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Existence and Stability of Weakly Cooperative Equilibria and Strong Cooperative Equilibria of Multi-Objective Population Games

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Abstract: Motivated by the concept of cooperative equilibria with a single objective, we introduce the concepts of weakly cooperative equilibria and strong cooperative equilibria of multi-objective population games. We give some examples to explain the difference between the cooperative equilibrium point and noncooperative equilibrium point of multi-objective population games. Under appropriate assumptions, we study the existence and stability of weakly cooperative equilibria and strong cooperative equilibria of multi-objective population games.

Keywords: cooperative equilibria; population game; vector payoff; multi-objective

MSC: 91A10; 91A40; 54C60

1. Introduction

The study of the subject of population games originated from the mass action interpretation of equilibrium points with the work of Nash (see [1]). Compared with the traditional game problem, population game theory has superior and has been widely used in military, social science, economic theory, and so on. As we know, the existence and stability of equilibrium points in games are two important topics in theory, algorithm, and application. However, a large number of game problems cannot be guaranteed to exist. Sandholm [2] proposed the concept of noncooperative equilibria of population games and obtained an equivalent characterization of this equilibrium point, which was a general form of Nash equilibria. Moreover, the author investigated the existence result of the solution of a population game with the continuity assumption of payoff functions. By virtue of the Ky Fan inequality of vector-valued mapping, Yang and Yang [3] investigated the existence theorem of weakly Pareto–Nash equilibria of a multi-objective population game. Simultaneously, they also considered the continuity of the parametric set of weakly Pareto–Nash equilibria, partly upper semicontinuous and almost lower semicontinuous of the parametric set of Pareto–Nash equilibria. In reality, there exist many cooperative behaviors between different populations. So, Yang and Zhang [4] first introduced the model of cooperative population games with finite pure strategies for each population, and investigated the existence and generic stability of the set of cooperative equilibria of population games. They showed the difference between cooperative equilibria and noncooperative equilibria in population games. Later, Yang and Zhang [5] introduced the coalitional population game problem with infinite pure strategies and considered the existence theorem for this game problem.

The stability analysis of the parametric solution mappings for game problems is another important topic in game theory. In [6], Yu considered the stability of the parametric set of noncooperative equilibria of n-person Nash games with three types of payoff function.

It should be pointed out that cooperative behaviors are of frequent occurrence between different populations and the goals considered by each population are more than one, such as personal profit, social position and life satisfaction, and so on. Hence, it is very necessary to study the cooperative equilibria of multi-objective population games. However, the research on the cooperative equilibria of population games still remains single-objective.

In this paper, we introduce the weakly cooperative equilibria and strong cooperative equilibria of multi-objective population games. We obtain some existence and stability results of weakly cooperative equilibria and strong cooperative equilibria of these games. We also give some examples to illustrate the difference between cooperative equilibria and noncooperative equilibria of multi-objective population games.

2. Preliminaries

We assume that \( V \) is a Banach space and \( C \) is a pointed closed convex cone in \( V \) with its interior \( \text{int}C \neq \emptyset \).

**Definition 1** ([18]). Let \( X \) be a nonempty convex set. A vector-valued map \( f : X \to V \) is said to properly \( C \)-quasi concave if for any \( x_1, x_2 \in X \) and \( l \in [0, 1] \) such that

\[
    f(lx_1 + (1 - l)x_2) \in f(x_1) + C
\]

or

\[
    f(lx_1 + (1 - l)x_2) \in f(x_2) + C.
\]

**Definition 2** ([19]). Assume that \( F : X \to 2^V \) is a set-valued mapping.

(i) The compact valued \( F \) is said to upper semicontinuous (u.s.c.) at \( x_0 \in E \) if for \( \{ x_n \} \subset E, x_n \to x_0 \) and \( \forall y_n \in F(x_n), \exists y_0 \in F(x_0) \) and a subsequence \( \{ y_{n_k} \} \) of \( \{ y_n \} \) such that \( y_{n_k} \to y_0 \).

(ii) \( F : X \to 2^V \) is said to lower semicontinuous (l.s.c.) at \( x_0 \in E \) if for \( \{ x_n \} \subset E, x_n \to x_0 \) and \( \forall y_0 \in F(x_0), \exists y_n \in F(x_n) \) such that \( y_n \to y_0 \).

(iii) \( F : X \to 2^V \) is said to continuous at \( x_0 \in E \), if for \( F \) is both u.s.c. and l.s.c. at \( x_0 \).

We restate the generalized Scarf’s theorem in [20].

**Theorem 1** ([20]). Let \( \Omega = \{1, 2, \ldots, n\} \) be a set of agents. Let \( X = \prod_{i \in \Omega} X_i \) and \( K_i \) be a set-valued mapping from \( X \) to \( X \). Let \( S \subseteq \Omega \) be a coalition, \( X_S = \prod_{i \in S} X_i \) and \( G_S \) be a set-valued mapping from \( X \) to \( X_S \). Suppose that

(i) for each \( i \in \Omega, X_i \) is nonempty compact convex set;

(ii) for any \( x \in X, x \not\in K_i(x), K_i(x) \) is a convex set and \( \text{Graph}K_i \) is open;

(iii) \( G_S \) is continuous and for each \( x \in X, G_{\Omega}(x) \) is nonempty closed convex set;
Axioms 2022, 11, 196

(iv) for any balanced family \( \gamma \) of coalition \( S \) and the balancing weight \( I_S, S \in \gamma \) with \( \sum_{S \in \gamma} I_S = 1 \),
\[ I_S \geq 0, \text{ for each } x \in X, \text{ if } y_S \in G_S(x), \forall S \in \gamma, \text{ then } \hat{y} \in G_\Omega(x) \text{ where } \hat{y} = (y_1, y_2, \ldots, y_n) \text{ with } y_i = \sum_{S \in \gamma} I_S y^*_S, \text{ and } y^*_S \text{ is the } i\text{th component of } y_S.

Then, \( \exists x \in G_\Omega(x) \) such that for any coalition \( S \subset \Omega \), there exists no \( y_S \in G_S(x) \) satisfying \( y \in K_i(x), \forall i \in S \), where \( y_S \) is the component in \( S \) of \( y \).

Next, we state the concept of the weakly cooperative equilibrium point and strong cooperative equilibrium point of multi-objective population games.

Let \( \Omega = \{1, 2, \ldots, N\} (N \in N_+) \) be the set of populations. For each \( p \in \Omega \), the number of agents of \( p \)th population is large but limited and they can choose pure strategies in the finite set \( T_p = \{1, 2, \ldots, n_p\} (n_p \in N_+) \). Let \( m_p \) be a continuum of mass of agents of \( p \)th population, \( x_{p,i} \) be the mass of agents of \( p \)th population choosing pure strategies \( i \in T_p \), and \( X_p \) be social state set of \( p \)th population by

\[ X_p = \{x_p = (x_{p,1}, x_{p,2}, \ldots, x_{p,n_p}) : x_{p,i} \geq 0, \forall i \in T_p, \sum_{i=1}^{n_p} x_{p,i} = m_p\}. \]

Let \( X = \prod_{p \in \Omega} X_p \) and \( f_{p,i} : X \rightarrow V \) be the vector payoff function of the \( p \)th population with respect to \( i \in T_p \). Let \( f_p = (f_{p,1}, f_{p,2}, \ldots, f_{p,n_p}) \), \( p \in \Omega \) be the payoff function matrix of the \( p \)th population, and \( f = (f_1, f_2, \ldots, f_N) \) be the payoff function tensor of multi-objective population game. For any population coalition \( S \subset \Omega \), assume that \( \overline{X}_S \) is the social state set of coalition \( S \) by

\[ \overline{X}_S = \{x_S \in \prod_{p \in S} R_+^{n_p} : \sum_{p \in S} \sum_{i=1}^{n_p} x_{p,i} = \sum_{p \in S} m_p\}, \]

where \( \overline{x}_p = (\overline{x}_{p,1}, \overline{x}_{p,2}, \ldots, \overline{x}_{p,n_p}) \), \( \forall p \in S \).

A social state \( x^* = (x^*_1, x^*_2, \ldots, x^*_N) \in \overline{X}_\Omega \) is called the weakly cooperative equilibrium point of multi-objective population games with respect to \( (\Omega, \overline{X}_\Omega, f) \) if for any coalition \( S \subset \Omega \), there exists no \( y_S \in \overline{X}_S \) such that

\[ \sum_{i=1}^{n_p} y_{s,i} f_{p,i}(x^*) \in \sum_{i=1}^{n_p} x^*_{p,i} f_{p,i}(x^*) + int C, \quad \forall p \in S, \]

where \( x^*_{p,i} \) is the \( i\text{th component of } x^*_p \) and \( y_s = (y_{s,1}, y_{s,2}, \ldots, y_{s,n_p}) \).

A social state \( x^* = (x^*_1, x^*_2, \ldots, x^*_N) \in \overline{X}_\Omega \) is called the strong cooperative equilibrium point of multi-objective population games with respect to \( (\Omega, \overline{X}_\Omega, f) \) if for any coalition \( S \subset \Omega \), there exists no \( y_S \in \overline{X}_S \) such that

\[ \sum_{i=1}^{n_p} y_{s,i} f_{p,i}(x^*) \not\in \sum_{i=1}^{n_p} x^*_{p,i} f_{p,i}(x^*) - C, \quad \forall p \in S, \]

where \( x^*_{p,i} \) is the \( i\text{th component of } x^*_p \) and \( y_s = (y_{s,1}, y_{s,2}, \ldots, y_{s,n_p}) \).

Remark 1. When \( V = R, C = R_+ \) and the vector payoff functions \( f_{p,i}, \forall p \in \Omega, i \in T_p \) reduce to real-valued payoff functions, the concepts of the weakly cooperative equilibrium point and strong cooperative equilibrium point of population games with multi-objective frameworks become the corresponding one in [4].

Remark 2. Clearly, the strong cooperative equilibria point of multi-objective population games requires that each population maximizes all goals. Hence, the weakly cooperative equilibrium point and strong cooperative equilibria point of multi-objective population games are different. If \( x^* \) is a strong cooperative equilibria point, then \( x^* \) is also a weakly cooperative equilibria point, but not vice versa. We give an example to illustrate this case.
Example 1. Let $\Omega = \{1, 2\}$, $T_1 = T_2 = \{1, 2\}$ and $m_1 = m_2 = 1$. Let $X_1 = \overline{X_1} = \{x_1 = (x_1^1, x_1^2) \in R^d_+ : x_1^1 + x_1^2 = 1\}$, $X_2 = \overline{X_2} = \{x_1 = (x_2^1, x_2^2) \in R^d_+ : x_2^1 + x_2^2 = 1\}$, and $\Omega = \overline{X_\Omega} = \{x = (x_1, x_2) \in R^4_+ : x_1^1 + x_1^2 + x_2^1 + x_2^2 = 2\}$. Denote the cone $C = R^d_+$ and the vector payoff functions as follows:

$$f_{1,1}(x) = f_{1,2}(x) = f_{2,1}(x) = f_{2,2}(x) = (1, -1); \ x \in R^d_+.$$  

It is easy to verify that $x^* = (1, 0, 1, 0) \in \overline{X_\Omega}$ is a weakly cooperative equilibria point of a multi-objective population game. However, $x_\star = (1, 0, 1, 0) \in \overline{X_\Omega}$ is not a strong cooperative equilibria point of a multi-objective population game. For the coalition $S = \{1, 2\}$, we deduce that

$$\sum_{i=1}^{2} x_{p,i} f_{p,i}(x^*) = (1, -1), \forall p \in S,$$

and

$$\sum_{i=1}^{2} y_{1,i} f_{1,i}(x^*) = (0, 0), \quad \sum_{i=1}^{2} y_{2,i} f_{2,i}(x^*) = (2, -2)$$

by letting $(y_{1,1}, y_{1,2}, y_{2,1}, y_{2,2}) = (0, 0, 1, 1) \in \overline{X_S}$. Obviously,

$$\sum_{i=1}^{2} y_{1,i} f_{1,i}(x^*) \notin \sum_{i=1}^{2} x_{1,i} f_{1,i}(x^*) - R^d_+$$

and

$$\sum_{i=1}^{2} y_{2,i} f_{2,i}(x^*) \notin \sum_{i=1}^{2} x_{2,i} f_{2,i}(x^*) - R^d_+.$$  

Namely, $x^*$ is not the strong cooperative equilibria of a multi-objective population game.

3. Existence

In this section, we investigate the existence of cooperative equilibria of population games with a multi-objective framework.

Next, we first show that the weakly cooperative equilibria of multi-objective population games are nonempty.

**Theorem 2.** Suppose that for each $p \in \Omega$ and $i \in T_p$, the vector payoff function $f_{p,i} : X \rightarrow V$ is continuous. Then weakly cooperative equilibria of multi-objective population games with respect to $(\Omega, X_\Omega, f)$ is nonempty.

**Proof.** Let $t \geq \sum_{p \in \Omega} m_p$ be a positive real number and

$$X^t_p = \{x_p = (x_{p,1}, x_{p,2}, \ldots, x_{p,n_p}) : 0 \leq x_{p,i} \leq t\}, \ X^t = \prod_{p \in \Omega} X^t_p.$$  

For every $p \in \Omega$, it could be easily seen that $X^t_p$ is bounded in Euclidean space $R^{n_p}$. Hence, $X^t_p$ is a nonempty compact convex set. For each $p \in \Omega$, define the set-valued mapping $K_p : X^t \rightarrow 2^{2^{n_p}}$ by the rule:

$$K_p(x) = \{y \in X^t : \sum_{i=1}^{n_p} y_{p,i} f_{p,i}(x) \in \sum_{i=1}^{n_p} x_{p,i} f_{p,i}(x) + intC\},$$

where $x = (x_1, x_2, \ldots, x_N), y = (y_1, y_2, \ldots, y_N), x_i = (x_{p,1}, x_{p,2}, \ldots, x_{p,n_p}), y_i = (y_{p,1}, y_{p,2}, \ldots, y_{p,n_p}), i = 1, 2, \ldots, N$.

Obviously, for each $x \in X^t, x \notin K_p(x)$ owing to $\theta \notin intC$. We can also prove that for each $x \in X^t, K_p(x)$ is convex owing to the convexity of $intC$. 

Next, we should prove that $K_p(x)$ is continuous on $X^t$.
Next, we show that for any \( p \in \Omega \), Graph\( K_p \) is open. Let \( \{ x_n, y_n \} \not\subset \text{Graph} K_p \) and \((x_n, y_n) \to (x_0, y_0) \in X^1 \times X^1 \). Then,

\[
\sum_{i=1}^{n_p} y_{p,i}^n f_{p,i}(x_n) \in \sum_{i=1}^{n_p} x_{p,i}^n f_{p,i}(x_n) + V\{\text{int}C\},
\]

where \( x_n = (x_1^n, x_2^n, \ldots, x_N^n), y_n = (y_1^n, y_2^n, \ldots, y_{N_p}^n), x_i^n = (x_{p,1}^n, x_{p,2}^n, \ldots, x_{p,n_p}^n), y_i^n = (y_{p,1}^n, y_{p,2}^n, \ldots, y_{p,n_p}^n), i = 1, 2, \ldots, N \). Since \( x_n \to x_0 \), by the continuity of the vector-valued function \( f_{p,i} \),

\[
f_{p,i}(x_n) \to f_{p,i}(x_0), \quad \text{as } n \to \infty.
\]

By the closeness of the set \( V\{\text{int}C\} \),

\[
\sum_{i=1}^{n_p} y_{p,i}^n f_{p,i}(x_0) \in \sum_{i=1}^{n_p} x_{p,i}^n f_{p,i}(x_0) + V\{\text{int}C\},
\]

where \( x_0 = (x_1^0, x_2^0, \ldots, x_N^0), y_0 = (y_1^0, y_2^0, \ldots, y_{N_p}^0), x_i^0 = (x_{p,1}^0, x_{p,2}^0, \ldots, x_{p,n_p}^0), y_i^0 = (y_{p,1}^0, y_{p,2}^0, \ldots, y_{p,n_p}^0), i = 1, 2, \ldots, N \). Hence, Graph\( K_p \) is open.

For any coalition \( S \), define the set-valued mapping \( G_S : X^1 \to 2^{X^1_S} \) by the rule:

\[
G_S(x) = \overline{X_S} x \in X^1,
\]

where \( X^1_S = \prod_{p \in S} X^1_p \). Now let us show that \( G_S \) is well-defined. For any \( \bar{x} \in \overline{X_S} \), we deduce that

\[
\overline{x_{p,i}} \geq 0, \forall p \in S, i \in T_p
\]

and

\[
\sum_{p \in S} \sum_{i=1}^{n_p} \overline{x_{p,i}} = \sum_{p \in S} m_p,
\]

where \( \overline{x_{p,i}} \) is component of \( \bar{x} \). Then,

\[
\sum_{p \in S} \sum_{i=1}^{n_p} x_{p,i} = \sum_{p \in S} m_p \leq \sum_{p \in \Omega} m_p \leq l.
\]

It follows that \( \bar{x} \in X^1_S \). Hence, \( \overline{X_S} \subset X^1_S \) and \( G_S \) is well-defined. Since \( \overline{X_P} \) is simplex, \( G_S \) is continuous and for each \( x \in X^1 \), \( G_O(x) \) is nonempty closed convex set.

Let \( \gamma \) be a balanced family of coalition \( S \) and \( l_S, S \in \Gamma \) be the balancing weight with \( \Sigma_{S \in \Gamma} l_S = 1 \) and \( l_S \geq 0 \). For any \( x \in X^1 \), if \( y_S \in G_S(x) = \overline{X_S}, \forall S \in \Gamma \). By the proof in Theorem 2.2 of [4], we can deduce that \( y \in \overline{G_O(x)} = \overline{X_O} \) where \( y = (y_1, y_2, \ldots, y_n) \) with \( y_i = \Sigma_{S \in \Gamma} l_S y_{p,i}^0 \) and \( y_{p,i}^0 \) is the \( i \)th component of \( y_S \).

Thus, all conditions are satisfied in Theorem 2.1. By applying Theorem 2.1, \( \exists x^* = (x_1^*, x_2^*, \ldots, x_N^*) \in X^1 \) such that \( x^* \in \overline{X_O} \) and for any coalition \( S \subset \Omega \), there exists no \( y_S \in \overline{X_S} \) satisfying

\[
y \in K_p(x^*), \forall p \in S,
\]

i.e., for any coalition \( S \subset \Omega \), there exists no \( y_S \in \overline{X_S} \) such that

\[
\sum_{i=1}^{n_p} y_{p,i} f_{p,i}(x^*) \in \sum_{i=1}^{n_p} x_{p,i}^* f_{p,i}(x^*) + \text{int}C, \quad \forall p \in S,
\]

where \( x_{p,i}^* \) is the \( i \)th component of \( x_p^* \). It follows that \( x^* \) is a weakly cooperative equilibrium point of multi-objective population games with respect to \( (\Omega, \overline{X_O}, f) \). This completes the proof. \( \square \)
Remark 3. The weakly cooperative equilibrium point and weakly noncooperative equilibrium point (see [3]) of multi-objective population games are different. Now we give the following example to explain the difference.

Example 2. Let \( \Omega = \{1, 2\} \), \( T_1 = T_2 = \{1, 2\} \) and \( m_1 = m_2 = 1 \). Let \( X_1 = X_1^\Omega = \{x_1 = (x_1^1, x_1^2) \in R_+^2 : x_1^1 + x_1^2 = 1\} \) and \( X_2 = \overline{X_2} = \{x_2 = (x_2^1, x_2^2) \in R_+^2 : x_2^1 + x_2^2 = 1\} \). Denote the cone \( C = R_+^2 \) and the vector payoff functions as follows:

\[
f_{1,1}(x) = (1, x_1^1); \quad f_{1,2}(x) = (1, x_2^1); \quad f_{2,1}(x) = (0, x_1^2); \quad f_{2,2}(x) = (0, x_2^2); \quad x \in R_+^2.
\]

For the coalition \( S = \{1, 2\} \), we obtain the social state set

\[
\overline{X_S} = \overline{X_{\Omega}} = \{x = (x_1, x_2) \in R_+^4 : x_1^1 + x_1^2 + x_2^1 + x_2^2 = 2\}.
\]

Note that the vector payoff functions are continuous. Thus, by Theorem 2, the weakly cooperative equilibrium point of multi-objective population games is nonempty. Next, we claim that \( x^* = (2, 0, 0, 0) \in \overline{X_{\Omega}} \) is a weakly cooperative equilibrium point of a multi-objective population game. In fact, by calculation, for any \( y_S = (y_{1,1}, y_{1,2}, y_{2,1}, y_{2,2}) \in \overline{X_S} \),

\[
y_{1,1}f_{1,1}(x^*) + y_{1,2}f_{1,2}(x^*) = (y_{1,1} + y_{1,2}, 2y_{1,1})
\]

and

\[
x^*_{1,1}f_{1,1}(x^*) + x^*_{1,2}f_{1,2}(x^*) = (2, 4).
\]

Hence,

\[
y_{1,1}f_{1,1}(x^*) + y_{1,2}f_{1,2}(x^*) \notin x^*_{1,1}f_{1,1}(x^*) + x^*_{1,2}f_{1,2}(x^*) + \text{int} R_+^2.
\]

Then, we have that there exists no \( y_S \in \overline{X_S} \) such that

\[
\sum_{i=1}^{2} y_{i,i}f_{i,i}(x^*) \notin \sum_{i=1}^{2} x^*_{i,i}f_{i,i}(x^*) + \text{int} C, \quad \forall p \in S,
\]

that is, \( x^* = (2, 0, 0, 0) \in \overline{X_{\Omega}} \) is a weakly cooperative equilibrium point of a multi-objective population game. Nevertheless, \( x^* = (2, 0, 0, 0) \notin X_1 \times X_2 \) and \( x^* \) is not weakly noncooperative equilibrium point of multi-objective population game.

Next, we show that the strong cooperative equilibria of multi-objective population games are nonempty.

Theorem 3. Let \( X_p^l \) and \( X_l^l \) be the same sets as Theorem 2. Suppose that the following assumptions hold:

(i) for each \( p \in \Omega \) and \( i \in T_p \), the vector payoff function \( f_{p,i} : X \to V \) is continuous;

(ii) for each population \( p \in \Omega \) and \( x \in X_l^l \), the vector function \( \sum_{i=1}^{m_p} y_{p,i}f_{p,i}(x) \) is properly \( C \)-quasi concave on \( X_p^l \).

Then strong cooperative equilibria of multi-objective population games with respect to \( (\Omega, \overline{X_{\Omega}}, f) \) are nonempty.

Proof. For each \( p \in \Omega \), define the set-valued mapping \( R_p : X^l \to 2^{X^l} \) by the rule:

\[
R_p(x) = \{y \in X^l : \sum_{i=1}^{n_p} y_{p,i}f_{p,i}(x) \notin \sum_{i=1}^{n_p} x_{p,i}f_{p,i}(x) - C\},
\]

where \( x = (x_1, x_2, \ldots, x_N), y = (y_1, y_2, \ldots, y_N), x_i = (x_{p,1}, x_{p,2}, \ldots, x_{p,n_p}), y_i = (y_{p,1}, y_{p,2}, \ldots, y_{p,n_p}), i = 1, 2, \ldots, N.\)
Obviously, for each \( x \in X^l, x \not\in R_p(x) \) owing to \( \theta \not\in V\{C\} \). Now, we can prove that for each \( x \in X^l, R_p(x) \) is convex. Let \( y_1, y_2 \in R_p(x) \) and \( \lambda \in [0, 1] \). Thus,

\[
\sum_{i=1}^{n_p} \lambda y^{1}_{p,i} f_{p,i}(x) + \sum_{i=1}^{n_p} (1 - \lambda) y^{2}_{p,i} f_{p,i}(x) \in \sum_{i=1}^{n_p} x_{p,i} f_{p,i}(x) + V\{C\}
\]

and

\[
\sum_{i=1}^{n_p} y^{1}_{p,i} f_{p,i}(x) + \sum_{i=1}^{n_p} y^{2}_{p,i} f_{p,i}(x) \in \sum_{i=1}^{n_p} x_{p,i} f_{p,i}(x) + V\{C\}.
\]

By assumption (ii), for each \( \lambda \in [0, 1] \),

\[
\sum_{i=1}^{n_p} (\lambda y^{1}_{p,i} + (1 - \lambda) y^{2}_{p,i}) f_{p,i}(x) \in \sum_{i=1}^{n_p} x_{p,i} f_{p,i}(x) + V\{C\} + C = \sum_{i=1}^{n_p} x_{p,i} f_{p,i}(x) + V\{-C\}.
\]

Hence, \( \lambda y_1 + (1 - \lambda) y_2 \in R_p(x) \) and \( R_p(x) \) is convex.

By assumptions and similar proof in Theorem 2, we obtain that all conditions are satisfied in Theorem 1. By applying Theorem 1, \( \exists x^* = (x^n_1, x^n_2, \ldots, x^n_M) \in X \) such that \( x^* \in X_{\Omega} \) and for any coalition \( S \subset \Omega \), there exists no \( y_s \in X_S \) satisfying

\[
y \in R_i(x^*), \forall i \in S,
\]

i.e., for any coalition \( S \subset \Omega \), there exists no \( y_s \in X_S \) such that

\[
\sum_{i=1}^{n_p} y^{1}_{p,i} f_{p,i}(x^*) \not\in \sum_{i=1}^{n_p} x^{*}_{p,i} f_{p,i}(x^*) - C, \forall p \in S,
\]

where \( x^{*}_{p,i} \) is the \( i \)-th component of \( x^*_{p} \). It follows that \( x^* \) is a strong cooperative equilibrium point of multi-objective population games with respect to \( (\Omega, X_{\Omega}, f) \). This completes the proof. \( \square \)

**Remark 4.** When \( S = R_+ \) and the vector payoff functions reduce to real-valued functions, assumption (ii) always holds. However, assumption (ii) is absolutely necessary in Theorem 3. We give the following example to explain this case.

**Example 3.** Let \( \Omega = \{1, 2\}, T_1 = T_2 = \{1, 2\} \) and \( m_1 = m_2 = 1 \). Let \( X_1 = X_{\Omega} = \{x_1 = (x^n_1, x^n_2) \in R^n_2 : x^n_1 + x^n_2 = 1\} \) and \( X_2 = X_{\Omega} = \{x_1 = (x^n_1, x^n_2) \in R^n_2 : x^n_1 + x^n_2 = 1\} \). Denote the cone \( C = R^n_+ \) and the vector payoff functions as follows:

\[
f_{1,1}(x) = f_{1,2}(x) = f_{2,1}(x) = f_{2,2}(x) = (1, -1); \ x \in R_+^4.
\]

By simple computing, the vector-valued function is

\[
\sum_{i=1}^{2} y_{i,1} f_{i,1}(x) = (y_{1,1} + y_{1,2}) \quad \text{and} \quad \sum_{i=1}^{2} y_{i,2} f_{i,2}(x) = y_{1,1} - y_{1,2}.
\]

Obviously, for each \( x \in X^l \), the vector-valued function \( \sum_{i=1}^{2} y_{i,1} f_{1,i}(x) \) is not properly \( R^n_+ \)-quasi concave. In addition, for each \( x^* \in X_{\Omega} \), we have

\[
y_{p,1} f_{p,1}(x^*) + y_{p,2} f_{p,2}(x^*) = (y_{p,1} + y_{p,2}) - (y_{p,1} - y_{p,2}), \forall p \in \Omega
\]

and

\[
x^*_{p,1} f_{p,1}(x^*) + x^*_{p,2} f_{p,2}(x^*) = (x^*_{p,1} + x^*_{p,2}) - (x^*_{p,1} - x^*_{p,2}), \forall p \in \Omega.
\]
4. Stability

In this section, we investigate the stability of weakly cooperative equilibria and strong cooperative equilibria of multi-objective population games when the payoff function tensor is perturbed.

Denote the distance \( \rho : X \times X \to R \) as follow:

\[
\rho(f, f') = \max_{p \in \Omega} \max_{i \in I} \sup_{x \in \overline{X}_S} ||f_{p,i}(x) - f'_{p,i}(x)||.
\]

It is easy to see that \((F, \rho)\) is a complete metric space. We assume that \(J(f)\) and \(Q(f)\) are the all weakly cooperative equilibrium point set and strong cooperative equilibrium point set when the payoff function tensor is \(f \in F\), respectively.

**Theorem 4.** Let \( F \) be the payoff tensor set satisfying all assumptions of Theorem 2. The parametric mapping \( J : F \to 2^{X_n} \) is u.s.c. and compact valued.

**Proof.** Since \( \overline{X}_\Omega \) is compact, it suffices to verify that \( \text{Graph} J \) is closed (see [19]). Let \( \{(f^n, x^n)\} \subset \text{Graph} J \) and \((f^n, x^n) \to (f^0, x^0)\). We only need to show \( x_0 \notin J(f^0) \), namely, there exist coalition \( \tilde{S} \) and \( \overline{y_S} \in X_\tilde{S} \) such that

\[
\sum_{i=1}^{n_p} \overline{y_{p,i}} f^0_{p,i}(x_0) - \sum_{i=1}^{n_p} x^0_{p,i} f^0_{p,i}(x_0) \in \text{int} C, \quad \forall p \in \tilde{S},
\]

where \( x_0 = (x^0_1, x^0_2, \ldots, x^0_N) \), \( x^0_i = (x^0_{p,1}, x^0_{p,2}, \ldots, x^0_{p,n_p}) \). \( \overline{y_S} = (\overline{y_{p,1}}, \overline{y_{p,2}}, \ldots, \overline{y_{p,n_p}}) \), \( i \in S \); \( \overline{x_S} \) is the ith component of \( \overline{y_S} \). Since \( \text{int} C \) is an open neighborhood of the vector \( \sum_{i=1}^{n_p} \overline{y_{p,i}} f^0_{p,i}(x_0) - \sum_{i=1}^{n_p} x^0_{p,i} f^0_{p,i}(x_0) \), by the continuity of the payoff function \( f^0_{p,i} \) for enough large \( n \),

\[
\sum_{i=1}^{n_p} \overline{y_{p,i}} f^0_{p,i}(x^n) - \sum_{i=1}^{n_p} x^0_{p,i} f^0_{p,i}(x^n) \in \text{int} C, \quad \forall p \in S,
\]

where \( x_n = (x^n_1, x^n_2, \ldots, x^n_N) \), \( x^n_i = (x^n_{p,1}, x^n_{p,2}, \ldots, x^n_{p,n_p}) \). Thus, there exists an open neighborhood \( O \) of zero element in \( V \) such that

\[
\sum_{i=1}^{n_p} \overline{y_{p,i}} f^n_{p,i}(x_n) - \sum_{i=1}^{n_p} x^n_{p,i} f^n_{p,i}(x_n) + O \subset \text{int} C, \quad \forall p \in \tilde{S}.
\]

(1)

Since \( f^n \to f^0 \), for the open neighborhood \( O \) of zero element, we have that

\[
\sum_{i=1}^{n_p} f^n_{p,i}(x_n)(\overline{y_{p,i}} - x^n_{p,i}) - \sum_{i=1}^{n_p} f^0_{p,i}(x_n)(\overline{y_{p,i}} - x^0_{p,i}) \in O, \quad \forall p \in \tilde{S},
\]

(2)
when \( n \) is enough large. From these conclusions (1) and (2) above, we can deduce that for any \( p \in S \), taking enough large \( n \),

\[
\sum_{i=1}^{n} f_{p,i}^u(x_n) (\overline{y}_{p,i} - x_{p,i}^u) = \sum_{i=1}^{n} f_{p,i}^0(x_n) (\overline{y}_{p,i} - x_{p,i}^0) - \sum_{i=1}^{n} f_{p,i}^0(x_n) (\overline{y}_{p,i} - x_{p,i}^u) + \sum_{i=1}^{n} f_{p,i}^0(x_n) (\overline{y}_{p,i} - x_{p,i}^u)
\]

\[
\in \bigcup_{i=1}^{n} f_{p,i}^0(x_n) (\overline{y}_{p,i} - x_{p,i}^u) + O \subset \text{int} \mathcal{C},
\]

which is absurd with \( \{(f^n, x_n)\} \subset \text{Graph} Q \). Therefore, the parametric mapping \( J : F \rightarrow 2^{\mathcal{X}_\Omega} \) is u.s.c.

For any fixed tensor \( f \in F \), we show that \( J(f) \) is compact. Since \( X_{\Omega} \) is compact, we only need to prove that \( J(f) \) is closed. Let \( \{x_n\} \subset J(f) \) and \( x_n \rightarrow x_0 \). Now, we show that \( x_0 \in J(f) \). We assume that \( x_0 \not\in J(f) \), namely, there exist coalition \( S \) and \( \overline{x} \in X_{S} \) such that

\[
\sum_{i=1}^{n} y_{p,i} f_{p,i}(x_0) - \sum_{i=1}^{n} x_{p,i} f_{p,i}(x_0) \not\in \text{int} \mathcal{C}, \quad \forall \ p \in S,
\]

where \( x_0 = (x_0^1, x_0^2, \ldots, x_0^n), y_i^0 = (y_{p,1}, x_0^2, \ldots, x_0^n), y_i = (y_{p,1}, y_{p,2}, \ldots, y_{p,n}), i \in S, \overline{y}_i \) is the \( i \)th component of \( \overline{y} \). By the similar proof as above, we can obtain \( x_0 \in J(f) \). Hence, the parametric mapping \( J : F \rightarrow 2^{\mathcal{X}_\Omega} \) is compact valued. This completes the proof. \( \square \)

**Theorem 5.** Let \( F \) be the payoff tensor set satisfying all assumptions of Theorem 3. The parametric mapping \( Q : F \rightarrow 2^{\mathcal{X}_\Omega} \) is u.s.c. and compact valued.

**Proof.** Since \( X_{\Omega} \) is compact, it suffices to verify that Graph\( Q \) is closed (see [19]). Let \( \{(f^n, x_n)\} \subset \text{Graph} Q \) and \( (f^n, x_n) \rightarrow (f^0, x_0) \). We only need to show \( x_0 \in Q(f^0) \). In fact, if we assume that \( x_0 \not\in Q(f^0) \), then there exist coalition \( S \) and \( \overline{x} \in X_{S} \) such that

\[
\sum_{i=1}^{n} y_{p,i} f_{p,i}(x_0) - \sum_{i=1}^{n} x_{p,i} f_{p,i}(x_0) \not\in -C, \quad \forall \ p \in S,
\]

where \( x_0 = (x_0^1, x_0^2, \ldots, x_0^n), y_i^0 = (y_{p,1}, x_0^2, \ldots, x_0^n), y_i = (y_{p,1}, y_{p,2}, \ldots, y_{p,n}), i \in S, \overline{y}_i \) is the \( i \)th component of \( \overline{y} \). Since \( V\{\overline{-C}\} \) is an open neighborhood of the vector \( \sum_{i=1}^{n} y_{p,i} f_{p,i}(x_0) - \sum_{i=1}^{n} x_{p,i} f_{p,i}(x_0) \), by the continuity of the payoff function \( f_{p,i}^0 \) for enough large \( n \),

\[
\sum_{i=1}^{n} y_{p,i} f_{p,i}(x_n) - \sum_{i=1}^{n} x_{p,i} f_{p,i}(x_n) \not\in -C, \quad \forall \ p \in S,
\]

where \( x_n = (x_n^1, x_n^2, \ldots, x_n^n), x_i^0 = (x_{n,1}, x_n^2, \ldots, x_n^n), i \in S, \overline{y}_i \) is the \( i \)th component of \( \overline{y} \). Since the set \( V\{\overline{-C}\} \) is open, \( \sum_{i=1}^{n} y_{p,i} f_{p,i}(x_n) - \sum_{i=1}^{n} x_{p,i} f_{p,i}(x_n) \) is an interior point in \( V\{\overline{-C}\} \). There is an open neighborhood \( U \) of zero element in \( V \) such that

\[
\sum_{i=1}^{n} y_{p,i} f_{p,i}(x_n) - \sum_{i=1}^{n} x_{p,i} f_{p,i}(x_n) + U \subset V\{\overline{-C}\}, \quad \forall \ p \in S.
\]

(3)

Since \( f^n \rightarrow f^0 \), we have that

\[
\sum_{i=1}^{n} f_{p,i}(x_n)(\overline{y}_{p,i} - x_{p,i}^u) - \sum_{i=1}^{n} f_{p,i}^0(x_n)(\overline{y}_{p,i} - x_{p,i}^u) + U \subset V\{\overline{-C}\}, \quad \forall \ p \in S,
\]

(4)
when \( n \) is enough large. By (3) and (4), we obtain that for any \( p \in S \), taking enough large \( n \),

\[
\sum_{i=1}^{n_p} f^x_{p,i}(x_n) (y^x_{p,i} - x^x_{p,i}) = \sum_{i=1}^{n_p} f^0_{p,i}(x_n) (y^x_{p,i} - x^x_{p,i}) + \sum_{i=1}^{n_p} f^0_{p,i}(x_n) (\overline{y}^x_{p,i} - x^x_{p,i}) \\
\in \sum_{i=1}^{n_p} f^0_{p,i}(x_n) (\overline{y}^x_{p,i} - x^x_{p,i}) + \mathbb{U} \\
\subset V \setminus \{ -C \}.
\]

This is a contradiction. Therefore, the parametric mapping \( Q : F \to 2^{\mathcal{X}_\Omega} \) is u.s.c.

For any fixed tensor \( f \in F \), since \( \mathcal{X}_\Omega \) is compact, we only need to prove that \( Q(f) \) is closed. By the above proof, it is easy to verify that \( Q(f) \) is closed and the parametric mapping \( J : F \to 2^{\mathcal{X}_\Omega} \) is compact valued. This completes the proof. \( \square \)

**Lemma 1** ([21]). Suppose that \( J : X \to 2^V \) is u.s.c. with nonempty compact values. Then, a dense residual subset \( Q \) of \( X \) exists such that \( F \) is lower semicontinuous on \( Q \).

**Lemma 2** ([9]). Let \( f \in F \). \( f \) is said to be essential on \( J \) if and only if \( J \) is l.s.c. on \( f \).

We can obtain the essential theorem of the parametric mappings \( J : F \to 2^{\mathcal{X}_\Omega} \) and \( Q : F \to 2^{\mathcal{X}_\Omega} \) on a dense residual subset of \( F \), respectively.

**Theorem 6.** There exists a dense residual set \( F' \subseteq F \) such that \( f \) is essential with respect to \( J \) for any \( f \in F' \).

**Theorem 7.** There exists a dense residual set \( F'' \subseteq F \) such that \( f \) is essential with respect to \( Q \) for any \( f \in F'' \).

5. Concluding Remark

Motivated by the idea of Yang and Zhang in [4], we introduced the concepts of weakly cooperative equilibria and strong cooperative equilibria of multi-objective population games. First, under the continuity assumption of the vector payoff function, we obtained the existence of weakly cooperative equilibria of multi-objective population games. Then, under the continuity and properly cone quasi concave assumptions of the vector payoff function, we obtained the existence of strong cooperative equilibria of multi-objective population games, which requires that each population maximizes all goals. Finally, we studied the stability of weakly cooperative equilibria and strong cooperative equilibria of multi-objective population games when the vector payoff functions are perturbed. Additionally, we gave some examples to explain the difference between the cooperative equilibrium point and noncooperative equilibrium point of multi-objective population games.

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