On Constrained Set-Valued Semi-Infinite Programming Problems with $\rho$-Cone Arcwise Connectedness

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Abstract: In this paper, we establish sufficient Karush–Kuhn–Tucker (for short, KKT) conditions of a set-valued semi-infinite programming problem (SP) via the notion of contingent epiderivative of set-valued maps. We also derive duality results of Mond–Weir (MWD), Wolfe (WD), and mixed (MD) types of the problem (SP) under $\rho$-cone arcwise connectedness assumptions.

Keywords: set-valued map; convex cone; duality; contingent epiderivative

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1. Introduction

Over the last few decades, many authors, such as Hanson [1], Craven [2], Corley [3], Zalmai [4] etc., studied different types of optimization problems. One of such optimization problems is semi-infinite programming problem. In recent times, many authors, such as Goberna and Lopez [5], Shapiro [6], etc., have studied optimality conditions of semi-infinite programming problems. In 2005, Shapiro [7] studied the Lagrangian-type duality of semi-infinite programming problems under convexity assumption. In 2010, Kostyukova and Tchemisova [8] established sufficient optimality conditions of semi-infinite programming problems using convexity assumption. In 2012, Mishra and Jaiswal [9] established the sufficient optimality conditions of semi-infinite programming problems using generalized convexity assumption. They also formulated duality model and proved the corresponding theorems of Mond–Weir type dual.


In this paper, we consider a set-valued semi-infinite programming problem (SP), where the objective function and constraints are set-valued maps. We are mainly interested in establishing the sufficient KKT optimality conditions of the problem (SP) in terms of...
generalized cone arcwise connectedness. We also study the duality theorems of Mond–Weir (MWD), Wolfe (WD), and mixed (MD) types associated with the problem (SP).

This paper is organized as follows. In Section 2, we state some definitions and preliminary concepts of set-valued maps. The concept of $\rho$-cone arcwise connectedness is included in Section 3. In Section 4, we establish (see Section 5) the sufficient KKT optimality conditions for the problem (SP). We also study the duality results of various types using generalized cone arcwise connectedness assumptions (see Sections 6–8). Section 9 concludes the paper.

2. Definitions and Preliminaries

Let $V$ be a real normed space and $\Omega$ be a nonempty subset of $V$. Then, $\Omega$ is said to be a cone if $\lambda v \in \Omega$, for all $v \in \Omega$ and $\lambda \geq 0$. Furthermore, $\Omega$ is called non-trivial if $\Omega \neq \{0\}$, proper if $\Omega \neq V$, pointed if $\Omega \cap (-\Omega) = \{0\}$, solid if $\text{int}(\Omega) \neq \emptyset$, closed if $\overline{\Omega} = \Omega$, and convex if $\lambda \Omega + (1 - \lambda)\Omega \subseteq \Omega$, for all $\lambda \in [0,1]$, where $\text{int}(\Omega)$ and $\overline{\Omega}$ denote the interior and closure of $\Omega$, respectively, and $\theta_V$ is the zero element of $V$.

Aubin [17,18] introduced the notion of contingent cone in set-valued optimization theory. Moreover, Aubin [17,18] and Cambini et al. [19] introduced the notion of second-order contingent set in a set-valued optimization theory.

**Definition 1** ([17,18]). Let $V$ be a real normed space, $\emptyset \neq B \subseteq V$, and $v' \in \overline{B}$. The contingent cone to $B$ at $v'$ is denoted by $T(B,v')$ and is defined as follows: an element $v \in T(B,v')$ if there are sequences $\{t_n\} \in \mathbb{R}$, with $t_n \to 0^+$ and $\{v_n\}$ in $V$, with $v_n \to v$, such that

$$v' + t_n v_n \in B, \quad \forall n \in \mathbb{N},$$

or, there exist sequences $\{t_n\}$ in $\mathbb{R}$, with $t_n > 0$ and $\{v_n'\}$ in $B$, with $v_n' \to v'$, such that

$$t_n(v_n' - v') \to v.$$

Let $U$, $V$ be real normed spaces, $2^V$ be the set of all subsets of $V$, and $\Omega$ be a solid pointed convex cone in $V$. Let $F : U \to 2^V$ be a set-valued map from $U$ to $V$, i.e., $F(u) \subseteq V$, for all $u \in U$. The domain, image, graph, and epigraph of $F$ are defined by

$$\text{dom}(F) = \{u \in U : F(u) \neq \emptyset\},$$

$$F(A) = \bigcup_{u \in A} F(u), \text{ for any } \emptyset \neq A \subseteq U,$$

$$\text{gr}(F) = \{(u,v) \in U \times V : v \in F(u)\},$$

and

$$\text{epi}(F) = \{(u,v) \in U \times V : v \in F(u) + \Omega\}.$$

Jahn and Rauh [20] introduced the notion of contingent epiderivative of set-valued maps. It plays a vital role in various aspects of set-valued optimization problems.

**Definition 2** ([20]). A single-valued map $D_I F(u',v') : U \to V$ whose epigraph coincides with the contingent cone to the epigraph of $F$ at $(u',v')$, i.e.,

$$\text{epi}(D_I F(u',v')) = T(\text{epi}(F),(u',v')),$$

is said to be the contingent epiderivative of $F$ at $(u',v')$.

We now turn our attention to the notion of cone convexity of set-valued maps which was introduced by Borwein [21] in 1977.
Definition 3 ([21]). Let A be a nonempty convex subset of a real normed space U. A set-valued map $F: U \to 2^U$, with $A \subseteq \text{dom}(F)$, is called $\Omega$-convex on A if for all $u_1, u_2 \in A$ and $\lambda \in [0, 1]$, 

$$\lambda F(u_1) + (1 - \lambda) F(u_2) \subseteq F(\lambda u_1 + (1 - \lambda) u_2) + \Omega.$$ 

In 1976, Avriel [10] introduced the notion of arcwise connectedness of set-valued maps.

Definition 4. A subset $A$ of a real normed space $U$ is said to be an arcwise connected subset of $U$ if for all $u_1, u_2 \in A$ there is a continuous arc $H_{u_1,u_2}(\lambda)$ defined on $[0, 1]$ with a value in $A$ such that $H_{u_1,u_2}(0) = u_1$ and $H_{u_1,u_2}(1) = u_2$.


Definition 5 ([11,12]). Let $A$ be an arcwise connected subset of a real normed space $U$ and $F: U \to 2^U$ be a set-valued map, with $A \subseteq \text{dom}(F)$. Then, $F$ is said to be $\Omega$-arcwise connected on $A$ if

$$(1 - \lambda)F(u_1) + \lambda F(u_2) \subseteq F(H_{u_1,u_2}(\lambda)) + \Omega, \quad \forall u_1, u_2 \in A \text{ and } \forall \lambda \in [0, 1].$$

Peng and Xu [16] introduced the notion of cone subarcwise connectedness of set-valued maps.

Definition 6 ([16]). Let $A$ be an arcwise connected subset of a real normed space $U$, $e \in \text{int}(\Omega)$, and $F: U \to 2^U$ be a set-valued map, with $A \subseteq \text{dom}(F)$. Then, $F$ is said to be $\Omega$-subarcwise connected on $A$ if

$$(1 - \lambda)F(u_1) + \lambda F(u_2) + ee \subseteq F(H_{u_1,u_2}(\lambda)) + \Omega, \quad \forall u_1, u_2 \in A, \forall e > 0, \text{ and } \forall \lambda \in [0, 1].$$

3. $\rho$-Cone Arcwise Connectedness

Das and Nahak [22–25] and Treanţă and Das [26] introduced the notion of $\rho$-cone convexity of set-valued maps. They establish the sufficient KKT optimality conditions for set-valued optimization problems under contingent epiderivative and $\rho$-cone convexity assumptions. They also construct various duality models and prove the associated duality theorems. For $\rho = 0$, we get the usual notion of cone convexity of set-valued maps introduced by Borwein [21] in the year of 1977.

We introduce the notion of $\rho$-cone arcwise connectedness of set-valued maps as a generalization of cone arcwise connected set-valued maps.

Definition 7. Let $A$ be an arcwise connected subset of a real normed space $U$, $e \in \text{int}(\Omega)$, and $F: U \to 2^U$ be a set-valued map, with $A \subseteq \text{dom}(F)$. Then, $F$ is said to be $\rho$-$\Omega$-arcwise connected on $A$ with respect to $e$ if there exists $\rho \in \mathbb{R}$, such that

$$(1 - \lambda)F(u_1) + \lambda F(u_2) \subseteq F(H_{u_1,u_2}(\lambda)) + \rho \lambda (1 - \lambda)\|u_1 - u_2\|^2 e + \Omega, \quad \forall u_1, u_2 \in A \text{ and } \forall \lambda \in [0, 1].$$

Remark 1. If $\rho > 0$, then $F$ is called strongly $\rho$-$\Omega$-arcwise connected; if $\rho = 0$, we have the notion of $\Omega$-arcwise connectedness; and if $\rho < 0$, then $F$ is called weakly $\rho$-$\Omega$-arcwise connected. Obviously, strongly $\rho$-$\Omega$-arcwise connectedness $\Rightarrow$ $\Omega$-arcwise connectedness $\Rightarrow$ weakly $\rho$-$\Omega$-arcwise connectedness.

Further, we formulate an example of $\rho$-cone arcwise connected set-valued map, which is not cone arcwise connected.
Example 1. Let $U = \mathbb{R}^2$, $V = \mathbb{R}$, $\Omega = \mathbb{R}_+$ and

$$A = \left\{ u = (u_1, u_2) \mid u_1 + u_2 \geq \frac{1}{2}, u_1 \geq 0, u_2 \geq 0 \right\} \subset U.$$ 

Define $\mathcal{H}_{u,s}(\lambda) = (1 - \lambda)u + \lambda s$, where $u = (u_1, u_2)$, $s = (s_1, s_2)$, and $\lambda \in [0, 1]$. Clearly, $A$ is an arcwise connected set. For the set-valued map $F : U \to 2^V$, defined as follows: $F(u) = [0, 2]$, $u_1 + u_2 \geq \frac{1}{2}, u_1 \neq u_2$, and $F(u) = [3, 5]$ for $\{u_1 + u_2 < \frac{1}{2}\} \cup \{u_1 + u_2 \geq \frac{1}{2}, u_1 = u_2\}$, we find that $F$ is not $\Omega$-arcwise connected for $u = (1, 0)$, $s = (0, 1)$, and $\lambda = \frac{1}{2}$. On the other hand, by considering $\rho = -2$ and $e = [3, 3] = (3)$, we get that $F$ is a $\rho$-$\Omega$-arcwise connected set-valued map for $u = (1, 0)$, $s = (0, 1)$.

Theorem 1. Let $A$ be an arcwise connected subset of a real normed space $U, e \in \text{int}(\Omega)$, and $F : U \to 2^V$ be $\rho$-$\Omega$-arcwise connected on $A$ with respect to $e$. Let $u' \in A$ and $v' \in F(u')$. Then,

$$F(u) - v' \subseteq D_+ F(u', v')(\mathcal{H}_{u',u}(0+)) + \rho\|u - u'\|^2e + \Omega, \quad \forall u \in A,$$

where

$$\mathcal{H}_{u',u}(0+) = \lim_{\lambda \to 0^+} \frac{\mathcal{H}_{u',u}(\lambda) - \mathcal{H}_{u',u}(0)}{\lambda},$$

assuming that $\mathcal{H}_{u',u}(0+)$ exists for all $u, u' \in A$.

Proof. Let $u \in A$. As $F$ is $\rho$-$\Omega$-arcwise connected on $A$ with respect to $e$, we have

$$(1 - \lambda)F(u') + \lambda F(u) \subseteq F(\mathcal{H}_{u',u}(\lambda)) + \rho \lambda (1 - \lambda)\|u - u'\|^2e + \Omega, \quad \forall \lambda \in [0, 1].$$

Let $v \in F(u)$. Consider a real sequence $\{\lambda_n\}$, with $\lambda_n \in (0, 1)$, $n \in \mathbb{N}$, such that $\lambda_n \to 0^+$ when $n \to \infty$. Suppose

$$u_n = \mathcal{H}_{u',u}(\lambda_n)$$

and

$$v_n = (1 - \lambda_n)v' + \lambda_n v - \rho \lambda_n (1 - \lambda_n)\|u - u'\|^2e.$$ 

Therefore,

$$v_n \in F(u_n) + \Omega.$$

It is clear that

$$u_n = \mathcal{H}_{u',u}(\lambda_n) \to \mathcal{H}_{u',u}(0) = u', v_n \to v', \quad \text{when} \quad n \to \infty,$$

$$\frac{u_n - u'}{\lambda_n} = \frac{\mathcal{H}_{u',u}(\lambda_n) - \mathcal{H}_{u',u}(0)}{\lambda_n} \to \mathcal{H}_{u',u}(0+), \quad \text{when} \quad n \to \infty,$$

and

$$\frac{v_n - v'}{\lambda_n} = v - v' - \rho (1 - \lambda_n)\|u - u'\|^2e \to v - v' - \rho\|u - u'\|^2e, \quad \text{when} \quad n \to \infty.$$ 

Therefore,

$$\left(\mathcal{H}_{u',u}(0+), v - v' - \rho\|u - u'\|^2e \right) \in T(epi(F), (u', v')) = epi(D_+ F(u', v')).$$

Consequently,

$$v - v' - \rho\|u - u'\|^2e \in D_+ F(u', v')(\mathcal{H}_{u',u}(0+)) + \Omega,$$
which is true, for all $v \in \mathcal{F}(u)$. Hence,

$$
\mathcal{F}(u) - v' \subseteq D_1 \mathcal{F}(u', v') (\mathcal{H}_{u', v'}^I (0^+)) + \rho \| u - u' \|^2 e + \Omega, \quad \forall u \in A.
$$

Hence, the theorem follows.

\[ \Box \]

4. Formulation of the Main Problem

Let $U$ be a countably infinite subset of $\mathbb{R}^n$, $\emptyset \neq A \subseteq \mathbb{R}^n$, and $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_m) : \mathbb{R}^n \rightarrow 2^{\mathbb{R}^m}$, $\mathcal{G} : \mathbb{R}^n \times U \rightarrow 2^{\mathbb{R}^m}$ be two set-valued maps with

$$
A \subseteq \text{dom}(\mathcal{F}) \text{ and } A \times U \subseteq \text{dom}(\mathcal{G}).
$$

Let $B_1, B_2, \ldots, B_m$ be $n \times n$ positive semi-definite symmetric real matrices. We consider a set-valued multi-objective semi-infinite programming problem (SP).

\[
\begin{align*}
\text{minimize} & \quad (\mathcal{F}_1(x) + (x^T B_1 x) \frac{1}{2}, \mathcal{F}_2(x) + (x^T B_2 x) \frac{1}{2}, \ldots, \mathcal{F}_m(x) + (x^T B_m x) \frac{1}{2}) \\
\text{subject to} & \quad \mathcal{G}(x, u) \cap (-\mathbb{R}_+) \neq \emptyset, \forall u \in U.
\end{align*}
\]

We define the feasible set of the problem (SP) by

$$
S = \{x \in A : \mathcal{G}(x, u) \cap (-\mathbb{R}_+) \neq \emptyset, \forall u \in U\}.
$$

**Definition 8.** A point $(x', y') \in \mathbb{R}^n \times \mathbb{R}^m$, with $x' \in S$ and $y' = (y'_1, y'_2, \ldots, y'_m) \in \mathcal{F}(x')$, is said to be a minimizer of the problem (SP) if for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$, with $x \in S$ and $y = (y_1, y_2, \ldots, y_m) \in \mathcal{F}(x)$,

$$
\begin{align*}
(y_1 + (x^T B_1 x) \frac{1}{2}, y_2 + (x^T B_2 x) \frac{1}{2}, \ldots, y_m + (x^T B_m x) \frac{1}{2}) \\
-(y'_1 + (x^T B_1 x') \frac{1}{2}, y'_2 + (x^T B_2 x') \frac{1}{2}, \ldots, y'_m + (x^T B_m x') \frac{1}{2}) \notin (-\mathbb{R}_+) \setminus \{\emptyset\}.
\end{align*}
$$

**Definition 9.** A point $(x', y') \in \mathbb{R}^n \times \mathbb{R}^m$, with $x' \in S$ and $y' = (y'_1, y'_2, \ldots, y'_m) \in \mathcal{F}(x')$, is said to be a weak minimizer of the problem (SP) if for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$, with $x \in S$ and $y = (y_1, y_2, \ldots, y_m) \in \mathcal{F}(x)$,

$$
\begin{align*}
(y_1 + (x^T B_1 x) \frac{1}{2}, y_2 + (x^T B_2 x) \frac{1}{2}, \ldots, y_m + (x^T B_m x) \frac{1}{2}) \\
-(y'_1 + (x^T B_1 x') \frac{1}{2}, y'_2 + (x^T B_2 x') \frac{1}{2}, \ldots, y'_m + (x^T B_m x') \frac{1}{2}) \notin (\text{int}(\mathbb{R}^m_+)).
\end{align*}
$$

Let $I$ be the index set, such that $U = \{u_j : j \in I\}$. Let $x' \in A$. We denote a set $J(x')$ by

$$
J(x') = \{j \in I : 0 \in \mathcal{G}(x', u_j)\}.
$$

We assume that $J(x') \neq \emptyset$.

For special case, when $f = (f_1, f_2, \ldots, f_m) : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $g : \mathbb{R}^n \times U \rightarrow \mathbb{R}$ are single-valued maps, we can have multi-objective semi-infinite programming problem ([9]) as

$$
\begin{align*}
\text{minimize} & \quad (f_1(x) + (x^T B_1 x) \frac{1}{2}, f_2(x) + (x^T B_2 x) \frac{1}{2}, \ldots, f_m(x) + (x^T B_m x) \frac{1}{2}) \\
\text{subject to} & \quad g(x, u) \in (-\mathbb{R}_+), \forall u \in U,
\end{align*}
$$

by considering $\mathcal{F}_i(x) = \{f_i(x), i = 1, 2, \ldots, m\}$ and $\mathcal{G}(x, u) = \{g(x, u)\}$ in the problem (SP).

5. Optimality Conditions

Let $\xi_i \in \mathbb{R}^n$, $i = 1, 2, \ldots, m$. Define maps $^T B_i \xi_i : \mathbb{R}^n \rightarrow \mathbb{R}$, $i = 1, 2, \ldots, m$, by

$$
(^T B_i \xi_i)(x) = x^T B_i \xi_i, \forall x \in \mathbb{R}^n.
$$
Then, \( (\text{Sufficient optimality conditions}) \)

Let \( x' \in A \) and \( j \in J(x') \). Define a set-valued map \( G(.,u_j) : \mathbb{R}^n \to 2^{\mathbb{R}} \) by

\[
G(.,u_j)(x) = G(x,u_j), \forall x \in \text{dom}(G).
\]

We establish the sufficient KKT optimality conditions of the set-valued semi-infinite programming problem (SP) under \( p \)-cone arcwise connectedness assumption.

**Theorem 2** (Sufficient optimality conditions). Let \( A \) be an arcwise connected subset of \( \mathbb{R}^n \), \( x' \in S \), and \( y' = (y'_1,y'_2, \ldots, y'_m) \in \mathcal{F}(x') \). Let \( \mathcal{X}, \mathcal{Y} \in \mathbb{R}^n \), \( i = 1,2,\ldots, m \) and \( z'_j \in \mathcal{G}(x',u_j) \cap (-\mathbb{R}^+) \). Let \( \rho, \rho'_i, \rho''_i \in \mathbb{R} \), for \( i = 1,2,\ldots, m \) and \( j \in J(x') \). Suppose that \( F_{\mu}^i B_{i} \mathcal{X}, i = 1,2,\ldots, m \), and \( G(.,u_j) \), \( j \in J(x') \), are \( \rho_i-\mathbb{R}_+ \)-arcwise connected, \( \rho'_i-\mathbb{R}_+ \)-arcwise connected, and \( \rho''_i-\mathbb{R}_+ \)-arcwise connected set-valued maps, respectively, on \( A \) with respect to 1.

Further, we suppose that the contingent epiderivatives \( D_i F_j(x', y'_j) \) and \( D_j G(., u_j)(x', z'_j) \) exist. If there exist \( y'_i > 0, i = 1,2,\ldots, m \), and \( z'_j \geq 0, j \in J(x') \), with \( z'_j \neq 0 \), for finitely many \( j \) and

\[
\sum_{i=1}^{m} y'_i (\rho_i + \rho'_i) + \sum_{j \in J(x')} z'_j \rho''_j \geq 0,
\]

satisfying the following conditions

\[
\left( \sum_{i=1}^{m} y'_i (D_i F_j(x', y'_j) + (B_i \mathcal{X})^T) + \sum_{j \in J(x')} z'_j D_j G(., u_j)(x', z'_j) \right) (H'_{x',x}(0+)) \geq 0, \forall x \in A, \tag{2}
\]

\[
z'_j z'_j = 0, \forall j \in J(x'), \tag{3}
\]

\[
\mathcal{X}_i^T B_i \mathcal{X}_i \leq 1, i = 1,2,\ldots, m, \tag{4}
\]

and

\[
(x'^T B_i x')^{\frac{1}{2}} = x'^T B_i \mathcal{X}_i, i = 1,2,\ldots, m. \tag{5}
\]

Then, \( (x', y') \) is a weak minimizer of (SP).

**Proof.** Let \( (x', y') \) be not a weak minimizer of (SP). Then there exist \( x \in S \) and \( y = (y_1, \ldots, y_m) \in \mathcal{F}(x) \), such that

\[
(y_1 + (x'^T B_1 x')^{\frac{1}{2}} y_2 + (x'^T B_2 x')^{\frac{1}{2}} \ldots, y_m + (x'^T B_m x')^{\frac{1}{2}}),\]

\[
< (y'_1 + (x'^T B_1 x')^{\frac{1}{2}} y'_2 + (x'^T B_2 x')^{\frac{1}{2}} \ldots, y'_m + (x'^T B_m x')^{\frac{1}{2}}).
\]

As \( y^* \in \mathbb{R}_+^m \setminus \{\theta_{\mathbb{R}^n}\} \), we have

\[
\sum_{i=1}^{m} y'_i (y_i + (x'^T B_i x')^{\frac{1}{2}}) > \sum_{i=1}^{m} y'_i (y'_i + (x'^T B_i x')^{\frac{1}{2}}). \tag{6}
\]

Since \( F_{\mu}^i, i = 1,2,\ldots, m \), is \( \rho_i-\mathbb{R}_+ \)-arcwise connected on \( A \) with respect to 1 and \( (x', y'_j) \in \text{gr}(F_j) \), we have

\[
F_j(x) - y'_j - \rho_i \|x - x'\|^2 \subseteq D_i F_j(x', y'_j) (H'_{x',x}(0+)) + \mathbb{R}^+. \]

Hence,

\[
y_j - y'_j - \rho_i \|x - x'\|^2 \subseteq D_i F_j(x', y'_j) (H'_{x',x}(0+)) + \mathbb{R}^+.
\]
Therefore,
\[ y_i^*(y_i - y'_i) - \rho_i \|x - x'\|^2 y_i^* \geq y_i^* D_i F_i(x', y'_i)(H'_{x,x}(0+)). \] (7)

Again, as \( T B_i x_i, i = 1, 2, \ldots, m \) and \( G(., u_j), j \in J(x'), \) are \( \rho_i^j \in \mathbb{R}_+ \)-arcwise connected and \( \rho_i^j \in \mathbb{R}_+ \)-arcwise connected, respectively, on \( A \), with respect to 1, we have
\[ x^T B_i x_i - x'^T B_i x_i - \rho_i^j \|x - x'\|^2 \geq (B_i x_i)^T (H'_{x,x}(0+)) \]
and
\[ G(x, u_j) - z_i^j - \rho_i^j \|x - x'\|^2 \subseteq D_i G(., u_j)(x', z'_i)(H'_{x,x}(0+)) + \mathbb{R}_+. \]
As \( x \in S \), there exists \( z_j \in G(x, u_j) \cap (-\mathbb{R}_+), j \in J(x') \). So, we have
\[ z_j - z'_j - \rho_i^j \|x - x'\|^2 \in D_i G(., u_j)(x', z'_i)(H'_{x,x}(0+)) + \mathbb{R}_+. \]

Therefore,
\[ y_i^*(x^T B_i x_i - x'^T B_i x_i) - \rho_i^j \|x - x'\|^2 y_i^* \geq y_i^* (B_i x_i)^T (H'_{x,x}(0+)) \] (8)
and
\[ z_j^*(z_j - z'_j) - \rho_i^j \|x - x'\|^2 z_j^* \geq z_j^* D_i G(., u_j)(x', z'_i)(H'_{x,x}(0+)). \] (9)

From (7)–(9), we have
\[
\sum_{i=1}^{m} y_i^*(y_i - y'_i + x^T B_i x_i - x'^T B_i x_i) + \sum_{j \in J(x')} z_j^* (z_j - z'_j) \\
- \|x - x'\|^2 \sum_{i=1}^{m} y_i^*(\rho_i + \rho'_i) - \|x - x'\|^2 \sum_{j \in J(x')} z_j^* \rho_i^j \\
\geq \left( \sum_{i=1}^{m} y_i^*(D_i F_i(x', y'_i) + (B_i x_i)^T) + \sum_{j \in J(x')} z_j^* D_i G(., u_j)(x', z'_i) \right)(H'_{x,x}(0+)) \geq 0.
\]

From (1), we have
\[
\sum_{i=1}^{m} y_i^*(y_i - y'_i + x^T B_i x_i - x'^T B_i x_i) + \sum_{j \in J(x')} z_j^* (z_j - z'_j) \\
\geq \|x - x'\|^2 \left( \sum_{i=1}^{m} y_i^*(\rho_i + \rho'_i) + \sum_{j \in J(x')} z_j^* \rho_i^j \right) \\
\geq 0.
\]

As \( z_j^* z'_j = 0 \) and \( z_j \in (-\mathbb{R}_+), \forall j \in J(x') \), we have
\[ \sum_{j \in J(x')} z_j^* (z_j - z'_j) \leq 0. \]

Hence,
\[ \sum_{i=1}^{m} y_i^*(y_i - y'_i + x^T B_i x_i - x'^T B_i x_i) \geq 0. \]

Using the generalized Schwarz inequality, we have
\[ (x^T B_i x)^{\frac{1}{2}} (x_i^T B_i x_i)^{\frac{1}{2}} \geq x^T B_i x_i. \]
Again, from (5), we have
\[
(x^TB_i'x')^{\frac{1}{2}} = x^TB_i\bar{x}_i, i = 1, 2, \ldots, m.
\]
Therefore,
\[
\sum_{i=1}^{m} y_i^*(y_i - y_i') + (x^TB_ix)^{\frac{1}{2}}(\bar{x}_i^T B_i \bar{x}_i)^{\frac{1}{2}} - (x^TB_i'x')^{\frac{1}{2}} \geq 0.
\]
From (4), we have \(\bar{x}_i^TB_i\bar{x}_i \leq 1, i = 1, 2, \ldots, m\). So, we have
\[
(x^TB_i)x^{\frac{1}{2}} \geq (x^TB_i\bar{x})^{\frac{1}{2}}(\bar{x}_i^T B_i \bar{x}_i)^{\frac{1}{2}}.
\]
Hence,
\[
\sum_{i=1}^{m} y_i^*(y_i + (x^TB_ix)^{\frac{1}{2}}) \geq \sum_{i=1}^{m} y_i^*(y_i' + (x^TB_i'x')^{\frac{1}{2}}),
\]
which contradicts (6). Hence, \((x', y')\) is a weak minimizer of (SP).

\[\square\]

6. Mond–Weir Type Dual

We consider the Mond–Weir type dual (MWD) of the problem (SP), where \(F_i\) and \(G(., u_j)\) are contingent epiderivable set-valued maps:

\[
\text{maximize} \quad (y'_1 + (x^TB_1x_1), y'_2 + (x^TB_2x_2), \ldots, y'_m + (x^TB_m\bar{x}_m))
\]
\text{subject to}
\[
\left(\sum_{i=1}^{m} y_i^*(D_iF_i(x', y_i') + (B_i\bar{x}_i)^T) + \sum_{j \in I(x')} z_j^* D_i G(., u_j)(x', z_j') (H'_{x', y}(0+))\right) \geq 0, \forall x \in A,
\]
\[
\sum_{j \in I(x')} z_j^* z_j' \geq 0,
\]
\[
\bar{x}_i^T B_i \bar{x}_i \leq 1, \bar{x}_i \in \mathbb{R}^n, i = 1, 2, \ldots, m,
\]
\[
x' \in A, y' = (y'_1, y'_2, \ldots, y'_m) \in F(x'), z' = (z'_j)_{j \in J}, z'_j \in G(x', u_j), j \in J,
\]
\[
y'_i > 0, i = 1, 2, \ldots, m, \sum_{i=1}^{m} y'_i = 1, z^* = (z'_j)_{j \in J}, z'_j \geq 0, j \in J,
\]
and \(z'_j \neq 0\), for finitely many \(j \in J\).

**Definition 10.** A feasible point \((x', y', \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_m, z', y^*, z^*)\) of (MWD) is said to be a weak maximizer of (MWD) if for all feasible points \((x, y, \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_m, z, y'_1, z'_1)\) of (MWD),
\[
(y'_1 + (x^TB_1\bar{x}_1), y'_2 + (x^TB_2\bar{x}_2), \ldots, y'_m + (x^TB_m\bar{x}_m))
\]
\[
\preceq (y_1 + (x^TB_1\bar{x}_1), y_2 + (x^TB_2\bar{x}_2), \ldots, y_m + (x^TB_m\bar{x}_m)),
\]
where \(y = (y_1, y_2, \ldots, y_m), y' = (y'_1, y'_2, \ldots, y'_m) \in \mathbb{R}^m\).

**Theorem 3** (Weak duality). Let \(A\) be an arcwise connected subset of \(\mathbb{R}^n\) and \(x_0\) be an element of the feasible set \(S\) of (SP). Let \((x', y', \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_m, z', y^*, z^*)\) be a feasible solution of (MWD). Let \(\rho_i, \rho'_i \in \mathbb{R}, \rho_i^\prime \in \mathbb{R}_+\), for \(i = 1, 2, \ldots, m\) and \(j \in I(x')\). Suppose that \(F_i^\prime, i = 1, 2, \ldots, m\), and \(G(., u_j), j \in I(x')\), are \(\rho_i^\prime\)-\(\mathbb{R}_+\)-arcwise connected, \(\rho'_i\)-\(\mathbb{R}_+\)-arcwise connected, and \(\rho''_i\)-\(\mathbb{R}_+\)-arcwise
connected set-valued maps, respectively, on \( A \), with respect to 1, satisfying (5.1). Further, we suppose that the contingent epiderivatives \( D_1 \mathcal{F}_i(x', y') \) and \( D_1 \mathcal{G}(., u_j)(x', z'_j) \) exist. Then,

\[
(\mathcal{F}_1(x_0) + (x_0^T B_1 x_0)^{\frac{1}{2}}) \mathcal{F}_2(x_0) + (x_0^T B_2 x_0)^{\frac{1}{2}}, \ldots, \mathcal{F}_m(x_0) + (x_0^T B_m x_0)^{\frac{1}{2}}
\]

\[- (y'_1 + (x^T B_1 x_1), y'_2 + (x^T B_2 x_2), \ldots, y'_m + (x^T B_m x_m)) \subseteq \mathbb{R}^m \setminus (-\text{int}(\mathbb{R}_+^m)).
\]

**Proof.** Assume that there is point \( y_i \in \mathcal{F}_i(x_0), i = 1, 2, \ldots, m \), such that

\[
(y_1 + (x_0^T B_1 x_0)^{\frac{1}{2}}, y_2 + (x_0^T B_2 x_0)^{\frac{1}{2}}, \ldots, y_m + (x_0^T B_m x_0)^{\frac{1}{2}}) \mathcal{F}_i(x_0) - (y'_1 + (x^T B_1 x_1), y'_2 + (x^T B_2 x_2), \ldots, y'_m + (x^T B_m x_m)) \subseteq (-\text{int}(\mathbb{R}_+^m)).
\]

As \( y^* \in \mathbb{R}_+^m \setminus \{\theta_{\mathbb{R}_+}\} \), we have

\[
\sum_{i=1}^m y_i' (y_i + (x_0^T B_i x_0)^{\frac{1}{2}}) < \sum_{i=1}^m y_i' (y_i' + (x^T B_i x_i)).
\]

As \( \mathcal{F}_i, i = 1, 2, \ldots, m \), is \( \rho_i^{(-\mathbb{R}_+)} \)-arcwise connected on \( A \) with respect to 1 and \( (x', y') \in \text{gr}(\mathcal{F}_i) \), we have

\[
\mathcal{F}_i(x_0) - y'_i - \rho_i ||x_0 - x'||^2 \subseteq D_1 \mathcal{F}_i(x', y')(H'_{x', y_0}(0^+)) + \mathbb{R}_+
\]

Hence,

\[
y_i - y'_i - \rho_i ||x_0 - x'||^2 \in D_1 \mathcal{F}_i(x', y')(H'_{x', y_0}(0^+)) + \mathbb{R}_+.
\]

Therefore,

\[
y_i' (y_i - y'_i) - \rho_i ||x_0 - x'||^2 y_i^* \geq y_i' D_1 \mathcal{F}_i(x', y')(H'_{x', y_0}(0^+)).
\]

As \( \mathcal{T} B_i \xi_i, i = 1, 2, \ldots, m \), is \( \rho_i^{(-\mathbb{R}_+)} \)-arcwise connected on \( A \) with respect to 1, we have

\[
x_0^T B_i x_i - x'^T B_i x_i - \rho_i ||x_0 - x'||^2 \geq (B_i \xi_i)^T (H'_{x', y_0}(0^+)).
\]

Therefore,

\[
y_i' (x_0^T B_i x_i - x'^T B_i x_i) - \rho_i ||x_0 - x'||^2 y_i^* \geq y_i' (B_i \xi_i)^T (H'_{x', y_0}(0^+)).
\]

Again, as \( \mathcal{G}(., u_j), j \in f(x') \), is \( \rho_j^{(-\mathbb{R}_+)} \)-arcwise connected on \( A \) with respect to 1 and \( z'_j \in \mathcal{G}(x', u_j), j \in f(x') \), we have

\[
\mathcal{G}(x_0, u_j) - z'_j - \rho_j^{(-\mathbb{R}_+)} ||x_0 - x'||^2 \subseteq D_1 \mathcal{G}(., u_j)(x', z'_j)(H'_{x', y_0}(0^+)) + \mathbb{R}_+.
\]

As \( x_0 \in S \), there exists \( z_j \in \mathcal{G}(x_0, u_j) \cap (-\mathbb{R}_+), j \in f(x') \), we have

\[
z_j - z'_j - \rho_j^{(-\mathbb{R}_+)} ||x_0 - x'||^2 \in D_1 \mathcal{G}(., u_j)(x', z'_j)(H'_{x', y_0}(0^+)) + \mathbb{R}_+.
\]

Hence,

\[
z_j^* (z_j - z'_j) - \rho_j^{(-\mathbb{R}_+)} ||x_0 - x'||^2 z_j^* \geq z_j^* D_1 \mathcal{G}(., u_j)(x', z'_j)(H'_{x', y_0}(0^+)).
\]
From (11)–(13), we have
\[
\sum_{i=1}^{m} y_i' (y_i - y_i') + x_0^T B_i x_i - x'^T B_i x_i) + \sum_{j \in J(x')} z_j^* (z_j - z_j') \\
- \|x_0 - x'\|^2 \sum_{i=1}^{m} y_i' (\rho_i + \rho_i') - \|x_0 - x'\|^2 \sum_{j \in J(x')} z_j^* \rho_j'' \\
\geq \left( \sum_{i=1}^{m} y_i' (D_i F_i(x', y_i')) + \sum_{j \in J(x')} z_j^* D_j G_i (u_j) (x', z_j')) \right) (H_{x', x_0} (0+)) \\
\geq 0.
\]

From (1), we have
\[
\sum_{i=1}^{m} y_i' (y_i - y_i') + x_0^T B_i x_i - x'^T B_i x_i) + \sum_{j \in J(x')} z_j^* (z_j - z_j') \\
\geq \|x_0 - x'\|^2 \left( \sum_{i=1}^{m} y_i' (\rho_i + \rho_i') + \sum_{j \in J(x')} z_j^* \rho_j'' \right) \\
\geq 0.
\]

As \( \sum_{j \in J(x')} z_j^* z_j' \geq 0, z_j^* \geq 0, \) and \( z_j \in (-\mathbb{R}_+), j \in J(x'), \) we have
\[
\sum_{j \in J(x')} z_j^* (z_j - z_j') \leq 0.
\]

Hence,
\[
\sum_{i=1}^{m} y_i' (y_i - y_i') + x_0^T B_i x_i - x'^T B_i x_i) \geq 0.
\]

Using the generalized Schwarz inequality, we have
\[
(x_0^T B_i x_0)^{\frac{1}{2}} (x_i^T B_i x_i)^{\frac{1}{2}} \geq x_0^T B_i x_i.
\]

Again, from the constraints of (MWD), we have
\[
\overline{x}_i^T B_i x_i \leq 1, i = 1, 2, \ldots, m.
\]

Hence, \( (x_0^T B_i x_0)^{\frac{1}{2}} \geq x_0^T B_i x_i. \) It shows that
\[
\sum_{i=1}^{m} y_i' (y_i - (x_0^T B_i x_0)^{\frac{1}{2}}) \geq \sum_{i=1}^{m} y_i' (y_i' + (x'^T B_i x_i)),
\]

which contradicts (10). Therefore,
\[
(y_1 + (x_0^T B_1 x_0)^{\frac{1}{2}}, y_2 + (x_0^T B_2 x_0)^{\frac{1}{2}}, \ldots, y_m + (x_0^T B_m x_0)^{\frac{1}{2}}) \\
- (y_1' + (x'^T B_1 x_1), y_2' + (x'^T B_2 x_2), \ldots, y_m' + (x'^T B_m x_m)) \notin (-\mathbb{R}_+^m).
\]

Hence,
\[
(F_1(x_0) + (x_0^T B_1 x_0)^{\frac{1}{2}}, F_2(x_0) + (x_0^T B_2 x_0)^{\frac{1}{2}}, \ldots, F_m(x_0) + (x_0^T B_m x_0)^{\frac{1}{2}}) \\
- (y_1' + (x'^T B_1 x_1), y_2' + (x'^T B_2 x_2), \ldots, y_m' + (x'^T B_m x_m)) \subseteq \mathbb{R}^m \setminus (-\mathbb{R}_+^m).
\]

It completes the proof of the theorem. \( \square \)
Theorem 4 (Strong duality). Suppose that \((x', y')\) is a weak minimizer of (SP), \(z' = (z'_j)_{j \in J}\), \(z'_j \in G(x', u_j) \cap (-\mathbb{R}_+), j \in J\), and \(\mathbf{1} \in \mathbb{R}^n, i = 1, 2, \ldots, m\). Suppose that, for some \(y^*_i > 0\), \(i = 1, 2, \ldots, m\), with \(\sum_{i=1}^m y^*_i = 1\) and \(z^*_j \geq 0\), with \(z^*_j \neq 0\) for finitely many \(j \in J\) and \(z^*_j = 0\), \(\forall j \in J \setminus J(x')\). Equations (5.2)–(5.5) are satisfied at the point \((x', y', x_1, x_2, \ldots, x_n, z', y^*, z^*)\). Then, \((x', y', x_1, x_2, \ldots, x_n, z', y^*, z^*)\) is a feasible solution of (MWD). If the Theorem 6.1 holds between the problems (SP) and (MWD), then \((x', y', x_1, x_2, \ldots, x_n, z', y^*, z^*)\) is a weak maximizer of (MWD).

Proof. As (2)–(5) are satisfied at the point \((x', y', x_1, x_2, \ldots, x_n, z', y^*, z^*)\), we have

\[
\left( \sum_{i=1}^m y^*_i \left( D_i F_i(x', y'_i) + (B_i x)_T \right) + \sum_{j \in J(x')} z^*_j \left( D_j G.(u_j)(x', z'_j) \right) \right) (H'_{x', x}(0+)) \geq 0, \forall x \in A,
\]

and

\[
(x^T B_j x')^{\frac{1}{2}} = x^T B_j x_i, i = 1, 2, \ldots, m.
\]

Hence, \((x', y', x_1, x_2, \ldots, x_n, z', y^*, z^*)\) is a feasible solution of (MWD). Assume that the weak duality Theorem 3 holds between the problems (SP) and (MWD). Let \((x', y', x_1, x_2, \ldots, x_n, z', y^*, z^*)\) be not a weak maximizer of (MWD). Then, there exists a feasible solution \((x, y, x_1, x_2, \ldots, x_n, z, y^*, z^*)\) of (MWD), such that

\[
\begin{align*}
(y'_1 + (x^T B_1 x_1), y'_2 + (x^T B_2 x_2), \ldots, y'_m + (x^T B_m x_n)) &- (y_1 + (x^T B_1 x_1), y_2 + (x^T B_2 x_2), \ldots, y_m + (x^T B_m x_n)) \\
&\in (- \text{int} \mathbb{R}_m^+) \]
\end{align*}
\]

where \(y = (y_1, \ldots, y_m) \in \mathbb{R}^m\). As \((x^T B_j x')^{\frac{1}{2}} = x^T B_j x_i, i = 1, 2, \ldots, m\), we have

\[
\begin{align*}
(y'_1 + (x^T B_1 x'), y'_2 + (x^T B_2 x'), \ldots, y'_m + (x^T B_m x')) &- (y_1 + (x^T B_1 x), y_2 + (x^T B_2 x), \ldots, y_m + (x^T B_m x_n)) \\
&\in (- \text{int} \mathbb{R}_m^+) \]
\end{align*}
\]

which contradicts the Theorem 3 between (SP) and (MWD). Hence, \((x', y', x_1, x_2, \ldots, x_n, z', y^*, z^*)\) is a weak maximizer of (MWD).

Theorem 5 (Converse duality). Let \(A\) be an arcwise connected subset of \(\mathbb{R}^n\). Suppose that \((x', y', x_1, x_2, \ldots, x_n, z', y^*, z^*)\) is a feasible solution of (MWD), with

\[
(x^T B_j x')^{\frac{1}{2}} = x^T B_j x_i, i = 1, 2, \ldots, m.
\]

Let \(\rho_i, \rho'_i, \rho''_i \in \mathbb{R}_+\) for \(i = 1, 2, \ldots, m\) and \(j \in J(x')\). Suppose that \(F_i, T B_j x_i, i = 1, 2, \ldots, m\), and \(G.(u_j), \in J(x')\), are \(\rho_i \mathbb{R}_+\)-arcwise connected, \(\rho'_i \mathbb{R}_+\)-arcwise connected, and \(\rho''_i \mathbb{R}_+\)-arcwise connected set-valued maps, respectively, on \(A\), with respect to \(1\), satisfying (1). We suppose that the contingent epiderivatives \(D_i F_i(x', y'_i)\) and \(D_j G.(u_j)(x', z'_j)\) exist. If \(x' \in S\), then \((x', y')\) is a weak minimizer of (SP).

Proof. Let \((x', y')\) be not a weak minimizer of (SP). Then, \(x \in S\) and \(y = (y_1, y_2, \ldots, y_m) \in F(x)\) exist, such that

\[
\begin{align*}
(y_1 + (x^T B_1 x)^{\frac{1}{2}}, y_2 + (x^T B_2 x)^{\frac{1}{2}}, \ldots, y_m + (x^T B_m x)^{\frac{1}{2}}) &- (y'_1 + (x^T B_1 x')^{\frac{1}{2}}, y'_2 + (x^T B_2 x')^{\frac{1}{2}}, \ldots, y'_m + (x^T B_m x')^{\frac{1}{2}}) \\
&\in (- \text{int} \mathbb{R}_m^+)
\end{align*}
\]
As \( y^* \in \mathbb{R}_+^m \setminus \{ \theta \} \), we have
\[
\sum_{i=1}^{m} y_i^* (y_i + (x^T B_i x)^{\frac{1}{2}}) < \sum_{i=1}^{m} y_i^* (x^T B_i x')^{\frac{1}{2}}.
\] (14)

Since \( F_i \), \( i = 1, 2, \ldots, m \), is \( \rho_i, \mathbb{R}_+ \)-arcwise connected on \( A \) with respect to 1 and \((x', y') \in \text{gr}(F_i)\), we have
\[
F_i(x) - y_i' - \rho_i \| x - x' \|^2 \subseteq D_i F_i(x', y_i') (H_{x',x}'(0+)) + \mathbb{R}_+.
\]

Hence,
\[
y_i - y_i' - \rho_i \| x - x' \|^2 \in D_i F_i(x', y_i') (H_{x',x}'(0+)) + \mathbb{R}_+.
\]

Therefore,
\[
y_i^* (y_i - y_i') - \rho_i \| x - x' \|^2 y_i^* \geq y_i^* D_i F_i(x', y_i') (H_{x',x}'(0+)).
\] (15)

Again, as \( .^T B_i \mathbb{R}_+, i = 1, 2, \ldots, m \), and \( \mathcal{G}(., u_i), j \in J(x') \), are \( \rho_i, \mathbb{R}_+ \)-arcwise connected and \( \rho_i' - \mathbb{R}_+ \)-arcwise connected, respectively, on \( A \), with respect to 1, we have
\[
x^T B_i x_i - x'^T B_i x_i - \rho_i' \| x - x' \|^2 \geq (B_i x_i)^T (H_{x,x}'(0+))
\]
and
\[
\mathcal{G}(x, u_j) - z_j' - \rho_j'' \| x - x' \|^2 \subseteq D_j \mathcal{G}(., u_j)(x', z_j')(H_{x',x}'(0+)) + \mathbb{R}_+.
\]

As \( x \in S \), there exists \( z_j \in \mathcal{G}(x, u_j) \cap (-\mathbb{R}_+), \forall j \in J(x') \). So,
\[
|z_j - z_j' - \rho_j'' \| x - x' \|^2 \in D_j \mathcal{G}(., u_j)(x', z_j')(H_{x',x}'(0+)) + \mathbb{R}_+.
\]

Therefore,
\[
y_i^* (x^T B_i x_i - x'^T B_i x_i) - \rho_i' \| x - x' \|^2 y_i^* \geq y_i^* (B_i x_i)^T (H_{x,x}'(0+))
\] (16)

and
\[
z_j^* (z_j - z_j') - \rho_j'' \| x - x' \|^2 z_j^* \geq z_j^* D_j \mathcal{G}(., u_j)(x', z_j')(H_{x',x}'(0+)).
\] (17)

From (15)–(17), we have
\[
\sum_{i=1}^{m} y_i^* (y_i - y_i' + x^T B_i x_i - x'^T B_i x_i) + \sum_{j \in J(x')} z_j^* (z_j - z_j') \\
- \| x - x' \|^2 \sum_{i=1}^{m} y_i^* (\rho_i + \rho_i') - \| x - x' \|^2 \sum_{j \in J(x')} z_j^* \rho_j'' \\
\geq \left( \sum_{i=1}^{m} y_i^* (D_i F_i(x', y_i') + (B_i x_i)^T) + \sum_{j \in J(x')} z_j^* D_j \mathcal{G}(., u_j)(x', z_j')(H_{x',x}'(0+)) \right) \geq 0.
\]

By (1), we have
\[
\sum_{i=1}^{m} y_i^* (y_i - y_i' + x^T B_i x_i - x'^T B_i x_i) + \sum_{j \in J(x')} z_j^* (z_j - z_j') \\
\geq \| x - x' \|^2 \left( \sum_{i=1}^{m} y_i^* (\rho_i + \rho_i') + \sum_{j \in J(x')} z_j^* \rho_j'' \right) \\
\geq 0.
\]
As \( \sum_{j \in I(x')} z^*_j z'_j \geq 0, z^*_j \geq 0, \) and \( z_j \in (-\mathbb{R}_+), j \in I(x'), \) we have
\[
\sum_{j \in I(x')} z^*_j (z_j - z'_j) \leq 0.
\]
Hence,
\[
\sum_{i=1}^{m} y^*_i (y_i - y'_i + x^T B_i \bar{x}_i - x^T B_i \bar{x}_i) \geq 0.
\]
From the generalized Schwarz inequality, we have
\[
(x^T B_i x)^{\frac{1}{2}} (x^T B_i \bar{x}_i)^{\frac{1}{2}} \geq x^T B_i \bar{x}_i.
\]
Again, by assumption, we have
\[
(x^T B_i x)^{\frac{1}{2}} = x^T B_i \bar{x}_i, i = 1, 2, \ldots, m.
\]
Therefore,
\[
\sum_{i=1}^{m} y^*_i (y_i - y'_i + (x^T B_i x)^{\frac{1}{2}} (x^T B_i \bar{x}_i)^{\frac{1}{2}} - (x^T B_i x')^{\frac{1}{2}}) \geq 0.
\]
As \( x^T B_i \bar{x}_i \leq 1, i = 1, 2, \ldots, m, \) (from the constraints of (MWD)), we have
\[
(x^T B_i x)^{\frac{1}{2}} \geq (x^T B_i x)^{\frac{1}{2}} (x^T B_i \bar{x}_i)^{\frac{1}{2}}.
\]
Hence,
\[
\sum_{i=1}^{m} y^*_i (y_i + (x^T B_i x)^{\frac{1}{2}}) \geq \sum_{i=1}^{m} y^*_i (y'_i + (x^T B_i x')^{\frac{1}{2}}),
\]
which contradicts (14). So, \((x', y')\) is a weak minimizer of (SP). \( \square \)

7. Wolfe Type Dual

We consider the Wolfe type dual (WD) of the problem (SP), where \( \mathcal{F}_i \) and \( \mathcal{G}(., u_j) \) are contingent epiderivable set-valued maps:

maximize \( (y'_1 + (x^T B_1 \bar{x}_1), y'_2 + (x^T B_2 \bar{x}_2), \ldots, y'_m + (x^T B_m \bar{x}_m)) + \left( \sum_{j \in I(x')} z^*_j z'_j \right) 1_{\mathbb{R}^m} \)
subject to
\[
\left( \sum_{i=1}^{m} y^*_i (D_i \mathcal{F}_i(x', y'_i) + (B_i \bar{x}_i)^T) + \sum_{j \in I(x')} z^*_j D_i \mathcal{G}(., u_j)(x', z'_j) \right) (H'_{x', x}(0+)) \\
\geq 0, \forall x \in A, \\
x^T B_i \bar{x}_i \leq 1, \bar{x}_i \in \mathbb{R}^n, i = 1, 2, \ldots, m, \\
x' \in A, y' = (y'_1, y'_2, \ldots, y'_m) \in \mathcal{F}(x'), z' = (z'_j)_{j \in I(x')}, z'_j \in \mathcal{G}(x', u_j), j \in J, \\
y^*_i > 0, i = 1, 2, \ldots, m, \sum_{i=1}^{m} y^*_i = 1, \text{ and } z^* = (z^*_j)_{j \in I}, z^*_j \geq 0, j \in I, \\
\text{and } z^*_j \neq 0, \text{ for finitely many } j \in I.
Definition 11. A feasible point $(x', y', x_1, x_2, \ldots, x_m, z', y^*, z^*)$ of (WD) is said to be a weak maximizer of (WD) if for all feasible points $(x, y, x_1, x_2, \ldots, x_m, z, y_1^*, z_1^*)$ of (WD),

$$
(y_1' + (x^T B_1 x_1), y_2' + (x^T B_2 x_2), \ldots, y_m' + (x^T B_m x_m)) + \left( \sum_{j \in J(x')} z_j^* z_j' \right) I_{R^m}
$$

$$
\not< (y_1 + (x^T B_1 x_1), y_2 + (x^T B_2 x_2), \ldots, y_m + (x^T B_m x_m)) + \left( \sum_{j \in J(x')} z_j^* z_j' \right) I_{R^m},
$$

where $y = (y_1, y_2, \ldots, y_m), y' = (y_1', y_2', \ldots, y_m') \in \mathbb{R}^m$.

We can prove the duality theorems of Wolfe type associated with the problem (SP). The proofs are very similar to Theorems 3–5, and hence we omit it.

Theorem 6 (Weak duality). Let $A$ be an arcwise connected subset of $\mathbb{R}^n$ and $x_0 \in S$. Let $(x', y', x_1, x_2, \ldots, x_m, z', y^*, z^*)$ be a feasible solution of (WD). Let $\rho_i, \rho_i', \rho_i'' \in \mathbb{R}$, for $i = 1, 2, \ldots, m$ and $j \in J(x')$. Suppose that $F_i, T B_i, i = 1, 2, \ldots, m$, and $G(., u_j), j \in J(x')$, are $\rho_i'$-arcwise connected, $\rho_i''$-arcwise connected and $\rho_i$-arcwise connected set-valued maps, respectively, on $A$, with respect to 1, satisfying (1). Suppose that the contingent epiderivatives $D_1 F_i (x', y_i')$ and $D_1 G(., u_j)(x', z_i')$ exist. Then,

$$
(F_1(x_0) + (x_1^T B_1 x_1)^{\frac{1}{2}}, F_2(x_0) + (x_2^T B_2 x_2)^{\frac{1}{2}}, \ldots, F_m(x_0) + (x_m^T B_m x_m)^{\frac{1}{2}})
$$

$$
\not< (y_1' + (x^T B_1 x_1), y_2' + (x^T B_2 x_2), \ldots, y_m' + (x^T B_m x_m)) + \left( \sum_{j \in J(x')} z_j^* z_j' \right) I_{R^m}.
$$

Theorem 7 (Strong duality). Suppose that $(x', y')$ is a weak minimizer of (SP), $z' = (z_j')_{j \in J}$, $z_j' \in G(x', u_j) \cap (-\mathbb{R}_+), j \in J$, and $x_i \in \mathbb{R}^n, i = 1, 2, \ldots, m$. Suppose that for some $y_i^* > 0$, $i = 1, 2, \ldots, m$, with $\sum_{i=1}^{m} y_i^* = 1$ and $z_j^* \geq 0$, with $z_j^* \neq 0$ for finitely many $j \in J$ and $z_j^* = 0, \forall j \in J \setminus J(x')$, Equations (2)–(5) are satisfied at the point $(x', y', x_1, x_2, \ldots, x_m, z', y', z^*)$. Then, $(x', y', x_1, x_2, \ldots, x_m, z', y', z^*)$ is a feasible solution of (WD). If the Theorem 6 holds between the problems (SP) and (WD), then $(x', y', x_1, x_2, \ldots, x_m, z', y^*, z^*)$ is a weak maximizer of (WD).

Theorem 8 (Converse duality). Let $A$ be an arcwise connected subset of $\mathbb{R}^n$. Suppose that $(x', y', x_1, x_2, \ldots, x_m, z', y^*, z^*)$ is a feasible solution of (WD), with

$$
(x^T B_i x_i)^{\frac{1}{2}} = x_i^T B_i x_i, i = 1, 2, \ldots, m
$$

and

$$
\sum_{j \in J(x')} z_j^* z_j' \geq 0.
$$

Let $\rho_i, \rho_i', \rho_i'' \in \mathbb{R}$, for $i = 1, 2, \ldots, m$ and $j \in J(x')$. Suppose that $F_i, T B_i, i = 1, 2, \ldots, m$, and $G(., u_j), j \in J(x')$, are $\rho_i'$-arcwise connected, $\rho_i''$-arcwise connected, and $\rho_i$-arcwise connected set-valued maps, respectively, on $A$, with respect to 1, satisfying (1). We suppose that the contingent epiderivatives $D_1 F_i (x', y_i')$ and $D_1 G(., u_j)(x', z_i')$ exist. If $x^* \in S$, then $(x', y')$ is a weak minimizer of (SP).
8. Mixed Type Dual

We consider the mixed type dual (MD) of the problem (SP), where $\mathcal{F}_i$ and $G(\cdot, u_i)$ are contingent epiderivatives set-valued maps:

maximize \( (y_1 + (x^T B_1 \bar{x}_1), y_2 + (x^T B_2 \bar{x}_2), \ldots, y_m + (x^T B_m \bar{x}_m)) + \left( \sum_{j \in J(x')} z_j^* z_j' \right) 1_{\mathbb{R}^m} \)

subject to

\[
\left( \sum_{i=1}^m y_i' (D_1 \mathcal{F}_i(x', y_i') + (B_i \bar{x}_i)^T) + \sum_{j \in J(x')} z_j^* D_1 G(\cdot, u_j)(x', z_j') \right) (H_{x',x}(0+)) \geq 0, \forall x \in A, \\
z_j^* z_j' \geq 0, \forall j \in J(x'), \\
x_i^T B_i \bar{x}_i \leq 1, x_i \in \mathbb{R}^n, i = 1, 2, \ldots, m, \\
x' \in A, y_i' = (y_1', y_2', \ldots, y_m') \in \mathcal{F}(x'), z_j' = (z_j')_{i \in J, j \in J(x')}, j \in J, \\
y_i^* > 0, i = 1, 2, \ldots, m, \sum_{i=1}^m y_i^* = 1, \text{ and } z^* = (z_j^*)_{i \in J, z_j^* \geq 0, j \in J,} \\
\text{and } z_j^* \neq 0, \text{ for finitely many } j \in J.
\]

Definition 12. A feasible point \((x', y', \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_m, z', y^*, z^*)\) of (MD) is said to be a weak maximizer of (MD) if for all feasible points \((x, y, \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_m, z, y', z'_1)\) of (MD),

\[
(y_1 + (x^T B_1 \bar{x}_1), y_2 + (x^T B_2 \bar{x}_2), \ldots, y_m + (x^T B_m \bar{x}_m)) + \left( \sum_{j \in J(x')} z_j^* z_j' \right) 1_{\mathbb{R}^m} \\
\preceq (y_1 + (x^T B_1 \bar{x}_1), y_2 + (x^T B_2 \bar{x}_2), \ldots, y_m + (x^T B_m \bar{x}_m)) + \left( \sum_{j \in J(x')} z_j^* z_j' \right) 1_{\mathbb{R}^m},
\]

where \(y = (y_1, y_2, \ldots, y_m), y' = (y_1', y_2', \ldots, y_m') \in \mathbb{R}^m\).

We develop the duality results of mixed type of the problem (SP). The proofs are very similar to Theorems 3–5, and hence we omit it.

Theorem 9 (Weak duality). Let \(A\) be an arcwise connected subset of \(\mathbb{R}^n\) and \(x_0 \in S\). Let \((x', y', \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_m, z', y^*, z^*)\) be a feasible solution of (MD). Let \(\rho_i, \rho_i', \rho''_i \in \mathbb{R}, \text{ for } i = 1, 2, \ldots, m \text{ and } j \in J(x')\). Suppose that \(\mathcal{F}_i, \mathcal{F}_i', \mathcal{F}_i'' \text{ and } G(\cdot, u_j), j \in J(x')\), are \(\rho_i, \mathbb{R}^+\)-arcwise connected, \(\rho_i', \mathbb{R}^+\)-arcwise connected, and \(\rho''_i, \mathbb{R}^+\)-arcwise connected set-valued maps, respectively, on \(A\), with respect to 1, satisfying (1). Let the contingent epiderivatives \(D_1 \mathcal{F}_i(x', y_i')\) and \(D_1 \mathcal{G}(\cdot, u_j)(x', z_j')\) exist. Then,

\[
(\mathcal{F}_1(x_0) + (x_1^T B_1 x_0)^{\frac{1}{2}}, \mathcal{F}_2(x_0) + (x_2^T B_2 x_0)^{\frac{1}{2}}, \ldots, \mathcal{F}_m(x_0) + (x_m^T B_m x_0)^{\frac{1}{2}}) \\
\preceq (y_1' + (x^T B_1 \bar{x}_1), y_2' + (x^T B_2 \bar{x}_2), \ldots, y_m' + (x^T B_m \bar{x}_m)) + \left( \sum_{j \in J(x')} z_j^* z_j' \right) 1_{\mathbb{R}^m}.
\]

Theorem 10 (Strong duality). Suppose that \((x', y')\) is a weak minimizer of (SP), \(z' = (z_j')_{i \in J}\), \(z_j^* \in \mathcal{G}(x', u_j) \cap (-\mathbb{R}^+), j \in J\), and \(\bar{x}_i \in \mathbb{R}^n, i = 1, 2, \ldots, m\). Suppose that for some \(y_i^* > 0, \text{ for finitely many } j \in J \text{ and } z_j^* = 0, \forall j \notin J(x')\). Equations (2)–(5) are satisfied at the point \((x', y', \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_m, z', y^*, z^*)\). Then, \((x', y', \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_m, z', y^*, z^*)\) is a feasible solution of (MD). If the Theorem 9 holds between the problems (SP) and (MD), then

\[
(x', y', \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_m, z', y^*, z^*)
\]
is a weak maximizer of (MD).

Theorem 11 (Converse duality). Let A be an arcwise connected subset of \( \mathbb{R}^n \). Suppose that 
\( (x', y', x_1, x_2, \ldots, x_m, z', y', z^+) \) is a feasible solution of (MD), with 
\( (x^T B_i x')^2 = x^T B_i x_i \)
for \( i = 1, 2, \ldots, m \). Let \( \rho_i, \rho'_i, \rho''_i \in \mathbb{R}_+ \), \( i = 1, 2, \ldots, m \), and \( j \in J(x') \). Suppose that 
\( F_i \), \( G(., u_j) \), \( j \in J(x') \), are \( \rho_i-\mathbb{R}_+\)-arcwise connected, \( \rho'_i-\mathbb{R}_+\)-arcwise connected, 
and \( \rho''_i-\mathbb{R}_+\)-arcwise connected set-valued maps, respectively, on A, with respect to 1, satisfying (1). 
We suppose that the contingent epiderivatives \( D_1 F_i(x', y'_j) \) and \( D_1 G(., u_j)(x', z'_j) \) exist. If \( x' \in S \),
then \( (x', y') \) is a weak minimizer of (SP).

9. Conclusions

In this paper, we have established the sufficient KKT optimality conditions of a set-valued semi-infinite programming problem (SP) under \( \rho \)-cone arcwise connectedness assumptions. We studied the duality theorems of Mond–Weir, Wolfe, and mixed types associated with the problem (SP).

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