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Statistically Convergent Sequences in Intuitionistic Fuzzy Metric Spaces

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Abstract: In this paper, we introduce the concepts of statistical convergence and statistical Cauchy sequences with respect to the intuitionistic fuzzy metric spaces inspired by the idea of statistical convergence in fuzzy metric spaces. Then, we give useful characterizations for statistically convergent sequences and statistically Cauchy sequences.

Keywords: statistical convergence; intuitionistic fuzzy metric; statistically convergent sequence; statistically Cauchy sequence

MSC: 40A05; 54A40; 54E50

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1. Introduction

Zadeh [1] introduced the theory of fuzzy sets and after that many authors discussed concepts of fuzzy sets in different areas, one of them being fuzzy metric space [2]. By using continuous t-norms George and Veeramani [3] modified the concept of fuzzy metric space introduced by Kramosil and Michalek [2]. Many researchers have studied in this field [4–6]. In 2004, using the idea of the intuitionistic fuzzy set [7], the concept of fuzzy metric space [3] was extended to the concept of intuitionistic fuzzy metric space by Park [8]. Park defined this concept with the help of continuous t-norms and continuous t-conorms. A lot of developments such as fixed point theorems and convergence have been studied with fuzzy metric spaces and intuitionistic fuzzy metric spaces [9–16].

The notion of statistical convergence was introduced by Fast [17] and Steinhous [18] in 1951 independently, and this idea drew attention from mathematicians working in both fields of pure and applied mathematics. As a generalization of the concept of convergence, statistical convergence is defined as: Let $K \subseteq IN$. $\forall n \in IN, K(n) = \{k \leqslant n : k \in K\}$. The natural (or asymptotic) density of K is defined by $\delta(K) = \lim_{n\to\infty} \frac{|K(n)|}{n}$ if the limit exists, where $|\ K(n)\ |$ denotes the cardinality of the set $K(n)$. $\delta(K) \in [0,1]$ and $\delta(IN \backslash K) = 1 - \delta(K)$ if $\delta(K)$ exists. For instance, $\delta(IN) = 1$, $\delta(A) = \frac{1}{2}$, where A is an even natural number and $\delta(B) = 0$, where B is a finite subset of IN. K is called statistically dense provided that $\delta(K) = 1$. A sequence $(x_n) \subset IR$ is called statistically convergent to $x_0 \in IR$ if $\delta(\{n \in IN : | x_n - x_0 | < \epsilon\}) = 1$ for each $\epsilon > 0$. There have been many important results on statistical convergence by many authors ([19–21]).

In 2020, Changqing et al. [22] introduced statistically convergent sequences in fuzzy metric spaces. In view of this, we pay attention to statistical convergence on intuitionistic fuzzy metric spaces with this study. Then, we analyze relations of convergence and statistical convergence on intuitionistic fuzzy metric spaces. Further, we study statistical Cauchy sequences and statistical completeness on intuitionistic fuzzy metric spaces.

2. Intuitionistic Fuzzy Metric Space

In this section, we give some basic definitions and notions to explain main results. Throughout the paper, IR and IN will denote the set of all real numbers and the set of all positive integer numbers, respectively.

Definition 1 ([23]). *A binary operation* $* : [0,1] \times [0,1] \to [0,1]$ *is called a continuous t-norm if* $*$ *satisfies the following:*

(1) $a * 1 = a,\ \forall a \in [0,1]$;
(2) $a * b = b * a$ *and* $a * (b * c) = (a * b) * c\ \forall a, b, c \in [0,1]$;
(3) *If* $a \leq c$ *and* $b \leq d$, *then* $a * b \leq c * d,\ \forall a, b, c, d \in [0,1]$;
(4) $*$ *is continuous.*

Definition 2 ([23]). *A binary operation* $\diamond : [0,1] \times [0,1] \to [0,1]$ *is called a continuous t-conorm if* $\diamond$ *satisfies the following:*

(1) $a \diamond 0 = a,\ \forall a \in [0,1]$;
(2) $a \diamond b = b \diamond a$ *and* $a \diamond (b \diamond c) = (a \diamond b) \diamond c\ \forall a, b, c \in [0,1]$;
(3) *If* $a \leq c,\ b \leq d$, *then* $a \diamond b \leq c \diamond d,\ \forall a, b, c, d \in [0,1]$;
(4) $\diamond$ *is continuous.*

Note that $a * b = min\{a, b\}, a \diamond b = max\{a, b\}, a * b = ab$ and $a \diamond b = min\{a + b, 1\}$ are basic examples of continuous t-norms and continuous t-conorms for all $a, b \in [0,1]$.

From the previous two definitions, we see that if $r_1 > r_2$, then there exist $r_3, r_4 \in (0,1)$ such that $r_1 * r_3 \geq r_2$ and $r_2 \diamond r_4 \leq r_1$.

Definition 3 ([7]). *An intuitionistic fuzzy set A is defined by* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ *where* $\mu_A : X \to [0,1]$ *and* $\nu_A : X \to [0,1]$ *denote membership and nonmembership functions respectively.* $\mu_A(x)$ *and* $\nu_A(x)$ *are membership and nonmembership degrees of each element* $x \in X$ *to the intuitionistic fuzzy set A and* $\mu_A(x) + \nu_A(x) \leq 1$ *for each* $x \in X$.

Definition 4 ([8]). *Let M and N be fuzzy sets on* $X^2 \times (0, \infty)$, $*$ *be a continuous t-norm,* $\diamond$ *be a continuous t-conorm. If M and N satisfy the following conditions, we say that* (M, N) *is intuitionistic fuzzy metric on X:*

(IF1) $M(x, y, t) + N(x, y, t) \leq 1$;
(IF2) $M(x, y, t) > 0$;
(IF3) $M(x, y, t) = 1$ *if and only if* $x = y$;
(IF4) $M(x, y, t) = M(y, x, t)$;
(IF5) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;
(IF6) $M(x, y, .) : (0, \infty) \to (0,1]$ *is continuous;*
(IF7) $N(x, y, t) > 0$;
(IF8) $N(x, y, t) = 0$ *if and only if* $x = y$;
(IF9) $N(x, y, t) = N(y, x, t)$;
(IF10) $N(x, y, t) \diamond N(y, z, s) \geq N(x, z, t + s)$;
(IF11) $N(x, y, .) : (0, \infty) \to (0,1]$ *is continuous.*

A 5-tuple $(X, M, N, *, \diamond)$ *is called intuitionistic fuzzy metric space.*

The functions $M(x, y, t)$ *and* $N(x, y, t)$ *denote the degree of nearness and the degree of non-nearness between x and y with respect to t, respectively.*

Remark 1. *Let* $(X, M, N, *, \diamond)$ *be an intuitionistic fuzzy metric space. Then* $(X, M, *)$ *is a fuzzy metric space. Conversely, if* $(X, M, *)$ *is a fuzzy metric space, then* $(X, M, 1 - M, *, \diamond)$ *is an intuitionistic fuzzy metric space, where* $a \diamond b = 1 - ((1 - a) * (1 - b)),\ \forall a, b \in [0,1]$.

Definition 5 ([8]). *Let* $(X, M, N, *, \diamond)$ *be an intuitionistic fuzzy metric space and* $t > 0, r \in (0,1)$ *and* $x \in X$. *The set* $B_x(r, t) = \{y \in X : M(x, y, t) > 1 - r, N(x, y, t) < r\}$ *is said to be an open ball with center x and radius r with respect to t.*

$\{B_x(r, t) : x \in X, r \in (0,1), t > 0\}$ *generates a topology* $\tau_{(M,N)}$ *called the (M,N) topology.*

Definition 6 ([8]). *Let* $(X, M, N, *, \diamond)$ *be an intuitionistic fuzzy metric space.*

(i) (x_n) *is called convergent to* x *if for all* $t > 0$ *and* $r \in (0,1)$ *there exists* $n_0 \in IN$ *such that* $M(x_n, x, t) > 1 - r$ *and* $N(x_n, x, t) < r$ *for all* $n \geq n_0$.
It is denoted by $x_n \to x$ *as* $n \to \infty$.
$*$ $M(x_n, x, t) \to 1$ *and* $N(x_n, x, t) \to 0$ *as* $n \to \infty$ *for each* $t > 0$.
(ii) (x_n) *is called a Cauchy sequence if, for* $t > 0$ *and* $r \in (0,1)$, *there exists* $n_0 \in IN$ *such that* $M(x_n, x_m, t) > 1 - r$ *and* $N(x_n, x_m, t) < r$ *for all* $n, m \geq n_0$.
(iii) $(X, M, N, *, \diamond)$ *is called (M,N)-complete if every Cauchy sequence is convergent.*

Definition 7 ([22])**.** *Let* $(X, M, *)$ *be a fuzzy metric space.*

(i) *A sequence* $(x_n) \subset X$ *is called statistically convergent to* $x_0 \in X$ *if* $\delta(\{n \in IN : M(x_n, x_0, t) > 1 - r\}) = 1$ *for every* $r \in (0,1)$ *and* $t > 0$.
(ii) *A sequence* $(x_n) \subset X$ *is called a statistically Cauchy sequence if, for every* $r \in (0,1)$ *and* $t > 0$, *there exists* $m \in IN$ *such that* $\delta(\{n \in IN : M(x_n, x_m, t) > 1 - r\}) = 1$.

3. Statical Convergence in Intuitionistic Fuzzy Metric Space

In this section, we study statistically convergent sequences on intuitionistic fuzzy metric spaces.

Definition 8. *Let* $(X, M, N, *, \diamond)$ *be an intuitionistic fuzzy metric space. A sequence* $(x_n) \subset X$ *is called statistically convergent to* $x_0 \in X$ *with respect to the intuitionistic fuzzy metric provided that, for every* $r \in (0,1)$ *and* $t > 0$,

$\delta(\{n \in IN : M(x_n, x_0, t) > 1 - r, N(x_n, x_0, t) < r\}) = 1.$

We say that (x_n) *is statically convergent to* x_0. *We see that* $\delta(\{n \in IN : M(x_n, x_0, t) > 1 - r, N(x_n, x, t) < r\}) = 1 \Leftrightarrow \lim_{n\to\infty} \frac{|\{k \leq n : M(x_k, x_0, t) > 1 - r,\ N(x_k, x_0, t) < r\}|}{n} = 1$

Example 1. *Let* $X = IR$, $a * b = ab$ *and* $a \diamond b = min\{a + b, 1\}$ *for all* $a, b \in [0,1]$. *Define* M *and* N *by* $M(x, y, t) = \frac{t}{t+|x-y|}$ *and* $N(x, y, t) = \frac{|x-y|}{t+|x-y|}$ *for all* $x, y \in X$ *and* $t > 0$. *Then* $(IR, M, N, *, \diamond)$ *is an intuitionistic fuzzy metric space.*

Now define a sequence (x_n) *by*

$$x_n = \begin{cases} 1, & n = k^2, k \in IN; \\ 0, & otherwise \end{cases}.$$

Then, for every $r \in (0,1)$ *and for any* $t > 0$, *let* $K = \{n \leq m : M(x_n, 0, t) \leq 1 - r, N(x_n, 0, t) \geq r\} = \{n \leq m : \frac{t}{t+|x_n|} \leq 1 - r, \frac{|x_n|}{t+|x_n|} \geq r\} = \{n \leq m : |x_n| \geq \frac{rt}{1-r} > 0\} = \{n \leq m : x_n = 1\} = \{n \leq m : n = k^2, k \in IN\}$, *and we obtain* $\frac{1}{m}|K| \leqslant \frac{1}{m}|\{n \leqslant m : n = k^2, n \in IN\}| \leqslant \frac{\sqrt{m}}{m} \to 0, m \to \infty$. *Hence, we obtain that* (x_n) *is statistically convergent to* 0 *with respect to the intuitionistic fuzzy metric space* $(X, M, N, *, \diamond)$.

Lemma 1. *Let* $(X, M, N, *, \diamond)$ *be an intuitionistic fuzzy metric space. The, for every* $r \in (0,1)$ *and* $t > 0$, *the following are equivalent:*

(i) (x_n) *is statistically convergent to* x_0;
(ii) $\delta(\{n \in IN : M(x_n, x_0, t) \leq 1 - r\}) = \delta(\{N(x_n, x_0, t) \geq r\}) = 0$;
(iii) $\delta(\{n \in IN : M(x_n, x_0, t) > 1 - r\}) = \delta(\{N(x_n, x_0, t) < r\}) = 1$.

Proof. Using Definition 8 and properties of density, we have the lemma. □

Theorem 1. *Let* $(X, M, N, *, \diamond)$ *be an intuitionistic fuzzy metric space. If a sequence* (x_n) *is statistically convergent with respect to the intuitionistic fuzzy metric, then the statistically convergent limit is unique.*

Proof. Suppose that (x_n) is statistically convergent to x_1 and x_2. For a given $r \in (0,1)$, chose $t > 0$ such that $(1 - t) * (1 - t) > 1 - r$ and $t \diamond t < r$.

Then define the following sets for any $\epsilon > 0$:
$K_{M1}(t,\epsilon) := \{n \in IN : M(x_n, x_1, \epsilon) > 1 - t\}$
$K_{M2}(t,\epsilon) := \{n \in IN : M(x_n, x_2, \epsilon) > 1 - t\}$
$K_{N1}(t,\epsilon) := \{n \in IN : N(x_n, x_1, \epsilon) < t\}$
$K_{N2}(t,\epsilon) := \{n \in IN : N(x_n, x_2, \epsilon) < t\}$
Since (x_n) is statistically convergent with respect to x_1 and x_2, we obtain
$\delta\{K_{M1}(t,\epsilon)\} = \delta\{K_{N1}(t,\epsilon)\} = 1$ and $\delta\{K_{M2}(t,\epsilon)\} = \delta\{K_{N2}(t,\epsilon)\} = 1$, for all $\epsilon > 0$.
Let $K_{MN}(t,\epsilon) := \{K_{M1}(t,\epsilon) \cup K_{M2}(t,\epsilon)\} \cap \{K_{N1}(t,\epsilon) \cup K_{N2}(t,\epsilon)\}$.
Hence, $\delta\{K_{MN}(t,\epsilon)\} = 1$ which implies that $\delta\{IN \backslash K_{MN}(t,\epsilon)\} = 0$.
If $n \in IN\backslash K_{MN}(t,\epsilon)$, then we have two options:
$n \in IN\backslash\{K_{M1}(t,\epsilon) \cup K_{M2}(t,\epsilon)\}$ or $n \in IN\backslash\{K_{N1}(t,\epsilon) \cup K_{N2}(t,\epsilon)\}$.
Let us consider $n \in IN\backslash\{K_{M1}(t,\epsilon) \cup K_{M2}(t,\epsilon)\}$. Then we obtain
$M(x_1, x_2, \epsilon) \geq M(x_1, x_n, \frac{\epsilon}{2}) * M(x_n, x_2, \frac{\epsilon}{2}) > (1-t) * (1-t) > 1 - r$.

Therefore, $M(x_1, x_2, \epsilon) > 1 - r$ and since $r > 0$ is arbitrary, we obtain $M(x_1, x_2, \epsilon) = 1$ for all $\epsilon > 0$, which implies $x_1 = x_2$.

Now let us consider $n \in IN\backslash\{K_{N1}(t,\epsilon) \cup K_{N2}(t,\epsilon)\}$. Then, $N(x_1, x_2, \epsilon) \leq N(x_1, x_n, \epsilon) \diamond N(x_n, x_2, \epsilon) < t \diamond t < r$. Since $r > 0$ is arbitrary, we obtain $N(x_1, x_2, \epsilon) = 0$ for all $\epsilon > 0$, which implies $x_1 = x_2$. This completes the proof. □

Theorem 2. *Let (x_n) be a sequence in an intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$. If (x_n) is convergent to x_0 with respect to the intuitionistic fuzzy metric, then (x_n) is statistically convergent to x_0 with respect to the intuitionistic fuzzy metric.*

Proof. Let (x_n) be convergent to x_0. Then for every $r \in (0,1)$ and $t > 0$, there exists $n_0 \in IN$ such that $M(x_n, x_0, t) > 1 - r$ and $N(x_n, x_0, t) < r$. We have $|\{k \leq n : M(x_n, x_0, t) > 1 - r \text{ and } N(x_n, x_0, t) < r\}| \geq n - n_0$.

Hence, the set $\{k \leq n : M(x_n, x_0, t) > 1 - r \textit{ and } N(x_n, x_0, t) < r\}$ has a finite number of terms.

Then, $\lim_{n\to\infty} \frac{|\{k \leq n: M(x_n, x_0, t) > 1-r, N(x_n, x_0, t) < r\}|}{n} \geq \lim_{n\to\infty} \frac{n - n_0}{n} = 1$.
Consequently, $\delta(\{n \in IN : M(x_n, x_0, t) > 1 - r, N(x_n, x_0, t) < r\}) = 1$. □

The converse of the theorem need not hold.

Example 2. *Let $X = [1,3]$, $a * b = ab$ and $a \diamond b = min\{a + b, 1\}$ for all $a, b \in [0,1]$. Define M and N by $M(x,y,t) = \frac{t}{t+|x-y|}$ and $N(x,y,t) = \frac{|x-y|}{t+|x-y|}$ for all $x, y \in X$ and $t > 0$. Then $(IR, M, N, *, \diamond)$ is an intuitionistic fuzzy metric space.*

Now define a sequence (x_n) by
$$x_n = \begin{cases} 2, & n = k^2, k \in IN; \\ 1, & otherwise \end{cases}.$$
We can see that (x_n) is not convergent to 1.

We need to show that (x_n) is statistically convergent to 1. Let $r \in (0,1)$ and $t > 0$. $K = \{n \in IN : M(x_n, 1, t) > 1 - r,\ N(x_n, 1, t) < r\}$.

Case 1. $r \in (0, \frac{1}{t+1}]$. If $n \neq k^2$ for all $k \in IN$, then $M(x_n, 1, t) = 1 > 1 - r$ and $N(x_n, 1, t) = 0 < r$. If $n = k^2$ for some $k \in IN$, then $M(x_n, 1, t) = \frac{t}{t+1} = 1 - \frac{1}{t+1} \leq 1 - r$ and $N(x_n, 1, t) = \frac{1}{t+1} \geq r$.

Now, let $n \in IN$. If $n = k_0^2$ for an $k_0 \in IN$, then $\lim_{n\to\infty} \frac{|K(n)|}{n} = \lim_{k_0\to\infty} \frac{k_0^2 - k_0}{k_0^2} = 1$. If $n \neq k^2$ for all $k \in IN$, then we can obtain $k_1 \in IN$ such that $n = k_1^2 - l$ with $l \in IN$ and $1 \leq l \leq k_1$. $\lim_{n\to\infty} \frac{|K(n)|}{n} = \lim_{k_1\to\infty} \frac{k_1^2 - l - (k_1 - l)}{k_1^2 - l} = \lim_{k_1\to\infty} \frac{k_1^2 - k_1 - l + 1}{k_1^2 - l} = 1$.

Case 2. $r \in (\frac{1}{t+1}, 1)$. If $n \neq k^2$ for all $k \in IN$, then $M(x_n, 1, t) = 1 > 1 - r$ and $N(x_n, 1, t) = 0 < r$. If $n = k^2$ for some $k \in IN$, then $M(x_n, 1, t) = \frac{t}{t+1} = 1 - \frac{1}{t+1} > 1 - r$ and $N(x_n, 1, t) = \frac{1}{t+1} < r$. Hence, $M(x_n, 1, t) > 1 - r$ and $N(x_n, 1, t) < r$ for all $n \in IN$. Therefore, $\lim_{n\to\infty} \frac{|K(n)|}{n} = \lim_{n\to\infty} \frac{n}{n} = 1$.

Therefore, $\delta(\{n \in IN : M(x_n, 1, t) > 1 - r,\ N(x_n, 1, t) < r\}) = 1$ *for all* $r \in (0,1)$ *and* $t > 0$.

Theorem 3. *Let* (x_n) *be a sequence in an intuitionistic fuzzy metric space* $(X, M, N, *, \diamond)$. *Then* (x_n) *statistically converges to* x_0 *if and only if there exists an increasing index sequence* $A = \{n_i\}_{i \in IN}$ *of the natural numbers such that* (x_{n_i}) *converges to* x_0 *and* $\delta(A) = 1$.

Proof. Assume that (x_n) statistically converges to x_0.

Let $K_{MN}(j,t) := \{n \in IN : M(x_n, x_0, t) > 1 - \frac{1}{j}$ *and* $N(x_n, x_0, t) < \frac{1}{j}\}$, for any $t > 0$ and $j \in IN$.

We show that $K_{MN}(j+1,t) \subset K_{MN}(j,t)$ for $t > 0$, $j \in IN$. Since (x_n) statistically converges to x_0,

$$\delta(K_{MN}(j,t)) = 1 \tag{1}$$

Take $s_1 \in K_{MN}(1,t)$. Since $\delta(K_{MN}(2,t)) = 1$ (by Equation (1)) we have a number $s_2 \in (K_{MN}(2,t)$ $(s_2 > s_1)$ such that

$\frac{|\{k \leq n : M(x_k, x_0, t) > 1 - \frac{1}{2},\ N(x_k, x_0, t) < \frac{1}{2}\}|}{n} > \frac{1}{2}$, for all $n \geq s_2$.

Again by Equation (1), $\delta(K_{MN}(3,t)) = 1$ and we can choose $s_3 \in K_{MN}(3,t)$ $(s_3 > s_2)$ such that

$\frac{|\{k \leq n : M(x_k, x_0, t) > 1 - \frac{1}{3},\ N(x_k, x_0, t) < \frac{1}{3}\}|}{n} > \frac{2}{3}$, for all $n \geq s_3$ and we continue like this. Then, we can obtain an increasing index sequence $\{s_j\}_{j \in IN}$ of the natural numbers such that $s_j \in (K_{MN}(j,t))$. We also have following;

$$\frac{|\{k \leq n : M(x_k, x_0, t) > 1 - \frac{1}{j},\ N(x_k, x_0, t) < \frac{1}{j}\}|}{n} > \frac{j-1}{j}, \textit{for all } n \geq s_j,\ j \in IN \tag{2}$$

Now we obtain the increasing index sequence A as

$A := \{n \in IN : 1 < n < s_1\} \cup \{\bigcup_{j \in IN}\{n \in (K_{MN}(j,t) : s_j \leq n < s_{j+1}\}\}$.

By Equation (2) and $K_{MN}(j+1,t) \subset K_{MN}(j,t)$, we write

$\frac{|\{k \leq n : k \in A\}|}{n} \geq \frac{|\{k \leq n : M(x_k, x_0, t) > 1 - \frac{1}{j},\ N(x_k, x_0, t) < \frac{1}{j}\}|}{n} > \frac{j-1}{j}$ for all n, $(s_j \leq n < s_{j+1})$.

Since $j \to \infty$, when $n \to \infty$, we have $\lim_{n \to \infty} \frac{|\{k \leq n : k \in A\}|}{n} = 1$, i.e., $\delta(A) = 1$.

Now we show that (x_{n_i}) converges to x_0. Let $r \in (0,1)$ and $t > 0$. Take $N_0 > s_2$ large enough that for some $l_0 \in IN$, $s_{l_0} \leq N_0 < s_{l_0+1}$ with $\frac{1}{l_0} < r$. Assume that $n_m \geq N_0$ with $n_m \in A$. By the definition of A, there exists $l \in IN$ such that $s_l \leq n_m < s_{l+1}$ with $n_m \in K_{MN}(l,t)$, $(l \geq l_0)$. Then, we obtain

$M(x_{n_m}, x_0, t) \geq M(x_{n_m}, x_0, \frac{1}{l_0}) \geq M(x_{n_m}, x_0, \frac{1}{l}) > 1 - \frac{1}{l} \geq 1 - \frac{1}{l_0} > 1 - r$ and $N(x_{n_m}, x_0, t) > \frac{1}{l_0} < r$. Therefore, (x_{ni}) converges to x_0.

Conversely, assume that there exists an increasing index sequence $A = \{n_i\}_{i \in IN}$ of the natural numbers such that $\delta(A) = 1$ and (x_{n_i}) converges to x_0. Let $r \in (0,1)$ and $t > 0$. Then, there is a number $n_0 \in IN$ such that for each $n \geq n_0$, the inequalities $M(x_{n_i}, x_0, t) > 1 - r$ and $N(x_{n_i}, x_0, t) < r$ are satisfied.

Let us define $K_{MN}(r,t) := \{n \in IN : M(x_{n_i}, x_0, t) \leq 1 - r$ *or* $N(x_{n_i}, x_0, t) \geq r\}$. We have

$K_{MN}(r,t) \subset IN \backslash \{n_{n_0}, n_{n_0+1}, n_{n_0+2}, ...\}$. Since $\delta(A) = 1$, we have $\delta(IN \backslash \{n_{n_0}, n_{n_0+1}, n_{n_0+2}, ...\}) = 0$, so we deduce $\delta(K_{MN}(r,t)) = 0$. Hence,

$\delta(\{n \in IN : M(x_n, x_0, t) < 1 - r$ *and* $n(x_n, x_0, t) < r\}) = 1$.

Therefore, (x_n) statistically converges to x_0. □

Corollary 1. *Let* (x_n) *be a sequence in an intuitionistic fuzzy metric space* $(X, M, N, *, \diamond)$. *If* (x_n) *is statistically convergent to* x_0 *and it is convergent, then* (x_n) *converges to* x_0.

Definition 9. *Let* $(X_1, M_1, N_1, *_1, \diamond_1)$ *and* $(X_2, M_2, N_2, *_2, \diamond_2)$ *be two intuitionistic fuzzy metric spaces.*

(i) *A mapping* $f : X_1 \to X_2$ *is called an isometry if for each* $x, y \in X_1$ *and* $t > 0$, $M_1(x, y, t) = M_2(f(x), f(y), t)$ *and* $N_1(x, y, t) = N_2(f(x), f(y), t)$.

(ii) $(X_1, M_1, N_1, *_1, \diamond_1)$ *and* $(X_2, M_2, N_2, *_2, \diamond_2)$ *are called isometric if there exists an isometry from* X_1 *onto* X_2.

(iii) *An intuitionistic fuzzy completion of* $(X_1, M_1, N_1, *_1, \diamond_1)$ *is a complete intuitionistic fuzzy metric space* $(X_2, M_2, N_2, *_2, \diamond_2)$ *such that* $(X_1, M_1, N_1, *_1, \diamond_1)$ *is isometric to a dense subspace of* X_2.

(iv) $(X_1, M_1, N_1, *_1, \diamond_1)$ *is called completable if it leads to an intuitionistic fuzzy metric completion.*

Proposition 1. *Let* (x_n) *be a sequence in a completable intuitionistic fuzzy metric space* $(X, M, N, *, \diamond)$. *If* (x_n) *is Cauchy sequence in X and it is statistically converges to* x_0, *then* (x_n) *converges to* x_0.

Proof. Let $(X_1, M_1, N_1, *_1, \diamond_1)$ be the completion of $(X, M, N, *, \diamond)$. Then $\exists\, x_1 \in X_1 :\ (x_n)$ converges to x_1. We have $M_1(x_n, x_0, t) = M(x_n, x_0, t)$ and $N_1(x_n, x_0, t) = N(x_n, x_0, t)$ for all $t > 0$ and $n \in IN$.

Let $r \in (0, 1)$ and $t > 0$. Since $\delta(\{n \in IN : M(x_n, x_0, t) > 1 - r \textit{ and } N(x_n, x_0, t) < r\}) = 1$, we obtain $\delta(\{n \in IN : M_1(x_n, x_0, t) > 1 - r \textit{ and } N_1(x_n, x_0, t) < r\}) = 1$. Hence, we see that (x_n) statistically converges to $x_0 \in X_1$ with respect to (M_1, N_1). By Corollary 1, we have $x_1 = x_0$. □

4. Statically Complete Intuitionistic Fuzzy Metric Space

In this section, we give the concept of a statistical Cauchy sequence on an intuitionistic fuzzy metric space and study a characterization.

Definition 10. *Let* $(X, M, N, *, \diamond)$ *be an intuitionistic fuzzy metric space. A sequence* $(x_n) \subset X$ *is called a statistically Cauchy sequence if, for every* $r \in (0, 1)$ *and* $t > 0$, *there exists* $m \in IN$ *such that* $\delta(\{n \in IN : M(x_n, x_m, t) > 1 - r, N(x_n, x_m, t) < r\}) = 1$.

Theorem 4. *Let* (x_n) *be a sequence in an intuitionistic fuzzy metric space* $(X, M, N, *, \diamond)$. *Then the following are equivalent:*

(i) (x_n) *is statistically Cauchy.*

(ii) *There exists an increasing index sequence* $K = \{n_i\}_{i \in IN}$ *of the natural numbers such that* (x_{ni}) *is Cauchy and* $\delta(K) = 1$.

Proof. Straightforward. □

Theorem 5. *Let* (x_n) *be a sequence in an intuitionistic fuzzy metric space* $(X, M, N, *, \diamond)$. *If* (x_n) *is statistically convergent with respect to the intuitionistic fuzzy metric, then* (x_n) *is statistically Cauchy with respect to the intuitionistic fuzzy metric.*

Proof. Let (x_n) be statistically convergent to x_0 and $r \in (0, 1), t > 0$. Then, $\exists r_1 \in (0, 1) : (1 - r_1) * (1 - r_1) > 1 - r$ and $r_1 \diamond r_1 < r$. We have $\delta(\{n \in IN : M(x_n, x_0, t) > 1 - r, N(x_n, x_0, t) < r\}) = 1$. From Theorem 1, there exists an increasing index sequence $\{n_i\}_{i \in IN}$ such that (x_{n_i}) converges to x_0. Hence, $\exists n_{i_0} \in \{n_i\}_{i \in IN} : M(x_{n_i}, x_0, \frac{t}{2}) > 1 - r_1$ and $N(x_{n_i}, x_0, \frac{t}{2}) < r_1$ for all $n_i \geq n_{i_0}$. Since

$M(x_n, x_{n_{i_0}}, t) \geq M(x_n, x_0, \frac{t}{2}) * M(x_0, x_{n_{i_0}}, \frac{t}{2}) \geq (1 - r_1) * (1 - r_1) > 1 - r$ and

$N(x_n, x_{n_{i_0}}, t) \leq N(x_n, x_0, \frac{t}{2}) \diamond N(x_0, x_{n_{i_0}}, \frac{t}{2}) < r_1 \diamond r_1 < r$, we have $\delta(\{n \in IN : M(x_n, x_{n_{i_0}}, t) > 1 - r, N(x_n, x_{n_{i_0}}, t)\}) = 1$. Therefore, (x_n) is statistically Cauchy with respect to the intuitionistic fuzzy metric. □

Remark 2. *If a sequence is Cauchy in an intuitionistic fuzzy metric space, then it is statistically Cauchy.*

Definition 11. *The intuitionistic fuzzy metric space* $(X, M, N, *, \diamond)$ *is called statistically complete if every statistically Cauchy sequence in* X *is statistically convergent.*

Theorem 6. *Let* $(X, M, N, *, \diamond)$ *be an intuitionistic fuzzy metric space. If* X *is statistically complete, then it is complete with respect to the intuitionistic fuzzy metric.*

Proof. The proof is similar to Theorem 5. □

5. Conclusions

Fast and Steinhaus introduced the concept of statistical convergence in 1951 independently, and then many authors became interested in the subject and researched it in different fields of mathematics. In 2020, Changqing et al. introduced the concept of statistical convergence in fuzzy metric spaces. In view of this, we have discussed generalizing this convergence to intuitionistic fuzzy metric spaces. We have defined the concepts of statistical convergence, statistical Cauchy sequences and statistical completeness with respect to intuitionistic fuzzy metric spaces. In addition, we have studied characterizations for statistically convergent sequences and statistically Cauchy sequences.

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