axioms

Article

Sequential Riemann–Liouville and Hadamard–Caputo Fractional Differential Equation with Iterated Fractional Integrals Conditions

Sotiris K. Ntouyas [1,2,†], Surang Sitho [3,†], Teerasak Khoployklang [4,†] and Jessada Tariboon [5,*,†]

1 Department of Mathematics, University of Ioannina, 451 10 Ioannina, Greece; sntouyas@uoi.gr
2 Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia
3 Department of Social and Applied Science, College of Industrial Technology, King Mongkut's University of Technology North Bangkok, Bangkok 10800, Thailand; surang.s@cit.kmutnb.ac.th
4 Department of Mathematics, Faculty of Science, Chandrakasem Rajabhat University, Bangkok 10900, Thailand; teerasak.k@chandra.ac.th
5 Intelligent and Nonlinear Dynamic Innovations Research Center, Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Bangkok 10800, Thailand
* Correspondence: jessada.t@sci.kmutnb.ac.th
† These authors contributed equally to this work.

check for updates

Citation: Ntouyas, S.K.; Sitho, S.; Khoployklang, T.; Tariboon, J. Sequential Riemann–Liouville and Hadamard–Caputo Fractional Differential Equation with Iterated Fractional Integrals Conditions. *Axioms* **2021**, *10*, 277. https://doi.org/10.3390/axioms10040277

Academic Editors: Feliz Manuel Minhós and João Fialho

Received: 31 August 2021
Accepted: 25 October 2021
Published: 27 October 2021

Publisher's Note: MDPI stays neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Copyright: © 2021 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https://creativecommons.org/licenses/by/4.0/).

Abstract: In the present research, we initiate the study of boundary value problems for sequential Riemann–Liouville and Hadamard–Caputo fractional derivatives, supplemented with iterated fractional integral boundary conditions. Firstly, we convert the given nonlinear problem into a fixed point problem by considering a linear variant of the given problem. Once the fixed point operator is available, we use a variety of fixed point theorems to establish results regarding existence and uniqueness. Some properties of iteration that will be used in our study are also discussed. Examples illustrating our main results are also constructed. At the end, a brief conclusion is given. Our results are new in the given configuration and enrich the literature on boundary value problems for fractional differential equations.

Keywords: fractional differential equations; Riemann–Liouville fractional derivative; Hadamard–Caputo fractional derivative; boundary value problems; iterated boundary conditions; existence; uniqueness; fixed point theorems

1. Introduction

Differential equations of fractional order have been of great interest in recent years because they play a vital role in describing many phenomena concerning biology, ecology, physics, chemistry, economics, chaotic synchronization, control theory and so on; for instance, see [1,2]. This is because fractional differential equations describe many real world processes related to memory and hereditary properties of various materials more accurately as compared to classical order differential equations. For a systematic development on the topic we refer to the monographs as [3–10]. Fractional order boundary value problems attracted considerable attention and the literature on the topic was enriched with a huge number of articles, for instance, see [11–23] and references cited therein. In the literature there are several kinds of fractional derivatives, such as Riemann–Liouville, Caputo, Hadamard, Hilfer, Katugampola, and so on. In many papers in the literature the authors studied existence and uniqueness results for boundary value problems and coupled systems of fractional differential equations by using mixed types of fractional derivatives. For example Riemann–Liouvile and Caputo fractional derivatives are used in the papers [14,19,21], Riemann–Liouville and Hadamard–Caputo fractional derivatives in the papers [15] and Caputo–Hadamard fractional derivatives in the papers [20,22]. Multi-term fractional differential equations also gained considerable importance in view of their occurrence in the mathematical models of certain real world problems, such as behavior of real materials [24], continuum and statistical mechanics [25], an inextensible pendulum with fractional damping terms [26], etc.

In [20] the authors studied the existence and uniqueness of solutions for two sequential Caputo–Hadamard and Hadamard–Caputo fractional differential equations subject to separated boundary conditions as

$$\begin{cases} {}^{C}D^{p}({}^{H}D^{q}x)(t) = f(t,x(t)), & t \in (a,b), \\ \alpha_1 x(a) + \alpha_2({}^{H}D^{q}x)(a) = 0, & \beta_1 x(b) + \beta_2({}^{H}D^{q}x)(b) = 0, \end{cases} \tag{1}$$

and

$$\begin{cases} {}^{H}D^{q}({}^{C}D^{p}x)(t) = f(t,x(t)), & t \in (a,b), \\ \alpha_1 x(a) + \alpha_2({}^{C}D^{p}x)(a) = 0, & \beta_1 x(b) + \beta_2({}^{C}D^{p}x)(b) = 0, \end{cases} \tag{2}$$

where ${}^{C}D^{p}$ and ${}^{H}D^{q}$ are the Caputo and Hadamard fractional derivatives of orders p and q, respectively, $0 < p, q \le 1$, $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $a > 0$ and $\alpha_i, \beta_i \in \mathbb{R}$, $i = 1,2$.

In a recent paper [15] the authors investigated the existence and uniqueness of solutions for the following coupled system of sequential Riemann–Liouville and Hadamard–Caputo fractional differential equations supplemented with nonlocal coupled fractional integral boundary conditions

$$\begin{cases} {}^{RL}D^{p_1}\left({}^{HC}D^{q_1}x\right)(t) = f(t,x(t),y(t)), & t \in [0,T], \\ {}^{RL}D^{p_2}\left({}^{HC}D^{q_2}y\right)(t) = g(t,x(t),y(t)), & t \in [0,T], \\ {}^{HC}D^{q_1}x(0) = 0, \quad x(T) = \sum_{i=1}^{m} \alpha_i {}^{RL}I^{\beta_i} y(\xi_i), \\ {}^{HC}D^{q_2}y(0) = 0, \quad y(T) = \sum_{j=1}^{k} \lambda_j {}^{RL}I^{\delta_j} x(\eta_j), \end{cases} \tag{3}$$

where ${}^{RL}D^{p_r}$ and ${}^{HC}D^{q_r}$ are the Riemann–Liouville and Hadamard–Caputo fractional derivatives of orders p_r and q_r, respectively, $0 < p_r, q_r < 1$, $r = 1,2$, the nonlinear continuous functions $f, g : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$, ${}^{RL}I^{\phi}$ is the Riemann–Liouville fractional integral of orders $\phi > 0$, $\phi \in \{\beta_i, \delta_j\}$ and given constants $\alpha_i, \lambda_j \in \mathbb{R}$, $\xi_i, \eta_j \in (0,T)$, $i = 1,\ldots,m$, $j = 1,\ldots,k$.

Inspired by the above-mentioned papers, our goal in this paper is to enrich the problems concerning sequential Riemann–Liouville and Hadamard–Caputo fractional derivatives with a new research area—iterated boundary conditions. Thus, in this work, we initiate the study of boundary value problems containing sequential Riemann–Liouville and Hadamard–Caputo fractional derivatives, supplemented with iterated fractional integral conditions of the form:

$$\begin{cases} {}^{RL}D^{p}\left({}^{HC}D^{q}x\right)(t) = f(t,x(t)), \quad t \in [0,T], \\ {}^{HC}D^{q}x(0) = 0, \\ x(T) = \lambda_1 \widetilde{R}^{(\alpha_n,\beta_{n-1},\ldots,\beta_1,\alpha_1)}x(\xi_1) + \lambda_2 \widehat{R}^{(\delta_m,\gamma_m,\ldots,\delta_1,\gamma_1)}x(\xi_2), \end{cases} \tag{4}$$

where ${}^{RL}D^{p}$ and ${}^{HC}D^{q}$ are the Riemann–Liouville and Hadamard–Caputo fractional derivatives of orders p and q, respectively, $0 < p, q < 1$, $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $m, n \in \mathbb{Z}^{+}$, the given constants $\lambda_1, \lambda_2 \in \mathbb{R}$ and

$$\widetilde{R}^{(\alpha_n,\ \ldots,\beta_1,\alpha_1)}x(t) = {}^{RL}I^{\alpha_n}\, {}^{H}I^{\beta_{n-1}}\, {}^{RL}I^{\alpha_{n-1}}\, {}^{H}I^{\beta_{n-2}}\ \ldots\ {}^{H}I^{\beta_2}\, {}^{RL}I^{\alpha_2}\, {}^{H}I^{\beta_1}\, {}^{RL}I^{\alpha_1}x(t),$$

and

$$\widehat{R}^{(\delta_m,\ \ldots,\delta_1,\gamma_1)}x(t) = {}^{H}I^{\delta_m}\, {}^{RL}I^{\gamma_m}\, {}^{H}I^{\delta_{m-1}}\, {}^{RL}I^{\gamma_{m-1}}\ \ldots\ {}^{H}I^{\delta_2}\, {}^{RL}I^{\gamma_2}\, {}^{H}I^{\delta_1}\, {}^{RL}I^{\gamma_1}x(t),$$

are the iterated fractional integrals, where $t = \xi_1$ and $t = \xi_2$, respectively, $\xi_1, \xi_2 \in (0, T)$, ${}^{RL}I^{\phi}$, ${}^{H}I^{\psi}$ are the Riemann–Liouville and Hadamard fractional integrals of orders $\phi, \psi > 0$, respectively, $\phi \in \{\alpha_{(\cdot)}, \gamma_{(\cdot)}\}$, $\psi \in \{\beta_{(\cdot)}, \delta_{(\cdot)}\}$.

Observe that $\widetilde{R}^{(\cdot)}(\cdot)(t)$ and $\widehat{R}^{(\cdot)}(\cdot)(t)$ are odd and even iterations, for example,

$$\widetilde{R}^{(\frac{1}{4},\frac{1}{3},\frac{1}{2})}x(t) = {}^{RL}I^{\frac{1}{4}}\, {}^{H}I^{\frac{1}{3}}\, {}^{RL}I^{\frac{1}{2}}x(t),$$

and

$$\widehat{R}^{(\frac{1}{8},\frac{1}{7},\frac{1}{6},\frac{1}{5})}x(t) = {}^{H}I^{\frac{1}{8}}\, {}^{RL}I^{\frac{1}{7}}\, {}^{H}I^{\frac{1}{6}}\, {}^{RL}I^{\frac{1}{5}}x(t),$$

respectively. In addition, these notations can be reduced to a single fractional integral of Riemann–Liouville and Hadamard types by $\widetilde{R}^{(\alpha_1)}x(t) = {}^{RL}I^{\alpha_1}x(t)$ and $\widehat{R}^{(\delta_1,0)}x(t) = {}^{H}I^{\delta_1}x(t)$. Furthermore, this is the first paper to define the iteration notation alternating between two different types of fractional integrals.

We establish existence and uniqueness results for the boundary value problem (4) by applying a variety of fixed point theorems. More precisely, the existence of a unique solution is proved by using Banach's contraction mapping principle, Banach's contraction mapping principle combined with Hölder's inequality and Boyd–Wong fixed point theorem for nonlinear contractions, while the existence result is established via Leray–Schauder nonlinear alternative.

Comparing problem (4) with the previous problem studied (3), in which sequential Riemann–Liouville and Hadamard–Caputo fractional derivatives were also used, we note that, except for the fact that both problems deal with sequential Riemann–Liouville and Hadamard–Caputo fractional derivatives, they are entirely different. Problem (3) concerns a coupled system subject to nonlocal coupled fractional integral boundary conditions, while problem (4) concerns a boundary value problem supplemented with iterated fractional boundary conditions. The methods of study are based on applications of fixed point theorems and are obviously different. As far as we know, this is the first paper in the literature which concerns iterated boundary conditions, and in this fact lies the novelty of the paper.

The rest of the paper is arranged as follows: Section 2 contain some preliminary notations and definitions from fractional calculus. The main results are presented in Section 3. Some special cases are discussed in Section 4, while illustrative examples are constructed in the final Section 5. The paper closes with a brief conclusion.

2. Preliminaries

Let us introduce some notations and definitions of fractional calculus [4,27] in the sense of Riemann–Liouville and Hadamard–Caputo and present preliminary results needed in our proofs later.

Definition 1. *The Riemann–Liouville fractional derivative of order $p > 0$ of a continuous function $f : (0, \infty) \to \mathbb{R}$ is defined by*

$${}^{RL}D^{p}f(t) = \frac{1}{\Gamma(n-p)}\left(\frac{d}{dt}\right)^{n}\int_0^t (t-s)^{n-p-1}f(s)ds, \quad n-1 < p < n,$$

where $n = [p] + 1$, $[p]$ denotes the integer part of a real number p and Γ is the Gamma function defined by $\Gamma(p) = \int_0^{\infty} e^{-s}s^{p-1}ds$.

Definition 2. *The Riemann–Liouville fractional integral of order p of a function* $f : (0, \infty) \to \mathbb{R}$ *is defined as*

$$^{RL}I^p f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{p-1} f(s) ds, \quad p > 0,$$

provided the right side is pointwise defined on $\mathbb{R}_+$.

Definition 3. *For an at least n-times differentiable function* $g : (0, \infty) \to \mathbb{R}$, *the Hadamard–Caputo derivative of fractional order* $q > 0$ *is defined as*

$$^{HC}D^q g(t) = \frac{1}{\Gamma(n-q)} \int_0^t \left(\log \frac{t}{s}\right)^{n-q-1} \delta^n g(s) \frac{ds}{s}, \quad n-1 < q < n, \quad n = [q]+1,$$

where $\delta = t\frac{d}{dt}$ *and* $\log(\cdot) = \log_e(\cdot)$.

Definition 4. *The Hadamard fractional integral of order* $q > 0$ *is defined as*

$$^{H}I^q g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \left(\log \frac{t}{s}\right)^{q-1} g(s) \frac{ds}{s},$$

provided the integral exists.

Lemma 1 (see [4])**.** *Let* $p > 0$. *Then for* $y \in C(0, T) \cap L(0, T)$ *it holds*

$$^{RL}I^p \left(^{RL}D^p y\right)(t) = y(t) + c_1 t^{p-1} + c_2 t^{p-2} + \cdots + c_n t^{p-n},$$

where $c_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$ *and* $n - 1 < p < n$.

Lemma 2 ([27])**.** *Let* $u \in AC^n_\delta[0, T]$ *or* $C^n_\delta[0, T]$ *and* $q \in \mathbb{C}$, *where* $X^n_\delta[0, T] = \{g : [0, T] \to \mathbb{C} : \delta^{n-1} g(t) \in X[0, T]\}$. *Then, we have*

$$^{H}I^q(^{HC}D^q)u(t) = u(t) + c_0 + c_1 \log t + c_2 (\log t)^2 + \cdots + c_{n-1} (\log t)^{n-1},$$

where $c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n-1$ $(n = [q]+1)$.

Lemma 3 ([4], page 113)**.** *Let* $q > 0$ *and* $\beta > 0$ *be given constants. Then the following formula*

$$^{H}I^q t^\beta = \beta^{-q} t^\beta,$$

holds.

Next we establish two new formulas for iteration of fractional integrals of Riemann–Liouville and Hadamard types.

Lemma 4. *Let* $m > -1$, $\alpha_j, \beta_i > 0$, $i = 1, 2, \ldots, n$, *be constants. Then we have*

(i)

$$\widetilde{R}^{(\alpha_n, \beta_{n-1}, \ldots, \beta_1, \alpha_1)} t^m = \Gamma(m+1) \frac{\prod_{i=1}^{n-1} \left(m + \sum_{j=1}^{i} \alpha_j\right)^{-\beta_i}}{\Gamma\left(1 + m + \sum_{j=1}^{n} \alpha_j\right)} t^{m + \sum_{j=1}^{n} \alpha_j}$$

(ii)

$$\widehat{R}^{(\beta_n, \alpha_n, \ldots, \beta_1, \alpha_1)} t^m = \Gamma(m+1) \frac{\prod_{i=1}^{n} \left(m + \sum_{j=1}^{i} \alpha_j\right)^{-\beta_i}}{\Gamma\left(1 + m + \sum_{j=1}^{n} \alpha_j\right)} t^{m + \sum_{j=1}^{n} \alpha_j}.$$

Proof. From ${}^{RL}I^{\alpha_1}t^m = \dfrac{\Gamma(m+1)}{\Gamma(1+m+\alpha_1)}t^{m+\alpha_1}$, we get that (i) holds for $n=1$. Applying the Lemma 3, we have

$$\begin{aligned} {}^{H}I^{\beta_1}\left({}^{RL}I^{\alpha_1}(1)\right)(t) &= \frac{\Gamma(m+1)}{\Gamma(1+m+\alpha_1)}\,{}^{H}I^{\beta_1}t^{m+\alpha_1} \\ &= \frac{\Gamma(m+1)}{\Gamma(1+m+\alpha_1)}(m+\alpha_1)^{-\beta_1}t^{m+\alpha_1}, \end{aligned} \tag{5}$$

which implies that (ii) is true for $n=1$. In the next step, we suppose that (i)–(ii) are fulfilled for $n=k$. Then, from

$${}^{RL}I^{\alpha_{k+1}}t^{m+\sum_{j=1}^{k}\alpha_j} = \frac{\Gamma(1+m+\sum_{j=1}^{k}\alpha_j)}{\Gamma(1+m+\sum_{j=1}^{k+1}\alpha_j)}t^{m+\sum_{j=1}^{k+1}\alpha_j},$$

we have

$$\begin{aligned} {}^{RL}I^{\alpha_{k+1}}\widehat{R}^{(\beta_k,\alpha_k,\ldots,\beta_1,\alpha_1)}t^m &= \Gamma(m+1)\frac{\prod_{i=1}^{k}\left(m+\sum_{j=1}^{i}\alpha_j\right)^{-\beta_i}}{\Gamma\left(1+m+\sum_{j=1}^{k}\alpha_j\right)}\,{}^{RL}I^{\alpha_{k+1}}\,t^{m+\sum_{j=1}^{k}\alpha_j} \\ &= \Gamma(m+1)\frac{\prod_{i=1}^{k}\left(m+\sum_{j=1}^{i}\alpha_j\right)^{-\beta_i}}{\Gamma\left(1+m+\sum_{j=1}^{k+1}\alpha_j\right)}t^{m+\sum_{j=1}^{k+1}\alpha_j}, \\ &= \widetilde{R}^{(\alpha_{k+1},\beta_k,\ldots,\beta_1,\alpha_1)}t^m, \end{aligned}$$

which yields that (i) holds when $n=k+1$.

Further, we get

$$\begin{aligned} {}^{H}I^{\beta_{k+1}}\widetilde{R}^{(\alpha_{k+1},\beta_k,\ldots,\beta_1,\alpha_1)}t^m &= \Gamma(m+1)\frac{\prod_{i=1}^{k}\left(m+\sum_{j=1}^{i}\alpha_j\right)^{-\beta_i}}{\Gamma\left(1+m+\sum_{j=1}^{k+1}\alpha_j\right)}\,{}^{H}I^{\beta_{k+1}}\,t^{m+\sum_{j=1}^{k+1}\alpha_j} \\ &= \Gamma(m+1)\frac{\prod_{i=1}^{k}\left(m+\sum_{j=1}^{i}\alpha_j\right)^{-\beta_i}}{\Gamma\left(1+m+\sum_{j=1}^{k+1}\alpha_j\right)} \\ &\quad\times\left(m+\sum_{j=1}^{k+1}\alpha_j\right)^{-\beta_{k+1}}t^{m+\sum_{j=1}^{k+1}\alpha_j} \\ &= \Gamma(m+1)\frac{\prod_{i=1}^{k+1}\left(m+\sum_{j=1}^{i}\alpha_j\right)^{-\beta_i}}{\Gamma\left(1+m+\sum_{j=1}^{k+1}\alpha_j\right)}t^{m+\sum_{j=1}^{k+1}\alpha_j} \\ &= \widehat{R}^{(\beta_{k+1},\alpha_{k+1}\ldots,\beta_1,\alpha_1)}t^m, \end{aligned}$$

which yields that (ii) is true for $n=k+1$. Therefore, by mathematical induction, (i) and (ii) hold and the proof is completed. □

Corollary 1. *If we put $m=0$ in Lemma 4, we obtain*

(i)

$$\widetilde{R}^{(\alpha_n,\beta_{n-1},\ldots,\beta_1,\alpha_1)}(1)(t) = \frac{\prod_{i=1}^{n-1}\left(\sum_{j=1}^{i}\alpha_j\right)^{-\beta_i}}{\Gamma\left(1+\sum_{j=1}^{n}\alpha_j\right)}t^{\sum_{j=1}^{n}\alpha_j}$$

(ii)

$$\widehat{R}^{(\beta_n,\alpha_n,\ldots,\beta_1,\alpha_1)}(1)(t) = \frac{\prod_{i=1}^{n}\left(\sum_{j=1}^{i}\alpha_j\right)^{-\beta_i}}{\Gamma\left(1+\sum_{j=1}^{n}\alpha_j\right)} t^{\sum_{j=1}^{n}\alpha_j}.$$

The following lemma dealing with a linear variant of problem (4) plays a key role in defining the solution of problem (4).

Lemma 5. *Let $z : [0,T] \to \mathbb{R}$ be a continuous function and*

$$\Omega = 1 - \lambda_1 \frac{\prod_{i=1}^{n-1}\left(\sum_{j=1}^{i}\alpha_j\right)^{-\beta_i}}{\Gamma\left(1+\sum_{j=1}^{n}\alpha_j\right)} \xi_1^{\sum_{j=1}^{n}\alpha_j} - \lambda_2 \frac{\prod_{i=1}^{m}\left(\sum_{j=1}^{i}\gamma_j\right)^{-\delta_i}}{\Gamma\left(1+\sum_{j=1}^{m}\gamma_j\right)} \xi_2^{\sum_{j=1}^{m}\gamma_j} \neq 0.$$

Then x is a solution of the boundary value problem

$$\begin{cases} {}^{RL}D^p\left({}^{HC}D^q x\right)(t) = z(t), \qquad t \in [0,T], \\ {}^{HC}D^q x(0) = 0, \\ x(T) = \lambda_1 \widetilde{R}^{(\alpha_n,\beta_{n-1},\ldots,\beta_1,\alpha_1)} x(\xi_1) + \lambda_2 \widehat{R}^{(\delta_m,\gamma_m,\ldots,\delta_1,\gamma_1)} x(\xi_2), \end{cases} \tag{6}$$

if and only if

$$\begin{aligned} x(t) &= \frac{\lambda_1}{\Omega}\widetilde{R}^{(\alpha_n,\beta_{n-1},\ldots,\alpha_1,q,p)} z(\xi_1) + \frac{\lambda_2}{\Omega}\widehat{R}^{(\delta_m,\gamma_m,\ldots,\gamma_1,q,p)} z(\xi_2) \\ &\quad - \frac{1}{\Omega}{}^{H}I^{q}\,{}^{RL}I^{p} z(T) + {}^{H}I^{q}\,{}^{RL}I^{p} z(t). \end{aligned} \tag{7}$$

Proof. For $t \in [0,T]$, taking Riemann–Liouville fractional integral of order p in (4), results in

$${}^{HC}D^q x(t) = c_1 t^{p-1} + {}^{RL}I^p z(t), \quad c_1 \in \mathbb{R}. \tag{8}$$

Since $0 < p < 1$, the condition ${}^{HC}D^q x(0) = 0$ implies $c_1 = 0$. Applying the Hadamard fractional integral of order q to (8) and substituting the value of c_1, we obtain

$$x(t) = c_0 + {}^{H}I^{q}\,{}^{RL}I^{p} z(t). \tag{9}$$

Now, we consider the terms

$$\begin{aligned} \widetilde{R}^{(\alpha_n,\beta_{n-1},\ldots,\beta_1,\alpha_1)} x(t) &= c_0 \widetilde{R}^{(\alpha_n,\ldots,\beta_1,\alpha_1)}(1)(t) + \widetilde{R}^{(\alpha_n,\ldots,\beta_1,\alpha_1)}\,{}^{H}I^{q}\,{}^{RL}I^{p} z(t) \\ &= c_0 \frac{\prod_{i=1}^{n-1}\left(\sum_{j=1}^{i}\alpha_j\right)^{-\beta_i}}{\Gamma\left(1+\sum_{j=1}^{n}\alpha_j\right)} t^{\sum_{j=1}^{n}\alpha_j} + \widetilde{R}^{(\alpha_n,\ldots,\alpha_1,q,p)} z(t), \end{aligned}$$

and

$$\begin{aligned} \widehat{R}^{(\delta_m,\gamma_m,\ldots,\delta_1,\gamma_1)} x(t) &= c_0 \widehat{R}^{(\delta_m,\ldots,\delta_1,\gamma_1)}(1)(t) + \widehat{R}^{(\delta_m,\ldots,\delta_1,\gamma_1)}\,{}^{H}I^{q}\,{}^{RL}I^{p} z(t) \\ &= c_0 \frac{\prod_{i=1}^{m}\left(\sum_{j=1}^{i}\gamma_j\right)^{-\delta_i}}{\Gamma\left(1+\sum_{j=1}^{m}\gamma_j\right)} t^{\sum_{j=1}^{m}\gamma_j} + \widehat{R}^{(\delta_m,\ldots,\gamma_1,q,p)} z(t), \end{aligned}$$

Then the second condition in (4) yields

$$c_0 = \frac{\lambda_1}{\Omega}\widetilde{R}^{(\alpha_n,\beta_{n-1},\ldots,\alpha_1,q,p)} z(\xi_1) + \frac{\lambda_2}{\Omega}\widehat{R}^{(\delta_m,\gamma_m,\ldots,\gamma_1,q,p)} z(\xi_2) - \frac{1}{\Omega}{}^{H}I^{q}\,{}^{RL}I^{p} z(T),$$

which leads to the integral Equation (7) by substituting the value of c_0 in (9).

Conversely, by taking the Hadamard–Caputo of order q to (7), we have ${}^{HC}D^q x(t) = {}^{RL}I^p z(t)$, which implies ${}^{HC}D^q x(0) = 0$. Applying $\widetilde{R}^{(\alpha_n,\beta_{n-1},\ldots,\alpha_1)}$ and $\widehat{R}^{(\delta_m,\gamma_m,\ldots,\gamma_1)}$ to (7) at points ξ_1, ξ_2, and multiplying constants λ_1 and λ_2, respectively, we obtain $x(T)$. Therefore, the proof is completed. □

3. Main Results

Let $\mathcal{C} = C([0,T],\mathbb{R})$ be the set of all continuous functions from $[0,T]$ to $\mathbb{R}$. Then, $\mathcal{C}$ is a Banach space endowed with the supremum norm defined as $\|x\| = \sup_{t\in[0,T]} |x(t)|$.

By Lemma 5 we define an operator $\mathcal{K} : \mathcal{C} \to \mathcal{C}$ by

$$\begin{aligned} \mathcal{K}x(t) &= \frac{\lambda_1}{\Omega}\widetilde{R}^{(\alpha_n,\beta_{n-1},\ldots,\alpha_1,q,p)} f_x(\xi_1) + \frac{\lambda_2}{\Omega}\widehat{R}^{(\delta_m,\gamma_m,\ldots,\gamma_1,q,p)} f_x(\xi_2) \\ &\quad - \frac{1}{\Omega}{}^{H}I^{q}\,{}^{RL}I^{p} f_x(T) + {}^{H}I^{q}\,{}^{RL}I^{p} f_x(t), \end{aligned} \tag{10}$$

where $f_x(\phi)$ is the abbreviation of the nonlinear function $f(\phi, x(\phi))$, $\phi \in \{t, T, \xi_1, \xi_2\}$. The existence and uniqueness theorems will be established by considering the operator equation $x = \mathcal{K}x$ and using fixed point theorems. Let us set a constant

$$\begin{aligned} \Phi_1 &= \frac{|\lambda_1|}{|\Omega|}\widetilde{R}^{(\alpha_n,\beta_{n-1},\ldots,\alpha_1,q,p)}(1)(\xi_1) + \frac{|\lambda_2|}{|\Omega|}\widehat{R}^{(\delta_m,\gamma_m,\ldots,\gamma_1,q,p)}(1)(\xi_2) \\ &\quad + \left(\frac{1}{|\Omega|} + 1\right)\widehat{R}^{(q,p)}(1)(T), \end{aligned} \tag{11}$$

where

$$\begin{aligned} \widetilde{R}^{(\alpha_n,\beta_{n-1},\ldots,\alpha_1,q,p)}(1)(\xi_1) &= \frac{\prod_{i=0}^{n-1}\left(\sum_{j=0}^{i}\alpha_j\right)^{-\beta_i}}{\Gamma\left(1+\sum_{j=0}^{n}\alpha_j\right)}\xi_1^{\sum_{j=0}^{n}\alpha_j}, \\ \widehat{R}^{(\delta_m,\gamma_m,\ldots,\gamma_1,q,p)}(1)(\xi_2) &= \frac{\prod_{i=0}^{m}\left(\sum_{j=0}^{i}\gamma_j\right)^{-\delta_i}}{\Gamma\left(1+\sum_{j=0}^{m}\gamma_j\right)}\xi_2^{\sum_{j=0}^{m}\gamma_j}, \\ \widehat{R}^{(q,p)}(1)(T) &= \frac{p^{-q}T^p}{\Gamma(p+1)}, \end{aligned}$$

with $\alpha_0 = \gamma_0 = p$ and $\beta_0 = \delta_0 = q$.

3.1. Existence and Uniqueness Result via Banach's Fixed Point Theorem

Theorem 1. *Suppose that the nonlinear function $f : [0,T] \times \mathbb{R}$ satisfies the following condition:*

(H_1) there exists a function $\omega(t) > 0$ with

$$|f(t,x) - f(t,y)| \le \omega(t)|x - y|,$$

for each $t \in [0,T]$ and $x, y \in \mathbb{R}$.

If $L\Phi_1 < 1$, where $L = \sup_{t\in[0,T]} |\omega(t)|$ and Φ_1 given by (11), then the sequential Riemann–Liouville and Hadamard–Caputo fractional differential equation with iterated fractional integral conditions (4) has a unique solution on $[0,T]$.

Proof. Let us start by setting $B_r = \{x \in \mathcal{C} : \|x\| \le r\}$ such that

$$r \ge \frac{\Phi_1 M}{1 - L\Phi_1} \tag{12}$$

and $M := \sup\{f(t,0) \,:\, t \in [0,T]\}$. Using relations $|f_x(t)| \leq |f_x(t) - f_0(t)| + |f_0(t)| \leq |\omega(t)|\|x\| + |f_0(t)| \leq Lr + M$ for all $t \in [0,T]$ and from Corollary 1, it follows that

$$\begin{aligned}
|\mathcal{K}x(t)| &\leq \frac{|\lambda_1|}{|\Omega|}\widetilde{R}^{(\alpha_n,\beta_{n-1},\ldots,\alpha_1,q,p)}|f_x|(\xi_1) + \frac{|\lambda_2|}{|\Omega|}\widehat{R}^{(\delta_m,\gamma_m,\ldots,\gamma_1,q,p)}|f_x|(\xi_2) \\
&\quad + \frac{1}{|\Omega|}{}^{H}I^{q}\,{}^{RL}I^{p}|f_x|(T) + {}^{H}I^{q}\,{}^{RL}I^{p}|f_x|(t) \\
&\leq \frac{|\lambda_1|}{|\Omega|}(Lr+M)\widetilde{R}^{(\alpha_n,\beta_{n-1},\ldots,\alpha_1,q,p)}(1)(\xi_1) \\
&\quad + \frac{|\lambda_2|}{|\Omega|}(Lr+M)\widehat{R}^{(\delta_m,\gamma_m,\ldots,\gamma_1,q,p)}(1)(\xi_2) \\
&\quad + \left(\frac{1}{|\Omega|}+1\right)(Lr+M)^{H}I^{q}\,{}^{RL}I^{p}(1)(T) \\
&\leq r,
\end{aligned}$$

which leads to $\mathcal{K}(B_r) \subseteq B_r$. To show that $\mathcal{K}$ is a contraction, for any $x, y \in B_r$, we get that

$$\begin{aligned}
|\mathcal{K}x(t) - \mathcal{K}y(t)| &\leq \frac{|\lambda_1|}{|\Omega|}\widetilde{R}^{(\alpha_n,\beta_{n-1},\ldots,\alpha_1,q,p)}|f_x - f_y|(\xi_1) + \frac{|\lambda_2|}{|\Omega|}\widehat{R}^{(\delta_m,\gamma_m,\ldots,\gamma_1,q,p)}|f_x - f_y|(\xi_2) \\
&\quad + \frac{1}{|\Omega|}{}^{H}I^{q}\,{}^{RL}I^{p}|f_x - f_y|(T) + {}^{H}I^{q}\,{}^{RL}I^{p}|f_x - f_y|(t) \\
&\leq \left\{\frac{|\lambda_1|}{|\Omega|}\widetilde{R}^{(\alpha_n,\beta_{n-1},\ldots,\alpha_1,q,p)}(1)(\xi_1) + \frac{|\lambda_2|}{|\Omega|}\widehat{R}^{(\delta_m,\gamma_m,\ldots,\gamma_1,q,p)}(1)(\xi_2)\right. \\
&\quad \left. + \left(\frac{1}{|\Omega|}+1\right)\widehat{R}^{(q,p)}(1)(T)\right\}\|\omega\|\|x-y\| \\
&= L\Phi_1\|x-y\|,
\end{aligned}$$

which yields $\|\mathcal{K}x - \mathcal{K}y\| \leq L\Phi_1\|x-y\|$. Since, by assumption, $L\Phi_1 < 1$, $\mathcal{K}$ is a contraction operator and then there exists a unique fixed point in B_r. Then the sequential Riemann–Liouville and Hadamard–Caputo fractional differential equation with iterated fractional integral conditions (4) has a unique solution on $[0,T]$. □

3.2. Existence and Uniqueness Result via Banach's Fixed Point Theorem and Hölder's Inequality

For convenience we put:

$$\begin{aligned}
\Phi_2 &= \frac{|\lambda_1|}{|\Omega|\Gamma(p)}\left(\frac{1-\sigma}{p-\sigma}\right)^{1-\sigma}(p-\sigma)^{-q}\Gamma((p-\sigma)+1) \\
&\quad \times \frac{\prod_{i=1}^{n-1}\left((p-\sigma)+\sum_{j=1}^{i}\alpha_j\right)^{-\beta_i}}{\Gamma\left(1+(p-\sigma)+\sum_{j=1}^{n}\alpha_j\right)}\xi_1^{(p-\sigma)+\sum_{j=1}^{n}\alpha_j} \\
&\quad + \frac{|\lambda_2|}{|\Omega|\Gamma(p)}\left(\frac{1-\sigma}{p-\sigma}\right)^{1-\sigma}(p-\sigma)^{-q}\Gamma((p-\sigma)+1) \\
&\quad \times \frac{\prod_{i=1}^{m}\left((p-\sigma)+\sum_{j=1}^{i}\gamma_j\right)^{-\delta_i}}{\Gamma\left(1+(p-\sigma)+\sum_{j=1}^{m}\gamma_j\right)}\xi_2^{(p-\sigma)+\sum_{j=1}^{m}\gamma_j} \\
&\quad + \left(\frac{1}{|\Omega|}+1\right)\frac{1}{\Gamma(p)}\left(\frac{1-\sigma}{p-\sigma}\right)^{1-\sigma}(p-\sigma)^{-q}T^{p-\sigma}. \qquad (13)
\end{aligned}$$

Theorem 2. *Assume that the function f satisfies condition (H_1) in Theorem 1 with $\omega \in L^{\frac{1}{\sigma}}([0,T],\mathbb{R}^+)$, where $\sigma \in (0,p)$. Denote $\|\omega\|_\sigma = \left(\int_0^t (\omega(s))^{\frac{1}{\sigma}}ds\right)^{\sigma}$.*

If $\|\omega\|_\sigma \Phi_2 < 1$*, where* Φ_2 *is given by (13), then the boundary value problem (4) has a unique solution on* $[0,T]$.

Proof. Setting $x, y \in \mathcal{C}$, for $t \in [0,T]$, we obtain by using (H_1) that

$$
\begin{aligned}
|\mathcal{K}x(t) - \mathcal{K}y(t)| \quad &\leq \quad \frac{|\lambda_1|}{|\Omega|}\widetilde{R}^{(\alpha_n,\beta_{n-1},\ldots,\alpha_1,q,p)}|f_x - f_y|(\xi_1) + \frac{|\lambda_2|}{|\Omega|}\widehat{R}^{(\delta_m,\gamma_m,\ldots,\gamma_1,q,p)}|f_x - f_y|(\xi_2) \\
&\quad + \frac{1}{|\Omega|}{}^H I^{q\ RL} I^p |f_x - f_y|(T) + {}^H I^{q\ RL} I^p |f_x - f_y|(t) \\
&\leq \frac{|\lambda_1|}{|\Omega|}\|x-y\|\widetilde{R}^{(\alpha_n,\beta_{n-1},\ldots,\alpha_1,q,p)}\omega(\xi_1) \\
&\quad + \frac{|\lambda_2|}{|\Omega|}\|x-y\|\widehat{R}^{(\delta_m,\gamma_m,\ldots,\gamma_1,q,p)}\omega(\xi_2) \\
&\quad + \left(\frac{1}{|\Omega|} + 1\right)\|x-y\|{}^H I^{q\ RL} I^p \omega(T).
\end{aligned} \tag{14}
$$

Now, we consider the application of Hölder's inequality as

$$
\begin{aligned}
{}^{RL}I^p\omega(t) \quad &= \quad \frac{1}{\Gamma(p)}\int_0^t (t-s)^{p-1}\omega(s)ds \\
&\leq \frac{1}{\Gamma(p)}\left(\int_0^t \left((t-s)^{p-1}\right)^{\frac{1}{1-\sigma}} ds\right)^{1-\sigma}\left(\int_0^t (\omega(s))^{\frac{1}{\sigma}} ds\right)^{\sigma} \\
&\leq \frac{\|\omega\|_\sigma}{\Gamma(p)}\left(\frac{1-\sigma}{p-\sigma}\right)^{1-\sigma} t^{p-\sigma},
\end{aligned}
$$

which yields

$$
{}^H I^{q RL} I^p \omega(t) \leq \frac{\|\omega\|_\sigma}{\Gamma(p)}\left(\frac{1-\sigma}{p-\sigma}\right)^{1-\sigma}(p-\sigma)^{-q} t^{p-\sigma}. \tag{15}
$$

Then we have

$$
\begin{aligned}
\widetilde{R}^{(\alpha_n,\beta_{n-1},\ldots,\alpha_1,q,p)}\omega(\xi_1) \quad &= \quad \widetilde{R}^{(\alpha_n,\beta_{n-1},\ldots,\alpha_1)}{}^H I^{q RL} I^p \omega(\xi_1) \\
&\leq \frac{\|\omega\|_\sigma}{\Gamma(p)}\left(\frac{1-\sigma}{p-\sigma}\right)^{1-\sigma}(p-\sigma)^{-q}\widetilde{R}^{(\alpha_n,\beta_{n-1},\ldots,\alpha_1)} t^{p-\sigma}(\xi_1) \\
&= \frac{\|\omega\|_\sigma}{\Gamma(p)}\left(\frac{1-\sigma}{p-\sigma}\right)^{1-\sigma}(p-\sigma)^{-q}\Gamma((p-\sigma)+1) \\
&\quad \times \frac{\prod_{i=1}^{n-1}\left((p-\sigma) + \sum_{j=1}^{i}\alpha_j\right)^{-\beta_i}}{\Gamma\left(1 + (p-\sigma) + \sum_{j=1}^{n}\alpha_j\right)}\xi_1^{(p-\sigma)+\sum_{j=1}^{n}\alpha_j},
\end{aligned} \tag{16}
$$

by applying Lemma 4. In the same way, we obtain

$$
\begin{aligned}
\widehat{R}^{(\delta_m,\gamma_m,\ldots,\gamma_1,q,p)}\omega(\xi_2) \quad &= \quad \widehat{R}^{(\delta_m,\gamma_m,\ldots,\gamma_1)}{}^H I^{q RL} I^p \omega(\xi_2) \\
&\leq \frac{\|\omega\|_\sigma}{\Gamma(p)}\left(\frac{1-\sigma}{p-\sigma}\right)^{1-\sigma}(p-\sigma)^{-q}\widehat{R}^{(\delta_m,\gamma_m,\ldots,\gamma_1)} t^{p-\sigma}(\xi_2) \\
&= \frac{\|\omega\|_\sigma}{\Gamma(p)}\left(\frac{1-\sigma}{p-\sigma}\right)^{1-\sigma}(p-\sigma)^{-q}\Gamma((p-\sigma)+1) \\
&\quad \times \frac{\prod_{i=1}^{m}\left((p-\sigma) + \sum_{j=1}^{i}\gamma_j\right)^{-\delta_i}}{\Gamma\left(1 + (p-\sigma) + \sum_{j=1}^{m}\gamma_j\right)}\xi_2^{(p-\sigma)+\sum_{j=1}^{m}\gamma_j}.
\end{aligned} \tag{17}
$$

Therefore, from (14)–(17), we have

$$\|\mathcal{K}x - \mathcal{K}y\| \leq \|\omega\|_\sigma \Phi_2 \|x-y\|,$$

which implies that $\mathcal{K}$ is a contraction operator. Hence, the Banach's fixed point theorem implies that $\mathcal{K}$ has a unique fixed point, which is the unique solution of the boundary value problem (4) on $[0,T]$. The proof is finished. □

3.3. Existence and Uniqueness Result via Nonlinear Contractions

Definition 5. *Assume that E is a Banach space. The operator $\mathcal{K}: E \to E$, is said to be a nonlinear contraction if there exists a continuous nondecreasing function $\Psi: \mathbb{R}^+ \to \mathbb{R}^+$ such that $\Psi(0)=0$ and $\Psi(u) < u$ for all $u > 0$ satisfying*

$$\|\mathcal{K}x - \mathcal{K}y\| \leq \Psi(\|x-y\|), \qquad \forall x, y \in E.$$

Lemma 6. *(Boyd and Wong) [28]. Assume that E is a Banach space and $\mathcal{K}: E \to E$ is a nonlinear contraction. Then $\mathcal{K}$ has a unique fixed point in E.*

Theorem 3. *Suppose that $f: [0,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying the assumption:*

(H_2) $|f(t,x) - f(t,y)| \leq g(t)\dfrac{|x-y|}{G^* + |x-y|}$, *for $t \in [0,T]$, $x, y \in \mathbb{R}$, where $g: [0,T] \to \mathbb{R}^+$ is a continuous function and G^* is a constant defined by*

$$\begin{aligned} G^* := & \frac{|\lambda_1|}{|\Omega|}\widetilde{R}^{(\alpha_n, \beta_{n-1}, \ldots, \alpha_1, q, p)} g(\xi_1) + \frac{|\lambda_2|}{|\Omega|}\widehat{R}^{(\delta_m, \gamma_m, \ldots, \gamma_1, q, p)} g(\xi_2) \\ & + \left(\frac{1}{|\Omega|} + 1\right) {}^{H}I^{q\, RL}I^{p} g(T). \end{aligned}$$

Then problem (4) has a unique solution on $[0,T]$.

Proof. Now, we will show that the operator $\mathcal{K}: \mathcal{C} \to \mathcal{C}$ defined in (10) is a nonlinear contraction. Next, we define a continuous nondecreasing function $\Psi: \mathbb{R}^+ \to \mathbb{R}^+$ by

$$\Psi(u) = \frac{G^* u}{G^* + u}, \qquad \forall u \geq 0.$$

It is easy to see that Ψ satisfies $\Psi(0) = 0$ and $\Psi(u) < u$ for all $u > 0$.
Then, for any $x, y \in \mathcal{C}$, $t \in [0,T]$, we obtain

$$\begin{aligned} & |(\mathcal{K}x)(t) - (\mathcal{K}y)(t)| \\ \leq & \frac{|\lambda_1|}{|\Omega|}\widetilde{R}^{(\alpha_n, \beta_{n-1}, \ldots, \alpha_1, q, p)} |f_x - f_y|(\xi_1) + \frac{|\lambda_2|}{|\Omega|}\widehat{R}^{(\delta_m, \gamma_m, \ldots, \gamma_1, q, p)} |f_x - f_y|(\xi_2) \\ & + \left(\frac{1}{|\Omega|} + 1\right) {}^{H}I^{q\, RL}I^{p} |f_x - f_y|(T) \\ \leq & \frac{|\lambda_1|}{|\Omega|}\widetilde{R}^{(\alpha_n, \beta_{n-1}, \ldots, \alpha_1, q, p)} \left(g(s)\frac{|x-y|}{G^* + |x-y|}\right)(\xi_1) \\ & + \frac{|\lambda_2|}{|\Omega|}\widehat{R}^{(\delta_m, \gamma_m, \ldots, \gamma_1, q, p)} \left(g(s)\frac{|x-y|}{G^* + |x-y|}\right)(\xi_2) \\ & + \left(\frac{1}{|\Omega|} + 1\right) {}^{H}I^{q\, RL}I^{p} \left(g(s)\frac{|x-y|}{G^* + |x-y|}\right)(T) \\ \leq & \frac{\Psi(\|x-y\|)}{G^*}\left(\frac{|\lambda_1|}{|\Omega|}\widetilde{R}^{(\alpha_n, \beta_{n-1}, \ldots, \alpha_1, q, p)} g(\xi_1) + \frac{|\lambda_2|}{|\Omega|}\widehat{R}^{(\delta_m, \gamma_m, \ldots, \gamma_1, q, p)} g(\xi_2)\right. \end{aligned}$$

$$
\begin{aligned}
&\quad + \left(\frac{1}{|\Omega|}+1\right){}^{H}I^{q}\,{}^{RL}I^{p}g(T)\Bigg) \\
&= \quad \Psi(\|x-y\|),
\end{aligned}
$$

which means that $\|\mathcal{K}x-\mathcal{K}y\| \leq \Psi(\|x-y\|)$. Thus we can deduce that $\mathcal{K}$ is a nonlinear contraction operator. Therefore, by applying Lemma 6 the operator $\mathcal{A}$ has a unique fixed point, which is the unique solution on $[0,T]$ of problem (4). The proof is completed. □

3.4. Existence Result via Leray–Schauder Nonlinear Alternative

Lemma 7 ((Nonlinear alternative for single valued maps) [29]). *Suppose that E is a Banach space, C is a closed, convex subset of E and U is an open subset of C with $0 \in U$. Assume that $\mathcal{K}: \overline{U} \to C$ is a continuous map and $\mathcal{K}(\overline{U})$ is a relatively compact subset of C. Then either*

(i) $\mathcal{K}$ has a fixed point in $\overline{U}$, or

(ii) there is a $x \in \partial U$ (the boundary of U in C) and $\theta \in (0,1)$ with $x = \theta\mathcal{K}(x)$.

Theorem 4. *Suppose that:*

(H_3) there exist a continuous nondecreasing function $h : [0,\infty) \to (0,\infty)$ and a function $v \in C([0,T],\mathbb{R}^+)$ such that

$$|f(t,x)| \leq v(t)h(\|x\|) \text{ for all } (t,x) \in [0,T]\times\mathbb{R};$$

(H_4) there exists a constant $M > 0$ such that

$$\frac{M}{\|v\|h(M)\Phi_1} > 1.$$

Then the boundary value problem (4) has at least one solution on $[0,T]$.

Proof. Let $B_r = \{x \in \mathcal{C} : \|x\| \leq r\}$. Then B_r is a closed and convex subset of $\mathcal{C}$. Define sequence $\{x_n\}$ in B_r converging to x. We can show the continuity of $\mathcal{K}$ as

$$
\begin{aligned}
& |(\mathcal{K}x)(t)-(\mathcal{K}x_n)(t)| \\
\leq\ & \frac{|\lambda_1|}{|\Omega|}\widetilde{R}^{(\alpha_n,\beta_{n-1},\ldots,\alpha_1,q,p)}|f_x-f_{x_n}|(\xi_1)+\frac{|\lambda_2|}{|\Omega|}\widehat{R}^{(\delta_m,\gamma_m,\ldots,\gamma_1,q,p)}|f_x-f_{x_n}|(\xi_2) \\
& +\left(\frac{1}{|\Omega|}+1\right){}^{H}I^{q}\,{}^{RL}I^{p}|f_x-f_{x_n}|(T) \\
\to\ & 0,
\end{aligned}
$$

which concludes that $\mathcal{K}$ is continuous.

The compactness of $\mathcal{K}$ can be proved as follows. Setting $x \in B_r$, we have

$$
\begin{aligned}
|\mathcal{K}x(t)| \leq\ & \frac{|\lambda_1|}{|\Omega|}\widetilde{R}^{(\alpha_n,\beta_{n-1},\ldots,\alpha_1,q,p)}|f_x|(\xi_1)+\frac{|\lambda_2|}{|\Omega|}\widehat{R}^{(\delta_m,\gamma_m,\ldots,\gamma_1,q,p)}|f_x|(\xi_2) \\
& +\frac{1}{\Omega}{}^{H}I^{q}\,{}^{RL}I^{p}|f_x|(T)+{}^{H}I^{q}\,{}^{RL}I^{p}|f_x|(t) \\
\leq\ & \|v\|h(r)\frac{|\lambda_1|}{|\Omega|}\widetilde{R}^{(\alpha_n,\beta_{n-1},\ldots,\alpha_1,q,p)}(1)(\xi_1) \\
& +\|v\|h(r)\frac{|\lambda_2|}{|\Omega|}\widehat{R}^{(\delta_m,\gamma_m,\ldots,\gamma_1,q,p)}(1)(\xi_2) \\
& +\|v\|h(r)\left(\frac{1}{|\Omega|}+1\right){}^{H}I^{q}\,{}^{RL}I^{p}(1)(T) \\
=\ & \|v\|h(r)\Phi_1 := \Phi_3,
\end{aligned}
$$

which leads to $\|\mathcal{K}x\| \leq \Phi_3$. Therefore, the set $\mathcal{K}B_r$ is a uniformly bounded set. By setting $\tau_1, \tau_2 \in B_r$ with $\tau_1 \to \tau_2$ and $x \in B_r$, the equicontinuity of $\mathcal{K}B_r$ can be considered as

$$\begin{aligned} |\mathcal{K}x(\tau_2) - \mathcal{K}x(\tau_1)| &= \left| {}^H I^q \, {}^{RL} I^p f_x(\tau_2) - {}^H I^q \, {}^{RL} I^p f_x(\tau_1) \right| \\ &\leq \|v\| h(r) \left| {}^H I^q \, {}^{RL} I^p (1)(\tau_2) - {}^H I^q \, {}^{RL} I^p (1)(\tau_1) \right| \\ &\leq \|v\| h(r) \frac{p^{-q}}{\Gamma(p+1)} \left| \tau_2^p - \tau_1^p \right| \to 0, \end{aligned}$$

independently of an unknown x. Then $\mathcal{K}B_r$ is an equicontinuous set. Hence we can conclude that $\mathcal{K}B_r$ is relatively compact. The benefit of the Arzelá–Ascoli theorem, implies that the operator $\mathcal{K}$ is completely continuous.

In the final step, we will show that the second condition of Lemma 7 does not hold. Let x be a solution of problem (4). Let us see the operator equation $x = \theta \mathcal{K}x$ for $\theta \in (0,1)$. From direct computation, we have

$$|x(t)| \leq \|v\| h(\|x\|) \Phi_1,$$

which means that

$$\frac{\|x\|}{\|v\| h(\|x\|) \Phi_1} \leq 1.$$

By the hypothesis (H_4), there exists M such that $\|x\| \neq M$. Now, we define the set $U = \{x \in B_r : \|x\| < M\}$. It is obvious that the operator $\mathcal{K} : \overline{U} \to \mathcal{C}$ is continuous and completely continuous. Then it is impossible that there exists $x \in \partial U$ such that $x = \theta \mathcal{K}x$ for any $\theta \in (0,1)$. Hence, by applying the nonlinear alternative of Leray–Schauder type, we can conclude that the operator $\mathcal{K}$ has a fixed point $x \in \overline{U}$ which is a solution on $[0, T]$, of the boundary value problem (4). □

4. Special Cases

Form (H_3) and (H_4), we can give the following three corollaries. Firstly, if we choose $\|v\| = N$ and $h(\cdot) \equiv 1$, then by (H_4), there exists a constant $M > \|v\|\Phi_1$.

Corollary 2. *If $|f(t,x)| \leq N$, for $(t,x) \in [0,T] \times \mathbb{R}$ and $N > 0$, then the boundary value problem (4) has at least one solution on $[0,T]$.*

Secondly if we set $v(\cdot) \equiv 1$ and $h(u) = Au + B$, $A \geq 0$, $B > 0$, then, by (H_4), there exists a constant $M > \frac{B\Phi_1}{1 - A\Phi_1}$.

Corollary 3. *Assume that $|f(t,x)| \leq A|x| + B$, for $(t,x) \in [0,T] \times \mathbb{R}$, $A \geq 0$ and $B > 0$. If $A\Phi_1 < 1$, then the nonlocal problem (4) has at least one solution on $[0,T]$.*

Finally, we recall the fact that if $Cx^2 - x + D < 0$, where $C > 0$, $D \geq 0$, then $x \in \left(\frac{1-\sqrt{1-4CD}}{2C}, \frac{1+\sqrt{1-4CD}}{2C} \right)$. Now, we choose $v(\cdot) \equiv 1$ and $h(u) = Pu^2 + Q$, $P > 0$, $Q \geq 0$. Then, by (H_4), there exists a constant $M \in \left(\frac{1-\sqrt{1-4PQ\Phi_1^2}}{2P\Phi_1}, \frac{1+\sqrt{1-4PQ\Phi_1^2}}{2P\Phi_1} \right)$.

Corollary 4. *Suppose that $|f(t,x)| \leq Px^2 + Q$, for $(t,x) \in [0,T] \times \mathbb{R}$, $P > 0$ and $Q \geq 0$. If $PQ\Phi_1^2 < \frac{1}{4}$, then problem (4) has at least one solution on $[0,T]$.*

5. Examples

Next, we present some examples to illustrate our results.

Example 1. *Consider the following sequential Riemann–Liouville and Hadamard–Caputo fractional differential equation with iterated fractional integral conditions*

$$\begin{cases} {}^{RL}D^{\frac{3}{4}}\left({}^{HC}D^{\frac{1}{2}}x\right)(t) = f(t,x(t)), \qquad t \in \left[0, \dfrac{3}{2}\right], \\ {}^{HC}D^{\frac{1}{2}}x(0) = 0, \\ x\left(\dfrac{3}{2}\right) = \dfrac{2}{17}\widetilde{R}^{\left(\frac{1}{6},\frac{1}{5},\frac{1}{4},\frac{1}{3},\frac{1}{2}\right)}x\left(\dfrac{6}{5}\right) + \dfrac{3}{19}\widehat{R}^{\left(\frac{7}{4},\frac{5}{4},\frac{3}{4},\frac{1}{4}\right)}x\left(\dfrac{13}{15}\right). \end{cases} \tag{18}$$

Here $p = 3/4 = \alpha_0 = \gamma_0$, $q = 1/2 = \beta_0 = \delta_0$, $T = 3/2$, $\lambda_1 = 2/17$, $\lambda_2 = 3/19$, $n = 3$, $\alpha_1 = 1/2$, $\alpha_2 = 1/4$, $\alpha_3 = 1/6$, $\beta_1 = 1/3$, $\beta_2 = 1/5$, $\xi_1 = 6/5$, $m = 2$, $\gamma_1 = 1/4$, $\gamma_2 = 5/4$, $\delta_1 = 3/4$, $\delta_2 = 7/4$ and $\xi_2 = 13/15$. From all detail, we can find that

$$\begin{aligned} \widetilde{R}^{(\alpha_n,\beta_{n-1},\ldots,\alpha_1,q,p)}(1)(\xi_1) &= \frac{\prod_{i=0}^{n-1}\left(\sum_{j=0}^{i}\alpha_j\right)^{-\beta_i}}{\Gamma\left(1+\sum_{j=0}^{n}\alpha_j\right)}\xi_1^{\sum_{j=0}^{n}\alpha_j} \\ &= \widetilde{R}^{\left(\frac{1}{6},\frac{1}{5},\frac{1}{4},\frac{1}{3},\frac{1}{2},\frac{1}{2},\frac{3}{4}\right)}(1)\left(\frac{6}{5}\right) \\ &= \frac{\left(\frac{3}{4}\right)^{-\frac{1}{2}}\left(\frac{3}{4}+\frac{1}{2}\right)^{-\frac{1}{3}}\left(\frac{3}{4}+\frac{1}{2}+\frac{1}{4}\right)^{-\frac{1}{5}}}{\Gamma\left(1+\frac{3}{4}+\frac{1}{2}+\frac{1}{4}+\frac{1}{6}\right)} \\ &\quad \times \left(\frac{6}{5}\right)^{\frac{3}{4}+\frac{1}{2}+\frac{1}{4}+\frac{1}{6}} \approx 0.8902306447, \end{aligned}$$

and

$$\begin{aligned} \widehat{R}^{(\delta_m,\gamma_m,\ldots,\gamma_1,q,p)}(1)(\xi_2) &= \frac{\prod_{i=0}^{m}\left(\sum_{j=0}^{i}\gamma_j\right)^{-\delta_i}}{\Gamma\left(1+\sum_{j=0}^{m}\gamma_j\right)}\xi_2^{\sum_{j=0}^{m}\gamma_j} \\ &= \widehat{R}^{\left(\frac{7}{4},\frac{5}{4},\frac{3}{4},\frac{1}{4},\frac{1}{2},\frac{3}{4}\right)}(1)\left(\frac{13}{15}\right) \\ &= \frac{\left(\frac{3}{4}\right)^{-\frac{1}{2}}\left(\frac{3}{4}+\frac{1}{4}\right)^{-\frac{3}{4}}\left(\frac{3}{4}+\frac{1}{4}+\frac{5}{4}\right)^{-\frac{7}{4}}}{\Gamma\left(1+\frac{3}{4}+\frac{1}{4}+\frac{5}{4}\right)} \\ &\quad \times \left(\frac{13}{15}\right)^{\frac{3}{4}+\frac{1}{4}+\frac{5}{4}} \approx 0.07941518582, \end{aligned}$$

and $\widehat{R}^{(q,p)}(1)(T) = \frac{p^{-q}T^p}{\Gamma(p+1)} \approx 1.702914149$, $\Omega \approx 0.6748988874$, $\Phi_1 \approx 4.399890599$. If we choose $\sigma = 1/4 \in (0,p)$, then we can compute that $\Phi_2 \approx 5.037579504$. Let the nonlinear Lipschitzian function $f : [0,3/2] \times \mathbb{R} \longrightarrow \mathbb{R}$ be given by

$$f(t,x) = \frac{1}{t+10}\left(\frac{x^2+2|x|}{1+|x|}\right) + \frac{1}{3}e^{-2t} + \frac{1}{4}. \tag{19}$$

Now, we see that $|f(t,x) - f(t,y)| \leq (2/(t+10))|x-y|$, for all $t \in [0,(3/2)]$, $x,y \in \mathbb{R}$. Setting $\omega(t) = 2/(t+10)$, we get $L = 1/5$, which gives the estimate $L\Phi_1 \approx 0.8799781198 < 1$. From the result in Theorem 1, problem (18) with f given by (19) has a unique solution on $[0,3/2]$.

Example 2. *Consider the following sequential Riemann–Liouville and Hadamard–Caputo fractional differential equation with iterated fractional integral conditions of Example 1, where the function* $f : [0,3/2] \times \mathbb{R} \longrightarrow \mathbb{R}$ *is defined by*

$$f(t,x) = \frac{e^{-175t}}{2}\left(\frac{x^2+2|x|}{1+|x|}\right) + \frac{1}{3}e^{-2t} + \frac{1}{4}. \tag{20}$$

Then we can derive that $|f(t,x) - f(t,y)| \leq e^{-175t}|x - y|$. Choosing $\omega(t) = e^{-175t}$, we get $L = 1$, which yields $L\Phi_1 \approx 4.399890599 > 1$. This means that the Theorem 1 can not be used to apply concerning problem (18) with f given by (20). However, we can compute that $\|\omega\|_\sigma \approx 0.1944130842$, which leads to $\|\omega\|_\sigma \Phi_2 \approx 0.9793713683 < 1$. By the benefit of Theorem 2, we deduce the conclusion that problem (18) with f given by (20) has a unique solution on $[0, 3/2]$.

Example 3. *Consider the following sequential Riemann–Liouville and Hadamard–Caputo fractional differential equation with iterated fractional integral conditions of Example 1, where the function $f : [0, 3/2] \times \mathbb{R} \longrightarrow \mathbb{R}$ is defined by*

$$f(t,x) = \frac{t^2+1}{7} F(x), \quad \textit{where} \quad F(x) = \begin{cases} \frac{1}{2}, & x > 1, \\ \frac{x}{2}, & 0 \leq x \leq 1, \\ 0, & x < 0. \end{cases} \tag{21}$$

Choosing $g(t) = (t^2 + 1)/7$, we find that $G^* \approx 0.9254866186$. Then we can show that

$$|f(t,x) - f(t,y)| \leq \frac{t^2+1}{7}\left(\frac{|x-y|}{2}\right) \leq \frac{t^2+1}{7}\left(\frac{|x-y|}{0.9254866186 + |x-y|}\right).$$

Therefore, condition (H_2) in Theorem 3 holds. We can get the conclusion by applying Theorem 3 so that problem (18) with f given by (21) has a unique solution on $[0, 3/2]$.

Example 4. *Consider the following sequential Riemann–Liouville and Hadamard–Caputo fractional differential equation with iterated fractional integral conditions of Example 1, where the function $f : [0, 3/2] \times \mathbb{R} \longrightarrow \mathbb{R}$ is defined by*

$$f(t,x) = \left(\frac{1}{t+5}\right)\frac{x^{32}}{|x|^{31}+1} + t + \frac{1}{2}\cos^8 x^3. \tag{22}$$

Then we can see the estimate

$$|f(t,x)| \leq \frac{1}{5}|x| + 2,$$

where $A = 1/5$ and $B = 2$. Since $A\Phi_1 \approx 0.8799781198 < 1$, we apply Corollary 3 to obtain that problem (18) with f given by (22) has at least one solution on $[0, 3/2]$.

Example 5. *Consider the following sequential Riemann–Liouville and Hadamard–Caputo fractional differential equation with iterated fractional integral conditions of Example 1, where the function $f : [0, 3/2] \times \mathbb{R} \longrightarrow \mathbb{R}$ is defined by*

$$f(t,x) = \frac{1}{(t+3)^2}\left(\frac{x^{36}}{x^{34}+1}\right) + \frac{1}{15}te^{-x^2}. \tag{23}$$

Observe that

$$|f(t,x)| \leq \frac{1}{9}x^2 + \frac{1}{10}.$$

Then we choose $P = 1/9$ and $Q = 1/10$. Hence all assumptions of Corollary 4 hold by computing $PQ\Phi_1^2 \approx 0.2151004142 < \frac{1}{4}$. Therefore, problem (18) with (23) has at least one solution on $[0, 3/2]$.

6. Conclusions

In this paper we initiated the study of fractional boundary value problems consisting of a differential equation with sequential Riemann–Liouville and Hadamard–Caputo fractional derivatives, supplemented with iterated fractional integral boundary conditions.

To the best of our knowledge, it is the first paper introducing iterated fractional boundary conditions. Firstly, we established two new formulas for iteration of fractional integrals of Riemann–Liouville and Hadamard types. Next, after proving an auxiliary lemma concerning a linear variant of the considered problem, we transformed the problem into a fixed point problem. By applying fixed point theorems, such as Banach's contraction mapping principle, Boyd–Wong fixed point theorem and Leray–Schauder nonlinear alternative, we established the existence and uniqueness of the solutions of the problem at hand. Some special cases are also discussed. The obtained results are well illustrated by numerical examples. Our results are new and enrich the literature on boundary value problems for fractional differential equations. We believe that it is an interesting and new problem that the upcoming researchers can offer similar results for different types of iterated boundary conditions or different kind of sequential fractional derivatives.

Author Contributions: Conceptualization, S.K.N. and J.T.; methodology, S.K.N., S.S., T.K. and J.T.; validation, S.K.N., S.S., T.K. and J.T.; funding acquisition, J.T. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by King Mongkut's University of Technology North Bangkok. Contract no. KMUTNB-62-KNOW-41.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Javidi, M.; Ahmad, B. Dynamic analysis of time fractional order phytoplankton-toxic phytoplankton–zooplankton system. *Ecol. Model.* **2015**, *318*, 8–18. [CrossRef]
2. Zhang, F.; Chen G.; Li C.; Kurths, J. Chaos synchronization in fractional differential systems. *Philos. Trans. R. Soc. A* **2013**, *371*, 201201553. [CrossRef] [PubMed]
3. Diethelm, K. *The Analysis of Fractional Differential Equations*; Lecture Notes in Mathematics; Springer: New York, NY, USA, 2010.
4. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of the Fractional Differential Equations*; North-Holland Mathematics Studies: Amsterdam, The Netherlands, 2006.
5. Lakshmikantham, V.; Leela, S.; Devi, J.V. *Theory of Fractional Dynamic Systems*; Cambridge Scientific Publishers: Cambridge, UK, 2009.
6. Miller, K.S.; Ross, B. *An Introduction to the Fractional Calculus and Differential Equations*; John Wiley: New York, NY, USA, 1993.
7. Podlubny, I. *Fractional Differential Equations*; Academic Press: New York, NY, USA, 1999.
8. Ahmad, B.; Alsaedi, A.; Ntouyas, S.K.; Tariboon, J. *Hadamard-Type Fractional Differential Equations, Inclusions and Inequalities*; Springer: Cham, Switzerland, 2017.
9. Zhou, Y. *Basic Theory of Fractional Differential Equations*; World Scientific: Singapore, 2014.
10. Ahmad, B.; Ntouyas, S.K. *Nonlocal Nonlinear Fractional-Order Boundary Value Problems*; World Scientific: Singapore, 2021.
11. Wang,Y.; Liang, S.; Wang, Q. Existence results for fractional differential equations with integral and multi-point boundary conditions. *Bound. Value Probl.* **2018**, *2018*, 4. [CrossRef]
12. Goodrich, C.S. Coercive nonlocal elements in fractional differential equations. *Positivity* **2017**, *21*, 377–394. [CrossRef]
13. Ahmad, B.; Alruwaily, Y.; Alsaedi, A.; Nieto, J.J. Fractional integro-differential equations with dual anti-periodic boundary conditions. *Differ. Integral Equ.* **2020**, *33*, 181–206.
14. Ahmad, B.; Alghanmi, M.; Alsaedi, A. Existence results for a nonlinear coupled system involving both Caputo and Riemann–Liouville generalized fractional derivatives and coupled integral boundary conditions. *Rocky Mt. J. Math.* **2020**, *50*, 1901–1922. [CrossRef]
15. Kiataramkul, C.; Yukunthorn, W.; Ntouyas, S.K.; Tariboon, J. Sequential Riemann–Liouville and Hadamard–Caputo fractional differential systems with nonlocal coupled fractional integral boundary conditions. *Axioms* **2021**, 10, 174. [CrossRef]
16. Benkerrouche, A.; Baleanu, D.; Souid, M.S.; Hakem, A.; Inc, M. Boundary value problem for nonlinear fractional differential equations of variable order via Kuratowski MNC technique. *Adv. Differ. Equ.* **2021**, *2021*, 365. [CrossRef]
17. Alam, M.; Zada, A.; Popa, I.-L.; Kheiryan, A.; Rezapour, S.; Kaabar, M.K. A. A fractional differential equation with multi-point strip boundary condition involving the Caputo fractional derivative and its Hyers-Ulam stability. *Bound. Value Probl.* **2021**, *2021*, 73. [CrossRef]
18. Alsaedi, A.; Ntouyas, S.K.; Agarwal, R.P.; Ahmad, B. On Caputo type sequential fractional differential equations with nonlocal integral boundary conditions. *Adv. Differ. Equ.* **2015**, *2015*, 33. [CrossRef]

19. Promsakon, C.; Phuangthong, N.; Ntouyas, S.K.; Tariboon, J. Nonlinear sequential Riemann–Liouville and Caputo fractional differential equations with generalized fractional integral conditions. *Adv. Differ. Equ.* **2018**, *2018*, 385. [CrossRef]
20. Tariboon, J.; Cuntavepanit, A. ; Ntouyas, S.K.; Nithiarayaphaks, W. Separated boundary value problems of sequential Caputo and Hadamard fractional differential equations. *J. Funct. Spaces* **2018**, *2018*, 6974046. [CrossRef]
21. Asawasamrit, S.; Phuangthong, N.; Ntouyas, S.K.; Tariboon, J. Nonlinear sequential Riemann–Liouville and Caputo fractional differential equations with nonlocal and integral boundary conditions. *Int. J. Anal. Appl.* **2019**, *17*, 47–63.
22. Asawasamrit, S.; Ntouyas, S.K.; Tariboon, J.; Nithiarayaphaks, W. Coupled systems of sequential Caputo and Hadamard fractional differential equations with coupled separated boundary conditions. *Symmetry* **2018**, *10*, 701. [CrossRef]
23. Alsaedi, A.; Albideewi, A.F.; Ntouyas, S.K.; Ahmad, B. Existence results for a multipoint boundary value problem of nonlinear sequential Hadamard fractional differential equations. *CUBO* **2021**, *23*, 225–237.
24. Torvik, P.J.; Bagley, R.L. On the appearance of the fractional derivative in the behavior of real materials. *J. Appl. Mech.* **1984**, *51*, 294–298. [CrossRef]
25. Mainardi, F. Some basic problems in continuum and statistical mechanics. In *Fractals and Fractional Calculus in Continuum Mechanics*; Carpinteri, A., Mainardi, F., Eds.; Springer: Berlin, Germany, 1997; pp. 291–348.
26. Yin, C.; Liu, F.; Anh, V. Numerical simulation of the nonlinear fractional dynamical systems with fractional damping for the extensible and inextensible pendulum. *J. Algorithm Comput. Technol.* **2007**, *1*, 427–447. [CrossRef]
27. Jarad, F.; Abdeljawad, T.; Baleanu, D. Caputo-type modification of the Hadamard fractional derivatives. *Adv. Differ. Equ.* **2012**, *2012*, 142. [CrossRef]
28. Boyd, D.W.; Wong, J.S.W. On nonlinear contractions. *Proc. Am. Math. Soc.* **1969**, *20*, 458–464. [CrossRef]
29. Granas, A.; Dugundji, J. *Fixed Point Theory*; Springer: New York, NY, USA, 2003.