) = \partial(S) + 1 > \partial(G)$ , a contradiction.  $\square$

The following lemma was proved in [16].

**Lemma 2 ([16]).** *Let  $G = (V, E)$  be a graph. If  $S$  is a minimum (respectively, maximum)  $\beta$ -differential set of  $G$ , then  $|B(S)| \geq (\lfloor \beta \rfloor + 1)|S|$  (respectively,  $|B(S)| \geq \lceil \beta \rceil |S|$ ).*

**Proposition 2.** *Let  $G$  be a graph of order  $n$ .*

- (1) *If  $S$  is a minimum  $\beta$ -differential set, then  $|S| \leq \frac{n}{\lfloor \beta \rfloor + 2}$ .*
- (2) *If  $S$  is a maximum  $\beta$ -differential set, then  $|S| \leq \frac{n}{\lceil \beta \rceil + 1}$ .*

**Proof.** If  $S$  is a minimum  $\beta$ -differential set of  $G$ , then

$$|B(S)| \geq (\lfloor \beta \rfloor + 1)|S|.$$

Moreover,  $n - |S| \geq |B(S)| \geq (\lfloor \beta \rfloor + 1)|S|$ . Hence,  $|S| \leq \frac{n}{\lfloor \beta \rfloor + 2}$ .

The same arguments allow us to prove the second part.  $\square$

**Proposition 3.** *Let  $G$  be a graph of order  $n$  and maximum degree  $\Delta \geq 2$ . If  $S \subseteq V$  is a differential set of  $G$ , then*

$$\left\lceil \frac{\partial(G)}{\Delta - 1} \right\rceil \leq |S| \leq \left\lfloor \frac{n - \partial(G)}{2} \right\rfloor.$$

**Proof.** Let  $S$  be a differential set of  $G$ . Since  $n - |S| \geq |B(S)|$ , we have that  $n - 2|S| \geq |B(S)| - |S| = \partial(S) = \partial(G)$ . Thus,

$$|S| \leq \left\lfloor \frac{n - \partial(G)}{2} \right\rfloor.$$

Moreover, note that for every differential set  $S$  we have

$$\partial(G) = |B(S)| - |S| \leq \Delta|S| - |S| = |S|(\Delta - 1).$$

Thus, we have

$$\left\lceil \frac{\partial(G)}{\Delta - 1} \right\rceil \leq |S|.$$

$\square$

In [6] it was proved that if  $G$  is a graph of order  $n \geq 3$ , then  $\partial(G) \geq \frac{n}{5}$ . As a consequence of the above proposition we have the following.

**Corollary 1.** *If  $G$  is a graph of order  $n \geq 3$  and maximum degree  $\Delta$ , then*

$$\left\lceil \frac{n}{5(\Delta - 1)} \right\rceil \leq |S| \leq \left\lfloor \frac{2n}{5} \right\rfloor.$$

**3. Relationships between  $\beta$ ,  $B(S)$  and  $C(S)$**

We said that  $S \subseteq V$  is a  $\beta$ -differential set of  $G$  if  $\partial_\beta(S) = \partial_\beta(G)$ . Now, we will see some relationships between the value of  $\beta$  and the degree of the elements in the sets  $B(S)$  and  $C(S)$ .

**Lemma 3.** *Let  $G$  be a graph. If  $S$  is a  $\beta$ -differential set, then*

- (1) *for all  $v \in B(S)$ ,  $\delta_{C(S)}(v) \leq \lfloor \beta \rfloor + 1$ .*
- (2) *for all  $v \in C(S)$ ,  $\delta_{C(S)}(v) \leq \lfloor \beta \rfloor$ .*

**Proof.** Let  $S$  be a  $\beta$ -differential set of  $G$ . (1) If there exists a  $v \in B(S)$  such that  $\delta_{C(S)}(v) \geq \lfloor \beta \rfloor + 2$ , we obtain  $\partial_\beta(S \cup \{v\}) \geq |B(S)| - 1 + \lfloor \beta \rfloor + 2 - \beta|S| - \beta = \partial_\beta(G) + \lfloor \beta \rfloor + 1 - \beta > \partial_\beta(G)$ , a contradiction.

(2) If there exists a  $v \in C(S)$  such that  $\delta_{C(S)}(v) \geq \lfloor \beta \rfloor + 1$ , then  $\partial_\beta(S \cup \{v\}) > \partial_\beta(G)$ , a contradiction.  $\square$

**Proposition 4.** *Let  $G$  be a graph of order  $n$ , minimum degree  $\delta \geq 2$  and maximum degree  $\Delta$ . For any differential set  $S$  the following holds:*

$$|S| \geq \frac{n(\delta - 1)}{\Delta\delta + \Delta + \delta - 1}.$$

**Proof.** Let  $S$  be a differential set, and let  $A$  be the number of edges from  $B(S)$  to  $C(S)$ . By Lemma 3 every vertex in  $C(S)$  has at most one neighbor in  $C(S)$ , and so we have that  $(\delta - 1)|C(S)| \leq A$ . By Lemma 3  $A \leq 2|B(S)|$ , and using that  $|C(S)| \leq \frac{2|B(S)|}{\delta - 1}$ , we have

$$\begin{aligned} n &= |S| + |B(S)| + |C(S)| \leq |S| + |B(S)| + \frac{2|B(S)|}{\delta - 1} \\ &= \frac{|S|(\delta - 1) + |B(S)|(\delta - 1) + 2|B(S)|}{\delta - 1} \\ &\leq \frac{|S|(\delta - 1) + |S|\Delta(1 + \delta)}{\delta - 1} = \frac{(\Delta\delta + \Delta + \delta - 1)|S|}{\delta - 1}, \end{aligned}$$

or, equivalently,  $|S| \geq \frac{n(\delta - 1)}{\Delta\delta + \Delta + \delta - 1}$ .  $\square$

**Proposition 5.** *Let  $G$  be a graph of order  $n$ , size  $m$ , maximum degree  $\Delta$ , and  $\beta \in (0, \delta)$ . Then*

$$\partial_\beta(G) \geq \frac{2m(\lfloor \beta \rfloor - \beta + 1)}{(\lfloor \beta \rfloor + 1)^3 + \Delta(\lfloor \beta \rfloor + 2)}.$$

**Proof.** Let  $S$  be a minimum  $\beta$ -differential set of  $G$ . Since  $\beta \in (0, \delta)$ , we know that  $B(S)$  is a dominating set of  $G$ . This and Lemma 3 (1) imply that  $|B(S)|(\lfloor \beta \rfloor + 1) \geq |C(S)|$ . Furthermore, by Lemmas 2 and 3 we have that

$$\begin{aligned}
 2m &\leq |S|\Delta + |B(S)|\Delta + \sum_{c \in C(S)} \delta(c) \leq |S|\Delta + |B(S)|\Delta + |B(S)|(\lfloor \beta \rfloor + 1) + |C(S)|\lfloor \beta \rfloor \\
 &\leq |S|\Delta + |B(S)|\Delta + |B(S)|(\lfloor \beta \rfloor + 1) + |B(S)|\lfloor \beta \rfloor(\lfloor \beta \rfloor + 1) \\
 &= |S|\Delta + |B(S)|(\Delta + \lfloor \beta \rfloor + 1 + \lfloor \beta \rfloor(\lfloor \beta \rfloor + 1)) \\
 &= |S|\Delta + |B(S)|((\lfloor \beta \rfloor + 1)^2 + \Delta) \\
 &= |S|\Delta + |S|\beta((\lfloor \beta \rfloor + 1)^2 + \Delta) + \partial_\beta(S)((\lfloor \beta \rfloor + 1)^2 + \Delta) \\
 &\leq \partial_\beta(G)\left(\frac{\beta(\lfloor \beta \rfloor + 1)^2 + (\beta + 1)\Delta}{\lfloor \beta \rfloor - \beta + 1}\right) + \partial_\beta(G)((\lfloor \beta \rfloor + 1)^2 + \Delta) \\
 &= \partial_\beta(G)\left(\frac{\beta(\lfloor \beta \rfloor + 1)^2 + (\beta + 1)\Delta + (\lfloor \beta \rfloor + 1)^2(\lfloor \beta \rfloor - \beta + 1) + \Delta(\lfloor \beta \rfloor - \beta + 1)}{\lfloor \beta \rfloor - \beta + 1}\right) \\
 &= \partial_\beta(G)\left(\frac{(\lfloor \beta \rfloor + 1)^3 + \Delta(\lfloor \beta \rfloor + 2)}{\lfloor \beta \rfloor - \beta + 1}\right).
 \end{aligned}$$

Thus,

$$\partial_\beta(G) \geq \frac{2m(\lfloor \beta \rfloor - \beta + 1)}{(\lfloor \beta \rfloor + 1)^3 + \Delta(\lfloor \beta \rfloor + 2)}.$$

□

A block in  $G$  is a maximal connected subgraph without a cut-vertex. An end-block is a block with exactly one cut-vertex of  $G$ . Let  $eb(G)$  and  $Ph(G)$  be the number of end-blocks and the maximum number of disjoint 3-vertex paths in  $G$ . We said that  $G$  is a *claw free* graph if no vertex has three pairwise nonadjacent neighbors.

**Proposition 6.** *If  $G$  is a claw free graph of order  $n$  and  $eb(G) \geq 2$ , then*

$$\partial_\beta(G) \geq \left\lfloor \frac{n - eb(G) + 2}{3} \right\rfloor (2 - \beta).$$

**Proof.** Let  $S$  be the set formed by the vertices of degree 2 associated to the maximum number of disjoint 3-vertex paths in  $G$ . Hence,

$$\partial_\beta(G) \geq \partial_\beta(S) = |B(S)| - \beta|S| \geq 2Ph(G) - \beta Ph(G).$$

$$\partial_\beta(G) \geq (2 - \beta)Ph(G).$$

In [17] it was proved that if  $G$  is a claw free graph of order  $n$  and  $eb(G) \geq 2$ , then

$$Ph(G) \geq \left\lfloor \frac{n - eb(G) + 2}{3} \right\rfloor.$$

Thus, we have

$$\partial_\beta(G) \geq \left\lfloor \frac{n - eb(G) + 2}{3} \right\rfloor (2 - \beta).$$

□

**Proposition 7.** *Let  $G$  be a graph of order  $n$ , let  $\delta_1 \geq \delta_2 \geq \dots \geq \delta_n$  be the sequence of degrees of vertices in  $G$ , and let  $\beta \in (0, \Delta)$ . If  $t = \min\{k : \sum_{r=1}^k \delta_r + k \geq n\}$ ,  $r_0 = \max\{r : \delta_r \geq \lceil \beta \rceil\}$  and  $t_0 = \min\{t, r_0\}$ , then*

$$\partial_\beta(G) + \beta t_0 \leq \sum_{r=1}^{t_0} \delta_r.$$

**Proof.** Let  $S$  be a  $\beta$ -differential set. If  $S = \{v_{\alpha_1}, \dots, v_{\alpha_j}\}$  such that  $\delta(v_{\alpha_r}) = \delta_{\alpha_r}$  for all  $r \in \{1, \dots, j\}$  and  $\delta_{\alpha_1} \geq \delta_{\alpha_2} \geq \dots \geq \delta_{\alpha_r}$ . We note that  $\alpha_r \geq r$  for every  $r \in \{1, \dots, j\}$ , thus

$$|B(S)| \leq \sum_{r=1}^j \delta_{\alpha_r} \leq \sum_{r=1}^j \delta_r,$$

and hence,  $\partial_\beta(G) = \partial_\beta(S) \leq \sum_{r=1}^j \delta_r - \beta j$ .

Now, if  $j \leq t_0$ , since  $\delta_r - \beta \geq 0$  for every  $r \leq t_0$ , we have

$$\partial_\beta(S) \leq \sum_{r=1}^j \delta_r - \beta j = \sum_{r=1}^j (\delta_r - \beta) \leq \sum_{r=1}^{t_0} (\delta_r - \beta) = \left( \sum_{r=1}^{t_0} \delta_r \right) - \beta t_0.$$

Finally, if  $j > t_0$  we have two cases. If  $t_0 = t$ , since  $\sum_{r=1}^{t_0} \delta_r + t_0 \geq n$ , then

$$\sum_{r=1}^{t_0} \delta_r - \beta t_0 \geq n - (1 + \beta)t_0 > n - (1 + \beta)j = n - |S| - \beta|S| \geq \partial_\beta(S).$$

If  $t_0 = r_0$ , then  $\delta_r - \beta \leq 0$  for every  $r > t_0$ , in consequence,

$$\partial_\beta(S) \leq \sum_{r=1}^j (\delta_r - \beta) = \sum_{r=1}^{t_0} (\delta_r - \beta) + \sum_{r=t_0+1}^j (\delta_r - \beta) \leq \sum_{r=1}^{t_0} (\delta_r - \beta) = \sum_{r=1}^{t_0} \delta_r - \beta t_0.$$

□

**Proposition 8.** Let  $G$  be a graph of order  $n$ , minimum degree  $\delta \geq 2$ , maximum degree  $\Delta$ , and diameter  $D(G)$ .

- (a) If  $D(G) = 2$ , then  $n - \delta(1 + \beta) \leq \partial_\beta(G) \leq \delta(\Delta - 1 - \beta) + 1$ .
- (b) If  $D(G) = 3$ , then  $2(\delta - \beta) \leq \partial_\beta(G) \leq (\Delta - \beta)(\Delta(\Delta - 1) + 1)$ .

**Proof.** (a) Since  $D(G) = 2$ , we have that  $\gamma(G) \leq \delta$ , thus  $n - \delta(1 + \beta) \leq n - \gamma(G)(1 + \beta) \leq \partial_\beta(G)$ . Now, we choose  $v \in V(G)$  such that  $\deg(v) = \delta$ , and let  $S := B(\{v\})$ . Then  $S$  is a dominating set,  $|S| = \delta$ , and  $|B(S) \setminus \{v\}| \leq \delta(\Delta - 1)$ . Therefore, by Lemma 1, we have  $\partial_\beta(G) \leq 1 + \delta(\Delta - 1) - \beta\delta = \delta(\Delta - 1 - \beta) + 1$ .

(b) Since  $D(G) = 3$ , there are two vertices  $u, v \in V(G)$  without common neighbors, so  $\partial(\{u, v\}) \geq 2\delta - 2\beta$ . If  $A = B(B(\{v\}))$  and  $S$  is a  $\beta$ -differential of  $G$ , we have that  $A$  is a dominating set of  $G$ , so, by Lemma 1, we have  $|S| \leq |A| = |A - v| + 1 \leq \Delta(\Delta - 1) + 1$ . Hence

$$\begin{aligned} \partial(G) &= \partial(S) = |B(S)| - \beta|S| \leq \Delta|S| - \beta|S| = (\Delta - \beta)|S| \\ &\leq (\Delta - \beta)|A| \leq (\Delta - \beta)(\Delta(\Delta - 1) + 1). \end{aligned}$$

□

**Proposition 9.** Let  $G$  be a graph of maximum degree  $\Delta$  and minimum degree  $\delta$ . If  $D(G) \geq 5$ , then

$$\partial_\beta(G) + 2\beta \geq \Delta + \delta.$$

**Proof.** Let  $v \in V(G)$  such that  $\deg(v) = \Delta$ . We suppose that every vertex  $u \in V(G) \setminus N[\{v\}]$  has a neighbor in  $B(\{v\})$ . In such a case, for every two vertices  $x, y \in V(G)$  we have  $d(x, y) \leq 4$ , a contradiction. Let  $u \in V(G) \setminus N[\{v\}]$  having no neighbor in  $B(\{v\})$ , then for  $S = \{u, v\}$ , we have

$$\partial_\beta(G) \geq \partial_\beta(S) = |B(S)| - \beta|S| \geq \Delta + \delta - 2\beta.$$

□

The following result allows us to establish a recurrence relationship between  $\partial_\beta(G)$  and  $\partial_{\beta-1}(G)$ .

**Theorem 1.** *Let  $G$  be a graph of order  $n$  and minimum degree  $\delta$ , and  $\beta \in (0, \delta)$ . Then,*

$$n \leq (\beta + 2)\partial_\beta(G) + \beta(\beta + 1)\partial_{\beta-1}(G).$$

**Proof.** Let  $S$  be a  $\beta$  differential set in  $G$ . Since  $\beta < \delta$ ,  $B(S)$  is a dominating set. Otherwise, if  $u \in V(G) \setminus N[B(S)]$ , then  $\partial_\beta(S) < \partial_\beta(S \cup \{u\})$ .

Note that  $\partial_\beta(B(S)) = |S| + |C(S)| - \beta|B(S)|$ . As  $\partial_\beta(B(S)) \leq \partial_\beta(S) = |B(S)| - \beta|S|$ , we obtain that

$$|C(S)| \leq |B(S)|(\beta + 1) - |S|(\beta + 1).$$

In consequence,

$$\begin{aligned} n &= |S| + |B(S)| + |C(S)| \leq |S| + |B(S)| + |B(S)|(\beta + 1) - |S|(\beta + 1) \\ &= \partial_\beta(G) + |B(S)|(\beta + 1) = (\beta + 2)\partial_\beta(G) + |S|\beta(\beta + 1). \end{aligned}$$

Using now that

$$|S| = \beta|S| - (\beta - 1)|S| \leq |B(S)| - (\beta - 1)|S| \leq \partial_{\beta-1}(G),$$

we obtain the result.  $\square$

In [16], the following result was proved.

**Proposition 10.** *Let  $G$  be a graph with order  $n$  and maximum degree  $\Delta(G)$ , then*

- (a)  $\Delta(G) = n - 1$  if and only if  $\partial_\beta(G) = n - (1 + \beta)$ .
- (b)  $\Delta(G) = n - 2$  if and only if  $\partial_\beta(G) = n - (2 + \beta)$ .
- (c) If  $\beta > 1$ , then  $\Delta(G) = n - 3$ , if and only if  $\partial_\beta(G) = n - (3 + \beta)$ .

The following result complements the previous proposition.

**Proposition 11.** *Let  $G$  be a graph with order  $n$ . For any  $k \geq 1$  and  $\beta > k$ , we have*

$$\Delta = n - k - 2 \Leftrightarrow \partial_\beta(G) = n - (2 + k + \beta).$$

**Proof.** Let  $v$  be a vertex such that  $\delta(v) = \Delta = n - k - 2$ , then

$$\partial_\beta(G) \geq \partial_\beta(\{v\}) = n - k - 2 - \beta = n - (2 + k + \beta).$$

If we suppose that there exists  $S \subseteq V$  such that  $|S| = t \geq 2$  and  $\partial(S) = \alpha - t\beta > n - (2 + k + \beta)$  with  $\alpha \leq n - 2$ . Then,

$$t < \frac{\alpha - (n - 2) + k + \beta}{\beta} \leq \frac{k + \beta}{\beta} < 2,$$

a contradiction.

If  $\partial_\beta(G) = n - (2 + k + \beta)$  and a  $\beta$ -differential set  $S$  has cardinality  $t \geq 2$ , then  $\partial(S) = \alpha - t\beta = n - (2 + k + \beta)$  with  $\alpha \leq n - 2$ . Therefore,  $(t - 1)\beta = \alpha - n + 2 + k < \alpha - n + 2 + \beta$ , consequently,  $t < \frac{\alpha - (n - 2)}{\beta} + 2 \leq 2$ , a contradiction. Hence, any differential set contains only one vertex with degree  $n - k - 2$ .  $\square$

Now, we can consider that each hospital  $v_i \in V$  serves  $w_i$  patients. The following results study the benefit obtained in this case.

If we consider a weighted graph  $G = (V, W, E)$ , where  $W = \{w_1, \dots, w_n\}$  are the corresponding weight of the vertices  $\{v_1, \dots, v_n\}$  in the graph, if  $\alpha$  is the cost of giving a

service in a vertex, the benefit of giving that service in a set of vertices  $S$  will be the value of  $P(B(S)) + P(S) - \alpha|S| = P(N[S]) - \alpha|S|$ , where  $P(A) = \sum_{k=1}^{|A|} w_{i_k}$  if  $A = \{v_{i_1}, \dots, v_{i_{|A|}}\}$ , with  $A \subseteq V$ . We will denote

$$\partial_{\alpha-1}^W(G) = \max\{P(N[S]) - \alpha|S| : S \subseteq V\}.$$

Given a weighted graph  $G = (V, W, E)$  we construct a graph  $G' = (V', E)$  without weights in the following way. For every  $v_i \in V$  we consider a complete graph  $K_{w_i}$  with vertices  $\{u_1^i, \dots, u_{w_i}^i\}$ , and  $u_j^i$  is adjacent to  $u_s^i$  if and only if  $v_i$  is adjacent to  $v_k$ . The graph  $G'$  will be called the *associated simple graph* of  $G$ . We have the following consequences, where  $n, \delta$  and  $\Delta$  denote the order and the minimum and maximum degree in  $G$ , respectively.

**Proposition 12.** *Let  $G = (V, W, E)$  be a weighted graph of order  $n$  and let  $G'$  be its associated simple graph. Then*

- (a)  $(\delta + 1)(\min\{w_i\}) - 1 \leq \delta(G') \leq \Delta(G') \leq (\Delta + 1)(\max\{w_i\}) - 1.$
- (b)  $n(\min\{w_i\}) \leq n(G') = \sum_{i=1}^n w_i \leq n(\max\{w_i\}).$

**Proof.** If  $u_j^i$  is a vertex in  $V'$ , then

$$\begin{aligned} \delta_{G'}(u_j^i) &\leq w_i - 1 + \delta(v_i)(\max\{w_i\}) \leq \max\{w_i\} - 1 + \Delta(\max\{w_i\}) \\ &= (\Delta + 1)(\max\{w_i\}) - 1. \end{aligned}$$

We also have that

$$\begin{aligned} (\delta + 1)(\min\{w_i\}) - 1 &\leq \min\{w_i\} - 1 + \delta(\min\{w_i\}) \\ &\leq w_i - 1 + \delta(v_i)(\min\{w_i\}) \\ &\leq \delta_{G'}(u_j^i). \end{aligned}$$

□

The following theorem let us use most of the results on the differential of a graph without weights on the differential of weighted graph.

**Theorem 2.** *Let  $G = (V, W, E)$  be a weighted graph and let  $G'$  be its associated simple graph. Then, for every  $\alpha > 0$ ,  $\partial_{\alpha-1}^W(G) = \partial_{\alpha-1}(G')$ . Moreover, if  $S \subseteq V$  such that  $\partial_{\alpha-1}^W(S) = \partial_{\alpha-1}^W(G)$ , there exists  $S' \subseteq V'$  such that  $\partial_{\alpha-1}(S') = \partial_{\alpha-1}(G')$  and  $|S'| = |S|$ .*

**Proof.** Let  $S' \subseteq V'$  be a set such that  $\partial_{\alpha-1}(S') = \partial_{\alpha-1}(G')$ . If  $u_j^i, u_k^i \in S'$ , then  $u_j^i$  and  $u_k^i$  are adjacent and  $B(\{u_j^i\}) \setminus \{u_k^i\} = B(\{u_k^i\}) \setminus \{u_j^i\}$ , so  $\partial_{\alpha-1}(S' \setminus \{u_k^i\}) = \partial_{\alpha-1}(S') + \alpha$ , a contradiction. We denote  $S' = \{u_{j_1}^{i_1}, \dots, u_{j_t}^{i_t}\}$ , where  $i_s \neq i_l$  if  $s \neq l$ , then

$$|B(S')| = \sum_{s=1}^t (w_{i_s} - 1) + P(B(\{v_{i_1}, \dots, v_{i_t}\})) = P(N[\{v_{i_1}, \dots, v_{i_t}\}]) - t.$$

Therefore,  $|B(S')| - (\alpha - 1)|S'| = P(N[S]) - \alpha|S|$  where  $S = \{v_{i_1}, \dots, v_{i_t}\}$ .

Finally, if  $S = \{v_{i_1}, \dots, v_{i_t}\} \subseteq V$  satisfies  $\partial_{\alpha-1}^W(S) = \partial_{\alpha-1}^W(G)$ , we can consider the set  $S' = \{u_{j_1}^{i_1}, \dots, u_{j_t}^{i_t}\}$  to obtain  $|B(S')| - (\alpha - 1)|S'| = P(N[S]) - \alpha|S|$ . □

#### 4. Conclusions

Through the differential of a graph  $G$ , we can know the sets of vertices of this graph that maximizes the difference between the cardinality of its border and the cardinality of the set itself, this has direct applications since it can be seen as the process of maximizing



the cost–benefit by properly selecting sets of vertices of  $G$ . The  $\beta$ -differential maximizes that difference when the cardinality of the differential set is penalized. Knowing upper and lower bounds for this parameter as a function of the invariants of the graph allows us to have a deeper understanding of the graph itself, which can represent communities whose needs must be met by optimizing resources, for example. The applications of this type of problem are very wide, the theoretical development in this field can be used in the optimization of resources, even when each node does not have the same weight in the graph (heavy vertices). In particular, the following results were obtained:

1. Let  $G$  be a graph of order  $n$  and minimum degree  $\delta$ , and  $\beta \in (0, \delta)$ . Then

$$n \leq (\beta + 2)\partial_{\beta}(G) + \beta(\beta + 1)\partial_{\beta-1}(G).$$

2. Let  $G$  be a graph with order  $n$ . For any  $k \geq 1$  and  $\beta > k$ , we have

$$\Delta = n - k - 2 \Leftrightarrow \partial_{\beta}(G) = n - (2 + k + \beta).$$

3. Let  $G = (V, W, E)$  be a weighted graph and let  $G'$  be its associated simple graph. Then, for every  $\alpha > 0$ ,  $\partial_{\alpha-1}^W(G) = \partial_{\alpha-1}(G')$ . Moreover, if  $S \subseteq V$  such that  $\partial_{\alpha-1}^W(S) = \partial_{\alpha-1}^W(G)$ , there exists  $S' \subseteq V'$  such that  $\partial_{\alpha-1}(S') = \partial_{\alpha-1}(G')$  and  $|S'| = |S|$ .

**Author Contributions:** All authors contributed equally to the development of this work. All authors have read and agreed to the published version of the manuscript.

**Funding:** This research received no external funding.

**Acknowledgments:** We would like to thank the reviewers for their careful reading of the manuscript and their suggestions which have improved the presentation of this work. This work was supported by a grant from Agencia Estatal de Investigación (PID2019-106433GB-I00/AEI /10.13039/501100011033), Spain.

**Conflicts of Interest:** The authors declare no conflict of interest.

## References

1. Bermudo, S.; Fernau, H. Computing the differential of a graph: Hardness, approximability and exact algorithms. *Discret. Appl. Math.* **2014**, *165*, 69–82. [[CrossRef](#)]
2. Mashburn, J.L.; Haynes, T.W.; Hedetniemi, S.M.; Hedetniemi, S.T.; Slater, P.J. Differentials in graphs. *Util. Math.* **2006**, *69*, 43–54.
3. Abreu-Blaya, R.; Bermudo, S.; Rodríguez, J.M.; Tourís, E. Topological Indices and f-Polynomials on Some Graph Products. *Symmetry* **2021**, *13*, 292. [[CrossRef](#)]
4. Basilio, L.A.; Bermudo, S.; Sigarreta, J.M. Bounds on the differential of a graph. *Util. Math.* **2017**, *103*, 319–334.
5. Bermudo, S.; De la Torre, L.; Martín-Caraballo, A.M.; Sigarreta, J.M. The differential of the strong product graphs. *Int. J. Comput. Math.* **2015**, *92*, 1124–1134. [[CrossRef](#)]
6. Bermudo, S.; Fernau, H. Lower bound on the differential of a graph. *Discret. Math.* **2012**, *312*, 3236–3250. [[CrossRef](#)]
7. Bermudo, S.; Fernau, H. Combinatorics for smaller kernels: The differential of a graph. *Theor. Comput. Sci.* **2015**, *562*, 330–345. [[CrossRef](#)]
8. Bermudo, S.; Rodríguez, J.M.; Sigarreta, J.M. On the differential in graphs. *Util. Math.* **2015**, *97*, 257–270.
9. Pushpam, P.R.L.; Yokesh, D. Differential in certain classes of graphs. *Tamkang J. Math.* **2010**, *41*, 129–138. [[CrossRef](#)]
10. Sigarreta, J.M. Differential in Cartesian Product Graphs. *Ars Comb.* **2016**, *126*, 259–267.
11. Goddard, W.; Henning, M.A. Generalised domination and independence in graphs. *Congr. Numer.* **1997**, *123*, 161–172.
12. Zhang, C.Q. Finding critical independent sets and critical vertex subsets are polynomial problems. *SIAM J. Discret. Math.* **1990**, *3*, 431–438. [[CrossRef](#)]
13. Slater, P.J. Enclaveless sets and MK-systems. *J. Res. Natl. Bur. Stand.* **1977**, *82*, 197–202. [[CrossRef](#)] [[PubMed](#)]
14. Haynes, T.W.; Hedetniemi, S.; Slater, P.J. *Fundamentals of Domination in Graphs*; Marcel Dekker: New York, NY, USA, 1998.
15. Haynes, T.W.; Hedetniemi, S.; Slater, P.J. *Domination in Graphs: Advanced Topics*; Marcel Dekker: New York, NY, USA, 1998.
16. Basilio, L.A.; Bermudo, S.; Leanós, J.; Sigarreta, J.M.  $\beta$ -Differential of a Graph. *Symmetry* **2017**, *9*, 205. [[CrossRef](#)]
17. Kaneko, A.; Kelmans, A.; Nishimura, T. On packing 3-vertex paths in a graph. *J. Graph Theory* **2001**, *36*, 175–197. [[CrossRef](#)]