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Strong Chromatic Index of Outerplanar Graphs

Ying Wang 1,†, Yiqiao Wang 2,‡, Weifan Wang 3,*,# and Shuyu Cui 3

1 School of Mathematics and Information Technology, Hebei Normal University of Science and Technology, Qinhuangdao 066004, China; zhuti@163.com
2 School of Management, Beijing University of Chinese Medicine, Beijing 100029, China; yqwang@bucm.edu.cn
3 Department of Mathematics, Zhejiang Normal University, Jinhua 321004, China; cuishuyu@zjnu.edu.cn

* Correspondence: wwf@zjnu.cn
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‡ Research Supported Partially by NSFC (No. 12071048) and Science and Technology Commission of Shanghai Municipality (No. 18dz2271000).
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Abstract: The strong chromatic index \( \chi_s'(G) \) of a graph \( G \) is the minimum number of colors needed in a proper edge-coloring so that every color class induces a matching in \( G \). It was proved in 2013 that every outerplanar graph \( G \) with \( \Delta \geq 3 \) has \( \chi_s'(G) \leq 3\Delta - 3 \). In this paper, we give a characterization for an outerplanar graph \( G \) to have \( \chi_s'(G) = 3\Delta - 3 \). We also show that if \( G \) is a bipartite outerplanar graph, then \( \chi_s'(G) \leq 2\Delta \); and \( \chi_s'(G) = 2\Delta \) if and only if \( G \) contains a particular subgraph.

Keywords: strong edge-coloring; strong chromatic index; outerplanar graph; bipartite graph

MSC: Graph Theory with Applications

1. Introduction

Only simple graphs are considered in this paper. For a graph \( G \), we use \( V(G) \), \( E(G) \), and \( \Delta(G) \) to denote its vertex set, edge set and maximum degree, respectively. A vertex \( v \) is called a \( k \)-vertex (or \( k^+ \)-vertex) if the degree \( d_G(v) \) of \( v \) is \( k \) (or at least \( k \)). Let \( N_G(v) \) denote the set of vertices adjacent to \( v \) in \( G \). If no ambiguity arises in the context, \( \Delta(G) \), \( d_G(v) \), and \( N_G(v) \) are simply written as \( \Delta \), \( d(v) \), and \( N(v) \), respectively. A subgraph of \( G \) is called a clique if any two of its vertices are adjacent in \( G \). A subset \( I \subset V(G) \) of a connected graph \( G \) is called a clique-cut if \( |I| \) is a clique and \( G - I \) is disconnected.

A proper edge-\( k \)-coloring of a graph \( G \) is a mapping \( \phi : E(G) \to \{1, 2, \ldots, k\} \) such that \( \phi(e) \neq \phi(e') \) for any two adjacent edges \( e \) and \( e' \). The chromatic index \( \chi'(G) \) of \( G \) is the smallest \( k \) such that \( G \) has a proper edge-\( k \)-coloring. An edge coloring of the graph \( G \) is called strong if every color class induces a matching in \( G \). The strong chromatic index of \( G \), denoted \( \chi_s'(G) \), is the smallest \( k \) such that \( G \) has a strong edge-\( k \)-coloring.

The strong edge-coloring of graphs was introduced by Fouquet and Jolivet [1]. In 1985, Erdős and Nešetřil raised the following conjecture and showed that the upper bounds are tight:

**Conjecture 1.** For a graph \( G \),

\[
\chi_s'(G) \leq \begin{cases} 
1.25\Delta^2, & \text{if } \Delta \text{ is even;} \\
1.25\Delta^2 - 0.5\Delta + 0.25, & \text{if } \Delta \text{ is odd.}
\end{cases}
\]

Using probabilistic method, Molloy and Reed [2] showed that \( \chi_s'(G) \leq 1.998\Delta^2 \) when \( \Delta \) is sufficiently large. This result was further improved in [3] to that \( \chi_s'(G) \leq 1.93\Delta^2 \) for any graph \( G \). Using Four-Colour Theorem and Vizing Theorem, Faudree et al. [4] showed...
that every planar graph \( G \) has \( \chi'_s(G) \leq 4\Delta + 4 \); and constructed a planar graph \( G \) such that \( \chi'_s(G) = 4\Delta - 4 \).

A planar graph is called outplanar if it has a plane embedding such that all the vertices lie on the boundary of the unbounded face. It was shown in [5] that a graph \( G \) is outplanar if and only if \( G \) is \( K_4 \)-minor-free and \( K_{2,3} \)-minor-free. Hence outplanar graphs are special \( K_4 \)-minor-free graphs. Wang et al. [6] showed that every \( K_4 \)-minor-free graph \( G \) with \( \Delta \geq 3 \) has \( \chi'_s(G) \leq 3\Delta - 2 \) and the upper bound is tight. Hocquard et al. [7] proved that every outplanar graph \( G \) with \( \Delta \geq 3 \) has \( \chi'_s(G) \leq 3\Delta - 3 \) and the upper bound is tight.

In this paper we will give a characterization for an outplanar graph \( G \) with \( \Delta \geq 3 \) to have \( \chi'_s(G) = 3\Delta - 3 \).

2. Sun-Graphs

Suppose that \( G \) is an outplanar graph. We embed \( G \) in the plane so that all the vertices occur in the boundary of unbounded face. Let \( F(G) \) denote the set of faces in \( G \). The unbounded face, denoted by \( f_0(G) \), of \( G \) is called outer face, and other faces inner faces. For a face \( f \in F(G) \), the boundary of \( f \) is denoted by \( b(f) \). A 3-face with \( x, y, z \) as boundary vertices is written as \([xyz]\). The edges lying in the outer face are called outer edges and other edges inner edges. An inner face \( f \) is called an end-face if \( b(f) \) contains at most one inner edge. A leaf of \( G \) is a vertex of degree 1, and a pendant edge is an edge incident to a leaf. For a vertex \( v \in V(G) \), let \( L(v) \) denote the set of pendant edges at vertex \( v \). For a cycle \( C \), an edge \( xy \in E(G) \setminus E(C) \) is called a chord of \( C \) if \( x, y \in V(G) \).

Let \( F_1 \) denote a subgraph of \( G \), which consists of a 3-cycle \( C_3 = x_0x_1x_2x_0 \) with \( d_G(x_i) = \Delta \geq 3 \) for \( i = 0, 1, 2 \).

Let \( F_2 \) denote a subgraph of \( G \), which consists of a 4-cycle \( C_4 = x_0x_1x_2x_3x_0 \) with \( d_G(x_0) = d_G(x_1) = \Delta \geq 3 \).

Let \( F_3 \) denote a subgraph of \( G \), which consists of a 7-cycle \( C_7 = x_0x_1\cdots x_6x_0 \) with \( d_G(x_i) = 3 \) for \( i = 0, 1, \ldots, 6 \).

We assume that \( C_4 \) in \( F_2 \) and \( C_7 \) in \( F_3 \) have no chord.

The configurations \( F_1, F_2, F_3 \) are depicted in Figure 1. By the outplanarity of \( G \), for \( F, F_j \in \{1, 2, 3\} \), some vertex \( y_i \in N(x_i) \setminus \{x_{i-1}, x_{i+1}\} \) may identify with some vertex \( y_{i+1} \in N(x_{i+1}) \setminus \{x_i, x_{i+2}\} \), but there is at most one such pair \( \{y_i, y_{i+1}\} \) satisfying \( y_i = y_{i+1} \), where indices \( i \) are taken as modulo \( n \).

![Figure 1](image)

**Figure 1.** Configurations \( F_1, F_2, \) and \( F_3 \).

**Lemma 1** ([7]). If \( G \) is an outplanar graph with \( \Delta \geq 3 \), then \( \chi'_s(G) \leq 3\Delta - 3 \).

**Lemma 2.** Let \( F_1, F_2, F_3 \) are defined as above. Then

1. \( \chi'_s(F_1) = 3\Delta - 3 \).
2. \( \chi'_s(F_2) = 3\Delta - 3 \).
3. \( \chi'_s(F_3) = 6 \).

**Proof.** (1) Since \( |E(F_1)| = 3\Delta - 3 \) and it is easy to check that any two edges of \( F_1 \) have distance at most two, so it follows that \( \chi'_s(F_1) = 3\Delta - 3 \).
(2) Applying the similar analysis as in (1), we can derive that $\chi'_v(F_2) = 3\Delta - 3$.

(3) It is evident that $\chi'_v(F_3) \leq 6$ by Lemma 1. Conversely, assume that $F_3$ admits a strong edge-5-coloring $\phi$ using the color set $C = \{1, 2, \ldots, 5\}$. Let $E_i$ denote the set of edges colored with the color $i$ under the coloring $\phi$. Set $E^* = E(F_3) - E(C)$. First, it is easy to inspect that $|E_i| \leq 3$ for each $i \in C$. Next, because $|E(F_3)| = 14$ and $|C| = 5$, we can assume that $|E_i| = 3$ for $i = 1, 2, 3, 4$ and $|E_5| = 2$. Since $|E^*| = 7$, some $E_i$ for $i \in \{1, 2, 3, 4\}$, say $i = 1$, satisfies $|E_1 \cap E^*| \leq 1$. It implies that $|E_1 \cap E(C)| \geq 2$. On the other hand, it is easy to inspect that $|E_1 \cap E(C)| \leq 2$. So $|E_1 \cap E(C)| = 2$ and $|E_1 \cap E^*| = 1$, however such coloring is impossible, a contradiction. This shows that $\chi'_v(F_3) \geq 6$. Consequently, $\chi'_v(F_3) = 6$.

Let $C_n = x_0 x_1 \cdots x_{n-1} x_0$ be a cycle with $n \geq 3$. Let $k \geq 3$ be an integer. At each vertex $x_i$, we glue $k - 2$ leaves and write the resultant graph as $S^k_n$. Then $S^k_n$ is an outerplanar graph with maximum degree $k$ and order $n (k - 1)$. We call $S^k_n$ a sun-graph with parameters $n$ and $k$. If $k = 3$, then we use $y_i$ to denote a leaf adjacent to $x_i$ for $i = 0, 1, \ldots, n - 1$.

As an easy observation, we have the following:

**Lemma 3.** Let $C_n$ be a cycle with $n \geq 3$. Then

$$\chi'_v(C_n) = \begin{cases} 5, & \text{if } n = 5; \\ 3, & \text{if } n \equiv 0 \pmod{3}; \\ 4, & \text{otherwise}. \end{cases}$$

**Lemma 4.** Let $S^3_n$ be a sun-graph with $n \geq 3$. Then

$$\chi'_v(S^3_n) = \begin{cases} 6, & \text{if } n = 3, 4, 7; \\ 5, & \text{otherwise}. \end{cases}$$

**Proof.** If $n = 3, 4, 7$, the conclusion follows immediately from Lemma 2. So suppose that $n \neq 3, 4, 7$. It holds trivially that $\chi'_v(S^3_n) \geq 5$ since $S^3_n$ contains two adjacent $3$-vertices. To show that $\chi'_v(S^3_n) \leq 5$, we make use of induction on $n$. It remains to construct a strong edge-5-coloring $\phi$ of $S^3_n$ using the color set $C = \{1, 2, \ldots, 5\}$.

- If $n = 5$, then we color the edges in $\{x_i y_i, x_{i+2} x_{i+3}\}$ with $i + 1$ for $i = 0, 1, 2, 3, 4$, where indices are taken as modulo 5.
- If $n \equiv 0 \pmod{6}$, then we alternatively color the edges in $E(C_n)$ with 1, 2, 3, and color alternatively pendant edges with 4, 5.
- If $n = 8$, then we color $\{x_1 y_1, x_3 x_4, x_6 x_7\}$ with 1, $\{x_3 y_3, x_0 x_1, x_5 x_6\}$ with 2, $\{x_5 y_5, x_0 x_7\}$ $\{x_2 x_3\}$ with 3, $\{x_7 y_7, x_1 x_2, x_4 x_5\}$ with 4, and $\{x_0 y_0, x_2 y_2, x_4 y_4, x_6 y_6\}$ with 5.
- If $n = 9$, then we color $\{x_1 y_1, x_3 y_3, x_8 y_8, x_5 x_6\}$ with 1, $\{x_0 y_0, x_5 y_5, x_7 y_7, x_2 x_3\}$ with 2, $\{x_2 y_2, x_4 y_4, x_6 y_6, x_0 x_8\}$ with 3, $\{x_0 x_1, x_3 x_4, x_6 x_7\}$ with 4, and $\{x_1 x_2, x_4 x_5, x_7 x_8\}$ with 5.

Now assume that $n \geq 10$ and $n \equiv 0 \pmod{6}$. Consider the graph $S^3_{n-5}$. Note that $n - 5 \geq 5$, and $n - 5 \neq 7$. By the induction hypothesis, $\chi'_v(S^3_{n-5}) = 5$. Let $\phi$ be a strong edge-5-coloring of $S^3_{n-5}$, so that $\phi(x_0 x_1) = 1$, $\phi(x_0 y_0) = 2$, $\phi(x_{n-7} x_{n-6}) = 3$, $\phi(x_{n-6} y_{n-6}) = 4$, and $\phi(x_0 x_{n-6}) = 5$. Clearly, $S^3_n$ can be obtained from $S^3_{n-5}$ by inserting five vertices $x_{n-5}$, $x_{n-4}$, $x_{n-3}$, $x_{n-2}$, $x_{n-1}$ to the edge $x_0 x_{n-6}$ and adding a leaf $y_j$ at $x_j$ for $j = n - 5, n - 4, \ldots, n - 1$. We extend $\phi$ to $S^3_n$ by coloring $\{x_{n-3} x_{n-2}, x_{n-5} y_{n-5}\}$ with 1, $\{x_{n-5} x_{n-4}, x_{n-2} y_{n-2}\}$ with 2, $\{x_{n-2} x_{n-1}, x_{n-4} y_{n-4}\}$ with 3, $\{x_{n-4} x_{n-3}, x_{n-1} y_{n-1}\}$ with 4, and $\{x_0 x_{n-1}, x_{n-6} x_{n-5}, x_{n-3} y_{n-3}\}$ with 5. It is easy to testify that the extended coloring is a strong edge-5-coloring of $S^3_n$. □

For a sun-graph $S^3_n$ with $C_n = x_0 x_1 \cdots x_{n-1} x_0$, we set $L(x_i) = \{e_{0}^i, e_{1}^i, \ldots, e_{n-2}^i\}$ for $i = 0, 1, \ldots, n - 1$. Recall that $L(x_i)$ stands for the set of pendant edges incident to $x_i$. 

Lemma 5. Let $S_n^k$ be a sun-graph with $k,n \geq 4$ and $n$ being even. Then
\[
\chi'_s(S_n^k) = \begin{cases} 
2k, & \text{if } n = 4; \\
2k - 1, & \text{if } n \geq 6.
\end{cases}
\]

Proof. Since $k \geq 4$, it follows that $k - 2 \geq 2$. The proof is split into the following two cases.

• Assume that $n = 4$. Color $x_0x_1, x_1x_2, x_2x_3, x_3x_0$ with $1, 2, 3, 4$, respectively; For $i = 0, 2, \ldots, n - 2$, we color $k - 2$ pendant edges in $L(x_i)$ with colors $5, 6, \ldots, k + 2$; For $i = 1, 3, \ldots, n - 1$, we color $k - 2$ pendant edges in $L(x_i)$ with colors $k + 3, k + 4, \ldots, 2k$. It is easy to see that the defining coloring is a strong edge-$2k$-coloring of $S_n^4$. Hence $\chi'_s(S_n^4) \leq 2k$. Conversely, we note that every pendant edge of $S_n^4$ has distance at most two to any edge in $E(C_4)$. This implies that, for any strong edge coloring of $S_n^4$, the color of any pendant edge is distinct from that of edges in $C_4$. Moreover, at least $2(k - 2)$ colors are needed when we color the $4(k - 2)$ pendant edges of $S_n^4$. It follows therefore that $\chi'_s(S_n^4) = 4 + 2(k - 2) = 2k$. This yields that $\chi'_s(S_n^4) = 2k$.

• Assume that $n \geq 6$. It is straightforward to conclude that $\chi'_s(S_n^k) \geq 2k - 1$ since $S_n^k$ contains two adjacent $k$-vertices. Conversely, we notice that $S_n^5$ is a spanning subgraph of $S_n^k$. By Lemma 4, $S_n^3$ has a strong edge-$5$-coloring $\phi$ using colors $1, 2, 3, 4, 5$. Based on $\phi$, we can color the remaining $k - 3$ pendant edges in $L(x_i)$ with colors $6, 7, \ldots, k + 2$ for each $i = 0, 2, \ldots, n - 2$; and color the remaining $k - 3$ pendant edges in $L(x_i)$ with colors $k + 3, k + 4, \ldots, 2k - 1$ for each $i = 1, 3, \ldots, n - 1$. The extended coloring is a strong edge-$2k$-coloring of $S_n^5$. It therefore turns out that $\chi'_s(S_n^k) \leq 2k - 1$. Consequently, $\chi'_s(S_n^k) = 2k - 1$.

Lemma 6. Let $n \geq 4$ be an odd number. Then

1. $\chi'_s(S_n^4) \leq 8$.
2. $\chi'_s(S_n^5) \leq 11$.

Proof. We first prove (1), by discussing two cases below.

• Assume that $n = 7$. Give a strong edge-$7$-coloring $\phi$ of $S_n^3$ as follows: $\phi(x_i, x_{i+1}) = i + 1$ for $i = 0, 1, \ldots, 6$, where indices are taken as modulo $7$; then we color $L(x_0)$ with $3, 5, L(x_1)$ with $4, 6$, $L(x_2)$ with $5, 7$, $L(x_3)$ with $1, 6$, $L(x_4)$ with $2, 7$, $L(x_5)$ with $1, 3$, and $L(x_6)$ with $2, 4$.

• Assume that $n \neq 7$. By Lemma 4, $S_n^3$ admits a strong edge-$5$-coloring $\phi$ using the colors $1, 2, \ldots, 5$. Let $e_0^1, e_1^1, \ldots, e_{n-1}^1$ have been colored. Afterward, we extend $\phi$ to the remaining edges of $S_n^3$ by coloring $e_0^2$ with $6$, $\{e_1^2, e_2^2, \ldots, e_{n-2}^2\}$ with $7$, and $\{e_0^3, e_2^3, \ldots, e_{n-1}^3\}$ with $8$. It is easily seen that the resultant coloring is a strong edge-$5$-coloring of $S_n^4$.

Next we prove (2). By the result of (1), $S_n^5$ has a strong edge-$8$-coloring $\phi$ using the colors $1, 2, \ldots, 8$. Based on $\phi$, we can color $e_0^3$ with $9$, $\{e_1^3, e_2^3, \ldots, e_{n-2}^3\}$ with $10$, and $\{e_0^4, e_2^4, \ldots, e_{n-1}^4\}$ with $11$. This leads to a strong edge-$11$-coloring of $S_n^5$. We first establish a useful claim:

Claim 1. Let $A_i = \{e_i^l, e_i^r\} \subseteq L(x_i)$ for $i = 0, 1, \ldots, n - 1$. Let $A = A_0 \cup A_1 \cup \cdots \cup A_{n-1}$. Then $A$ can be strongly edge-$5$-colored on the graph $S_n^5$.

Proof. Since $n \geq 5$ is odd, we can give an edge $5$-coloring $\pi$ of $A$ as follows: coloring $A_1$ with $2, 4$; $A_2$ with $3, 5$; $A_3$ with $1, 4$; each of $A_0, A_5, A_7, \ldots, A_{n-2}$ with $1, 3$; and each of $A_4, A_6, A_8, \ldots, A_{n-1}$ with $2, 5$. It is easy to confirm that $\pi$ is a strong edge-$5$-coloring of $A$ restricted in the graph $S_n^5$.

Lemma 7. Let $k \geq 6$, and let $n \geq 5$ be odd. Then $\chi'_s(S_n^k) \leq \lceil 2.5k - 2 \rceil$. 

Theorem 1. Assume the contrary, let $H$ be a strong edge-$l$-coloring of $G$ using the color set $C = \{1, 2, \ldots, l\}$. By the definition of $G$, we deduce that $E(I) \subseteq E(G)_i$ for $i = 1, 2$, and $E(G_1) \cap E(G_2) = E(I)$. Observe that any edge in $E(G_1) \setminus E(I)$ and any edge in $E(G_2) \setminus E(I)$ have distance at least three in $G$. Moreover, since the distance between any two edges in $E(I)$ is less than three, no two edges in $E(I)$ are assigned same color in both $\phi_1$ and $\phi_2$. So we may assume that $\phi_1(e) = \phi_2(e)$ for each $e \in E(I)$. Combining $\phi_1$ and $\phi_2$, we get a strong edge-$l$-coloring of $G$. This shows that $\chi'_s(G) \leq l$. On the other hand, since $G_1$ is a subgraph of $G$, we have naturally that $\chi'_s(G) \geq \max\{\chi'_s(G_1), \chi'_s(G_2)\} = l$. Consequently, $\chi'_s(G) = l$. □

Theorem 1. Let $G$ be an outerplanar graph with $\Delta \geq 4$. If $G$ does not contain $F_1$ as a subgraph, then $\chi'_s(G) \leq 3\Delta - 4$.

Proof. Assume the contrary, let $G$ be a counterexample with $|E(G)|$ being as small as possible. Then $G$ is connected, $|E(G)| \geq 3\Delta - 3$, and possesses the following properties:

\begin{itemize}
  \item [(P1)] No $F_1$ is contained in $G$ or its subgraphs.
  \item [(P2)] $G$ is not strongly edge-$(3\Delta - 4)$-colorable, but any subgraph $H$ of $G$ with $|E(H)| < |E(G)|$ is strongly edge-$(3\Delta - 4)$-colorable.
  \item [(P3)] $G$ is not a tree; otherwise $\chi'_s(G) \leq 2\Delta - 1 < 3\Delta - 4$, contradicting (P2).
\end{itemize}

By Lemma 8, the following claim holds:

Claim 2. $G$ does not contain a separable clique-cut $I \subseteq V(G)$ with $1 \leq |I| \leq 2$.

Embed $G$ to the plane so that all the vertices lie in the boundary of $f_0(G)$. Let $H$ denote the graph obtained from $G$ by removing all leaves. By (P3) and Claim 2, we can easily deduce Claims 3 and 4 below.

Claim 3. $H$ is 2-connected, and $b(f_0(H))$ forms a Hamiltonian cycle. This furthermore implies that all vertices in $V(G) \setminus V(H)$ are leaves.

Claim 4. Every inner edge $uv$ of $H$ is incident to an end-3-face $[uvw]$ such that $d_G(w) = d_H(w) = 2$.

Claim 4 implies that $2 \leq \Delta(H) \leq 4$; for otherwise $H$ will contain an inner edge $xy$ with $d_H(x) \geq 5$ and $\{x, y\}$ is a separable clique-cut of $G$. 

3. Outerplanar Graphs

Suppose that $G$ is a connected outerplanar graph. Let $I \subseteq V(G)$ be a clique-cut of $G$ with $1 \leq |I| \leq 2$; that is, $G[I]$ is $K_1$ or $K_2$ such that $G - I$ is disconnected. If $G - I$ has at least two components each containing at least one edge, then $I$ is said to be a separable clique-cut.

For a separable clique-cut $I$ of $G$, let $H_1, H_2, \ldots, H_s (s \geq 2)$ denote the components of $G - I$ with $|E(H_i)| \geq 1$ and $|E(H_i)| \geq 1$. We set $G_1 = G[E(H_1) \cup E(I)]$ and $G_2 = G[E(H_2) \cup \cdots \cup E(H_s) \cup E(I)]$, where $E(I)$ denotes the set of edges in $G$ which are incident to at least one vertex in $I$.

The following lemma plays a crucial role in the proof of our main results.

Lemma 8. Let $G$ be a connected outerplanar graph with a separable clique-cut $I$. Suppose that $G_1$ and $G_2$ are defined as above. Then

$$
\chi'_s(G) = \max\{\chi'_s(G_1), \chi'_s(G_2)\}
$$

Proof. Let $l_1 = \chi'_s(G_1), l_2 = \chi'_s(G_2)$, and $l = \max\{l_1, l_2\}$. For $i = 1, 2$, let $\phi_i$ be a strong edge-$l$-coloring of $G_i$ using the color set $C = \{1, 2, \ldots, l\}$. By the definition of $G_i$, we deduce that $E(I) \subseteq E(G)_i$ for $i = 1, 2$, and $E(G_1) \cap E(G_2) = E(I)$. Observe that any edge in $E(G_1) \setminus E(I)$ and any edge in $E(G_2) \setminus E(I)$ have distance at least three in $G$. Moreover, since the distance between any two edges in $E(I)$ is less than three, no two edges in $E(I)$ are assigned same color in both $\phi_1$ and $\phi_2$. So we may assume that $\phi_1(e) = \phi_2(e)$ for each $e \in E(I)$. Combining $\phi_1$ and $\phi_2$, we get a strong edge-$l$-coloring of $G$. This shows that $\chi'_s(G) \leq l$. On the other hand, since $G_1$ is a subgraph of $G$, we have naturally that $\chi'_s(G) \geq \max\{\chi'_s(G_1), \chi'_s(G_2)\} = l$. Consequently, $\chi'_s(G) = l$. □
Let $G^*$ denote the graph obtained from $G$ by carrying out repeatedly the following operation:

(*) If $x$ is a 2-vertex of $H$ incident to an end-3-face $[xyz]$, then we split $x$ into two new vertices $y_1$ and $z_1$ so that $y_1$ joins with $y$, and $z_1$ joins with $z$.

Intuitively speaking, every 2-vertex of $H$ which is incident to an end-3-face is replaced by two leaves in $G$. It is easy to see that $\Delta(G^*) = \Delta(G)$, and $\phi$ is a strong edge-$k$-coloring of $G^*$ if and only if $\phi$ is a strong edge-$k$-coloring of $G$.

It is easily observed that $G^*$ is a spanning subgraph of some sun-graph $S^k_n$, where $k = \Delta(G)$ and $n$ is the total number of $3^+$-vertices in $G$ and the number of 2-vertices in $G$ which are not on any 3-face. As an example, we observe the graphs $G$ and $G^*$ depicted in Figure 2.

![Figure 2](image_url)

Figure 2. $G^*$ is obtained from $G$ by carrying out (*), and $G^*$ is a subgraph of $S^k_n$.

Noting that $3k - 4 \geq \max\{2k, [2.5k - 2]\}$, we deduce by Lemmas 5 and 7 that $\chi'_d(G) = \chi'_d(G^*) \leq \chi'_d(S^k_n) \leq 3k - 4 = 3\Delta - 4$. This completes the proof of the theorem. □

Combining Theorem 1 and Lemmas 1 and 2(1), the following theorem holds:

**Theorem 2.** Let $G$ be an outerplanar graph with $\Delta \geq 4$. Then $\chi'_d(G) \leq 3\Delta - 3$; and $\chi'_d(G) = 3\Delta - 3$ if and only if $G$ contains $F_1$ as a subgraph.

**Theorem 3.** Let $G$ be an outerplanar graph with maximum degree $\Delta = 3$. If $G$ does not contain $F_1, F_2$ or $F_3$ as a subgraph, then $\chi'_d(G) \leq 5$.

**Proof.** Assume the contrary, let $G$ be a counterexample with $|E(G)|$ being as small as possible. Then $G$ is connected, $|E(G)| \geq 6$, and possesses the following properties:

(Q1) None of $F_1, F_2, F_3$ is contained in $G$ or its subgraphs.

(Q2) $G$ is not strongly edge-5-colorable, but any subgraph $H$ of $G$ with $|E(H)| < |E(G)|$ is strongly edge-5-colorable. Actually, if $\Delta(H) \leq 2$, then by Lemma 3, $\chi'_d(H) \leq 5$. If $\Delta(H) = 3$, then by the minimality of $G$, we obtain that $\chi'_d(H) \leq 5$.

(Q3) $G$ is not a tree; otherwise $\chi'_d(G) \leq 5$, contradicting (Q2).

By Lemma 8, $G$ does not contain a separable clique-cut $I \subseteq V(G)$ with $1 \leq |I| \leq 2$. Embed $G$ to the plane so that all the vertices lie in $b(f_0(G))$. Removing all the leaves of $G$, we get a subgraph $H$ of $G$. Similarly to the proof of Theorem 1, we conclude the following:

- $H$ is 2-connected, and all vertices in $V(G) \setminus V(H)$ are leaves.
- Every inner edge $uv$ of $H$ is incident to an end-3-face $[uvw]$ such that $d_C(w) = d_H(w) = 2$.

Let $G^*$ be the graph obtained from $G$ by doing repeatedly the following operation:

(*) If $x$ is a 2-vertex of $H$ incident to an end-3-face $[xyz]$ in $H$, then we split $x$ into two new vertices $y_1$ and $z_1$ so that $y_1$ joins with $y$, and $z_1$ joins with $z$. 
Then $\Delta(G^*) = \Delta(G)$, and $\chi'_s(G) = \chi'_s(G^*)$. Note that $G^*$ is a spanning subgraph of some sun-graph $S^k_n$, where $n$ is the total number of 3-vertices in $G$ and the number of 2-vertices in $G$ which are not on any 3-face. By Lemma 4, we derive immediately that $\chi'_s(G) = \chi'_s(G^*) \leq \chi'_s(S^k_n) \leq 5$. \hfill $\square$

Combining Theorem 3 and Lemmas 1 and 2, we have the following:

**Theorem 4.** Let $G$ be an outerplanar graph with $\Delta = 3$. Then $\chi'_s(G) \leq 6$; and $\chi'_s(G) = 6$ if and only if $G$ contains at least one of $F_1, F_2, F_3$ as a subgraph.

When restricted to the family of bipartite outerplanar graphs $G$, smaller and tight upper bounds for $\chi'_s(G)$ can be obtained.

**Theorem 5.** Let $G$ be a bipartite and outerplanar graph with maximum degree $\Delta \geq 3$. Then $\chi'_s(G) \leq 2\Delta$; moreover, $\chi'_s(G) = 2\Delta$ if and only if $G$ contains $F_2$ as a subgraph.

**Proof.** We first show that $\chi'_s(G) \leq 2\Delta$. Assume the contrary, let $G$ be a counterexample with $|E(G)|$ being as small as possible. Then $G$ is connected, other than a tree, and is not strongly edge-2$\Delta$-colorable, but any subgraph $H$ of $G$ with $|E(H)| < |E(G)|$ is strongly edge-2$\Delta$-colorable. Moreover, by Lemma 8, there is no separable clique-cut $I \subseteq V(G)$ with $1 \leq |I| \leq 2$.

Embed $G$ to the plane so that all the vertices occur in $b(f_0(G))$. Removing all the leaves of $G$, we obtain a subgraph $H$ of $G$. Then $H$ is a Hamiltonian cycle without chords, and $V(G) \setminus V(H)$ are all leaves. So $G$ is a subgraph of some $S^k_n$ where $n = |V(H)|$ is even and $k = \Delta(G)$. By Lemma 5, $\chi'_s(G) \leq \chi'_s(S^k_n) \leq 2k = 2\Delta$.

If $G$ contains $F_2$ as a subgraph, then $\chi'_s(G) \geq \chi'_s(F_2) = |E(F_2)| = 2\Delta$. Using the foregoing proof, we get that $\chi'_s(G) = 2\Delta$. Conversely, if $G$ does not contain $F_2$ as a subgraph, then similarly to the above discussion we can show that $\chi'_s(G) \leq 2\Delta - 1$. \hfill $\square$

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