A New Solution to a Cubic Diophantine Equation

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Abstract: A positive integer, which can be written as the sum of two positive cubes in two different ways, is known as a “Ramanujan number”. The most famous example is 1729 = 10^3 + 9^3 = 12^3 + 1^3, which was identified by Ramanujan as the lowest such number. In this paper, we consider the homogeneous cubic Diophantine equation \( x^3 + y^3 = u^3 + v^3 \), where there is no restriction on the signs of the integers \( x, y, u, v \). We show that every solution can be written in terms of two parameters in the ring \( \mathbb{Z}(\sqrt{-3}) \). It is also shown that solutions with arbitrarily high values of \( \max(|x|, |y|, |u|, |v|) \) arise amongst the primitive solutions.

Keywords: Diophantine equations; cubic; Ramanujan numbers.

MSC: 11Axx

1. Introduction

The aim of this paper is to give a new solution to the Diophantine equation

\[
    x^3 + y^3 = u^3 + v^3, \quad x, y, u, v \in \mathbb{Z}.
\]

(1)

The result found in this paper gives the general solution in terms of two parameters in the ring \( R := \mathbb{Z}(\sqrt{-3}) \).

Depending on the signs of the integers in any particular solution of (1), we obtain a solution to one or the other of

\[
    x^3 + y^3 = u^3 + v^3, \quad x, y, u, v \in \mathbb{Z}^+. \tag{2}
\]

\[
    x^3 + y^3 + z^3 = w^3, \quad x, y, z, w \in \mathbb{Z}^+. \tag{3}
\]

For example, the solution \( [x, y, u, v] = [10, -1, -9, 12] \) to (1) implies the solution \( [x, y, u, v] = [10, 9, 1, 12] \) to (2), and the solution \( [x, y, u, v] = [6, -4, 5, 3] \) to (1) implies the solution \( [x, y, z, w] = [3, 4, 5, 6] \) to (3). The values of \( x^3 + y^3 \), in solutions to (2), are known as “Ramanujan numbers”. The historical event that made (1) into a famous problem was recounted by C. P. Snow in his foreword to [1]. In [2], a method was given for generating an unlimited set of solutions by iteration. This differs markedly from the present paper, which generates all solutions parametrically. The problem was also discussed in [3].

In Section 2, we consider some properties of \( R \), and in Section 3, we present a solution to (1). This is followed by Section 3, where the cubic Diophantine Equation (3) is considered, and this leads to an algorithm for generating all possible solutions in Section 4. Finally, in Section 5, we show how to construct primitive solutions with arbitrarily large values.

2. The Ring \( \mathbb{Z}(\sqrt{-3}) \)

For \( \xi = a + b\sqrt{-3} \in R \), the norm is defined as \( \|\xi\| = \xi\overline{\xi} = a^2 + 3b^2 \). If \( \xi = a + b\sqrt{-3}, \eta = c + d\sqrt{-3} \), then \( \xi\eta = (ac - 3bd) + (ad + bc)\sqrt{-3} \).

We consider the question: “which odd primes greater than 3 are equal to \( \|\xi\| \) for some \( \xi \)?”
Lemma 1. Let $p$ be a prime greater than 3; then, there exists $a + b\sqrt{-3} \in R$ with $p \nmid a$, $p \nmid b$, such that

$$a^2 + 3b^2 \equiv 0 \mod p,$$

if and only if $p \equiv 1 \text{ mod } 6$.

Proof. Choose the integer $c$ such that $bc = 1 \mod p$ and multiply (4) by $c$ and write $d = ac$. It follows that $-3 \equiv d^2 \mod p$, implying that, using Legendre symbols,

$$\left(\frac{-3}{p}\right) = \left(\frac{-1}{p}\right) \left(\frac{3}{p}\right) = (-1)^{\frac{p-1}{2}} \left(\frac{3}{p}\right) = \left(\frac{3}{p}\right) = \begin{cases} 1, & p \equiv 1 \mod 6, \\ -1, & p \equiv 5 \mod 6. \end{cases}$$

Denote $P$ as the set of primes that are congruent to 1 mod 6. For a given $p \in P$, denote $N = \{1, 2, \ldots, \frac{p-1}{2}\}$.

Lemma 2. Let $p \in P$; then, there exist integers $a, b \in N$ such that

$$a^2 + 3b^2 \equiv 0 \mod p.$$  

Proof. Replace $a$ by $a'$ in (5), where $a = np \pm a'$ with $a' \in N$ and similarly for $b$.

Theorem 1. Let $p \in P$; then, a unique $|a|$ and $|b|$ exist such that

$$a^2 + 3b^2 = p,$$  

Proof. The proof is by induction for $p = 7, 13, 19, \ldots \in P$. For $p = 7$, it is verified that $2^2 + 3(1)^2 = 7$. It remains for us to prove the result for $p \in P$, $p > 7$, where it is assumed to hold for all $q \in P$, such that $q < p$. From Lemma 2, $a, b$ exist such that $a^2 + 3b^2 = pQ$, with $Q$ a product of prime powers. We show that this statement is also true with $Q = 1$. It is not possible that $p \mid Q$ because

$$a^2 + 3b^2 \leq (\frac{p-1}{2})^2 + 3(\frac{p-1}{2})^2 < p^2.$$  

The proof continues by considering a series of propositions that make it possible to exclude all possible prime divisors of $Q$. In each case, for a possible divisor $g$, we show that there exist $a', b'$ such that $a'^2 + 3b'^2 = pQ'$ with $Q' < Q$. In this proof only, the symbol $\Rightarrow$ is used to introduce values of $a', b', Q'$.

1. $\gcd(a, b) = g > 1 \Rightarrow a' = a/g, b' = b/q. Q' = Q/g^2$.  

2. $2 \mid Q$.
   (a) $a, b$ even. This reduces to 1.
   (b) $a, b$ odd.
   i. $4 \mid (a + b) \Rightarrow a' = (a - 3b)/4, b' = (a + b)/4. Q' = (Q/4)$
   ii. $4 \mid (a - b) \Rightarrow a' = (a + 3b)/4, b' = (a - b)/4. Q' = (Q/4)$

3. $3 \mid Q$. This implies $3 \mid a \Rightarrow a' = b, b' = a/3. Q' = Q/3$.

4. $q \mid Q$, for $q$ prime and $q \equiv 5 \mod 6$. It follows that case 1 applies with $g = q$.

5. $q \mid Q$, for $q \in P$.
   (a) $q \mid a, q \mid b$. Reduces to case 1.
   (b) $q \mid a, q \nmid b$. $(c + d\sqrt{-3})(c - d\sqrt{-3}) = pqQ$.
      i. $q \mid (c + d\sqrt{-3}) \Rightarrow a' = (ac - 3bd)/q, b' = ad + bc, Q' = Q/q$
      ii. $q \mid (c - d\sqrt{-3}) \Rightarrow a' = (ac + 3bd)/q, b' = ad - bc, Q' = Q/q$

\[\square\]
We now consider the primes of $R$ and factorization in $R$.

**Remark 1.** The primes of $R$ are

\[ 1 \pm \sqrt{-3}, \quad \frac{1}{2} \pm \sqrt{-3} \]

Prime numbers $p \equiv 5 \mod 6$

\[ a \pm b\sqrt{-3}, \text{ with } a, b \in \mathbb{Z}^+, \text{ and } a^2 + 3b^2 \in \mathbb{P}. \]

**Remark 2.** The product of two primes of $R$ has a unique factorization with the exception of

\[ 4 = 2^2 = (1 + \sqrt{-3})(1 - \sqrt{-3}). \]

3. **Solution to Cubic Equation**

We first state the solution to an introductory equation.

**Lemma 3.** The general solution to

\[ AB = CD, \quad A, B, C, D \in \mathbb{Z}^+ \quad (7) \]

is

\[ A = ij, B = k\ell, C = ik, D = j\ell, \quad i, j, k, \ell \in \mathbb{Z}^+. \quad (8) \]

**Proof.** Let $i = \gcd(A, C), j = A/i, \ell = \gcd(B, D), k = B/\ell$. □

Consider two Diophantine equations

\[ x^3 + y^3 = u^3 + v^3, \quad (9) \]
\[ a(a^2 + 3b^2) = c(c^2 + 3d^2) \quad (10) \]

In each case, we consider primitive solutions.

The next result is easily verified and is given without proof.

**Lemma 4.** $(x, y, u, v)$ is a primitive solution to (9) if and only if $(a, b, c, d)$ is a primitive solution to (10), where

\[
\begin{bmatrix}
a \\
b
\end{bmatrix} = T \begin{bmatrix}
x \\
y
\end{bmatrix}, \quad \begin{bmatrix}
c \\
d
\end{bmatrix} = T^{-1} \begin{bmatrix}
u \\
v
\end{bmatrix},
\]

with

\[
T = \begin{cases}
\begin{bmatrix}
1 & 1 \\
1 & -1
\end{bmatrix}, & x + y \text{ odd,} \\
\begin{bmatrix}
1 & 1 \\
1 & -1
\end{bmatrix}, & x + y \text{ even,}
\end{cases}
\]

\[
T^{-1} = \begin{cases}
\begin{bmatrix}
1 & 1 \\
1 & -1
\end{bmatrix}, & x + y \text{ odd,} \\
\begin{bmatrix}
1 & 1 \\
1 & -1
\end{bmatrix}, & x + y \text{ even.}
\end{cases}
\]

We now focus on (10), but written in a slightly different form.

\[
a(a^2 + 3b^2) = c(c^2 + 3d^2) \quad (11)
\]
\[
\tilde{a} = a, \quad \tilde{c} = c. \quad (12)
\]

We use Lemma 3 to solve (11) and then identify the free parameters by requiring them to satisfy (12).
Lemma 5. The general solution to (11) is
\[
\tilde{a} = rY, \\
\tilde{a}^2 + 3\tilde{b}^2 = XZ, \\
\tilde{c} = rX, \\
\tilde{c}^2 + 3\tilde{d}^2 = YZ,
\]
where
\[
X = \zeta^2 = \alpha^2 + 3\beta^2, \quad \zeta = \alpha + \beta\sqrt{-3}, \quad (17) \\
Y = \eta^2 = \gamma^2 + 3\delta^2, \quad \eta = \gamma + \delta\sqrt{-3}, \quad (18) \\
Z = \zeta^2 = s^2 + 3t^2, \quad \zeta = s + t\sqrt{-3}, \quad (19)
\]
and \(r, s, t, \alpha, \beta, \gamma, \delta\) are parameters.

Proof. Use Lemma 3 and note that \(Z = \gcd(\alpha^2 + 3\beta^2, \gamma^2 + 3\delta^2)\), which is necessarily of the form \(s^2 + 3t^2\).

Discussion 1. To eliminate \(r, s, t\), use the conditions \(\tilde{a} = a, \tilde{c} = c\) to obtain
\[
\tilde{a} = r\zeta = r(\gamma^2 + 3\delta^2), \quad (20) \\
\tilde{c} = r\zeta = r(\alpha^2 + 3\beta^2), \quad (21) \\
a = \Re \zeta = sa - 3t\beta, \quad (22) \\
c = \Re \eta = s\gamma - 3t\delta, \quad (23)
\]
so that \(r, s, t\) satisfy the homogeneous linear system
\[
\begin{bmatrix}
\gamma^2 + 3\delta^2 & -\alpha & 3\beta \\
\alpha^2 + 3\beta^2 & -\gamma & 3\delta
\end{bmatrix}
\begin{bmatrix}
r \\
s \\
t
\end{bmatrix} = 0,
\]
with solution
\[
\begin{bmatrix}
r \\
s \\
t
\end{bmatrix} = G^{-1}\begin{bmatrix}
3(\beta\gamma - \alpha\delta) \\
9(\beta^3 - \delta^3) + 3(\alpha^2\beta - \gamma^2\delta) \\
\alpha^3 - \gamma^3 + 3(\alpha^2\beta - \gamma^2\delta)
\end{bmatrix}, \quad (25)
\]
where \(G^{-1}\) is introduced to reduce \(\gcd(r, s, t)\) to 1, if necessary.

4. Algorithm and Sample Solutions

Using the results of Lemmas 4, 5 and Discussion 1, an algorithm can be constructed for finding solutions to
\[
x^3 + y^3 = u^3 + v^3.
\]
1. Choose \(a + \beta\sqrt{-3}, \gamma + \delta\sqrt{-3} \in R\);
2. Evaluate \(r, s, t\);
3. Evaluate \(a, b, c, d\);
4. Evaluate \(x, y, u, v\).

It is found computationally that there are exactly 25 solutions to (2) with \(x^3 + y^3 \leq 1000,000\) and 31 solutions to (3) with \(w \leq 100\). These are shown in Table 1.
Table 1. Solutions to (2) (left-hand table) and (3) (right-hand table).

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</table>

5. Extreme Values

**Theorem 2.** Primitive solutions to (2), with \(x, y, u, v > 0\), exist with arbitrarily high values of \(x^3 + y^3 = u^3 + v^3\).

**Proof.** For \([α, β, γ, δ] = [n, 1, n, 0]\) with \(3 ∤ n\), details of the solution are found to be

\[
[r, s, t] = [n, n^2 + 3, n],
\]

\[
x = n^3 + 2n^2 + 3,
\]

\[
y = n^3 - 2n^2 - 3,
\]

\[
u = n^3 + n^2 + 3n,
\]

\[
v = n^3 - n^2 + 3n,
\]

\[
x^3 + y^3 = u^3 + v^3 = 2n^3(n^2 + 3)(n^4 + 9n^2 + 9).
\]


**Theorem 3.** Primitive solutions to (3), with \(x, y, z > 0\), exist with arbitrarily high values of \(w\).

**Proof.** For \([α, β, γ, δ] = [n, 1, 0, n]\), with \(3 ∤ n\), details of the solution are found to be
\[ [r, s, t] = [-3n^2, 3n^2 + 9, n^2 + 3n], \]
\[ x = 6n^4 - 3n^3 - 9n^2 - 9n, \]
\[ y = 8n^4 + 9n^3 - 6n^2 - 9, \]
\[ z = 10n^4 - 9n^3 + 6n^2 + 9, \]
\[ w = 12n^4 - 3n^3 + 9n^2 - 9n, \]

6. Conclusions

A new parametric solution to the homogeneous cubic equation
\[ x^3 + y^3 = u^3 + v^3, \quad x, y, u, v \in \mathbb{Z}, \tag{26} \]
is derived using a pair of parameters in the ring \( \mathbb{Z}(\sqrt{-3}) \). As a consequence, it is shown that, amongst solutions of (26), there exist arbitrarily large values of \( N = x^3 + y^3, x, y, u, v > 0 \). Furthermore, it is shown that, amongst solutions of \( x^3 + y^3 + z^3 = w^3, x, y, z, w > 0 \), there exist arbitrarily large values of \( w \).

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References
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