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Vector-Valued Entire Functions of Several Variables: Some Local Properties

Andriy Ivanovych Bandura [1,*], Tetyana Mykhailivna Salo [2] and Oleh Bohdanovych Skaskiv [3]

[1] Department of Advanced Mathematics, Ivano-Frankivsk National Technical Unuversity of Oil and Gas, 76019 Ivano-Frankivsk, Ukraine
[2] Department of Mathematics, Lviv Polytechnic National University, 79000 Lviv, Ukraine; tetyan.salo@gmail.com
[3] Department of Mechanics and Mathematics, Ivan Franko National University of Lviv, 79000 Lviv, Ukraine; olskask@gmail.com
* Correspondence: andriykopanytsia@gmail.com

Abstract: The present paper is devoted to the properties of entire vector-valued functions of bounded L-index in join variables, where $\mathbf{L} : \mathbb{C}^n \to \mathbb{R}^n_+$ is a positive continuous function. For vector-valued functions from this class we prove some propositions describing their local properties. In particular, these functions possess the property that maximum of norm for some partial derivative at a skeleton of polydisc does not exceed norm of the derivative at the center of polydisc multiplied by some constant. The converse proposition is also true if the described inequality is satisfied for derivative in each variable.

Keywords: bounded index; bounded **L**-index in joint variables; entire function; local behavior; maximum modulus; sup-norm; vector-valued function

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1. Introduction

The present paper is devoted to the properties of entire vector-valued functions of bounded **L**-index in joint variables (see Definition 1 below) where $L : \mathbb{C}^n \to \mathbb{R}_{+}$ is some positive continuous function. Recently, F. Nuray and R. Patterson [1] introduced a concept of bounded index (i.e., $L(z) \equiv 1$) for entire bivariate functions from $\mathbb{C}^2$ onto $\mathbb{C}^n$ by replacing the absolute value in the definition of an entire function of bounded index by the maximum of the absolute values of the components. If the components of a $\mathbb{C}^n$-valued bivariate entire function are of bounded index, then the function is also of bounded index. They presented sufficient conditions providing index boundedness of bivariate vector-valued entire solutions of certain system of partial differential equations with polynomial coefficients.

This class of functions is interesting with its connections with value distribution theory [2,3] and analytic theory of differential equation [1,4,5]. For example, every entire function has bounded value distribution if and only if its derivative has bounded index [6].

S. Shah proved that any entire function of bounded index [7] is a function of exponential type. Generalizing a notion of bounded index for entire functions of two variables F. Nuray and R. Patterson [8] obtained some sufficient conditions that ensure that exponential type is preserved. Another interesting application of this notion concerns summability methods. Recently F. Nuray [9] presented necessary and sufficient conditions on four-dimensional matrix transformations that preserve entireness, bounded index, and absolute convergence of double sequences. He obtained general characterizations for four-dimensional RH-regular matrix transformations for the space of entire, bounded index, and absolutely summable double sequences.

Of course, there are papers on analytic curves of bounded l-index. This class of functions naturally appears if we consider systems of differential equations and investigate properties of their analytic solutions. A concept of bounded index for entire curves was introduced with the sup-norm [10] and with the Euclidean norm [11]. In these papers the authors replaced the modulus of function by the appropriate norm in the definition. Later there was proposed a definition of bounded ν-index [12] for entire curves with these norms. In this definition, R. Roy and S. M. Shah replaced $p!$ by $p!|z|^p$ and so on. Also M. T. Bordulyak and M. M. Sheremeta [13,14] studied curves of bounded l-index which are analytic in arbitrary bounded domain on a complex plane. These mathematicians found sufficient conditions providing l-index boundedness of every analytic solutions for some system of ordinary differential equations. They obtained some growth estimates and described local behavior of the solutions.

As we wrote above, the first attempt to study analytic vector-valued solutions of partial differential equations system by the notion of bounded index belongs to F. Nuray and R. Patterson [1]. They considered only systems with polynomial coefficients and used the notion of bounded index. In view of results from [7,8] such entire solutions are functions of exponential type. It is known [15] that for any entire function $F : \mathbb{C}^n \to \mathbb{C}$ with bounded multiplicities of zero points there exists a positive continuous function $\mathbf{L} : \mathbb{C}^n \to \mathbb{R}^n_+$ such that F has bounded $\mathbf{L}$-index in joint variables. Therefore, the usage of auxiliary functions $\mathbf{L}$ in the definition allows to study very wide class of functions. We hope that similar fact will be true for vector-valued entire functions. But for analog of the result and for application of the notion of bounded index to system of partial differential equations we need many propositions having a special separate interest in function theory. Therefore, there was posed a general problem in paper [16] to construct theory of bounded index for entire vector-valued functions. In this paper, we continue investigations from [16] and obtain some new local properties of vector-valued entire functions from this class. We assume that in future these results will help to study properties of entire vector-valued solutions for system of partial differential equations as in the case of scalar-valued entire functions of several complex variables (see details for the last case in [5]).

2. Notations and Definitions

We need some notations and definitions. Let us consider a class of vector-valued entire functions

$$F = (f_1, \dots, f_p) \colon \mathbb{C}^n \to \mathbb{C}^p.$$

For this class of functions there is introduced a notion of boundedness of $\mathbf{L}$-index in joint variables (see [16]).

Let $|\cdot|_p$ be a norm in $\mathbb{C}^p$. Let $\mathbf{L}(z) = (l_1(z), \dots, l_n(z))$, where $l_j(z) \colon \mathbb{C}^n \to \mathbb{R}_+$ is a positive continuous function.

Definition 1. *An entire vector-valued function $F \colon \mathbb{C}^n \to \mathbb{C}^p$ is said to be of bounded $\mathbf{L}$-index (in joint variables), if there exists $n_0 \in \mathbb{Z}_+$ such that for every $z \in C^n$ and for all $J \in \mathbb{Z}^n_+$ one has*

$$\frac{|F^{(J)}(z)|_p}{J!\mathbf{L}^J(z)} \le \max\left\{ \frac{|F^{(K)}(z)|_p}{K!\mathbf{L}^K(z)} : K \in \mathbb{Z}^n_+, \|K\| \le n_0 \right\}, \tag{1}$$

where $F^{(J)}(z) = (f_1^{(J)}(z), \dots, f_p^{(J)}(z)), f_k^{(J)}(z) = \dfrac{\partial^{\|J\|}}{\partial z_1^{j_1} \dots \partial z_n^{j_n}} f_k(z)$, $J = (j_1, \dots, j_n)$, $\|J\| = j_1 + \dots + j_n$, $J! = j_1! \cdot \dots \cdot j_n!$, $\mathbf{L}^J(z) = l_1^{j_1}(z) \cdot \dots \cdot l_n^{j_n}(z)$.

The least such integer n_0 is called the $\mathbf{L}$-index in joint variables and is denoted by $N(F, \mathbf{L})$.

We assume the function $\mathbf{L}\colon \mathbb{C}^n \to \mathbb{R}^n_+$ such that $0 < \lambda_{1,j}(R) \le \lambda_{2,j}(R) < \infty$ for any $j \in \{1, 2, \dots, n\}$ and $\forall R \in \mathbb{R}^n_+$, where

$$\lambda_{1,j}(R) = \inf_{z_0 \in \mathbb{C}^n} \inf\{l_j(z)/l_j(z_0)\colon z \in \mathbb{D}^n[z_0, R/\mathbf{L}(z_0)]\},$$

$$\lambda_{2,j}(R) = \sup_{z_0 \in \mathbb{C}^n} \sup\{l_j(z)/l_j(z_0)\colon z \in \mathbb{D}^n[z_0, R/\mathbf{L}(z_0)]\}$$

and $\mathbb{D}^n[z_0, R] = \{z = (z_1, \dots, z_n) \in \mathbb{C}^n\colon |z_1 - z_{0,1}| < r_1, \dots, |z_n - z_{0,n}| < r_n\}$ is the polydisc with $z_0 = (z_{0,1}, \dots, z_{0,n})$, $R = (r_1, \dots, r_n)$. The class of such functions $\mathbf{L}$ we will denote by Q^n. For simplicity, we will use the notation $\Lambda_s(R) = (\lambda_{s,1}(R), \lambda_{s,2}(R), \dots, \lambda_{s,n}(R))$, $s \in \{1, 2\}$.

In [16], V. Baksa and A. Bandura obtained the following result.

Theorem 1 ([16])**.** *Let $|A|_p = \max\{|a_j|\colon 1 \le j \le p\}$ for $A = (a_1, \dots, a_p) \in \mathbb{C}^p$, $\mathbf{L} \in Q^n$. An entire vector-valued function $F\colon \mathbb{C}^n \to \mathbb{C}^p$ has bounded $\mathbf{L}$-index in joint variables if and only if for every $R \in \mathbb{R}^n_+$ there exist $n_0 \in \mathbb{Z}_+$, $p_0 > 0$ such that for all $z_0 \in \mathbb{C}^n$ there exists $K_0 \in \mathbb{Z}^n_+$, $\|K_0\| \le n_0$, satisfying inequality*

$$\max\left\{ \frac{|F^{(K)}(z)|_p}{K!\mathbf{L}^K(z)}\colon \|K\| \le n_0, z \in \mathbb{D}^n[z_0, R/\mathbf{L}(z_0)] \right\} \le p_0 \frac{|F^{(K_0)}(z_0)|_p}{K_0!\mathbf{L}^{K_0}(z_0)}. \quad (2)$$

This theorem is an analog of Fricke's Theorem obtained for entire functions of bounded index of one complex variable in [17].

This theorem implies the following corollary.

Corollary 1 ([16])**.** *Let $\mathbf{L} \in Q^n$ and $\|\cdot\|_0$ be some norm in $\mathbb{C}^p$. An entire vector-function $F\colon \mathbb{C}^n \to \mathbb{C}^p$ has bounded $\mathbf{L}$-index in joint variables in the* sup*-norm $|\cdot|_p$ if and only if it has bounded $\mathbf{L}$-index in joint variables in the norm $\|\cdot\|_0$ where the* sup*-norm is defined as $|A|_p = \max\{|a_j|\colon 1 \le j \le p\}$ for $A = (a_1, \dots, a_p) \in \mathbb{C}^p$.*

Corollary 1 shows that a choice of norm has not influence by the boundedness of the $\mathbf{L}$-index in joint variables for entire vector-valued functions.

We will use Theorem 1 and Corollary 1 in our proofs.

For $A = (a_1, \dots, a_n) \in \mathbb{R}^n$, $B = (b_1, \dots, b_n) \in \mathbb{R}^n$, we will use formal notations without violation of the existence of these expressions: $AB = (a_1 b_1, \dots, a_n b_n)$, $A/B = (a_1/b_1, \dots, a_n/b_n)$, $A^B = a_1^{b_1} \cdot \ldots \cdot a_n^{b_n})$, and the notation $A < B$ means that $a_j < b_j$, $j \in \{1, \dots, n\}$; the relation $A \le B$ is defined in the similar way.

3. Connection between Scalar-Valued and Vector-Valued Entire Functions of Bounded L-Index

The following proposition was obtained for entire curves in [14]. Here, we deduce it for vector-valued entire functions $F\colon \mathbb{C}^n \to \mathbb{C}^p$.

Proposition 1. *Let $\mathbf{L}(z) = (l_1(z), \dots, l_n(z))$ be a positive continuous function in $\mathbb{C}^n$. If each component f_j of an entire vector-valued function $F = (f_1, \dots, f_p)\colon \mathbb{C}^n \to \mathbb{C}^p$ is of bounded $\mathbf{L}$-index $N(\mathbf{L}, f_s)$ in joint variables in the* sup*-norm, then F is of bounded $\mathbf{L}$-index in joint variables in every norm in $\mathbb{C}^n$, in particular, in the* sup*-norm and*

$$N(F, \mathbf{L}) \le \max\{N(f_s, \mathbf{L})\colon 1 \le s \le p\}$$

and also F is of bounded $\mathbf{L}_*$-index in the Euclidean norm with $\mathbf{L}_*(z, w) \ge \sqrt{p}\mathbf{L}(z, w)$ and

$$N_E(F, \mathbf{L}_*) \le \max\{N(f_s, \mathbf{L}) : 1 \le s \le p\},$$

where $N_E(F, \mathbf{L}_*)$ is the L_*-index in joint variables for the function F with the Euclidean norm $\|\cdot\|_E$ instead $|\cdot|_p$ in inequality (1).

Proof. For all $J = (j_1, \ldots, j_n), \|J\| \geq N = \max\{N(f_s, \mathbf{L}) : 1 \leq s \leq p\}$, and $K = (k_1, \ldots, k_n), \|K\| \leq N$, we have

$$\begin{aligned}\frac{|F^{(J)}(z)|_p}{J!\mathbf{L}^J(z)} &= \frac{\max\{|f_s^{(J)}(z)| : 1 \leq s \leq p\}}{J!\mathbf{L}^J(z)} \\ &\leq \max\left\{\max\left\{\frac{|f_s^{(K)}(z)|}{K!\mathbf{L}^K(z)} : 1 \leq s \leq p\right\} : 0 \leq \|K\| \leq N\right\} \\ &\leq \max\left\{\frac{|F^{(K)}(z)|_p}{K!\mathbf{L}^K(z)} : 0 \leq \|K\| \leq N\right\},\end{aligned}$$

that is $N(\mathbf{L}; F) \leq N = \max\{N(\mathbf{L}; f_s) : 1 \leq s \leq p\}$. Also by definition of $N(f_s, \mathbf{L})$

$$\begin{aligned}\frac{\|F^{(J)}(z)\|_E}{J!\mathbf{L}^J(z)} &= \frac{\sqrt{\sum_{s=1}^p |f_s^{(J)}(z)|^2}}{J!\mathbf{L}^J(z)} = \sqrt{\sum_{s=1}^p \left(\frac{|f_s^{(J)}(z)|}{J!\mathbf{L}^J(z)}\right)^2} \\ &\leq \sqrt{\sum_{s=1}^p \left(\max\left\{\frac{|f_s^{(K)}(z)|}{K!\mathbf{L}^K(z)} : 0 \leq \|K\| \leq N\right\}\right)^2}.\end{aligned}$$

Since $\|A\|_E \leq \sqrt{p}\max\{|a_j| : 1 \leq j \leq p\}$ for $A = (a_1, \ldots, a_p)$, $a_s = \max\limits_{0 \leq \|K\| \leq N} \frac{|f_s^{(K)}(z)|}{K!\mathbf{L}^K(z)}$, we have

$$\begin{aligned}\frac{\|F^{(J)}(z)\|_E}{J!\mathbf{L}^J(z)} &\leq \sqrt{p}\max\left\{\frac{|f_s^{(K)}(z)|}{K!\mathbf{L}^K(z)} : 0 \leq \|K\| \leq N, 1 \leq s \leq p\right\} \\ &\leq \sqrt{p}\max\left\{\frac{\|F^{(K)}(z)\|_E}{K!\mathbf{L}^K(z)} : 0 \leq \|K\| \leq N\right\}.\end{aligned} \tag{3}$$

We put $I_N := \{K \in \mathbb{Z}_+^n : \|K\| = N\}$, $B_K := \{J \in \mathbb{Z}_+^n : J \geq K, J \neq K\}$. Then,

$$\{J \in \mathbb{Z}_+^n : \|J\| \geq N+1\} = \bigcup_{K \in I_N} B_K.$$

Thus, for every $J = (j_1, \ldots, j_n) \in B_K, \|J\| \geq N+1$ and any $K = (k_1, \ldots, k_n), \|K\| \leq N$ there exists $m, 1 \leq m \leq n$ such that $j_m \geq k_m + 1$. Hence,

$$\frac{\mathbf{L}^J(z)}{\mathbf{L}_*^J(z)} \cdot \frac{\mathbf{L}_*^K(z)}{\mathbf{L}^K(z)} = \left(\frac{\mathbf{L}(z)}{\mathbf{L}_*(z)}\right)^{J-K} = \left(\frac{l_1(z)}{l_{1*}(z)}\right)^{j_1-k_1} \cdots \left(\frac{l_n(z)}{l_{n*}(z)}\right)^{j_n-k_n} \leq \frac{l_m(z)}{l_{m*}(z)} \leq p^{-1/2}, \tag{4}$$

where $\mathbf{L}(z) = (l_1(z), \ldots, l_p(z))$, $\mathbf{L}_*(z) = (l_{1*}(z), \ldots, l_{p*}(z))$. Therefore, inequality (4) holds for all $J, \|J\| \geq N+1$, and for every $K, \|K\| \leq N$. From inequality (3) by using of (4) we now obtain

$$\begin{aligned}\frac{\|F^{(J)}(z)\|_E}{J!\mathbf{L}_*^J(z)} &\leq \sqrt{p}\max\left\{\frac{\mathbf{L}^J(z)}{\mathbf{L}_*^J(z)} \cdot \frac{\mathbf{L}_*^K(z)}{\mathbf{L}^K(z)} \cdot \frac{\|F^{(K)}(z)\|_E}{K!\mathbf{L}^K(z)} : 0 \leq \|K\| \leq N\right\} \\ &\leq \max\left\{\frac{\|F^{(K)}(z)\|_E}{K!\mathbf{L}_*^K(z)} : 0 \leq \|K\| \leq N\right\},\end{aligned}$$

that is $N(\mathbf{L}_*, F) \leq \max\{N(\mathbf{L}, f_j) : 1 \leq j \leq p\}$. Thus, Proposition 1 is proved. □

Example 1. *Let us consider the following entire vector-valued function*

$$F(z,w) = (w + e^z, e^{wz}, e^{z^3}).$$

For the first component $f_1(z,w) = w + e^z$ of the function F we calculate partial derivatives:

$$\frac{\partial^j f_1(z,w)}{\partial z^j} = e^z \text{ for all } j \in \mathbb{N},$$

$$\frac{\partial f_1(z,w)}{\partial w} = 1, \quad \frac{\partial^j f_1(z,w)}{\partial w^j} = 0 \text{ for } j \geq 2.$$

$$\frac{\partial^{i+j} f_1(z,w)}{\partial z^i \partial w^j} = 0 \text{ for } i \geq 1, j \geq 1.$$

In view of Definition 1 the component f_1 has bounded $\mathbf{L}_1$-index in joint variables with $\mathbf{L}_1(z) = (1,1)$ and $N(f_1, \mathbf{L}_1) = 1$.

Similarly, it is easy to show that the second component $f_2(z,w) = e^{wz}$ has bounded $\mathbf{L}_2$-index in joint variables with $\mathbf{L}_2(z) = (|w| + 1, |z| + 1)$ and $N(f_2, \mathbf{L}_2) = 0$.

For the third component $f_3(z,w) = e^{z^3}$ its $\mathbf{L}_3$-index in joint variables is equal to zero (i.e., $N(f_3, \mathbf{L}_3) = 0$) with $\mathbf{L}_3(z,w) = (3|z|^2 + 1, 1)$.

Now we claim that the vector-valued function $F(z,w) = (w + e^z, e^{wz}, e^{z^3})$ has bounded $\mathbf{L}$-index in joint variables (in the sup*-norm) with*

$$\mathbf{L}(z,w) = \Big(\max\{1, |w| + 1, 3|z|^2 + 1\}, \max\{1, |z| + 1, 1\}\Big) = \Big(1 + \max\{|w|, 3|z|^2\}, |z| + 1\Big)$$

and $N(F, \mathbf{L}) = 1 \leq \max\{N(f_j, \mathbf{L}) : 1 \leq j \leq 3\} \leq \max\{N(f_j, \mathbf{L}_j) : 1 \leq j \leq 3\} = 1$.

4. Local Behavior of Entire Vector-Valued Functions at Skeleton of Polydisc

Theorem 2. *Let $\mathbf{L} \in Q^n$. In order that an entire vector-valued function $F\colon \mathbb{C}^n \to \mathbb{C}^p$ be of bounded $\mathbf{L}$-index in joint variables it is necessary that for all $R \in \mathbb{R}^n_+$ there exist $n_0 \in \mathbb{Z}_+$, $p_1 \geq 1$ such that for all $z_0 \in \mathbb{C}^n$ there exists $K_0 \in \mathbb{Z}^n_+$, $\|K_0\| \leq n_0$, satisfying inequality*

$$\max\{|F^{(K_0)}(z)|_p : z \in \mathbb{D}^n[z_0, R/\mathbf{L}(z_0)]\} \leq p_1 |F^{(K_0)}(z_0)|_p \tag{5}$$

and it is sufficiently that for all $R \in \mathbb{R}^n_+$ there exist $n_0 \in \mathbb{Z}_+$, $p_1 \geq 1$ $\forall z_0 \in \mathbb{C}^n$ $\exists K_1^0 = (k_1^0, 0, \ldots, 0)$, $\exists K_2^0 = (0, k_2^0, 0, \ldots, 0), \ldots, \exists K_n^0 = (0, \ldots, 0, k_n^0) : k_j^0 \leq n_0$, and $(\forall j, 1 \leq j \leq n)$:

$$\max\{|F^{(K_j^0)}(z)|_p : z \in \mathbb{D}^n[z_0, R/\mathbf{L}(z_0)]\} \leq p_1 |F^{(K_j^0)}(z_0)|_p. \tag{6}$$

Proof. Necessity. By Theorem 1 inequality (2)

$$\max\left\{ \frac{|F^{(K)}(z)|_p}{K!\mathbf{L}^K(z)} : \|K\| \leq n_0, z \in \mathbb{D}^n[z_0, R/\mathbf{L}(z_0)] \right\} \leq p_0 \frac{|F^{(K_0)}(z_0)|_p}{K_0!\mathbf{L}^{K_0}(z_0)}$$

is valid for some K_0. In view of definition class Q^n, the following inequality holds for all $z \in \mathbb{D}^n[z_0, R/\mathbf{L}(z_0)]$

$$\frac{\mathbf{L}^K(z)}{\mathbf{L}^K(z_0)} = \frac{{l_1}^{k_1}(z)}{{l_1}^{k_1}(z_0)} \cdot \ldots \cdot \frac{{l_p}^{k_n}(z)}{{l_p}^{k_n}(z_0)} \leq \lambda_{2,1}^{k_1}(R) \cdot \ldots \cdot \lambda_{2,n}^{k_n}(R) = \Lambda_2^K(R).$$

Hence, $\max\left\{\frac{\mathbf{L}^{K_0}(z)}{\mathbf{L}^{K_0}(z_0)} : z \in \mathbb{D}^n[z_0, R/\mathbf{L}(z_0)]\right\} \le \Lambda_2^{K_0}(R)$. Therefore, we obtain

$$\begin{aligned}&\max\{|F^{(K_0)}(z)|_p : z \in \mathbb{D}^n[z_0, R/\mathbf{L}(z_0)]\}\\&= K_0!\mathbf{L}^{K_0}(z_0)\max\left\{\frac{|F^{(K_0)}(z)|_p}{K_0!\mathbf{L}^{K_0}(z)}\cdot\frac{\mathbf{L}^{K_0}(z)}{\mathbf{L}^{K_0}(z_0)} : z \in \mathbb{D}^n[z_0, R/\mathbf{L}(z_0)]\right\}\\&\le K_0!\mathbf{L}^{K_0}(z_0)\Lambda_2^{K_0}(R)\max\left\{\frac{|F^{(K_0)}(z)|_p}{K_0!\mathbf{L}^{K_0}(z)} : z \in \mathbb{D}^n[z_0, R/\mathbf{L}(z_0)]\right\}.\end{aligned}$$

Hence, by inequality (2)

$$\begin{aligned}&\max\{|F^{(K_0)}(z)|_p : z \in \mathbb{D}^n[z_0, R/\mathbf{L}(z_0)]\}\\&\le p_0K_0!\mathbf{L}^{K_0}(z_0)\Lambda_2^{K_0}(R)\frac{|F^{(K_0)}(z_0)|_p}{K_0!\mathbf{L}^{K_0}(z_0)} = p_0\Lambda_2^{K_0}(R)|F^{(K_0)}(z_0)|_p.\end{aligned}$$

From this inequality it follows inequality (5) with $p_1 = p_0\Lambda_2^{K_0}(R)$. The necessity of condition (5) is proved.

Sufficiency. Now we prove the sufficiency of (6). Suppose that for each $R \in \mathbb{R}^n$ there exist $n_0 \in \mathbb{Z}_+$, $p_1 \ge 1$ such that for every $z_0 \in \mathbb{C}^n$ and some $k_j^0 \in \mathbb{Z}_+$ with $k_j^0 \le n_0$, $(1 \le j \le n)$ inequalities (6) hold.

For each $z_0 \in \mathbb{C}^n$ and for every $S \in \mathbb{Z}_+^n$ we write the Cauchy formula $(K_j^0 = (0,\dots,0,k_j^0,0,\dots,0))$

$$\frac{F^{(K_j^0+S)}(z_0)}{(K_j^0+S)!} = \frac{1}{(2\pi i)^n}\int_{\mathbb{T}^n(z_0,R/\mathbf{L}(z_0))}\frac{F^{(K_j^0)}(z)dz}{(z-z_0)^{S+\mathbf{1}}},$$

where $\mathbf{1} = (1,\dots,1) \in \mathbb{R}_+^n$, and $\mathbb{T}^n(z_0,R) = \{z \in \mathbb{C}^n : |z_1 - z_{0,1}| = r_1,\dots,|z_n - z_{0,n}| = r_n\}$ denotes the skeleton of polydisc. We obtain that

$$\begin{aligned}&\frac{|F^{(K_j^0+S)}(z_0)|_p}{(K_j^0+S)!} \le \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n(z_0,R/\mathbf{L}(z_0))}\frac{|F^{(K_j^0)}(z)|_p|dz|}{\left|(z-z_0)^{S+\mathbf{1}}\right|}\\&\le \frac{1}{(2\pi)^n}\max\left\{|F^{(K_j^0)}(z)|_p : z \in \mathbb{D}^n[z_0, R/\mathbf{L}(z_0)]\right\}\frac{\mathbf{L}^{S+\mathbf{1}}(z_0)}{R^{S+\mathbf{1}}}\int_{\mathbb{T}^n(z_0,R/\mathbf{L}(z_0))}|dz|\\&=\max\left\{|F^{(K_j^0)}(z)|_p : z \in \mathbb{D}^n[z_0, R/\mathbf{L}(z_0)]\right\}\frac{\mathbf{L}^S(z_0)}{R^S}.\end{aligned}$$

We choose $R > 1$ and s_0 such that $\frac{p_1K_j^0!}{R^S} \le 1$ for all $S \in \mathbb{Z}_+^n$, $\|S\| \ge s_0$. In view of (6)

$$\max\{|F^{(K_j^0)}(z)|_p : z \in \mathbb{D}^n[z_0, R/\mathbf{L}(z_0)]\} \le p_1|F^{(K_j^0)}(z_0)|_p,$$

we have

$$\begin{aligned}\frac{|F^{(K_j^0+S)}(z_0)|_p}{(K_j^0+S)!\mathbf{L}^{K_j^0+S}(z_0)} &\le p_1|F^{(K_j^0)}(z_0)|_p\cdot\frac{1}{R^S\mathbf{L}^{K_j^0}(z_0)}\\&= \frac{|F^{(K_j^0)}(z_0)|_p}{K_j^0!\mathbf{L}^{K_j^0}(z_0)}\cdot\frac{p_1K_j^0!}{R^S} \le \frac{|F^{(K_j^0)}(z_0)|_p}{K_j^0!\mathbf{L}^{K_j^0}(z_0)}.\end{aligned} \tag{7}$$

For each $J \in \mathbb{Z}_+^n$, one of the following two possibilities holds: either $J \le (k_1^0 + s_0, k_2^0 + s_0, \dots, k_n^0 + s_0)$, or there exists $j, 1 \le j \le n$, such that $J \ge K_j^0 + (s_0, \dots, s_0)$. Then from (7) for every $J \in \mathbb{Z}_+^n$ we obtain

$$\frac{|F^{(J)}(z_0)|_p}{J!\mathbf{L}^J(z_0)} \le \max\left\{ \frac{|F^{(K)}(z_0)|_p}{K!\mathbf{L}^K(z_0)} \colon \ 0 \le \|K\| \le \max\{k_j^0 + s_0 \colon 1 \le j \le n\} \right\}$$

Therefore, $N(F, \mathbf{L}) \le \max\{k_j^0 + s_0 \colon 1 \le j \le n\}$.

The proof of Theorem 2 is completed. □

Example 2. *We will use the results from Example 1. For the function* $F(z,w) = (w + e^z, e^{wz}, e^{z^3})$ *we choose* $n_0 = N(F, \mathbf{L}) = 1$. *In view of proof of Theorem 2, the parameter* p_1 *can be chosen as*

$$p_1 = p_0 \Lambda_2^{K_0}(R),$$

where $R \in \mathbb{R}_+^n$ *and the parameter* p_0 *is calculated in proof of Theorem 1 from [16]. There was proved that*

$$p_0 = 2^q \prod_{j=1}^{n} \Big((\lambda_{1,j}(R))^{-N} (\lambda_{2,j}(R))^{N} \Big)^q$$

where

$$q = q(R) = \Big[2(N+1) \prod_{j=1}^{n} \Big(\big(\lambda_{2,j}(R)\big)^{N+1} \big(\lambda_{1,j}(R)\big)^{-N} \Big) (r_1 + \ldots + r_n) \Big] + 1,$$

the notation $[x]$ *stands for the entire part of the real number* x *and* $N = N(F, \mathbf{L}) = 1$. *One should observe that* $\lambda_{1,j}(R) = 1$ *and* $\lambda_{2,j}(R) = 1$ *if* $\mathbf{L}(z,w) = \big(1 + \max\{|w|, 3|z|^2\}, |z| + 1\big)$. *Therefore, for* $R = (r_1, r_2)$ *one has* $q(R) = [4(r_1 + r_2)] + 1$, $p_0(R) = 2^{[4(r_1+r_2)]+1}$ *and* $p_1(R) = 2^{[4(r_1+r_2)]+1}$. *In view of Theorem 2, we claim that for any* $R = (r_1, r_2) \in \mathbb{R}_+^2$ *and for every* $(z_0, w_0) \in \mathbb{C}^2$ *at least one from the following inequalities holds*

$$\max\Big\{ |F^{(1,0)}(z,w)|_p \colon \ |z - z_0| = \frac{r_1}{1 + \max\{|w_0|, 3|z_0|^2\}}, |w - w_0| = \frac{r_2}{|w_0| + 1} \Big\}$$
$$\le 2^{[4(r_1+r_2)]+1} |F^{(1,0)}(z_0, w_0)|_p,$$
$$\max\Big\{ |F^{(0,1)}(z,w)|_p \colon \ |z - z_0| = \frac{r_1}{1 + \max\{|w_0|, 3|z_0|^2\}}, |w - w_0| = \frac{r_2}{|w_0| + 1} \Big\}$$
$$\le 2^{[4(r_1+r_2)]+1} |F^{(0,1)}(z_0, w_0)|_p,$$
$$\max\Big\{ |F(z,w)|_p \colon \ |z - z_0| = \frac{r_1}{1 + \max\{|w_0|, 3|z_0|^2\}}, |w - w_0| = \frac{r_2}{|w_0| + 1} \Big\}$$
$$\le 2^{[4(r_1+r_2)]+1} |F(z_0, w_0)|_p.$$

The notation $\mathbf{L} \asymp \widetilde{\mathbf{L}}$ means that there exist $\theta_1 \in \mathbb{R}_+, \theta_2 \in \mathbb{R}_+$ such that for all $z \in \mathbb{C}^n$

$$\theta_1 \widetilde{\mathbf{L}}(z) \le \mathbf{L}(z) \le \theta_2 \widetilde{\mathbf{L}}(z).$$

Proposition 2. *Let* $\mathbf{L} \in Q^n$, $\mathbf{L} \asymp \widetilde{\mathbf{L}}$. *An entire vector-valued function* $F \colon \mathbb{C}^n \to \mathbb{C}^p$ *has bounded* $\widetilde{\mathbf{L}}$*-index in joint variables if and only if it has bounded* $\mathbf{L}$*-index in joint variables.*

Proof. It is easy to prove that the conditions $\mathbf{L} \in Q^n$ and $\mathbf{L} \asymp \widetilde{\mathbf{L}}$ imply that the function $\widetilde{\mathbf{L}} \in \mathbb{Q}^n$. Without loss of generality, we believe that $\theta_1 \le 1 \le \theta_2$.

Let $N(F, \widetilde{\mathbf{L}}) = \widetilde{n}_0 < +\infty$. Then for the function F we obtain

$$\begin{aligned}
&\max\left\{ \frac{|F^{(K)}(z)|_p}{K!\mathbf{L}^K(z)} : \|K\| \leq \widetilde{n}_0, z \in \mathbb{D}^n[z_0, \widetilde{R}/\widetilde{\mathbf{L}}(z_0)] \right\} \\
&= \max\left\{ \frac{|F^{(K)}(z)|_p}{K!\widetilde{\mathbf{L}}^K(z)} \frac{\widetilde{\mathbf{L}}^K(z)}{\mathbf{L}^K(z)} : \|K\| \leq \widetilde{n}_0, z \in \mathbb{D}^n[z_0, \widetilde{R}/\widetilde{\mathbf{L}}(z_0)] \right\} \\
&\leq \theta_1^{-\widetilde{n}_0} \cdot \max\left\{ \frac{|F^{(K)}(z)|_p}{K!\widetilde{\mathbf{L}}^K(z)} : \|K\| \leq \widetilde{n}_0, z \in \mathbb{D}^n[z_0, \widetilde{R}/\widetilde{\mathbf{L}}(z_0)] \right\}. \qquad (8)
\end{aligned}$$

Since $N(F, \widetilde{\mathbf{L}}) = \widetilde{n}_0 < +\infty$, by Theorem 1 for each $\widetilde{R} \in \mathbb{R}^n_+$ there exists $\widetilde{p} \geq 1$ such that for all $z_0 \in \mathbb{C}^n$ and some K_0 with $\|K_0\| \leq \widetilde{n}_0$ inequality (2) is true with $\widetilde{\mathbf{L}}$ and $\widetilde{R}$ instead of $\mathbf{L}$ and R, respectively, i.e.

$$\max\left\{ \frac{|F^{(K)}(z)|_p}{K!\widetilde{\mathbf{L}}^K(z)} : \|K\| \leq \widetilde{n}_0, z \in \mathbb{D}^n[z_0, \widetilde{R}/\widetilde{\mathbf{L}}(z_0)] \right\} \leq \widetilde{p} \frac{\|F^{(K_0)}(z_0)\|}{K_0!\widetilde{\mathbf{L}}^{K_0}(z_0)}. \qquad (9)$$

Substituting (9) in (8), we deduce

$$\begin{aligned}
&\max\left\{ \frac{|F^{(K)}(z)|_p}{K!\mathbf{L}^K(z)} : \|K\| \leq \widetilde{n}_0, z \in \mathbb{D}^n[z_0, \widetilde{R}/\widetilde{\mathbf{L}}(z_0)] \right\} \\
&\leq \theta_1^{-\widetilde{n}_0} \cdot \widetilde{p} \cdot \frac{|F^{(K_0)}(z_0)|_p}{K_0!\widetilde{\mathbf{L}}^{K_0}(z_0)} = \theta_1^{-\widetilde{n}_0} \cdot \widetilde{p} \cdot \frac{|F^{(K_0)}(z_0)|_p}{K_0!\mathbf{L}^{K_0}(z_0)} \cdot \frac{\mathbf{L}^{K_0}(z_0)}{\widetilde{\mathbf{L}}^{K_0}(z_0)} \\
&\leq \theta_2^{\|K_0\|}\theta_1^{-\widetilde{n}_0} \cdot \widetilde{p} \cdot \frac{|F^{(K_0)}(z_0)|_p}{K_0!\mathbf{L}^{K_0}(z_0)}. \qquad (10)
\end{aligned}$$

For given $R \in \mathbb{R}^n_+$ we put $\widetilde{R} = \theta_2 R$. Then $R/\mathbf{L}(z_0) \leq \widetilde{R}/\widetilde{\mathbf{L}}(z_0)$, hence

$$\mathbb{D}^n[z_0, R/\mathbf{L}(z_0)] \subset \mathbb{D}^n[z_0, \widetilde{R}/\widetilde{\mathbf{L}}(z_0)].$$

Therefore, in view of (10) by Theorem 1 we conclude that the vector-valued function F has bounded **L**-index in joint variables. □

5. Conclusions

In the paper, we obtained some results describing local properties of vector-valued entire functions of several complex variables. We studied functions having bounded **L**-index in joint variables. New results are needed to deduce analog of Hayman's Theorem for this class of functions and demonstrate its application to study properties entire vector-valued solutions of partial differential equations system as it was done for entire scalar-valued solutions in [5].

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