**Neutral Differential Equations of Fourth-Order: New Asymptotic Properties of Solutions**

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1. Introduction

In this work, we study the oscillatory behavior of solutions of the differential equations of the form

\[ (r(\varrho)\varphi''')'' + f(\varrho, \varphi(\varrho))) = 0. \tag{1} \]

where \( \varrho \geq \varrho_0 \) and \( \varphi(\varrho) = u(\varrho) + p(\varrho)u(\varrho) \). During this study, we will assume the following conditions are satisfied:

\((H_1)\) \( r \in C^1([\varrho_0, \infty), \, r'(\varrho) \geq 0, \, \theta \in C([\varrho_0, \infty), \, \varphi(\varrho) \leq \varrho, \, \lim_{\varrho \to \infty} \theta(\varrho) = \infty, \, \varrho_0 := \{\varrho_0, \infty\} \) and \( \varphi_0(\varrho_0) < \infty \),

\[
\varphi_0(\varrho) := \int_{\varrho_0}^{\varrho} \frac{1}{r(\varrho)} \, d\varrho. \tag{2}
\]

\((H_2)\) \( f \in C^3([\varrho_0, \infty), \, \text{and there exists a function } h \in C([\varrho_0, \varrho_0]), \text{ such that } f(\varrho, u) \geq h(u) u. \)

\((H_3)\) \( p, \eta \in C([\varrho_0, \varrho_0], \, \eta(\varrho) \leq \varrho, \, \lim_{\varrho \to \infty} \eta(\varrho) = \infty \) and \( \varphi_2(\eta(\varrho)) > 0. \)

We say that a real-valued function \( u \in C^3([\varrho], \, \varrho_0 \geq \varrho_0, \text{ is a solution of } (1) \text{ if } r(\varphi''') \in C^1([\varrho]), \text{ and satisfies } (1) \text{ on } [\varrho_0, \infty]. \)

**Remark 1.** As usual, all occurring functional inequalities are assumed to hold finally, that is, they are satisfied for all \( \varrho \) large enough. Moreover, we evaluate some integrals on the extended real line.

Fourth-order differential equations are quite often encountered in mathematical models of various physical, biological, and chemical phenomena. Applications include, for in-
stance, problems of elasticity, deformation of structures, or soil settlement; see [1]. Questions related to the existence of oscillatory and nonoscillatory solutions play an important role in mechanical and engineering problems. In natural science and technology, neutral differential equations have a wide range of applications. They are often employed, for example, in the study of distributed networks with lossless transmission lines (see Hale [2]).

Many investigations on the oscillation and non oscillation of solutions of various types of neutral functional differential equations have been conducted in recent years. Because there is such a wide collection of relevant work on this topic, the reader is recommended to monographs [3–17] for a summary of numerous significant oscillation results. In the following, we show some previous results in the literature.

Many authors in [7–9] studied the asymptotic properties of the solutions of equation
\[
\left( r(\varrho)\left( u^{(n-1)}(\varrho)\right)^{\alpha}\right)^{'} + q(\varrho)u^{\beta}(\theta(\varrho)) = 0, \tag{3}
\]
where \( r'(\varrho) > 0 \) and
\[
\int_{\varrho_0}^{\infty} r^{-1/a}(s)ds = \infty. \tag{4}
\]

In [10,11], Zhang et al. studied the oscillation of (3) under the assumption that
\[
\int_{\varrho_0}^{\infty} r^{-1/a}(s)ds < \infty. \tag{5}
\]

El-Nabulsi et al. [12] investigated the oscillation properties of solutions to the fourth-order nonlinear differential equations
\[
\left( r(\varrho)\left( \phi''''(\varrho)\right)^{\alpha}\right)^{'} + q(\varrho)f(u(\theta(\varrho))) = 0, \tag{6}
\]
where \( f(\psi)/\phi^k \geq k > 0 \), for \( \phi \neq 0 \) and (4) holds. Zhang et al. [13] and Moaaz et al. [14] studied the oscillation of (6) under the condition (5). By using the technique of comparison with first order delay equations, Xing et al. [15] established some oscillation theorems for equation
\[
\left( r(\varrho)\left( \phi^{(n-1)}(\varrho)\right)^{\alpha}\right)^{'} + q(\varrho)\varrho^\alpha(\theta(\varrho)) = 0,
\]
under the condition (4). Chatzarakis et al. [16] established some oscillation criteria for neutral differential equation
\[
\left( r(\varrho)\left( \phi''''(\varrho)\right)^{\alpha}\right)^{'} + \int_{a}^{b} q(\varrho,s)f(u(\theta(\varrho,s)))ds = 0,
\]
under the assumption (4). Very recently, By using Riccati transform, Dassios and Bazighifan [17] proved that the fourth-order nonlinear differential equation
\[
\left( r(\varrho)\left( \phi''''(\varrho)\right)^{\alpha}\right)^{'} + q(\varrho)\varrho^\alpha(\theta(\varrho)) = 0
\]
is almost oscillatory, under the condition (5).

On the other hand, there are other techniques for studying the oscillatory behavior of differential equations by analysis of the characteristic equation and its roots. For example very recently Pedro in [18] obtains sufficient conditions under which the system has at least a nonoscillatory solution, based on the form of the system matrices, are obtained via the analysis of the characteristic equation. Also in [19] the numerical controllability of an integro-differential equation is briefly discussed. Additionally, the authors consider as key-tools: the Laplace transform, the Mittag-Leffler matrix function and the iterative scheme.

Our results here is based on creating new comparison theorems that compare the 4th-order equation with first-order delay differential equations. We establish new oscil-
lation criteria for a class of fourth-order neutral differential equations. This new results improves a number of results reported in the literature. Example is provided to illustrate the main results.

**Lemma 1 ([5]).** Assume that \( w \in C^n([0,T],\mathbb{R}^+), w^{(n)} \) is of one sign, eventually. Then, there exist a \( \varkappa \in I_0 \) and \( \kappa \in [0,n] \) is integer, with \((-1)^{n+\kappa}w^{(n)}(\varkappa) \geq 0 \) such that

\[
\kappa > 0 \text{ give } w^{(k)}(\varkappa) > 0, \text{ for } k = 0,1,\ldots,\kappa - 1
\]

and

\[
\kappa \leq n - 1 \text{ give } (-1)^{k+\kappa}w^{(k)}(\varkappa) > 0 \text{ for } k = \kappa,\kappa + 1,\ldots,n - 1,
\]

for all \( \varkappa \in I_0 \).

2. Main Results

For brevity, we define the operators \( \varphi_\kappa \) by

\[
\varphi_0(\rho) := \int_\rho^\infty r^{-1}(\varkappa)d\varkappa, \quad \varphi_\kappa(\rho) := \int_\rho^{\infty} \varphi_{\kappa-1}(\varkappa)d\varkappa, \text{ for } \kappa = 1,2.
\]

**Lemma 2.** Assume that \( u(\rho) \in C([0,\infty), (0,\infty)) \) is a solution of (1). Then \( \varphi(\rho) > 0, (r(\rho)\varphi''(\rho))' \leq 0 \), and one of the following cases holds, for \( \varphi \in [\varphi_1, \infty) \), \( \rho_1 \geq \rho_0 \):

(A) \( \varphi'(\rho), \varphi''(\rho) \) are positive and \( \varphi(\rho) \) is negative;
(B) \( \varphi'(\rho), \varphi''(\rho) \) are positive and \( \varphi'''(\rho) \) is negative;
(C) \( (-1)^{\alpha} \varphi(\rho) \) are positive for all \( \alpha = 1,2,3 \).

**Proof.** Assume that \( u \) is an eventually positive solution of (1). Then, there exists \( \rho_1 \geq \rho_0 \) such that \( u(\rho) \), \( u(\eta(\rho)) \) and \( u(\theta(\rho)) \) are positive for all \( \rho \geq \rho_1 \). Hence, we see that \( \varphi(\rho) > 0 \) for \( \rho \geq \rho_1 \). It follows from (1) that \( (r(\rho)\varphi''(\rho))' \leq 0 \). Now, using Lemma 1 with \( n = 4 \), we readily get the cases (A)–(C).

**Lemma 3.** Assume that \( u \) is a positive solution of (1) and \( \varphi \) satisfies case (C), then

\[
\left(\begin{array}{c}
\varphi(\rho) \\
\varphi_2(\rho)
\end{array}\right)' \geq 0.
\]

**Proof.** Assume that \( u \) is a positive solution of (1) and \( \varphi \) satisfies case (C). From (1) and \( (H_2) \), we have that \( r(\rho)\varphi''(\rho) \) is nonincreasing, and hence

\[
\begin{align*}
\varphi''(\rho) & \geq \frac{1}{r(\rho)} \int_\rho^\infty \frac{1}{r(\varkappa)}d\varkappa \\
& \geq \frac{1}{r(\rho)} \int_\rho^\infty \frac{1}{r(\varkappa)}r(\varkappa)\varphi''(\varkappa)d\varkappa = \lim_{\varkappa \to \infty} \varphi''(\varkappa) - \varphi''(\rho).
\end{align*}
\]

Since \( \varphi''(\rho) \) is a positive decreasing function, we have that \( \varphi''(\rho) \) converges to a nonnegative constant when \( \rho \to \infty \). Thus, (8) becomes

\[
-\varphi''(\rho) \leq r(\rho)\varphi''(\rho)\varphi_0(\rho),
\]

from (9), we get

\[
\left(\begin{array}{c}
\varphi''(\rho) \\
\varphi_0(\rho)
\end{array}\right)' = \frac{1}{\varphi_2(\rho)} \left(\varphi''(\rho)\varphi_0(\rho) + \varphi''(\rho)r(\rho)\right) \geq 0,
\]

which leads to

\[
-\varphi'(\rho) \geq \int_\rho^\infty \frac{\varphi''(\rho)}{\varphi_0(\rho)} \varphi_0(\rho) d\varkappa \geq \frac{\varphi''(\rho)}{\varphi_0(\rho)} \varphi_1(\rho).
\]

This implies
Theorem 1. Assume that \( u \) is a positive solution of (1) and \( \phi \) satisfies case (C) in Lemma 1. If
\[
\int_{\phi_{0}}^{\infty} \frac{1}{r(p)} \left( \int_{\phi_{2}}^{p} h(\varphi) \left( 1 - p(\varphi) \frac{\phi_{2}'(\varphi)}{\phi_{2}(\varphi)} \right) \varphi \right) d\varphi = \infty
\]
and there exists a \( \delta_{0} \in (0, 1) \) such that
\[
h(\varphi)\phi_{2}(\varphi)\varphi_{1}(\varphi) \left( 1 - p(\varphi) \frac{\phi_{2}'(\varphi)}{\phi_{2}(\varphi)} \right) \geq \delta_{0},
\]
then
\[
(B_1) \quad (-1)^{k+1} \phi^{(2-k)}(\varphi) \leq r(\varphi)\phi''(\varphi)\varphi_k(\varphi) \text{ for } k = 0, 1, 2;
\]
\[
(B_2) \quad \lim_{\varphi \to \infty} \phi(\varphi) = 0;
\]
\[
(B_3) \quad \phi / \phi_{2}^{\delta_{0}} \text{ is decreasing};
\]
\[
(B_4) \quad \lim_{\varphi \to \infty} \phi(\varphi) / \phi_{2}^{\delta_{0}}(\varphi) = 0.
\]

Proof. Assume that \( u \) is a positive solution of (1) and \( \phi \) satisfies case (C) for \( \varphi \geq \varphi_{1} \) for some \( \varphi_{1} \in I_{0} \). Then, there is a \( \varphi_{2} \geq \varphi_{1} \) with \( \phi(\varphi(\varphi)) > 0 \) for all \( \varphi_{2} \), and hence, from (1),
\[
(r(\varphi)\phi''(\varphi))' \leq -h(\varphi)u(\varphi(\varphi)) \leq 0.
\]

(B1): Using case (C), we have that
\[
\int_{\varphi_{0}}^{\infty} \frac{r(\varphi)\phi''(\varphi)}{r(\varphi)} \varphi \int_{\varphi_{2}}^{\varphi} h(\varphi) \left( 1 - p(\varphi) \frac{\phi_{2}'(\varphi)}{\phi_{2}(\varphi)} \right) \varphi \right) d\varphi = \infty
\]
or equivalently
\[
\phi''(\varphi) \geq -r(\varphi)\phi''(\varphi)\varphi_{0}(\varphi).
\]
Integrating this relationship twice over \([\varphi_{0}, \infty)\), and taking into account the behavior of derivatives in case (C), we arrive at \( B_1 \).

(B2): Since
\[
\begin{align*}
\phi_{1}(\varphi) = \phi(\varphi) - p(\varphi)u(\eta(\varphi)) \geq \phi(\varphi) - p(\varphi)\phi(\eta(\varphi)),
\end{align*}
\]
from (7), we have
\[
\int_{\varphi_{1}}^{\varphi_{2}} h(\varphi) \left( 1 - p(\varphi) \frac{\phi_{2}'(\varphi)}{\phi_{2}(\varphi)} \right) \varphi \right) d\varphi \leq \delta_{0}.
\]
Now, from (1), we get
\[
(r(\varphi)\phi''(\varphi))' \leq -h(\varphi) \left( 1 - p(\varphi) \frac{\phi_{2}'(\varphi)}{\phi_{2}(\varphi)} \right) \phi(\varphi(\varphi)).
\]
Since $\phi$ is positive decreasing, we get that $\lim_{q \to \infty} \phi(q) = k \geq 0$. Assume the contrary that $k > 0$. Then, there is a $q_2 \geq q_1$ with $\phi(q) \geq k$ for $q \geq q_2$. Thus (12) becomes

$$(r(q)\phi'''(q))^2 \leq -kh(q)\left(1 - p(\theta(q)) \frac{q_2(\eta(\theta(q)))}{q_2(\theta(q))}\right).$$

Integrating this inequality twice over $[q_2, q]$, we obtain

$$r(q)\phi'''(q) - r(q_2)\phi'''(q_2) \leq -k \int_{q_2}^{q} h(\kappa)\left(1 - p(\theta(\kappa)) \frac{q_2(\eta(\theta(\kappa)))}{q_2(\theta(\kappa))}\right) d\kappa.$$

Using case (C), we have $\phi'''(q) < 0$ for $q \geq q_1$. Then, $r(q_2)\phi'''(q_2) < 0$, and so

$$\phi'''(q) \leq -\frac{k}{r(q)} \int_{q_2}^{q} h(\kappa)\left(1 - p(\theta(\kappa)) \frac{q_2(\eta(\theta(\kappa)))}{q_2(\theta(\kappa))}\right) d\kappa,$$

and then

$$\phi''(q) \leq \phi''(q_2) - k \int_{q_2}^{q} \frac{1}{r(\kappa)} \left(1 - p(\theta(\kappa)) \frac{q_2(\eta(\theta(\kappa)))}{q_2(\theta(\kappa))}\right) d\kappa \phi(\theta(\kappa)) d\kappa,$$

which with (10) gives $\lim_{q \to \infty} \phi''(q) = -\infty$, a contradiction with the positivity of $\phi''(q)$. Therefore, $\phi(q)$ converges to $\phi_{\text{ero}}$.

(B$_3$): Integrating (12) over $[q_2, q]$, and using (11), we find

$$r(q)\phi'''(q) \leq r(q_2)\phi'''(q_2) - r(q_2)\phi'(q_2) \int_{q_2}^{q} h(\kappa)\left(1 - p(\theta(\kappa)) \frac{q_2(\eta(\theta(\kappa)))}{q_2(\theta(\kappa))}\right) d\kappa$$

$$\leq r(q_2)\phi'''(q_2) - \delta_0 \phi(q_2) \int_{q_2}^{q} \frac{q_1(\kappa)}{q_2(\kappa)} d\kappa$$

$$\leq r(q_2)\phi'''(q_2) + \delta_0 \frac{\phi(q_2)}{q_2(q_2)} - \delta_0 \frac{\phi(q_2)}{q_2(q_2)},$$

which, with (B$_2$), gives

$$r(q)\phi'''(q) \leq -\delta_0 \frac{\phi(q)}{q_2(q_2)}.$$  \hspace{1cm} (13)

Thus, from (B$_1$) at $\kappa = 1$, we obtain

$$\frac{\phi'(q_2)}{q_1(q_2)} \leq -\delta_0 \frac{\phi(q)}{q_2(q_2)}.$$

Consequently,

$$\frac{d}{dq} \frac{\phi(q)}{q_2(q_2)} = \frac{1}{q_2^{2+k}(q_2)} (q_2(q_2)\phi'(q) + \delta_0 \phi_1(q)\phi(q)) \leq 0.$$

(B$_4$): Now, since $\phi/q_2^{2+k}$ is a positive decreasing function, we see that $\lim_{q \to \infty} \phi(q)/q_2^{2+k} = k_1 \geq 0$. Assume the contrary that $k_1 > 0$. Then, there is a $q_2 \geq q_1$ with $\phi(q)/q_2^{2+k} \geq k_1$ for $q \geq q_2$. Next, we define

$$G(q) := \frac{\phi(q) + r(q)\phi'''(q)q_2(q_2)}{q_2^{2+k}(q_2)}.$$
Then, from \((B_1), G(q) > 0\) for \(q \geq q_2\). Differentiating \(G(q)\) and using \((11)\) and \((B_1)\), we get

\[
\begin{align*}
G'(q) &= \frac{1}{\varphi_2^{\delta_0}(q)} \left[ \frac{\delta_1}{\varphi_2^{\delta_1}(q)} \left( \varphi'(q) - r(q)\varphi''(q)\varphi_1(q) + (r(q)\varphi'''(q))'\varphi_2(q) \right) \\
& \quad \quad + \varphi_2^{\delta_0}(q) \left( \varphi'(q) - r(q)\varphi''(q)\varphi_1(q) + (r(q)\varphi'''(q))'\varphi_2(q) \right) \right] \\
\leq \frac{\delta_0}{\varphi_2^{\delta_0+1}(q)} \left[ \varphi_2^{\delta_0}(q) \left( \varphi'(q) - r(q)\varphi''(q)\varphi_1(q) + (r(q)\varphi'''(q))'\varphi_2(q) \right) \right]
\end{align*}
\]

Combining \((14)\) and \((15)\), we get

\[
G'(q) \leq -\delta_0^2 k_1 \frac{\varphi_1(q)}{\varphi_2(q)} < 0.
\]

Integrating this inequality over \([q_2, q]\), we find

\[
-G(q_2) \leq -\delta_0^2 k_1 \ln \frac{\varphi_2(q_2)}{\varphi_2(q)} \rightarrow \infty \text{ as } q \rightarrow \infty.
\]

Then, we arrive at a contradiction, and so \(k_1 = 0\).

Therefore, the proof is complete. \(\square\)

**Theorem 2.** Assume that \(u\) is a positive solution of \((1)\) and \(\varphi\) satisfies case \((C)\) in Lemma 1, and that \((10)\) and \((11)\) hold for some \(\delta_0 \in (0, 1)\). If \(\delta_{i-1} \leq \delta_i < 1\) for all \(i = 1, 2, \ldots, m - 1\), then

\[
\begin{align*}
(B_{1,m}) & \quad \text{\(\varphi(q)/\varphi_2^{\delta_m}\) is decreasing;} \\
(B_{2,m}) & \quad \lim_{q \to \infty} \frac{\varphi(q)}{\varphi_2^{\delta_m}(q)} = 0,
\end{align*}
\]

where

\[
\delta_i = \delta_0 \frac{\lambda_{\delta_i}}{1 - \delta_{i-1}}, \quad j = 1, 2, \ldots, m
\]

and

\[
\frac{\varphi_2(m(\varphi_2)))}{\varphi_2(q)} \geq \lambda, \text{ for all } q \geq q_0,
\]

for some \(\lambda \geq 1\).

**Proof.** Assume that \(u\) is a positive solution of \((1)\) and \(\varphi\) satisfies case \((C)\) for \(q \geq q_1\) for some \(q_1 \in I_0\). Then, from Theorem 1, we have that \((B_1) - (B_4)\) hold. Using induction, we have from Theorem 1 that \((B_{1,0})\) and \((B_{2,0})\) hold. Now, we assume that \((B_{1,m-1})\) and \((B_{2,m-1})\) hold. Integrating \((12)\) over \([q_2, q]\), we find

\[
r(q)\varphi'''(q) \leq r(q_2)\varphi'''(q_2) - \int_{q_2}^q h(x) \left( 1 - p(\varphi(x)) \frac{\varphi_2(\theta(x)))}{\varphi_2(\varphi(x)))} \right) \varphi(\varphi(x)) \, dx. \tag{17}
\]
Using \((B_{1,m-1})\), we have that
\[
\phi(\theta(t)) \geq \frac{\phi(t)}{\phi_2^{\delta_m-1}(t)}.
\]
Then, (17) becomes
\[
r(\varrho)\phi'''(\varrho) \leq r(\varrho_2)\phi'''(\varrho_2) - \int_{\varrho_2}^{\varrho} h(\varpi) \left( 1 - p(\varpi(\varpi)) \frac{\varrho_2(\theta(\varpi)))}{\varrho_2(\theta(\varpi)))} \right) \frac{\phi(\varpi)}{\phi_2^{\delta_m-1}(\varpi)} d\varpi,
\]
which, with the fact that \(\phi/\phi_2^{\delta_m-1}\) is a decreasing function, gives
\[
r(\varrho)\phi'''(\varrho) \leq r(\varrho_2)\phi'''(\varrho_2) - \frac{\phi(t)}{\phi_2^{\delta_m-1}(t)} \int_{\varrho_2}^{\varrho} h(\varpi) \left( 1 - p(\varpi(\varpi)) \frac{\varrho_2(\theta(\varpi)))}{\varrho_2(\theta(\varpi)))} \right) \frac{\phi(\varpi)}{\phi_2^{\delta_m-1}(\varpi)} d\varpi.
\]
Hence, from (11) and (16), we obtain
\[
r(\varrho)\phi'''(\varrho) \leq \delta_m \frac{\phi(t)}{\phi_2^{\delta_m-1}(t)} \int_{\varrho_2}^{\varrho} h(\varpi) \left( 1 - p(\varpi(\varpi)) \frac{\varrho_2(\theta(\varpi)))}{\varrho_2(\theta(\varpi)))} \right) \frac{\phi(\varpi)}{\phi_2^{\delta_m-1}(\varpi)} d\varpi.
\]
which, with the fact that \(\lim_{t \to \infty} \phi(t)/\phi_2^{\delta_m-1}(t) = 0\), gives
\[
r(\varrho)\phi'''(\varrho) \leq -\delta_m \frac{\phi(t)}{\phi_2(t)}.
\]
Thus, from (B_1) at \(\kappa = 1\), we obtain
\[
\frac{\phi'(t)}{\phi_1(t)} \leq -\delta_m \frac{\phi(t)}{\phi_2(t)}.
\]
Consequently,
\[
\frac{d}{dt} \frac{\phi(t)}{\phi_2^{\delta_m}(t)} = \frac{1}{\phi_2^{\delta_m+1}(t)} \left( \phi_2(t)\phi'(t) + \delta_m \phi_1(t)\phi(t) \right) \leq 0.
\]
Proceeding as in the proof of (B_4) in Theorem 1, we can prove that \(\lim_{t \to \infty} \phi(t)/\phi_2^{\delta_m}(t) = 0\).
Therefore, the proof is complete. □

**Theorem 3.** Assume that \(u\) is a positive solution of (1) and \(\phi\) satisfies case (C) in Lemma 1, and that (10) and (11) hold for some \(\delta_0 \in (0,1)\). If \(\delta_{i-1} \leq \delta_i < 1\) for all \(i = 1, 2, \ldots, m-1\), then the DDE
\[
H'(t) + \frac{1}{(1-\delta_m)} h(t) \varrho_2(t) \left( 1 - p(\theta(t)) \frac{\varrho_2(\theta(t)))}{\varrho_2(\theta(t)))} \right) H(\theta(t)) = 0,
\]
as a positive solution, where \(\delta_j\) and \(\lambda\) are defined as in Theorem 2.

**Proof.** Assume that \(u\) is a positive solution of (1) and \(\phi\) satisfies case (C) for \(q \geq q_1\) for some \(q_1 \in I_0\). Then, from Theorem 2, we have that \((B_{1,m})\) and \((B_{2,m})\) hold.

Now, we define
\[
H(t) := r(t)\phi'''(t)\varrho_2(t) + \phi(t).
\]
Then, from \( \text{(B}_1 \) at \( \kappa = 2 \), \( H(\epsilon) > 0 \) for \( \epsilon \geq \epsilon_2 \), and
\[
H'(\epsilon) = (r(\epsilon)\phi'''(\epsilon))' \phi_2(\epsilon) - r(\epsilon)\phi''(\epsilon) \phi_1(\epsilon) + \phi'(\epsilon),
\]
which, with \( \text{(B}_1 \) at \( \kappa = 1 \), leads to
\[
H'(\epsilon) \leq (r(\epsilon)\phi'''(\epsilon))' \phi_2(\epsilon) \leq -h(\epsilon)\phi_2(\epsilon) \left( 1 - p(\theta(\epsilon)) \frac{\phi_2(\eta(\theta(\epsilon)))}{\phi_2(\theta(\epsilon))} \right) \phi(\epsilon)). \tag{21}
\]
As in the proof of Theorem 2, we arrive at (18). From (20) and (18), we get
\[
H(\epsilon) \leq (1 - \delta_m)\phi(\epsilon).
\]
Thus, (21) becomes
\[
H'(\epsilon) + \frac{1}{(1 - \delta_m)} h(\epsilon) \phi_2(\epsilon) \left( 1 - p(\theta(\epsilon)) \frac{\phi_2(\eta(\theta(\epsilon)))}{\phi_2(\theta(\epsilon))} \right) H(\theta(\epsilon)) \leq 0. \tag{22}
\]
Hence, \( H \) is a positive solution of the differential inequality (22). Using [20] (Theorem 1), the Equation (19) has also a positive solution, and this completes the proof. \( \square \)

3. Applications in the Oscillation Theory

In the following, we use our results in the previous section to obtain the criteria of oscillation for the solutions of (1).

**Theorem 4.** Assume that (10) and (11) hold for some \( \delta_0 \in (0, 1) \), and that \( \delta_j, \lambda \) are defined as in Theorem 2. If, \( \delta_{i-1} \leq \delta_i < 1 \) for all \( i = 1, 2, \ldots, m - 1 \) and the first-order differential Equations (19) and
\[
y'(\epsilon) + h(\epsilon) \left( \frac{\lambda_0(1 - p(\theta(\epsilon)))\theta(\epsilon)}{3r(\theta(\epsilon))} \right) y(\theta(\epsilon)) = 0 \tag{23}
\]
is oscillatory for some \( \lambda_0, \delta_i \in (0, 1) \) and that
\[
\limsup_{\epsilon \to \infty} \int_{\theta(\epsilon)}^{\epsilon} \left( h(\sigma) (1 - p(\theta(\sigma))) \right) \left( \frac{\lambda_1 \theta^2(\sigma)}{2!} \right) \phi_0(\sigma) - \frac{r^{-1}(\sigma)}{4\phi_0(\sigma)} \right) d\sigma = \infty \tag{24}
\]
holds for some \( \lambda_1 \in (0, 1) \), then, every solution of (1) is oscillatory.

**Proof.** Assume that \( u \) is a positive solution of (1). Then from Lemma 2, we get the cases (A)–(C). In view of [21], the fact that the solutions of Equation (23) oscillate and the condition (24) is fulfilled, rules out the cases (A) and (B), respectively. Then, we have (C) hold. Using Theorem 3, we get that Equation (19) has a positive solution, a contradiction. Therefore, the proof is complete. \( \square \)

**Corollary 1.** Assume that (10) and (11) hold for some \( \delta_0 \in (0, 1) \), and that \( \delta_j, \lambda \) are defined as in Theorem 2. If \( \delta_{i-1} \leq \delta_i < 1 \) for all \( i = 1, 2, \ldots, m - 1 \), (24),
\[
\liminf_{\epsilon \to \infty} \int_{\theta(\epsilon)}^{\epsilon} h(\sigma) \frac{\lambda_0(1 - p(\theta(\sigma)))\theta(\sigma)}{r(\theta(\sigma))} d\sigma > \frac{3!}{e} \tag{25}
\]
and
\[
\liminf_{\epsilon \to \infty} \int_{\theta(\epsilon)}^{\epsilon} h(\sigma) \phi_2(\sigma) \left( 1 - p(\theta(\sigma)) \frac{\phi_2(\eta(\theta(\sigma)))}{\phi_2(\theta(\sigma))} \right) d\sigma > \frac{(1 - \delta_m)}{e}, \tag{26}
\]
hold for some \( \lambda_0, \delta_m \in (0, 1) \), then every solution of (1) is oscillatory.
Proof. In view of [22, Corollary 2.1], Conditions (25) and (26) imply oscillation of the solutions of (23) and (19) respectively, thus rules out the cases (A) and (C). Now, since the condition (24) is fulfilled, rules out the case (B). Therefore, from Theorem 4, every solution of (1) is oscillatory. □

Example 1. Consider the differential equation

$$\left(e^\theta(u(\varrho) + p_0 u(\varrho - \eta_0))'''\right)' + h_0 e^\theta u(\varrho - \theta_0) = 0,$$  
(27)

where $\varrho \geq 1$, $h_0 < 1/(1 - p_0 e^{\eta_0})$, $p_0 \in (0, e^{-\eta_0})$ and $\theta_0, \eta_0 > 0$. It is easy to verify that $\varphi_i(\varrho) = e^{-\varrho}$, $i = 0, 1, 2$, and then

$$\int_{\theta_0}^{\infty} \frac{1}{\varrho(\varrho)} \left( \int_{\varrho_2}^{\varrho} h(\varrho) \left(1 - p(\theta(\varrho)) \frac{\varphi_2'(\theta(\varrho)))}{\varphi_2'(\theta(\varrho)))} \right) d\varrho \right) d\varrho = \int_{\theta_0}^{\infty} \frac{1}{\varrho} \left( \int_{\varrho_2}^{\varrho} h_0 e^{\varrho}(1 - p_0 e^{\eta_0}) d\varrho \right) d\varrho = \infty.$$

By choosing $\delta_0 = h_0(1 - p_0 e^{\eta_0})$, we find that (11) holds. Moreover, by a simple computation, we have that (25) and (24) are satisfied.

Now, we note that (26) reduces to

$$\lim_{\varrho \to \infty} \int_{\varrho(\varrho)}^{\infty} \varphi_2(\varrho) \left(1 - p(\theta(\varrho)) \frac{\varphi_2'(\theta(\varrho)))}{\varphi_2'(\theta(\varrho)))} \right) d\varrho = \int_{\theta_0}^{\varrho_2} h_0(1 - p_0 e^{\eta_0}) d\varrho > \frac{(1 - \delta_0)}{e},$$

which is satisfied if

$$h_0(1 - p_0 e^{\eta_0}) \theta_0 > \frac{(1 - \delta_0)}{e}. \quad (28)$$

Thus, using Corollary 1, every solution of (27) is oscillatory if (28) holds.

Remark 2. By reviewing the results in [23] and by choosing $\eta_0 = 2$ and $p_0 = 0.12$ we have that (27) is oscillatory if $h_0 > 2.2063$. It is easy to note that this condition essentially neglects the influence of delay argument $\theta(1)$. However, our criterion (28) takes into account the influence of $\theta_0$. Furthermore, using (28) and $m = 0$, every solution of the differential equation:

$$\left(e^\theta(u(\varrho) + 0.12u(\varrho - 2))'''\right)' + 2e^\theta u(\varrho - 3) = 0$$

is oscillatory, despite the failure of the results in [23].

Remark 3. Consider the differential equation

$$\left(e^\theta(u(\varrho) + 0.12u(\varrho - 2))'''\right)' + 0.9e^\theta u(\varrho - 3) = 0, \quad (29)$$

Note that, the condition (28), with $m = 0$, reduces to

$$(0.9) \left(1 - (0.12)e^2\right) \varphi_2(\theta(\varrho)) = \frac{(1 - (0.9)(1 - (0.12)e^2))}{e},$$

which is not satisfied, and thus, the oscillatory behavior of (29) cannot be verified. However, using the iterative nature of (28), we find that

$$\frac{\varphi_2'(\theta(\varrho)))}{\varphi_2'(\varrho))} = \frac{e^{-e} \eta_0}{e^{-e}} = e^{\eta_0} > e\varrho_0 = \lambda$$
and }\delta_3 = 0.16953.\text{ Now, the condition (28) with } m = 3 \text{ reduces to }
(0.9)\left(1 - (0.12)e^2\right)3 > \frac{(1 - 0.16953)}{e},

which is satisfied. Hence, every solution of (29) is oscillatory.

Remark 4. In the non-neutral case, that is, }p_0 = 0\text{, the oscillation condition of the Equation (27) becomes: }
h_0\theta_0 > \frac{(1 - \delta_m)}{e},

(1) If } m = 0 \text{ and } \theta_0 = 1.10366, \text{ then we have the same condition obtained in [6,10].}
(2) If } m = 0 \text{ and } \theta_0 > 1.10366, \text{ then our results improve results in [6,10].}

4. Conclusions

In this work, we studied the oscillatory behavior of a class of fourth-order neutral differential equations were presented. In the noncanonical case, we obtained new criteria based on comparison principles that ensure the oscillation of all solutions of the studied equation. By comparing with previous results, we found that our results are easy to apply and do not require unknown functions. Moreover, the new criteria have an iterative nature. An interesting problem is to extend our results to even-order neutral differential equations.

Author Contributions: All authors contributed equally to this manuscript. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: Research Supporting Project number (RSP-2022/167), King Saud University, Riyadh, Saudi Arabia.

Conflicts of Interest: The authors declare no conflict of interest.

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