axioms

Article

On the Composition of Overlap and Grouping Functions

Songsong Dai, **Lei Du**, **Haifeng Song** and **Yingying Xu** *

School of Electronics and Information Engineering, Taizhou University, Taizhou 318000, China; ssdai@tzc.edu.cn (S.D.); dulei2109@tzc.edu.cn (L.D.); isshf@126.com (H.S.)

* Correspondence: yyxu@tzc.edu.cn

Abstract: Obtaining overlap/grouping functions from a given pair of overlap/grouping functions is an important method of generating overlap/grouping functions, which can be viewed as a binary operation on the set of overlap/grouping functions. In this paper, firstly, we studied closures of overlap/grouping functions w.r.t. $\circledast$-composition. In addition, then, we show that these compositions are order preserving. Finally, we investigate the preservation of properties like idempotency, migrativity, homogeneity, k-Lipschitz, and power stable.

Keywords: overlap functions; grouping functions; composition; closures; properties preservation

Citation: Dai, S.; Du, L.; Song, H.; Xu, Y. On the Composition of Overlap and Grouping Functions. *Axioms* **2021**, *10*, 272. https://doi.org/10.3390/axioms10040272

Academic Editor: Amit K. Shukla

Received: 9 September 2021
Accepted: 20 October 2021
Published: 24 October 2021

1. Introduction

Overlap function [1] is a special case of aggregation functions [2]. Grouping function [3] is the dual concept of overlap function. In recent years, overlap and grouping functions have attracted wide interest. In the field of application, they are used in image processing [1,4], classification [5,6], and decision-making [7,8]. In the field of theoretical research, the concepts of general, Archimedean, n-dimensional, interval-valued, and complex-valued overlap/grouping functions have been introduced [9–17]. In the literature about overlap/grouping functions, much attention have been recently paid to their properties, this study has enriched overlap/grouping functions. Bedregal [9] studied some properties such as migrativity, idempotency, and homogeneity of overlap/overlap functions. Gomez et al. [12] also considered these properties of N-dimensional overlap functions. Costa and Bedregal [18] introduced quasi-homogeneous overlap functions. Qian and Hu [19] studied the migrativity of uninorms and nullnorms over overlap/grouping functions. They [13,20,21] also studied multiplicative generators and additive generators of overlap/grouping functions and the distributive laws of fuzzy implication functions over overlap functions [9,12,13,18–21]. Moreover, overlap/grouping functions also can be viewed as binary connectives on [0, 1], then they can be used to construct other fuzzy connectives. Residual implication, (G, N)-implications, QL-implications, (IO, O)-fuzzy rough sets, and binary relations induced from overlap/grouping functions have been studied [22–27].

The construction of the following overlap/grouping functions was developed in many literature works [1,4,13,15,16,21,27,28]. Obtaining overlap/grouping functions from given overlap/grouping functions is one of the methods to generate overlap/grouping functions. We consider this work as a composition of two or more overlap/grouping functions. As mentioned above, some properties are important for overlap/grouping functions. Thus, it raises the question of whether the new generated overlap/grouping function still satisfies the properties of overlap/grouping functions. In this paper, we consider properties preservation of four compositions such as meet operation, join operation, convex combination, and $\circledast$-composition of overlap/grouping functions. These results might serve as a certain criteria for choices of generation methods of overlap/grouping functions from given overlap/grouping functions.

The paper is organized as follows: In Section 2, we recall the concepts of overlap/grouping functions and their properties. In Section 3, we studied the closures of overlap/grouping functions w.r.t. $\circledast$-composition. In Section 4, we study the order preservation of compositions. In Section 5, we study properties' preservation of compositions. In Section 6, conclusions are briefly summed up.

2. Preliminaries

2.1. Overlap and Grouping Functions

First, we recall the concepts of overlap/grouping functions and their properties; for details, see [1,9,12,13].

Definition 1 ([1])**.** *A bivariate function $O : [0,1]^2 \to [0,1]$ is an overlap function if it has the following properties:*

(O1) *It is commutative;*

(O2) $O(\eta,\xi) = 0$ *if and only if* $\eta\xi = 0$;

(O3) $O(\eta,\xi) = 1$ *if and only if* $\eta\xi = 1$;

(O4) *It is non-decreasing;*

(O5) *It is continuous.*

Definition 2 ([1])**.** *A bivariate function $G : [0,1]^2 \to [0,1]$ is a grouping function if it has the following properties:*

(G1) *It is commutative;*

(G2) $G(\eta,\xi) = 0$ *if and only if* $\eta = \xi = 0$;

(G3) $G(\eta,\xi) = 1$ *if and only if* $\eta = 1$ *or* $\xi = 1$.

(G4) *It is non-decreasing;*

(G5) *It is continuous.*

If O is an overlap function, then the function $G(\eta,\xi) = 1 - O(1-\eta, 1-\xi)$ is the dual grouping function of G.

2.2. Properties of Overlap and Grouping Functions

For any two overlap (or grouping) functions O and O', if $O(\eta,\xi) \leq O'(\eta,\xi)$ holds for all $(\eta,\xi) \in [0,1]^2$, then we say that O is weaker than O', denoted $O \preceq O'$. For example, consider the following three overlap functions $O_M(\eta,\xi) = \min(\eta,\xi)$, $O_P(\eta,\xi) = \eta\xi$ and $O_{Mid}(\eta,\xi) = \eta\xi\frac{\eta+\xi}{2}$, we get this ordering for these overlap functions:

$$O_{Mid} \preceq O_P \preceq O_M.$$

Some interesting properties for overlap (or grouping) functions are:

(ID) Idempotency:

$$O(\eta,\eta) = \eta$$

for all $\eta \in [0,1]$;

(MI) Migrativity:

$$O(\alpha\eta,\xi) = O(\eta,\alpha\xi)$$

for all $\alpha,\eta,\xi \in [0,1]$;

(HO-*k*) Homogeneous of order $k \in]0,\infty[$:

$$O(\alpha\eta,\alpha\xi) = \alpha^k O(\eta,\xi)$$

for all $\alpha \in [0,\infty[$ and $\eta,\xi \in [0,1]$ such that $\alpha\eta,\alpha\xi \in [0,1]$;

(k-LI) k-Lipschitz:

$$|O(\eta_1, \xi_1) - O(\eta_2, \xi_2)| \le k(|\eta_1 - \eta_2| + |\xi_1 - \xi_2|)$$

for all $\eta_1, \eta_2, \xi_1, \xi_2 \in [0, 1]$.

(PS) Power stable [29]:

$$O(\eta^r, \xi^r) = O(\eta, \xi)^r$$

for all $r \in]0, \infty[$ and $\eta, \xi \in [0, 1]$.

3. Compositions of Overlap and Grouping Functions and Their Closures

In the following, we list four compositions of overlap/grouping functions including meet, join, convex combination, and $\circledast$-composition. In addition, we then studied their closures.

3.1. Compositions of Overlap and Grouping Functions

For any two overlap (or grouping) functions O_1 and O_2, meet and join operations of O_1 and O_2 are defined by

$$(O_1 \vee O_2)(\eta, \xi) = \max\left(O_1(\eta, \xi), O_2(\eta, \xi)\right), \tag{1}$$
$$(O_1 \wedge O_2)(\eta, \xi) = \min\left(O_1(\eta, \xi), O_2(\eta, \xi)\right) \tag{2}$$

for all $(\eta, \xi) \in [0, 1]^2$.

For any two overlap (or grouping) functions O_1 and O_2, a convex combination of O_1 and O_2 is defined as

$$O_\lambda = \lambda O_1(\eta, \xi) + (1 - \lambda) O_2(\eta, \xi) \tag{3}$$

for all $(\eta, \xi) \in [0, 1]^2$ and $\lambda \in [0, 1]$.

For any two overlap (or grouping) functions O_1 and O_2, the $\circledast$-composition of O_1 and O_2 is defined as

$$(O_1 \circledast O_2)(\eta, \xi) = O_1\left(\eta, O_2(\eta, \xi)\right) \tag{4}$$

for all $(\eta, \xi) \in [0, 1]^2$.

3.2. Closures of the Compositions

Closures of the meet operation, join operation, and convex combination have been obtained in [1,3,9]. The $\circledast$-composition of two overlap functions is closed means $\circledast$-composition of two bivariate functions on [0, 1] preserves (***O*1**), (***O*2**), (***O*3**), (***O*4**) and (***O*5**). Similarly, the $\circledast$-composition of two grouping functions is closed means $\circledast$-composition of two bivariate functions on [0, 1] preserves (***G*1**), (***G*2**), (***G*3**), (***G*4**) and (***G*5**).

Theorem 1. *If two bivariate functions $O_1, O_2 : [0, 1]^2 \to [0, 1]$ satisfy (**O2**)$\Big($(**O3**), (**G2**), (**G3**), (**O4**), (**O5**)$\Big)$, then $(O_1 \circledast O_2)$ also satisfies (**O2**)$\Big($(**O3**), (**G2**), (**G3**), (**O4**), (**O5**)$\Big)$.*

Proof. First, we show that $\circledast$-composition preserves (***O*2**). If

$$(O_1 \circledast O_2)(\eta, \xi) = O_1\left(\eta, O_2(\eta, \xi)\right) = 0,$$

then, since O_1 satisfies (***O*2**), we have $\eta O_2(\eta, \xi)$=0. Case I, if $\eta = 0$ and $O_2(\eta, \xi) \neq 0$, then $\eta\xi = 0\xi = 0$; Case II, if $\eta = 0$ and $O_2(\eta, \xi) = 0$, then $\eta\xi = 0\xi = 0$; Case III, if $\eta \neq 0$ and $O_2(\eta, \xi) = 0$, since O_2 satisfies (***O*2**), then $\eta\xi = 0$.

Next, we show that $\circledast$-composition preserves (***O*3**). If

$$(O_1 \circledast O_2)(\eta, \xi) = O_1\left(\eta, O_2(\eta, \xi)\right) = 1,$$

then, since O_1 satisfies (***O*3**), we have $\eta O_2(\eta,\xi)$=1. Then, $\eta = 1$ and $O_2(\eta,\xi) = 1$, since O_2 satisfies (***O*3**), then $\eta\xi = 1$.

Then, we show that $\circledast$-composition preserves (***G*2**). If

$$(O_1 \circledast O_2)(\eta,\xi) = O_1\big(\eta, O_2(\eta,\xi)\big) = 0,$$

then, since O_1 satisfies (***G*2**), we have $\eta = O_2(\eta,\xi)$=0. Since O_2 satisfies (***G*2**), then $\eta = \xi = 0$.

Afterwards, we show that $\circledast$-composition preserves (***G*3**). If

$$(O_1 \circledast O_2)(\eta,\xi) = O_1\big(\eta, O_2(\eta,\xi)\big) = 1,$$

then, since O_1 satisfies (***G*3**), we have $\eta = 1$ or $O_2(\eta,\xi)$=1. Since O_2 satisfies (***G*3**), $O_2(\eta,\xi)$=1 means $\eta = 1$ or $\xi = 1$.

The case for (***O*4**) and (***O*5**) are straightforward. □

Unfortunately, $\circledast$-composition of two bivariate functions does not preserve (***O*1**). For example, let $O_1(\eta,\xi) = O_2(\eta,\xi) = \eta\xi$; then, $(O_1 \circledast O_2)(\eta,\xi) = \eta^2\xi$ is not commutative. This means $\circledast$-composition of two overlap/grouping functions is not closed.

However, it is possible to find an example that $\circledast$-composition of two overlap/grouping functions is also an overlap/grouping function. For example, for two given overlap functions $O_1(\eta,\xi) = O_2(\eta,\xi) = \min(\eta,\xi)$, their $\circledast$-composition $(O_1 \circledast O_2)(\eta,\xi) = \min(\eta,\xi)$ is an overlap function.

The summary of the closures of two bivariate functions w.r.t. these compositions is shown in Table 1.

Table 1. Closures of the compositions.

Property	O_1	O_2	$O_1 \vee O_2$	$O_1 \wedge O_2$	O_λ	$O_1 \circledast O_2$
O_1	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\times$
O_2	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
O_3	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
G_2	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
G_3	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
O_4	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
O_5	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$

4. Order Preservation

In the following we show that the meet operation, join operation, convex combination, and $\circledast$-composition of overlap/grouping functions are order preserving.

Theorem 2. *Suppose that four overlap functions have $O_1 \preceq O_2$ and $O_3 \preceq O_4$, then $(O_1 \vee O_3) \preceq (O_2 \vee O_4)$, $(O_1 \wedge O_3) \preceq (O_2 \wedge O_4)$ $(O_{1,3,\lambda}) \preceq (O_{2,4,\lambda})$ and $(O_1 \circledast O_3) \preceq (O_2 \circledast O_4)$, where $O_{1,3,\lambda} = \lambda O_1(\eta,\xi) + (1-\lambda)O_3(\eta,\xi)$ and $O_{2,4,\lambda} = \lambda O_2(\eta,\xi) + (1-\lambda)O_4(\eta,\xi)$.*

Proof. The case for meet operation, join operation, and convex combination are straightforward. We show only that $\circledast$-composition preserves order. For any $\eta,\xi \in [0,1]$, from $O_3 \preceq O_4$, we have $O_3(\eta,\xi) \leq O_4(\eta,\xi)$. Since O_1 is non-decreasing and $O_1 \preceq O_2$, we have

$$\begin{aligned}(O_1 \circledast O_3)(\eta,\xi) &= O_1\big(\eta, O_3(\eta,\xi)\big)\\ &\leq O_1\big(\eta, O_4(\eta,\xi)\big)\\ &\leq O_2\big(\eta, O_4(\eta,\xi)\big)\\ &= (O_2 \circledast O_4)(\eta,\xi).\end{aligned}$$

Thus, $(O_1 \circledast O_3) \preceq (O_2 \circledast O_4)$. □

Theorem 3. *Suppose that four grouping functions have $G_1 \preceq G_2$ and $G_3 \preceq G_4$, then $(G_1 \vee G_3) \preceq (G_2 \vee G_4)$, $(G_1 \wedge G_3) \preceq (G_2 \wedge G_4)$ $(G_{1,3,\lambda}) \preceq (G_{2,4,\lambda})$ and $(G_1 \circledast G_3) \preceq (G_2 \circledast G_4)$, where $G_{1,3,\lambda} = \lambda G_1(\eta,\xi) + (1-\lambda)G_3(\eta,\xi)$ and $G_{2,4,\lambda} = \lambda G_2(\eta,\xi) + (1-\lambda)G_4(\eta,\xi)$.*

5. Properties Preservation

In the following, we study properties preserved by meet operation, join operation, convex combination, and $\circledast$-composition of overlap/grouping functions.

5.1. Properties Preserved by Meet and Join Operations of Overlap/Grouping Functions

First, we consider the meet and join operations of overlap/grouping functions.

Theorem 4. *If two overlap functions O_1 and O_2 satisfy (**ID**)$\Big($(**MI**), (**HO-*k***), (***k*-LI**), (**PS**)$\Big)$, then $(O_1 \vee O_2)$ and $(O_1 \wedge O_2)$ also satisfy (**ID**)$\Big($(**MI**), (**HO-*k***), (***k*-LI**), (**PS**)$\Big)$.*

Proof. First, we show that meet operation preserves (**ID**). Assume that O_1 and O_2 satisfy (**ID**); then, for any $\lambda, \eta \in [0,1]$,

$$\begin{aligned}(O_1 \vee O_2)(\eta,\eta) &= \max\big(O_1(\eta,\eta), O_2(\eta,\eta)\big)\\ &= \max\big(\eta,\eta\big)\\ &= \eta.\end{aligned}$$

Next, we show that meet operation preserves (**MI**). Assume that O_1 and O_2 satisfy (**MI**), then, for any $\alpha, \eta, \xi \in [0,1]$,

$$\begin{aligned}(O_1 \vee O_2)(\alpha\eta,\xi) &= \max\big(O_1(\alpha\eta,\xi), O_2(\alpha\eta,\xi)\big)\\ &= \max\big(O_1(\eta,\alpha\xi), O_2(\eta,\alpha\xi)\big)\\ &= (O_1 \vee O_2)(\eta,\alpha\xi).\end{aligned}$$

Then, we show that the meet operation preserves (**HO-*k***). Assuming that O_1 and O_2 satisfy (**HO-*k***), then, for any $\alpha, \eta, \xi \in [0,1]$,

$$\begin{aligned}(O_1 \vee O_2)(\alpha\eta,\alpha\xi) &= \max\big(O_1(\alpha\eta,\alpha\xi), O_2(\alpha\eta,\alpha\xi)\big)\\ &= \max\big(\alpha^k O_1(\eta,\xi), \alpha^k O_2(\eta,\xi)\big)\\ &= \alpha^k \max\big(O_1(\eta,\xi), O_2(\eta,\xi)\big)\\ &= \alpha^k (O_1 \vee O_2)(\eta,\xi).\end{aligned}$$

Afterwards, we show that meet operation preserves (***k*-LI**). Assume that O_1 and O_2 satisfy (***k*-LI**), then, for any $\eta_1, \eta_2, \xi_1, \xi_2 \in [0,1]$,

$$\begin{aligned}&|(O_1 \vee O_2)(\eta_1,\xi_1) - (O_1 \vee O_2)(\eta_2,\xi_2)|\\ &= \big|\max\big(O_1(\eta_1,\xi_1), O_2(\eta_1,\xi_1)\big) - \max\big(O_1(\eta_2,\xi_2), O_2(\eta_2,\xi_2)\big)\big|\\ &\le \max\big(\big|O_1(\eta_1,\xi_1) - O_1(\eta_2,\xi_2)\big|, \big|O_2(\eta_1,\xi_1) - O_2(\eta_2,\xi_2)\big|\big)\\ &\le \max\big(k(|\eta_1-\eta_2| + |\xi_1-\xi_2|), k(|\eta_1-\eta_2| + |\xi_1-\xi_2|)\big|\big)\\ &= k(|\eta_1-\eta_2| + |\xi_1-\xi_2|).\end{aligned}$$

Finally we show that meet operation preserves (**PS**). Assume that O_1 and O_2 satisfy (**PS**), then, for any $r, \eta, \xi \in [0,1]$,

$$\begin{aligned}(O_1 \vee O_2)(\eta^r,\xi^r) &= \max\big(O_1(\eta^r,\xi^r), O_2(\eta^r,\xi^r)\big)\\ &= \max\big(O_1(\eta,\xi)^r, O_2(\eta,\xi)^r\big)\\ &= \Big(\max\big(O_1(\eta,\xi), O_2(\eta,\xi)\big)\Big)^r\\ &= (O_1 \vee O_2)(\eta,\xi)^r.\end{aligned}$$

Similarly, we can show that the join operation also preserves (**ID**)$\Big($(**MI**), (**HO-*k***), (***k*-LI**), (**PS**)$\Big)$. □

5.2. Properties Preserved by Convex Combination of Overlap/Grouping Functions

Second, we consider the convex combination of overlap/grouping functions.

Theorem 5. *If two overlap functions O_1 and O_2 satisfy (**ID**)$\Big($(**MI**), (**HO-k**), (**k-LI**)$\Big)$, then, for any $\lambda \in [0,1]$, their convex combination of O_λ also satisfies (**ID**)$\Big($(**MI**), (**HO-k**), (**k-LI**)$\Big)$.*

Proof. First, we show that convex combination preserves (**ID**). Assume that O_1 and O_2 satisfy (**ID**), then, for any $\lambda, \eta \in [0,1]$,

$$\begin{aligned} O_\lambda(\eta,\eta) &= \lambda O_1(\eta,\eta) + (1-\lambda)O_2(\eta,\eta) \\ &= \lambda\eta + (1-\lambda)\eta \\ &= \eta. \end{aligned}$$

Next, we show that convex combination preserves (**MI**). Assume that O_1 and O_2 satisfy (**MI**), then, for any $\lambda, \alpha, \eta, \xi \in [0,1]$,

$$\begin{aligned} O_\lambda(\alpha\eta,\xi) &= \lambda O_1(\alpha\eta,\xi) + (1-\lambda)O_2(\alpha\eta,\xi) \\ &= \lambda O_1(\eta,\alpha\xi) + (1-\lambda)O_2(\eta,\alpha\xi) \\ &= O_\lambda(\eta,\alpha\xi). \end{aligned}$$

Then, we show that convex combination preserves (**HO-*k***). Assume that O_1 and O_2 satisfy (**HO-*k***), then, for any $\lambda, \alpha, \eta, \xi \in [0,1]$,

$$\begin{aligned} O_\lambda(\alpha\eta,\alpha\xi) &= \lambda O_1(\alpha\eta,\alpha\xi) + (1-\lambda)O_2(\alpha\eta,\alpha\xi) \\ &= \lambda\alpha^k O_1(\eta,\xi) + (1-\lambda)\alpha^k O_2(\eta,\xi) \\ &= \alpha^k\big(\lambda O_1(\eta,\xi) + (1-\lambda)O_2(\eta,\xi)\big) \\ &= \alpha^k O_\lambda(\eta,\xi). \end{aligned}$$

Finally, we show that convex combination preserves (***k*-LI**). Assume that O_1 and O_2 satisfy (***k*-LI**), then, for any $\lambda, \alpha, \eta, \xi \in [0,1]$,

$$\begin{aligned} &|O_\lambda(\eta_1,\xi_1) - O_\lambda(\eta_2,\xi_2)| \\ &= \big|\lambda O_1(\eta_1,\xi_1) + (1-\lambda)O_2(\eta_1,\xi_1) - \lambda O_1(\eta_2,\xi_2) - (1-\lambda)O_2(\eta_2,\xi_2)\big| \\ &= \big|\lambda\big(O_1(\eta_1,\xi_1) - O_1(\eta_2,\xi_2)\big) + (1-\lambda)\big(O_2(\eta_1,\xi_1) - O_2(\eta_2,\xi_2)\big)\big| \\ &\le \big|\lambda k(|\eta_1-\eta_2| + |\xi_1-\xi_2|) + (1-\lambda)k(|\eta_1-\eta_2| + |\xi_1-\xi_2|)\big| \\ &= k(|\eta_1-\eta_2| + |\xi_1-\xi_2|). \end{aligned}$$

□

Note that convex combination does not preserve (**PS**), since we have

$$\begin{aligned} O_\lambda(\eta^r,\xi^r) &= \lambda O_1(\eta^r,\xi^r) + (1-\lambda)O_2(\eta^r,\xi^r) \\ &= \lambda O_1(\eta,\xi)^r + (1-\lambda)O_2(\eta,\xi)^r, \end{aligned}$$

and

$$\begin{aligned} O_\lambda(\eta,\xi)^r &= \Big(\lambda O_1(\eta,\xi) + (1-\lambda)O_2(\eta,\xi)\Big)^r \\ &\neq \lambda O_1(\eta,\xi)^r + (1-\lambda)O_2(\eta,\xi)^r \end{aligned}$$

for some $\lambda, r, \eta, \xi \in [0,1]$.

5.3. Properties Preserved by $\circledast$-Composition of Overlap/Grouping Functions

Third, we consider the $\circledast$-composition of overlap/grouping functions.

Theorem 6. *If two overlap functions O_1 and O_2 satisfy* (**ID**)$\Big($(**HO-1**), (**PS**)$\Big)$, *then, their* $\circledast$-*composition* $(O_1 \circledast O_2)$ *also satisfies* (**ID**)$\Big($(**HO-1**), (**PS**)$\Big)$.

Proof. First, we show that $\circledast$-composition preserves (**ID**). Assume that O_1 and O_2 satisfy (**ID**), then, for any $\lambda, \eta \in [0,1]$,

$$\begin{aligned}(O_1 \circledast O_2)(\eta,\eta) &= O_1\big(\eta, O_2(\eta,\eta)\big)\\ &= O_1\big(\eta,\eta\big)\\ &= \eta.\end{aligned}$$

Next, we show that $\circledast$-composition preserves (**HO-1**). Assume that O_1 and O_2 satisfy (**HO-1**), then, for any $\alpha, \eta, \xi \in [0,1]$,

$$\begin{aligned}(O_1 \circledast O_2)(\alpha\eta,\alpha\xi) &= O_1\big(\alpha\eta, O_2(\alpha\eta,\alpha\xi)\big)\\ &= O_1\big(\alpha\eta, \alpha O_2(\eta,\xi)\big)\\ &= \alpha O_1\big(\eta, O_2(\eta,\xi)\big)\\ &= \alpha(O_1 \circledast O_2)(\eta,\xi).\end{aligned}$$

Then, we show that $\circledast$-composition preserves (**PS**). Assume that O_1 and O_2 satisfy (**PS**), then, for any $r, \eta, \xi \in [0,1]$,

$$\begin{aligned}(O_1 \circledast O_2)(\eta^r,\xi^r) &= O_1\big(\eta^r, O_2(\eta^r,\xi^r)\big)\\ &= O_1\big(\eta^r, O_2(\eta,\xi)^r\big)\\ &= O_1\big(\eta, O_2(\eta,\xi)\big)^r\\ &= (O_1 \circledast O_2)(\eta,\xi)^r.\end{aligned}$$

□

Note that we only show that $\circledast$-composition preserves (**HO-1**), it does not preserve (**HO-*k***) for $k \in]0,\infty[$ and $k \neq 1$. For example, let $O_1(\eta,\xi) = O_2(\eta,\xi) = \eta^2\xi^2$, then $(O_1 \circledast O_2)(\eta,\xi) = \eta^6\xi^4$, we know that O_1 and O_2 satisfy (**HO-2**), i.e., $O_1(\alpha\eta,\alpha\xi) = \alpha^2 O_1(\eta,\xi)$, but $(O_1 \circledast O_2)(\eta,\xi)$ does not satisfy (**HO-2**) since $(O_1 \circledast O_2)(\alpha\eta,\alpha\xi) = \alpha^{10}\eta^6\xi^4 \neq \alpha^2\eta^6\xi^4 = \alpha^2(O_1 \circledast O_2)(\eta,\xi)$.

The $\circledast$-composition does not preserve (**MI**). Assume that O_1 and O_2 satisfy (**MI**), then

$$\begin{aligned}(O_1 \circledast O_2)(\eta,\alpha\xi) &= O_1\big(\eta, O_2(\eta,\alpha\xi)\big)\\ &= O_1\big(\eta, O_2(\alpha\eta,\xi)\big)\\ &\neq O_1\big(\alpha\eta, O_2(\alpha\eta,\xi)\big)\\ &= (O_1 \circledast O_2)(\alpha\eta,\xi)\end{aligned}$$

for some $\alpha, \eta, \xi \in [0,1]$.

The $\circledast$-composition does not preserve (***k*-LI**).

Example 1. *Let $O_1(\eta,\xi) = O_2(\eta,\xi) = \eta\xi$, then $(O_1 \circledast O_2)(\eta,\xi) = \eta^2\xi$,*

$$\begin{aligned}|O_1(\eta_1,\xi_1) - O_2(\eta_2,\xi_2)| &= |\eta_1\xi_1 - \eta_2\xi_2|\\ &= |\eta_1\xi_1 - \eta_1\xi_2 + \eta_1\xi_2 - \eta_2\xi_2|\\ &= |\eta_1(\xi_1 - \xi_2) + \xi_2(\eta_1 - \eta_2)|\\ &\leq |\eta_1(\xi_1 - \xi_2)| + |\xi_2(\eta_1 - \eta_2)|\\ &\leq |\xi_1 - \xi_2| + |\eta_1 - \eta_2|.\end{aligned}$$

Thus, O_1 and O_2 satisfy (**1-LI**). *Let $\eta_1 = \xi_1 = 0.8$ and $\eta_2 = \xi_2 = 1$, then $(O_1 \circledast O_2)(0.8, 0.8) - (O_1 \circledast O_2)(1,1) = 0.488 > 0.4 = (|0.8-1| + |0.8-1|)$, so $O_1 \circledast O_2$ does not satisfy* (**1-LI**).

However, we have the following result.

Theorem 7. *If two overlap functions O_1 and O_2 respectively satisfy (**k_1-LI**) and (**k_2-LI**), then their $\circledast$-composition $(O_1 \circledast O_2)$ satisfies ((**k_1 + k_1k_2)-LI**).*

Proof. Assume that O_1 and O_2 respectively satisfy (**k_1-LI**) and (**k_2-LI**), then, for any $\eta_1, \eta_2, \xi_1, \xi_2 \in [0,1]$, we have

$$\begin{aligned} |(O_1 \circledast O_2)(\eta_1, \xi_1) - (O_1 \circledast O_2)(\eta_2, \xi_2)| &= \big|O_1(\eta_1, O_2(\eta_1, \xi_1)) - O_1(\eta_2, O_2(\eta_2, \xi_2))\big| \\ &\leq k_1(|\eta_1 - \eta_2| + |O_2(\eta_1, \xi_1) - O_2(\eta_2, \xi_2)|) \\ &\leq k_1(|\eta_1 - \eta_2| + k_2|\eta_1 - \eta_2| + k_2|\xi_1 - \xi_2|) \\ &= (k_1 + k_1k_2)|\eta_1 - \eta_2| + k_1k_2|\xi_1 - \xi_2| \\ &\leq (k_1 + k_1k_2)\Big(|\eta_1 - \eta_2| + |\xi_1 - \xi_2|\Big). \end{aligned}$$

□

5.4. Summary

Thus far, we have studied the basic properties of overlap/grouping functions w.r.t. the meet operation, join operation, convex combination, and $\circledast$-composition. The summary of the properties of overlap/grouping functions w.r.t. the meet operation, join operation, convex combination, and $\circledast$-composition is shown in Table 2.

Table 2. Properties preservation of the compositions.

Property	O_1	O_2	$O_1 \vee O_2$	$O_1 \wedge O_2$	O_λ	$O_1 \circledast O_2$
ID	√	√	√	√	√	√
MI	√	√	√	√	√	×
HO-*k*	√	√	√	√	√	×
***k*-LI**	√	√	√	√	√	×
PS	√	√	√	√	×	√

6. Conclusions

This paper studies the properties preservation of overlap/grouping functions w.r.t. meet operation, join operation, convex combination, and $\circledast$-composition. The main conclusions are listed as follows.

(1) Closures of two bivariate functions w.r.t. meet operation, join operation, convex combination, and $\circledast$-composition have been obtained in Table 1. Note that $\circledast$-composition does not preserve (***O*1**), and $\circledast$-composition of overlap/grouping functions is not closed. In other words, $\circledast$-composition can not be used to generate new overlap/grouping functions.

(2) We show that meet operation, join operation, convex combination, and $\circledast$-composition of overlap/grouping functions are order preserving, see Theorems 2 and 3.

(3) We have investigated the preservation of the law of (**ID**), (**MI**), (**HO-*k***), (***k*-LI**), and (**PS**) w.r.t. meet operation, join operation, convex combination, and $\circledast$-composition, which can be summarized in Table 2.

These results can be served as a certain criteria for choices of generation methods of overlap/grouping functions from given overlap/grouping functions. For example, convex combination does not preserve (**PS**). Thus, we can not generate a power stable overlap function from two power stable overlap functions by their convex combination.

As we know, overlap/grouping functions have been extended to interval-valued and complex-valued overlap/grouping functions. Could similar results be carried over to the interval-valued and complex-valued settings? Moreover, special overlap/grouping functions such as Archimedean and multiplicatively generated overlap/grouping functions have been studied. In these cases, many restrictions have been added. For further works, it follows that we intend to consider properties preservation of these overlap/grouping functions w.r.t. different composition methods.

Author Contributions: All authors have read and agreed to the published version of the manuscript. Funding acquisition, S.D. and Y.X.; Writing—original draft, S.D. and Y.X.; Writing—review and editing, L.D. and H.S.

Funding: This research was funded by the National Science Foundation of China (Grant Nos. 62006168 and 62101375) and Zhejiang Provincial Natural Science Foundation of China (Grant Nos. LQ21A010001 and LQ21F020001).

Institutional Review Board Statement: Not applicable

Informed Consent Statement: Not applicable

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Bustince, H.; Fernández, J.; Mesiar, R.; Montero, J.; Orduna, R. Overlap functions. *Nonlinear Anal. Theory Methods Appl.* **2010**, *72*, 1488–1499.
2. Beliakov, G.; Pradera, A.; Calvo, T. *Aggregation Functions: A Guide for Practitioners*; Springer: Berlin, Germany, 2007.
3. Bustince, H.; Pagola, M.; Mesiar, R.; Hüllermeier, E.; Herrera, F. Grouping, overlaps, and generalized bientropic functions for fuzzy modeling of pairwise comparisons. *IEEE Trans. Fuzzy Syst.* **2012**, *20*, 405–415.
4. Jurio, A.; Bustince, H.; Pagola, M.; Pradera, A.; Yager, R. Some properties of overlap and grouping functions and their application to image thresholding. *Fuzzy Sets Syst.* **2013**, *229*, 69–90.
5. Elkano, M.; Galar, M.; Sanz, J.; Bustince, H. Fuzzy Rule-Based Classification Systems for multi-class problems using binary decomposition strategies: On the influence of n-dimensional overlap functions in the Fuzzy Reasoning Method. *Inf. Sci.* **2016**, *332*, 94–114.
6. Elkano, M.; Galar, M.; Sanz, J.; Fernández, A.; Barrenechea, E.; Herrera, F.; Bustince, H. Enhancing multi-class classification in FARC-HD fuzzy classifier: On the synergy between n-dimensional overlap functions and decomposition strategies. *IEEE Trans. Fuzzy Syst.* **2015**, *23*, 1562–1580.
7. Elkano, M.; Galar, M.; Sanz, J.A.; Schiavo, P.F.; Pereira, S.; Dimuro, G.P.; Borges, E.N.; Bustince, H. Consensus via penalty functions for decision making in ensembles in fuzzy rule-based classification systems. *Appl. Soft Comput.* **2018**, *67*, 728–740.
8. Santos, H.; Lima, L.; Bedregal, B.; Dimuro, G.P.; Rocha, M.; Bustince, H. Analyzing subdistributivity and superdistributivity on overlap and grouping functions. In Proceedings of the 8th International Summer School on Aggregation Operators (AGOP 2015), Katowice, Poland, 7–10 July 2015; pp. 211–216.
9. Bedregal, B.; Dimuro, G.P.; Bustince, H.; Barrenechea, E. New results on overlap and grouping functions. *Inf. Sci.* **2013**, *249*, 148–170.
10. Bedregal, B.; Bustince, H.; Palmeira, E.; Dimuro, G.; Fernandez, J. Generalized interval-valued OWA operators with interval weights derived from interval-valued overlap functions. *Int. J. Approx. Reason.* **2017**, *90*, 1–16.
11. Dimuro, G.P.; Bedregal, B. Archimedean overlap functions: The ordinal sum and the cancellation, idempotency and limiting properties. *Fuzzy Sets Syst.* **2014**, *252*, 39–54.
12. Gómez, D.; Rodríguez, J.T.; Montero, J.; Bustince, H.; Barrenechea, E. N-dimensional overlap functions. *Fuzzy Sets Syst.* **2016**, *287*, 57–75.
13. Qiao, J.; Hu, B.Q. On interval additive generators of interval overlap functions and interval grouping functions. *Fuzzy Sets Syst.* **2017**, *323*, 19–55.
14. Asmus, T.C.; Dimuro, G.P.; Bedregal, B.; Sanz, J.A.; Pereira, S.; Bustince, H. General interval-valued overlap functions and interval-valued overlap indices. *Inf. Sci.* **2020**, *527*, 27–50.
15. Chen, Y.; Bi, L.; Hu, B.; Dai, S. General Complex-Valued Overlap Functions. *J. Math.* **2021**, *2021*, 6613730.
16. Chen, Y.; Bi, L.; Hu, B.; Dai, S. General Complex-Valued Grouping Functions. *J. Math.* **2021**, *2021*, 5793151.
17. Santos, H.; Dimuro, G.P.; Asmus, T.C.; Lucca, G.; Bueno, E.; Bedregal, B.; Bustince, H. General grouping functions. In Proceedings of 18th International Conference on Information Processing and Management of Uncertainty in Knowledge-Based Systems, Lisbon, Portugal, 15–19 June 2020; Series Communications in Computer and Information Science; Springer: Cham, Switzerland, 2020.
18. Costa, L.M.; Bedregal, B.R.C. Quasi-homogeneous overlap functions. In *Decision Making and Soft Computing*; World Scientific: Joao Pessoa, Brazil, 2014; pp. 294–299.
19. Qiao, J.; Hu, B.Q. On the migrativity of uninorms and nullnorms over overlap and grouping functions. *Fuzzy Sets Syst.* **2018**, *354*, 1–54.
20. Qiao, J.; Hu, B.Q. On the distributive laws of fuzzy implication functions over additively generated overlap and grouping functions. *IEEE Trans. Fuzzy Syst.* **2017**, https://doi.org/10.1109/TFUZZ.2017.2776861.
21. Qiao, J.; Hu, B.Q. On multiplicative generators of overlap and grouping functions. *Fuzzy Sets Syst.* **2018**, *332*, 1–24.
22. Dimuro, G.P.; Bedregal, B. On residual implications derived from overlap functions. *Inf. Sci.* **2015**, *312*, 78–88.

23. Dimuro, G.P.; Bedregal, B. On the laws of contraposition for residual implications derived from overlap functions. In Proceedings of the 2015 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE), Los Alamitos, CA, USA, 2–5 August 2015; pp. 1–7.
24. Dimuro, G.P.; Bedregal, B.; Santiago, R.H.N. On (G, N)-implications derived from grouping functions. *Inf. Sci.* **2014**, *279*, 1–17.
25. Qiao, J. On binary relations induced from overlap and grouping functions. *Int. J. Approx. Reason.* **2019**, *106*, 155–171.
26. Qiao, J. On (IO, O)-fuzzy rough sets based on overlap functions. *Int. J. Approx. Reason.* **2021**, *132*, 26–48.
27. Dimuro, G.P.; Bedregal, B.; Bustince, H.; Asiáin, M.J.; Mesiar, R. On additive generators of overlap functions. *Fuzzy Sets Syst.* **2016**, *287*, 76–96.
28. Wang, H. Constructions of overlap functions on bounded lattices. *Int. J. Approx. Reason.* **2020**, *125*, 203–217.
29. Kolesarova, A.; Mesiar, R. 1-Lipschitz power stable aggregation functions. *Inf. Sci.* **2015**, *294*, 57–63,