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Certain Subclasses of Analytic Multivalent Functions Associated with Petal-Shape Domain

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Abstract: In this article, we introduce a new class of multivalent analytic functions associated with petal-shape region. Furthermore, some useful properties, such as the Fekete–Szegö inequality, and their consequences for some special cases are discussed. For some specific value of function f, we obtain sufficient conditions for multivalent starlike functions connected with petal-shape domain. Finally, in the concluding section, we draw the attention of the interested readers toward the prospect of studying the basic or quantum (or q-) generalizations of the results, which are presented in this paper. However, the $(\mathfrak{p}, q)$-variations of the suggested q-results will provide a relatively minor and inconsequential development because the additional (rather forced-in) parameter $\mathfrak{p}$ is obviously redundant.

Keywords: multivalent functions; starlike functions; petal-shape domain; differential subordination; Fekete–Szegö inequality

1. Introduction and Motivation

To understand our main results and the notations used in this paper in a better way, some basic literature of Geometric Function Theory is presented here. For this understanding, we start with the class $\mathcal{A}$, the class of all regular or analytic functions f in the open unit disk

$$\mathbb{D} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\} \tag{1}$$

and satisfying the conditions

$$f(0) = 0 \quad \text{and} \quad f'(0) = 1.$$

Further, the class $\mathcal{S}$ represents all regular univalent functions f of class $\mathcal{A}$. Let $\mathbb{B}_0$ be the class of Schwarz regular functions w such that $w : \mathbb{D} \to \mathbb{D}$ with property that $w(0) = 0$ and has power series expansion

$$w(z) = \sum_{n=1}^{\infty} c_n z^n. \tag{2}$$

Furthermore, we briefly discuss the notion of subordinations; let $\Lambda_1, \Lambda_2 \in \mathcal{A}$, then Λ_1, is said to subordinate to Λ_2 symbolically:

$$\Lambda_1 \prec \Lambda_2$$

if there exists an analytic function $w(z)$ with properties that

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1,$$

such that

$$\Lambda_1(z) = \Lambda_2(w(z)).$$

Utilizing the idea of subordinations, different subclasses of starlike functions were introduced by Ma and Minda [1], in which the quantity $z f'(z)/f(z)$ is subordinated to a more general maximal function Φ such that

$$\frac{z f'(z)}{f(z)} \prec \Phi(z) = \frac{1+z}{1-z}.$$

In some particular cases, if we put $\Phi(z) = \frac{1+Lz}{1+Mz}$, then such functions are known as Janowski functions with condition on L and M is $-1 \leq M < L \leq 1$. These functions map the open unit disk to the disk in the right half plane with center on real axis and the end points of diameter are $\frac{1+L}{1+M}$ and $\frac{1-L}{1-M}$ (see [2]). If we vary $\Phi(z)$ to $1 + \frac{4}{3}z + \frac{2}{3}z^2$, then the image of $\Phi(z)$ under open unit disk is bounded by cardioid; this case was studied by Sharma et al. [3]. Cho et al. [4] extended the idea to trigonometric function by replacing $\Phi(z)$ by $1 + \sin(z)$. Furthermore, we put $\Phi(z) = 2/1 + e^{-z}$; this class was introduced by Goel and Kumar [5], in which they connected starlike functions to modified sigmoid functions. For more details of the above-mentioned interesting topics, we refer the reader to [6–9]. Recently, Ali et al. [10] obtained some conditions on α, where α is any real number, such that

$$1 + \alpha \frac{z g'(z)}{g^i(z)} \prec \sqrt{1+z} \Rightarrow g(z) \prec \sqrt{1+z} \qquad (i = 0, 1, 2,),$$

where g is a regular function defined on $\mathbb{D}$ with $g(0) = 1$. Similar findings were carried out by various authors such as Kumar et al. [7,11,12], Paprocki and Sokół [13], Ahmad et al. [14], and Sharma et al. [15].

Let $\mathcal{A}_p$ denote the family of all regular p-valent functions f in the open disk $\mathbb{D}$ having series representation

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \qquad (z \in \mathbb{D}). \tag{3}$$

In recent years, many authors have studied the subclasses of multivalent (p-valent) functions from different viewpoints and perspectives. For example, recently, Khan et al. [16,17] were essentially motivated by the Srivastava's published review article [18] and defined certain new families of multivalent q-starlike functions and studied some of their entrusting properties, such as inclusion results, radius problems, and sufficient conditions. Furthermore, Rehman et al. [19] defined and studied generalized subclasses of multivalent starlike functions. Recently, many well-known mathematicians have obtained the Fekete–Szegö functional for different subclasses of analytic and bi-univalent functions, see for example [20–24].

Motivated by the above-mentioned work, we now introduce a new class $\mathcal{S}_p^*(pet)$ of analytic multivalent starlike functions associated with the petal-shape domain.

Definition 1. *A function $f \in \mathcal{A}_p$ will be in the class $\mathcal{S}_p^*(pet)$ if it satisfies*

$$\frac{zf'(z)}{pf(z)} \prec 1 + \sinh^{-1}(z). \tag{4}$$

The function $1 + \sinh^{-1}(z)$ is a multivalued function and has branch cuts along the line segments $(-\infty, -i) \cup (i, \infty)$ on the imaginary axis; hence, it is analytic in $\mathbb{D}$ and also the function $1 + \sinh^{-1}(z)$ maps the open unit disk onto petal-shape region, for example,

$$\{w \in \mathbb{C} : |\sinh(w-1)| < 1\}.$$

Remark 1. *If we take $p = 1$, we have the class of analytic starlike functions associated with petal-shape domain introduced and studied by Kumar and Arora [25].*

In this article, we first introduced a new class of multivalent analytic functions associated with petal-shape region. Some useful properties have been discussed, such as Fekete–Szegö inequality and their consequences to some special cases, and also evaluating conditions on α so that the following holds

$$1 + \frac{\alpha}{p} \frac{z^{1+p(i-1)} g'(z)}{g^i(z)} \prec \frac{1+Lz}{1+Mz}, \text{ for each } i = 0,1,2,3$$

then, $\frac{g(z)}{z^p}$ is proved to be subordinated to $1 + \sinh^{-1}(z)$. For some specific value of function g, we obtain sufficient conditions for multivalent starlike functions connected with the petal-shape domain.

2. A Set of Lemmas

To obtain our results, we need the following important Lemmas.

Lemma 1. [26] *Let $w \in \mathbb{B}_0$ of the form (2), then for all $\lambda \in \mathbb{C}$, we have*

$$\left| c_2 - \lambda c_1^2 \right| \le \max\{1, |\lambda|\}.$$

Lemma 2. [27] *Let w be a nonconstant regular function in $\mathbb{D}$ with $w(0) = 0$; if*

$$|w(z_0)| = \max\{|w(z)|, \ |z| \le |z_0|\}, \ z \in \mathbb{D}$$

then, there exists a real number $m(m \ge 1)$ such that

$$z_0 w'(z_0) = m w(z_0).$$

3. Main Results

In this section, we start with the Fekete–Szegö problem for this newly defined class and their consequences to some special cases. Throughout our discussion, we assume that

$$-1 \le M < L \le 1 \quad \text{and} \quad \alpha \in \mathbb{R}.$$

Theorem 1. *Let the function f be of the form (3) and $f \in \mathcal{S}_p^*(pet)$, then,*

$$\left| a_{p+2} - \lambda a_{p+1}^2 \right| \le \frac{p}{2} \max\{1, |2\lambda - 1|\} \tag{5}$$

and

$$\left|a_{p+2}\right| \le \frac{p}{2}. \tag{6}$$

Proof. For f to be in the class $\mathcal{S}_p^*(pet)$, there exists a Schwarz function $\varphi(z)$ such that

$$\frac{zf'(z)}{pf(z)} = 1 + \sinh^{-1}(\varphi(z)).$$

Now,

$$\frac{zf'(z)}{pf(z)} = 1 + \frac{1}{p}a_{p+1}z + \left(\frac{2}{p}a_{p+2} - \frac{1}{p}a_{p+1}^2\right)z^2 + \cdots \tag{7}$$

and

$$1 + \sinh^{-1}(\varphi(z)) = 1 + c_1 z + c_2 z^2 + \cdots. \tag{8}$$

From (7) and (8), we have

$$a_{p+1} = pc_1, \tag{9}$$

$$a_{p+2} = \frac{p}{2}c_2 + \frac{p^2}{2}c_1^2. \tag{10}$$

From (9) and (10), we have

$$\left|a_{p+2} - \lambda a_{p+1}^2\right| = \frac{p}{2}\left|c_2 - (2\lambda - 1)c_1^2\right|,$$

using Lemma 1, we obtain the desired result (5). To prove (6), put $\lambda = 0$ in (5)—the desired result is achieved. □

Corollary 1. *Let f of the form (3) be in the class $\mathcal{S}_p^*(pet)$. Then,*

$$\left|a_{p+2} - a_{p+1}^2\right| \le \frac{p}{2}. \tag{11}$$

Corollary 2. *Let $f \in \mathcal{A}$ be of the form (3) belongs to the class $\mathcal{S}^*(pet)$. Then,*

$$\left|a_3 - \lambda a_2^2\right| \le \frac{1}{2}\max\{1, |2\lambda - 1|\} \tag{12}$$

and

$$|a_3| \le \frac{1}{2}$$

$$\left|a_3 - a_2^2\right| \le \frac{1}{2}.$$

Theorem 2. *Let $g \in \mathcal{A}_p$ and be of the form (3) satisfying the condition*

$$1 + \frac{\alpha z^{1-p} g'(z)}{p} \prec \frac{1 + Lz}{1 + Mz}$$

with the restriction on α being

$$|\alpha| \ge \frac{2^{\frac{1}{2}} p(L - M)}{1 - |M| - 2^{\frac{1}{2}} p\left(1 + \sinh^{-1}(1)\right)(1 + |M|)} \tag{13}$$

then,

$$\frac{g(z)}{z^p} \prec 1 + \sinh^{-1}(z).$$

Proof. Let us define a function

$$h(z) = 1 + \frac{\alpha z^{1-p} g'(z)}{p}, \tag{14}$$

where $h(z)$ is analytic in $\mathbb{D}$ and $h(0) = 1$. Further, consider

$$\frac{g(z)}{z^p} = 1 + \sinh^{-1}(w(z)). \tag{15}$$

To prove our result, we are required to show that $|w(z)| < 1$. For this, consider

$$h(z) = 1 + \alpha\left(1 + \sinh^{-1}(w(z))\right) + \frac{\alpha z w'(z)}{p\sqrt{(w(z))^2 + 1}},$$

and

$$\left|\frac{h(z) - 1}{L - Mh(z)}\right| = \left|\frac{\alpha\left(1 + \sinh^{-1}(w(z))\right) + \frac{\alpha z w'(z)}{p\sqrt{(w(z))^2+1}}}{L - M\left(1 + \alpha\left(1 + \sinh^{-1}(w(z))\right) + \frac{\alpha z w'(z)}{p\sqrt{(w(z))^2+1}}\right)}\right|$$

$$= \left|\frac{\alpha\left(1 + \sinh^{-1}(w(z))\right)p\sqrt{(w(z))^2 + 1} + \alpha z w'(z)}{(L - M)p\sqrt{(w(z))^2 + 1} - M Y_1(p, \alpha, z)}\right|,$$

where

$$Y_1(p, \alpha, z) = \left(p\alpha\sqrt{(w(z))^2 + 1}\left(1 + \sinh^{-1}(w(z))\right) + \alpha z w'(z)\right).$$

Now, suppose that for a point $z_0 \in \mathbb{D}$ we have

$$\max_{|z| \le |z_0|} |w(z)| = |w(z_0)| = 1.$$

Further, by Lemma 2, a number $m \ge 1$ exists with

$$z_0 w'(z_0) = m w(z_0).$$

In addition, we also suppose that

$$w(z_0) = e^{i\theta} \qquad (\theta \in [-\pi, \pi]).$$

Then, we have

$$\left|\frac{h(z_0) - 1}{L - Mh(z_0)}\right| = \left|\frac{\alpha\left(1 + \sinh^{-1}(w(z_0))\right)p\sqrt{(w(z_0))^2 + 1} + \alpha z w'(z_0)}{(L - M)p\sqrt{(w(z_0))^2 + 1} - M Y_2(p, \alpha, z_0)}\right|$$

$$\ge \frac{|\alpha|m - |\alpha|\left(1 + \sinh^{-1}(|e^{i\theta}|)\right)p\sqrt{e^{2i\theta} + 1}}{(L - M)p\sqrt{e^{2i\theta} + 1} + |M|\left(p|\alpha|\left(1 + \sinh^{-1}(|e^{i\theta}|)\right)\sqrt{e^{2i\theta} + 1} + |\alpha|m\right)}$$

$$\ge \frac{|\alpha|\left(m - 2^{\frac{1}{2}}p\left(1 + \sinh^{-1}(1)\right)\right)}{2^{\frac{1}{2}}(L - M)p + |\alpha||M|\left[m + 2^{\frac{1}{2}}p\left(1 + \sinh^{-1}(1)\right)\right]},$$

where

$$Y_2(p, \alpha, z_0) = \left(p\alpha\sqrt{(w(z_0))^2 + 1}\left(1 + \sinh^{-1}(w(z_0))\right) + \alpha z w'(z_0)\right).$$

Now, let

$$k(m) = \frac{|\alpha|\left(m - 2^{\frac{1}{2}}p\left(1 + \sinh^{-1}(1)\right)\right)}{2^{\frac{1}{2}}(L-M)p + |\alpha||M|\left[m + 2^{\frac{1}{2}}p\left(1 + \sinh^{-1}(1)\right)\right]},$$

then,

$$k'(m) = \frac{2^{\frac{1}{2}}|\alpha|(L-M)p + |M||\alpha|^2 2^{\frac{3}{2}}p\left(1 + \sinh^{-1}(1)\right)}{\left(2^{\frac{1}{2}}(L-M)p + |\alpha||M|\left[m + 2^{\frac{1}{2}}p\left(1 + \sinh^{-1}(1)\right)\right]\right)^2} > 0.$$

Clearly, $k(m)$ is an increasing function; so,

$$\max k(m) \geq k(1) \qquad (m \geq 1),$$

and

$$\left|\frac{h(z_0) - 1}{L - Mh(z_0)}\right| \geq \frac{|\alpha|\left(1 - 2^{\frac{1}{2}}p\left(1 + \sinh^{-1}(1)\right)\right)}{2^{\frac{1}{2}}(L-M)p + |M||\alpha|\left[1 + 2^{\frac{1}{2}}p\left(1 + \sinh^{-1}(1)\right)\right]}.$$

From (13), we have

$$\left|\frac{h(z_0) - 1}{L - Mh(z_0)}\right| \geq 1,$$

which is contradictory to the fact that

$$h(z) \prec \frac{1 + Lz}{1 + Mz}.$$

Thus, $|w(z)| < 1$, and so, we obtain our desired result. $\square$

If we take $g(z) = \frac{z^{p+1}f'(z)}{pf(z)}$ in Theorem 2, we have the following Corollary.

Corollary 3. *If f is in the class $\mathcal{A}_p$, has the form* (3), *and satisfies the condition*

$$1 + \frac{\alpha z f'(z)}{p^2 f(z)}\left(p + 1 + \frac{z f''(z)}{f'(z)} - \frac{z f'(z)}{f(z)}\right) \prec \frac{1 + Lz}{1 + Mz},$$

where the limitation on α is

$$|\alpha| \geq \frac{2^{\frac{1}{2}}p(L-M)}{1 - |M| - 2^{\frac{1}{2}}p\left(1 + \sinh^{-1}(1)\right)(1 + |M|)},$$

then, $f \in \mathcal{S}_p^*(pet)$.

Proof. The proof is straightforward so it is left for the reader. $\square$

Theorem 3. *Let $g \in \mathcal{A}_p$ and be of the form* (3), *satisfying the condition*

$$1 + \frac{\alpha}{p}\frac{z g'(z)}{g(z)} \prec \frac{1 + Lz}{1 + Mz} \tag{16}$$

where the limitation on α is

$$|\alpha| \geq \frac{2^{\frac{1}{2}}p(L-M)\left(1 + \sinh^{-1}(1)\right)}{1 - |M| - 2^{\frac{1}{2}}p\left(1 + \sinh^{-1}(1)\right)(1 + |M|)} \tag{17}$$

then,

$$\frac{g(z)}{z^p} \prec 1 + \sinh^{-1}(z).$$

Proof. Let us define a function

$$h(z) = 1 + \frac{\alpha}{p}\frac{zg'(z)}{g(z)}, \tag{18}$$

where $h(z)$ is analytic in $\mathbb{D}$ and $h(0) = 1$. Moreover, consider

$$\frac{g(z)}{z^p} = 1 + \sinh^{-1}(w(z)). \tag{19}$$

To show our result, we are required to show that $|w(z)| < 1$. Now,

$$h(z) = 1 + \alpha + \frac{\alpha z w'(z)}{p\left(1 + \sinh^{-1}(w(z))\right)\sqrt{(w(z))^2 + 1}},$$

and

$$\left|\frac{h(z) - 1}{L - Mh(z)}\right| = \left|\frac{\alpha + \frac{\alpha z w'(z)}{p\left(1 + \sinh^{-1}(w(z))\right)\sqrt{(w(z))^2 + 1}}}{L - M\left(1 + \alpha + \frac{\alpha z w'(z)}{p\left(1 + \sinh^{-1}(w(z))\right)\sqrt{(w(z))^2 + 1}}\right)}\right|$$

$$= \left|\frac{\alpha p\left(1 + \sinh^{-1}(w(z))\right)\sqrt{(w(z))^2 + 1} + \alpha z w'(z)}{(L - M)p\left(1 + \sinh^{-1}(w(z))\right)\sqrt{(w(z))^2 + 1} - M Y_3(p, \alpha, z)}\right|,$$

where

$$Y_3(p, \alpha, z) = \left(\alpha p\left(1 + \sinh^{-1}(w(z))\right)\sqrt{(w(z))^2 + 1} + \alpha z w'(z)\right)$$

and suppose that any point $z_0 \in \mathbb{D}$ such that

$$\max_{|z| \le |z_0|} |w(z)| = |w(z_0)| = 1.$$

Further, by Lemma 2, a number $m \ge 1$ exists with

$$z_0 w'(z_0) = m w(z_0).$$

In addition, we also suppose that

$$w(z_0) = e^{i\theta} \quad (\theta \in [-\pi, \pi]).$$

Then, we have

$$\left|\frac{h(z_0) - 1}{L - Mh(z_0)}\right| = \left|\frac{\alpha p\left(1 + \sinh^{-1}(w(z_0))\right)\sqrt{(w(z_0))^2 + 1} + \alpha z w'(z_0)}{(L - M)p\left(1 + \sinh^{-1}(w(z_0))\right)\sqrt{(w(z_0))^2 + 1} - M Y_4(p, \alpha, z)}\right|$$

$$\ge \frac{|\alpha| m - |\alpha| p\left(1 + \sinh^{-1}(|e^{i\theta}|)\right)\sqrt{e^{2i\theta} + 1}}{(L - M)p\left(1 + \sinh^{-1}(|e^{i\theta}|)\right)\sqrt{e^{2i\theta} + 1} + |M| Y_5(p, \alpha, e)}$$

$$\ge \frac{|\alpha|\left(m - 2^{\frac{1}{2}}p\left(1 + \sinh^{-1}(1)\right)\right)}{2^{\frac{1}{2}}p(L - M)\left(1 + \sinh^{-1}(1)\right) + |\alpha||M|\left[m + 2^{\frac{1}{2}}p\left(1 + \sinh^{-1}(1)\right)\right]},$$

where
$$Y_4(p, \alpha, z) = \left(\alpha p \left(1 + \sinh^{-1}(w(z_0)) \right) \sqrt{(w(z_0))^2 + 1} + \alpha z w'(z_0) \right)$$

and
$$Y_5(p, \alpha, e) = \left(p|\alpha| \left(1 + \sinh^{-1} \left(\left| e^{i\theta} \right| \right) \right) \sqrt{e^{2i\theta} + 1} + |\alpha|m \right).$$

Now, let
$$k(m) = \frac{|\alpha| \left(m - 2^{\frac{1}{2}} p \left(1 + \sinh^{-1}(1) \right) \right)}{2^{\frac{1}{2}} p (L - M) \left(1 + \sinh^{-1}(1) \right) + |\alpha||M| \left[m + 2^{\frac{1}{2}} p \left(1 + \sinh^{-1}(1) \right) \right]},$$

then,
$$k'(m) = \frac{2^{\frac{1}{2}} |\alpha| p (L - M) \left(1 + \sinh^{-1}(1) \right) + |M||\alpha|^2 2^{\frac{3}{2}} p \left(1 + \sinh^{-1}(1) \right)}{\left(2^{\frac{1}{2}} p (L - M) \left(1 + \sinh^{-1}(1) \right) + |\alpha||M| \left[m + 2^{\frac{1}{2}} p \left(1 + \sinh^{-1}(1) \right) \right] \right)^2}$$
$$> 0.$$

Clearly, $k(m)$ is an increasing function; so,
$$\max k(m) \geq k(1) \qquad (m \geq 1),$$

and
$$\left| \frac{h(z_0) - 1}{L - Mh(z_0)} \right|$$
$$\geq \frac{|\alpha| \left(1 - 2^{\frac{1}{2}} p \left(1 + \sinh^{-1}(1) \right) \right)}{2^{\frac{1}{2}} p (L - M) \left(1 + \sinh^{-1}(1) \right) + |\alpha||M| \left[1 + 2^{\frac{1}{2}} p \left(1 + \sinh^{-1}(1) \right) \right]}.$$

From (17), we have
$$\left| \frac{h(z_0) - 1}{L - Mh(z_0)} \right| \geq 1,$$

which is contradictory to the fact that $h(z) \prec \frac{1 + Lz}{1 + Mz}$. Thus, $|w(z)| < 1$, and so, we obtain our desired result. $\square$

If we take $g(z) = \frac{z^{p+1} f'(z)}{p f(z)}$ in Theorem 3, we have the following Corollary.

Corollary 4. *If f is in the class $\mathcal{A}_p$, has the form (3), and satisfies the condition*
$$1 + \frac{\alpha}{p} \left(p + 1 + \frac{z f''(z)}{f'(z)} - \frac{z f'(z)}{f(z)} \right) \prec \frac{1 + Lz}{1 + Mz},$$

where the limitation on α is
$$|\alpha| \geq \frac{2^{\frac{1}{2}} p (L - M) \left(1 + \sinh^{-1}(1) \right)}{1 - |M| - 2^{\frac{1}{2}} p \left(1 + \sinh^{-1}(1) \right) (1 + |M|)},$$

then, $f \in \mathcal{S}_p^*(pet)$.

Proof. The proof is straightforward so it is left for the reader. $\square$

Theorem 4. *Let $g \in A_p$ be of the form* (3) *and satisfy the condition*

$$1 + \frac{\alpha}{p} \frac{z^{1+p} g'(z)}{(g(z))^2} \prec \frac{1 + Lz}{1 + Mz} \tag{20}$$

where the limitation on α is

$$|\alpha| \geq \frac{2^{\frac{1}{2}} p (L - M) \left(1 + \sinh^{-1}(1)\right)^2}{1 - |M| - 2^{\frac{1}{2}} p \left(1 + \sinh^{-1}(1)\right)(1 + |M|)} \tag{21}$$

then,

$$\frac{g(z)}{z^p} \prec 1 + \sinh^{-1}(z). $$

Proof. Let us define a function

$$h(z) = 1 + \frac{\alpha}{p} \frac{z^{1+p} g'(z)}{(g(z))^2}, \tag{22}$$

where $h(z)$ is analytic in $\mathbb{D}$ and $h(0) = 1$. Further, consider

$$\frac{g(z)}{z^p} = 1 + \sinh^{-1}(w(z)). \tag{23}$$

To show our result, we are required to show that $|w(z)| < 1$. Now,

$$h(z) = 1 + \frac{\alpha}{1 + \sinh^{-1}(w(z))} + \frac{\alpha z w'(z)}{p \left(1 + \sinh^{-1}(w(z))\right)^2 \sqrt{(w(z))^2 + 1}},$$

and

$$\left| \frac{h(z) - 1}{L - M h(z)} \right| = \left| \frac{\frac{\alpha}{1 + \sinh^{-1}(w(z))} + \frac{\alpha z w'(z)}{p \left(1 + \sinh^{-1}(w(z))\right)^2 \sqrt{(w(z))^2 + 1}}}{L - M \left(1 + \frac{\alpha}{1 + \sinh^{-1}(w(z))} + \frac{\alpha z w'(z)}{p \left(1 + \sinh^{-1}(w(z))\right)^2 \sqrt{(w(z))^2 + 1}}\right)} \right|$$

$$= \left| \frac{\alpha p \left(1 + \sinh^{-1}(w(z))\right) \sqrt{(w(z))^2 + 1} + \alpha z w'(z)}{(L - M) p \left(1 + \sinh^{-1}(w(z))\right)^2 \sqrt{(w(z))^2 + 1} - M Y_6(p, \alpha, z)} \right|,$$

where

$$Y_6(p, \alpha, z) = \left(\alpha p \left(1 + \sinh^{-1}(w(z))\right) \sqrt{(w(z))^2 + 1} + \alpha z w'(z) \right)$$

and suppose there occurs a point $z_0 \in \mathbb{D}$ such that

$$\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1.$$

Additionally, by Lemma 2, a number $m \geq 1$ exists with

$$z_0 w'(z_0) = m w(z_0).$$

We also suppose that

$$w(z_0) = e^{i\theta} \quad (\theta \in [-\pi, \pi]).$$

Then, we have

$$
\left| \frac{h(z_0) - 1}{L - Mh(z_0)} \right| = \left| \frac{\alpha p \left(1 + \sinh^{-1}(w(z_0))\right) \sqrt{(w(z_0))^2 + 1} + \alpha z w'(z_0)}{(L - M)p \left(1 + \sinh^{-1}(w(z_0))\right)^2 \sqrt{(w(z_0))^2 + 1} - M Y_7(p, \alpha, z)} \right|
$$

$$
\geq \frac{|\alpha|m - |\alpha|p \left(1 + \sinh^{-1}\left(|e^{i\theta}|\right)\right) \sqrt{e^{2i\theta} + 1}}{(L - M)p \left(1 + \sinh^{-1}\left(|e^{i\theta}|\right)\right)^2 \sqrt{e^{2i\theta} + 1} + |M| Y_8(p, \alpha, e)}
$$

$$
\geq \frac{|\alpha| \left(m - 2^{\frac{1}{2}} p \left(1 + \sinh^{-1}(1)\right)\right)}{2^{\frac{1}{2}} p(L - M) \left(1 + \sinh^{-1}(1)\right)^2 + |\alpha||M| \left[m + 2^{\frac{1}{2}} p \left(1 + \sinh^{-1}(1)\right)\right]},
$$

where

$$
Y_7(p, \alpha, z) = \left(\alpha p \left(1 + \sinh^{-1}(w(z_0))\right) \sqrt{(w(z_0))^2 + 1} + \alpha z w'(z_0) \right)
$$

and

$$
Y_8(p, \alpha, e) = \left(p|\alpha| \left(1 + \sinh^{-1}\left(|e^{i\theta}|\right)\right) \sqrt{e^{2i\theta} + 1} + |\alpha|m \right).
$$

Now, let

$$
k(m) = \frac{|\alpha| \left(m - 2^{\frac{1}{2}} p \left(1 + \sinh^{-1}(1)\right)\right)}{2^{\frac{1}{2}} p(L - M) \left(1 + \sinh^{-1}(1)\right)^2 + |\alpha||M| \left[m + 2^{\frac{1}{2}} p \left(1 + \sinh^{-1}(1)\right)\right]},
$$

then,

$$
k'(m) = \frac{2^{\frac{1}{2}} |\alpha| p(L - M) \left(1 + \sinh^{-1}(1)\right)^2 + |M||\alpha|^2 2^{\frac{3}{2}} p \left(1 + \sinh^{-1}(1)\right)}{\left(2^{\frac{1}{2}} p(L - M) \left(1 + \sinh^{-1}(1)\right)^2 + |\alpha||M| \left[m + 2^{\frac{1}{2}} p \left(1 + \sinh^{-1}(1)\right)\right]\right)^2}
$$

$$
> 0.
$$

Clearly, $k(m)$ is an increasing function; so, $\max k(m) \geq k(1)$ for $m \geq 1$, so

$$
\left| \frac{h(z_0) - 1}{L - Mh(z_0)} \right|
$$

$$
\geq \frac{|\alpha| \left(1 - 2^{\frac{1}{2}} p \left(1 + \sinh^{-1}(1)\right)\right)}{2^{\frac{1}{2}} p(L - M) \left(1 + \sinh^{-1}(1)\right)^2 + |\alpha||M| \left[1 + 2^{\frac{1}{2}} p \left(1 + \sinh^{-1}(1)\right)\right]}.
$$

From (17), we have

$$
\left| \frac{h(z_0) - 1}{L - Mh(z_0)} \right| \geq 1,
$$

which is contradictory to the fact that $h(z) \prec \frac{1 + Lz}{1 + Mz}$. Thus, $|w(z)| < 1$, and so, we obtain our desired result. □

Corollary 5. *If the function f is in the class $\mathcal{A}_p$, has the form* (3), *and satisfies the condition*

$$
1 + \frac{\alpha f(z)}{z^{2p+1} f'(z)} \left(p + 1 + \frac{z f''(z)}{f'(z)} - \frac{z f'(z)}{f(z)} \right) \prec \frac{1 + Lz}{1 + Mz},
$$

where the limitation on α is

$$|\alpha| \geq \frac{2^{\frac{1}{2}} p (L - M) \left(1 + \sinh^{-1}(1)\right)^2}{1 - |M| - 2^{\frac{1}{2}} p \left(1 + \sinh^{-1}(1)\right)(1 + |M|)},$$

then, $f \in S_p^*(pet)$.

Theorem 5. *Let $g \in A_p$ be of the form* (3) *satisfy the condition*

$$1 + \frac{\alpha}{p} \frac{z^{1+2p} g'(z)}{(g(z))^3} \prec \frac{1 + Lz}{1 + Mz} \tag{24}$$

where the condition on α is

$$|\alpha| \geq \frac{2^{\frac{1}{2}} p (L - M) \left(1 + \sinh^{-1}(1)\right)^3}{1 - |M| - 2^{\frac{1}{2}} p \left(1 + \sinh^{-1}(1)\right)(1 + |M|)} \tag{25}$$

then,

$$\frac{g(z)}{z^p} \prec 1 + \sinh^{-1}(z). $$

Proof. Let us define a function

$$h(z) = 1 + \frac{\alpha}{p} \frac{z^{1+p} g'(z)}{(g(z))^2}, \tag{26}$$

where $h(z)$ is analytic in $\mathbb{D}$ and $h(0) = 1$. Further, consider

$$\frac{g(z)}{z^p} = 1 + \sinh^{-1}(w(z)). \tag{27}$$

Now, to prove our result we will required to show that $|w(z)| < 1$. Now,

$$h(z) = 1 + \frac{\alpha}{1 + \sinh^{-1}(w(z))} + \frac{\alpha z w'(z)}{p \left(1 + \sinh^{-1}(w(z))\right)^2 \sqrt{(w(z))^2 + 1}},$$

and

$$\left| \frac{h(z) - 1}{L - Mh(z)} \right| = \left| \frac{\dfrac{\alpha}{1 + \sinh^{-1}(w(z))} + \dfrac{\alpha z w'(z)}{p \left(1 + \sinh^{-1}(w(z))\right)^2 \sqrt{(w(z))^2 + 1}}}{L - M \left(1 + \dfrac{\alpha}{1 + \sinh^{-1}(w(z))} + \dfrac{\alpha z w'(z)}{p \left(1 + \sinh^{-1}(w(z))\right)^2 \sqrt{(w(z))^2 + 1}} \right)} \right|$$

$$= \left| \frac{\alpha p \left(1 + \sinh^{-1}(w(z))\right) \sqrt{(w(z))^2 + 1} + \alpha z w'(z)}{(L - M) p \left(1 + \sinh^{-1}(w(z))\right)^2 \sqrt{(w(z))^2 + 1} - MY_9(p, \alpha, z)} \right|,$$

where

$$Y_9(p, \alpha, z) = \left(\alpha p \left(1 + \sinh^{-1}(w(z))\right) \sqrt{(w(z))^2 + 1} + \alpha z w'(z) \right).$$

Suppose there occurs a point $z_0 \in \mathbb{D}$ such that

$$\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1.$$

By Lemma [2], a number $m \geq 1$ exists with $z_0 w'(z_0) = mw(z_0)$. In addition, we also suppose that $w(z_0) = e^{i\theta}$ for $\theta \in [-\pi, \pi]$. Then, we have

$$\left| \frac{h(z_0) - 1}{L - Mh(z_0)} \right| = \left| \frac{\alpha p\left(1 + \sinh^{-1}(w(z_0))\right)\sqrt{(w(z_0))^2 + 1} + \alpha z w'(z_0)}{(L-M)p\left(1 + \sinh^{-1}(w(z_0))\right)^2 \sqrt{(w(z_0))^2 + 1} - MY_{10}(p,\alpha,z_0)} \right|$$

$$\geq \frac{|\alpha|m - |\alpha|p\left(1 + \sinh^{-1}(|e^{i\theta}|)\right)\sqrt{e^{2i\theta} + 1}}{(L-M)p\left(1 + \sinh^{-1}(|e^{i\theta}|)\right)^2 \sqrt{e^{2i\theta} + 1} + |M|Y_{11}(p,\alpha,e)}$$

$$\geq \frac{|\alpha|\left(m - 2^{\frac{1}{2}}p\left(1 + \sinh^{-1}(1)\right)\right)}{2^{\frac{1}{2}}p(L-M)\left(1 + \sinh^{-1}(1)\right)^2 + |\alpha||M|\left[m + 2^{\frac{1}{2}}p\left(1 + \sinh^{-1}(1)\right)\right]},$$

where

$$Y_{10}(p,\alpha,z_0) = \left(\alpha p\left(1 + \sinh^{-1}(w(z_0))\right)\sqrt{(w(z_0))^2 + 1} + \alpha z w'(z_0) \right)$$

and

$$Y_{11}(p,\alpha,e) = \left(p|\alpha|\left(1 + \sinh^{-1}(|e^{i\theta}|)\right)\sqrt{e^{2i\theta} + 1} + |\alpha|m \right).$$

Now, let

$$k(m) = \frac{|\alpha|\left(m - 2^{\frac{1}{2}}p\left(1 + \sinh^{-1}(1)\right)\right)}{2^{\frac{1}{2}}p(L-M)\left(1 + \sinh^{-1}(1)\right)^3 + |\alpha||M|\left[m + 2^{\frac{1}{2}}p\left(1 + \sinh^{-1}(1)\right)\right]},$$

then,

$$k'(m) = \frac{2^{\frac{1}{2}}|\alpha|p(L-M)\left(1 + \sinh^{-1}(1)\right)^2 + |M||\alpha|^2 2^{\frac{3}{2}}p\left(1 + \sinh^{-1}(1)\right)}{\left(2^{\frac{1}{2}}p(L-M)\left(1 + \sinh^{-1}(1)\right)^3 + |\alpha||M|\left[m + 2^{\frac{1}{2}}p\left(1 + \sinh^{-1}(1)\right)\right]\right)^2}$$

$$> 0.$$

Clearly, $k(m)$ is an increasing function; so,

$$\max k(m) \geq k(1) \qquad (m \geq 1)$$

and

$$\left| \frac{h(z_0) - 1}{L - Mh(z_0)} \right| \geq \frac{|\alpha|\left(1 - 2^{\frac{1}{2}}p\left(1 + \sinh^{-1}(1)\right)\right)}{2^{\frac{1}{2}}p(L-M)\left(1 + \sinh^{-1}(1)\right)^3 + |\alpha||M|\left[1 + 2^{\frac{1}{2}}p\left(1 + \sinh^{-1}(1)\right)\right]}.$$

From ([17]), we have

$$\left| \frac{h(z_0) - 1}{L - Mh(z_0)} \right| \geq 1$$

which is contradictory to the fact that $h(z) \prec \frac{1 + Lz}{1 + Mz}$. Thus, $|w(z)| < 1$, and so, we obtain our desired result. $\quad\square$

4. Conclusions

In this article, a subclass of regular multivalent functions in petal-shape domain has been introduced. These functions are then characterized with the help of some useful

properties such as Fekete–Szegö problems and consequences to some special cases are discussed. We also derived some differential subordination implementation results involving $1 + \sinh^{-1}(z)$. These results can be generalized if we consider some other regions such as Lemniscate of Bernoulli region, cardioid region, nephroid domain, etc., instead of the Janowski domain.

In concluding our present investigation, we draw the attention of the interested readers toward the prospect of studying the basic or quantum (or q-) generalizations of the results we have developed in this paper. This direction of research was indeed influenced and motivated by a recently published survey-cum-expository review article by Srivastava [18]. However, as already demonstrated by Srivastava (see [18], p. 340; ([28], Section 5, pp. 1511–1512)), the $(\mathfrak{p}, q)$-variations of the proposed q-results will lead trivially to inconsequential research, because the forced-in parameter $\mathfrak{p}$ is obviously redundant. Furthermore, in light of Srivastava's more recent expository article [28], the interested readers should be advised *not* to be misled to believe that the so-called k-Gamma function provides a "generalization" of the classical (Euler's) Gamma function. Similar remarks will apply also to all usages of the so-called k-Gamma function, including (for example) the so-called (k, s)-extensions of the Riemann–Liouville and other operators of fractional integral and fractional derivatives.

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