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Fuzzy Caratheodory's Theorem and Outer ∗-Fuzzy Measure

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Abstract: The goal of this paper is to introduce two new concepts $*$-fuzzy premeasure and outer $*$-fuzzy measure, and to further prove some properties, such as Caratheodory's Theorem, as well as the unique extension of $*$-fuzzy premeasure. This theorem is remarkable for it allows one to construct a $*$-fuzzy measure by first defining it on a small algebra of sets, where its $*$-additivity could be easy to verify, and then this theorem guarantees its extension to a sigma-algebra.

Keywords: $*$-outer fuzzy measure; t-norm; $*$-fuzzy premeasure; Caratheodory's theorem

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1. Introduction

The notion of $*$-fuzzy measure ($*$-FM) and its properties were defined and investigated in [1]; this version of fuzzy measure has a dynamic situation and can model new events, such as the COVID-19 disease, explained in [2]. Further, some results of $*$-FM are discussed in [3]. In fact, $*$-FM is a dynamic generalization of the classical measure theory. This generalization is obtained by replacing the non-negative real range and the additivity of classical measures with fuzzy sets and triangular norms. Our development of the fuzzy measure theory has been motivated by defining a new additivity property using triangular norms. Here, the classical additivity of measures based on the addition of real additivity is replaced by triangular norms-based aggregation. Our approach is related to the idea of fuzzy metric spaces [4–6]. Though our paper is purely theoretical, we expect several applications of our results in domains considering the development in time, e.g., in quantum physics or in color image filtering. Based on the obtained work, we are going to define two new notions $*$-fuzzy premeasure and outer $*$-fuzzy measure, and study their properties and the relationship between them.

2. $*$-Fuzzy Measure

We begin by giving some background and related results from $*$-fuzzy measure theory that we will use in this article. Let $X \neq \emptyset$ and $\mathcal{M}$ be a σ-algebra of subsets of X. Further, we use $I = [0,1]$ and $J = [0,+\infty)$.

Definition 1 ([7,8])**.** *A topological monoid*

$$*: I^2 \longrightarrow I,$$

such that

(i) $\wp * \wp' = \wp' * \wp$, *for all* $\wp, \wp' \in I$,

(ii) $\wp * (\wp' * \wp'') = (\wp * \wp') * \wp''$, *for all* $\wp, \wp', \wp'' \in I$,

(iii) $\wp * 1 = \wp$, *for all* $\wp \in I$,

(iv) *If* $\wp_1 \leq \wp_2$ *and* $\wp'_1 \leq \wp'_2$ *then* $\wp_1 * \wp'_1 \leq \wp_2 * \wp'_2$, *for all* $\wp_1 \wp_2, \wp'_1, \wp'_2 \in I$,
is said to be a ct-norm.

Example 1. *Now, we consider important ct-norms.*
(1) $\gamma *_P \gamma' = \gamma\gamma'$;
(2) $\gamma *_M \gamma' = \min\{\gamma, \gamma'\}$;
(3) $\gamma *_L \gamma' = \max\{\gamma + \gamma' - 1, 0\}$;
(4)
$$\gamma *_H \gamma' = \begin{cases} 0, & \text{if } \gamma = \gamma' = 0, \\ \frac{1}{\frac{1}{\gamma} + \frac{1}{\gamma'} - 1}, & \text{otherwise}, \end{cases}$$
(the Hamacher ct-norm).

When a ct-norm possesses an Archimedean property ($\gamma * \gamma < \gamma$ for every $\gamma \in I^0 = (0,1)$), we say that $*$ is a cat-norm. For example, $*_H, *_L, *_P$ are cat-norms but $*_M$ is not (for more details about the cat-norm we refer to [9]).

Definition 2 ([1–3]). *Consider the set* X, *σ-algebra* $\mathcal{M} \subseteq \mathcal{P}(X)$, *and cat-norm* $*$. *We define a* $*$*-fuzzy measure (*$*$*-FM)* μ *from* $\mathcal{M} \times J$ *to* I, *in which*

(1) μ *maps* $(\emptyset, t)$ *to* 1, *for each* $t \in J$;
(2) $\mu(v, .)$ *is left-continuous, increasing and* $\mu(v, t)$ *tends to 1 when* t *tends to* $+\infty$ *for every* $v \in \mathcal{M}$;
(3) *if* $v_\ell \in \mathcal{M}$, *in which* $v_\ell \cap v_k = \emptyset$ *for* $\ell \neq k$ *and* $\ell, k = 1, 2, \cdots$, *then*

$$\mu\left(\bigcup_{\ell=1}^{+\infty} v_\ell, t\right) = *_{\ell=1}^{+\infty} \mu(v_\ell, t), \text{ for every } t \in J.$$

It is clear that Item (3) of Definition 2 is a countable $*$-additivity. Further, a $*$-FM is finitely $*$-additive if

$$\mu\left(\bigcup_{\ell=1}^{n} v_\ell, t\right) = *_{\ell=1}^{n} \mu(v_\ell, t), \text{ for every } t \in J,$$

whenever $v_1, \cdots, v_n \in \mathcal{M}$ and $v_\ell \cap v_k = \emptyset, \ell \neq k$.

Observe that if $*$ is a strict cat (i.e., $*$ is strictly increasing on $(0,1]^2$), then it is additively generated by a decreasing bijection $f : I \to [0, +\infty]$, where $\wp * \wp' = f^{-1}(f(\wp) + f(\wp'))$. Then, for any $*$–FM μ, and any $t \in J$, the set function $m_t : \mathcal{M} \to J$ given by $f(\mu(., t))$ is a sigma-additive measure. Vice-versa, for any decreasing surjection $g : J \to J$, and any sigma-additive measure m, define $\mu(\nu, t) = f^{-1}(g(t).m(\nu))$ for $\nu \in \mathcal{M}$, which implies that μ is a $*$-FM.

Example 2. *Consider the measure space* $(X, \mathcal{M}, m)$*, and classical* σ*-additive measure* $m : \mathcal{M} \to [0, +\infty]$*. Put* $f(\wp) = \frac{1-\wp}{\wp}$ *for each* $\wp \in I$ *and* $g(t) = \frac{1}{t}$ *for every* $t > 0$*. Then* $* = *_H$ *because* $f^{-1}(\xi) = \frac{1}{1+\xi}$ *for every* $\xi \in [0, +\infty]$*, and hence*

$$\begin{aligned} \wp * \wp' &= f^{-1}\left(\frac{1-\wp}{\wp} + \frac{1-\wp'}{\wp'}\right) = \frac{1}{1 + \frac{1-\wp}{\wp} + \frac{1-\wp'}{\wp'}} \\ &= \frac{1}{\frac{1}{\wp} + \frac{1}{\wp'} - 1} = \wp *_H \wp'. \end{aligned}$$

Further,

$$\mu(v, t) = f^{-1}(g(t).m(v)) = f^{-1}\left(\frac{m(v)}{t}\right) = \frac{t}{t + m(v)},$$

for all $t \in J$*, and* μ *is a* $*$*-FM.*

A $*$-fuzzy measure space (abbreviated to $*$-FMS) is denoted by the tetrad $(X, \mathcal{M}, \mu, *)$. According to Definition 2, $\mu(v, .)$ is a left-continuous and increasing map (it is a left-continuous distance function in the sense of Rodabaugh and Klement's earlier works). Therefore, $\mu(v, .)$ is a fuzzy number. We claim μ is monotone because $\mu(v, .)$ is a decomposable measure with $*$, and $*$-decomposability implies the monotonicity [10]. From [11,12], we can extend $\mu : v \times (0, +\infty) \to I$ to $\mu : v \times (-\infty, +\infty) \to I$ with $\mu(v, t) = 0$ for every $t \leq 0$. Thus, $\mu(v, .)$ from $\mathbb{R}$ to I is a special L-fuzzy number [13–15] or is a distance function [8]. The fuzzy measure theory was initially introduced by Sugeno et al. in [16,17]. With new approaches, we have further defined $*$-FMS from fuzzy metric spaces and fuzzy normed spaces [4–6,10,13,18–34]. There are two classical references [35,36] in this area.

Definition 3. *Let the quadriple* $(X, \mathcal{M}, \mu, *)$ *be a* $*$*-FMS. Positivity of* $\mu(X, t)$ *for positive number* t *implies that* μ *is a bounded* $*$*–FM. Furthermore, when* $X = \bigcup_{\ell=1}^{+\infty} v_\ell$*, for* $v_\ell \in \mathcal{M}$*,* $\ell = 1, 2, \ldots$ *and* $\mu(v_\ell, t) > 0$*, we get* μ *as* σ*-bounded. If* μ *is a bounded* $*$*-FM we say the quadriple* $(X, \mathcal{M}, \mu, *)$ *is a bounded* $*$*-FMS. On the other hand,* σ*-boundedness* $*$*-FM,* μ *shows* σ*-boundedness of* $(X, \mathcal{M}, \mu, *)$*. Let* $t > 0$*. If for every* $v \in \mathcal{M}$ *with* $\mu(v, t) = 0$*, there exists a set* $\vartheta \in \mathcal{M}$ *such that* $\vartheta \subseteq v$ *and* $0 < \mu(\vartheta, t) < 1$*, we call* μ *a* $*$*-fuzzy pseudo bounded measure.*

Definition 4. *Let the quadriple* $(X, \mathcal{M}, \mu, *)$ *be a* $*$*-FMS. If* $v \in \mathcal{M}$ *and* $\mu(v, t) = 1$*, for each* $t > 0$*, then we say* v *is a* $*$*-fuzzy null set.*

The notion of a $*$-fuzzy null set should not be confused with the empty set as defined in set theory. Although for the empty set $\varnothing$ we have $\mu(\varnothing, t) = 1$, for each $t > 0$. Consider Example 2, for any non-empty countable set ν of real numbers, we have

$$\begin{aligned} \mu(\nu, t) &= \frac{t}{t + m(\nu)} \\ &= \frac{t}{t + 0} \\ &= 1, \end{aligned}$$

for each $t > 0$.

Definition 5. *A complete* $*$*-FMS is a* $*$*-FMS that contains all subsets of null sets.*

Note that a $*$-FMS $(X, \mathcal{M}, \mu, *)$ is complete if and only if $v \subset u \in \mathcal{M}$ and $\mu(u, t) = 1$ for each $t > 0$ implies that $v \in \mathcal{M}$.

Theorem 1 ([1]). *Let the quadriple $(X, \mathcal{M}, \mu, *)$ be a $*$-FMS. Let*

$$\mathcal{N}_\alpha = \{N_\gamma \in \mathcal{M} : \mu(N_\gamma, t) = 1, \text{ for every } t > 0\},$$

and

$$\overline{\mathcal{M}} = \{v \cup \vartheta : v \in \mathcal{M} \text{and } \vartheta \subset N_\gamma \text{ for some } N_\gamma \in \mathcal{N}_\alpha\},$$

such that it is not necessary $\vartheta \in \mathcal{M}$. Then, it is clear that $\overline{\mathcal{M}}$ is a σ-algebra and there exists a unique extension $\overline{\mu}$ of μ.

3. Outer $*$-Fuzzy Measure

Definition 6. *Consider $X \neq \emptyset$. A fuzzy set $\mu^\diamond : \mathcal{P}(X) \times (0, +\infty) \to I$ that satisfies the following for every $t > 0$,*

(i) $\mu^\diamond(\emptyset, t) = 1$,
(ii) *If $v \subseteq \vartheta$ then $\mu^\diamond(\vartheta, t) \le \mu^\diamond(v, t)$,*
(iii) $\mu^\diamond\left(\bigcup_{\ell=1}^{+\infty} v_\ell, t\right) \ge *_{\ell=1}^{+\infty} \mu^\diamond(v_\ell, t)$,

is called an outer $*$-FM.

For example, let $X = \mathbb{R}$ and define $\mu^\diamond : \mathcal{P}(\mathbb{R}) \times (0, +\infty) \to I$ by

$$\mu^\diamond(v, t) = \begin{cases} 1, & \text{if } v = \emptyset, \\ \frac{t}{t+1}, & \text{if } v \neq \emptyset, \end{cases}$$

for each $t > 0$ and let $* = *_H$. Then, $\mu^\diamond$ is an outer $*$-FM.

Definition 7. *Let $\xi \subseteq \mathcal{P}(X)$, we say ξ is an elementary family of subsets of X, if,*

(i) $\emptyset \in \xi$;
(ii) *If $v, \vartheta \in \xi$ then $v \cap \vartheta \in \xi$;*
(iii) *If $v \in \xi$ then v^c is a finite disjoint union of members of ξ.*

Now, we present a fact concerning elementary families [35].

Theorem 2. *Let ξ be an elementary family, then*

$$\mathcal{A} = \left\{ \bigcup_{\ell=1}^{n} v_\ell : v_\ell \cap v_k = \emptyset, \ell \neq k, v_\ell \in \xi \right\}$$

is an algebra.

We obtain outer $*$-FMs by a family ξ of elementary sets as follows:

Theorem 3. *Let $\xi \subseteq \mathcal{P}(X)$ such that $X, \emptyset \in \xi$, and $\rho : \xi \times J \to I$ satisfy $\rho(\emptyset, t) = 1$ for every $t > 0$. We define for $v \subset X$,*

$$\mu^\diamond(v, t) = \sup\left\{ *_{\ell=1}^{+\infty} \rho(\varrho_\ell, t) : \varrho_\ell \in \xi \text{ and } v \subset \bigcup_{\ell=1}^{+\infty} \varrho_\ell \right\}. \tag{1}$$

Therefore, $\mu^\diamond$ is an outer $*$-FM.

Proof. For any $v \subset X$ we can find $\{\varrho_\ell\}_1^{+\infty} \subseteq \xi$ such that $v \subset \bigcup_{\ell=1}^{+\infty} \varrho_\ell$ (take $\varrho_\ell = X$ for all ℓ) so the definition of $\mu^\diamond$ makes sense. Now, we show the outer $*$-fuzzy measure properties.

(i) It is clear $\mu^\diamond(\emptyset, t) = 1$.
(ii) If $v \subset \vartheta$ then $\mu^\diamond(\vartheta, t) \le \mu^\diamond(v, t)$.

(iii) To show property (iii) of Definition 6, we apply induction.
Let $\{v_1, v_2\} \subseteq \mathcal{P}(X)$ and $0 < \epsilon < 1$. Since

$$\mu^\diamond(v_1, t) = \sup\left\{ *_{\ell=1}^{+\infty} \rho(\varrho_\ell, t) : \varrho_\ell \in \xi \;,\; v_1 \subset \bigcup_{\ell=1}^{+\infty} \varrho_\ell \right\},$$

we have

$$\mu^\diamond(v_1, t) - \epsilon < *_{\ell=1}^{+\infty} \rho(\varrho_\ell, t). \tag{2}$$

Similarly, we have

$$\mu^\diamond(v_2, t) \le *_{k=1}^{+\infty} \rho(\varrho'_k, t), \quad \bigcup_{k=1}^{+\infty} \varrho'_k \subseteq v_2. \tag{3}$$

From (2) and (3) we get

$$\mu^\diamond(v_1, t) * \mu^\diamond(v_2, t) \le *_{j=1}^{+\infty} \rho(\varrho''_j, t), \tag{4}$$

where $\varrho''_j = \varrho_\ell$ or ϱ'_k.

On the other hand, $\cup\varrho''_j \subseteq v_1 \cup v_2$, $\varrho''_j \in \xi$ so

$$\mu^\diamond(v_1 \cup v_2, t) \ge *_{j=1}^{+\infty} \rho(\varrho''_j, t). \tag{5}$$

From (4) and (5) we can conclude that

$$\mu^\diamond(v_1 \cup v_2, t) \ge \mu^\diamond(v_1, t) * \mu^\diamond(v_2, t),$$

and the proof is complete. □

Note that $v \subset X$ is a $\mu^\diamond$-$*$-fuzzy measurable set if $\mu^\diamond$ is an outer $*$-FM on X and

$$\mu^\diamond(\varrho, t) = \mu^\diamond(\varrho \cap v, t) * \mu^\diamond(\varrho \cap v^c, t) \text{ for all } \varrho \subset X.$$

Clearly, the inequality $\mu^\diamond(\varrho, t) \ge \mu^\diamond(\varrho \cap v, t) * \mu^\diamond(\varrho \cap v^c, t)$ holds for any v and ϱ. To prove v is $\mu^\diamond$-$*$-fuzzy measurable, it suffices to prove the converse of the above inequality. If $\mu^\diamond(\varrho, t) = 0$, we claim v is $\mu^\diamond$-$*$-fuzzy measurable if and only if

$$\mu^\diamond(\varrho, t) \le \mu^\diamond(\varrho \cap v, t) * \mu^\diamond(\varrho \cap v^c, t), \text{ for all } \varrho \subset X \text{ such that } \mu^\diamond(\varrho, t) > 0.$$

Theorem 4 (Caratheodory's Theorem)**.** *Consider outer $*$–FM $\mu^\diamond$ on X, then the family $\mathcal{M}$ consisting of all $\mu^\diamond$-$*$-fuzzy measurable sets is a σ-algebra, and the restriction of $\mu^*|_{\mathcal{M}}$ is a complete $*$–FM.*

Proof. Clearly, $\mathcal{M}$ is closed under the complement operation. Furthermore, if $v, \vartheta \in \mathcal{M}$ and $\varrho \subset X$ we get

$$\begin{aligned} \mu^\diamond(\varrho, t) &= \mu^\diamond(\varrho \cap v, t) * \mu^\diamond(\varrho \cap v^c, t) \\ &= \mu^\diamond(\varrho \cap v \cap \vartheta, t) * \mu^\diamond(\varrho \cap v \cap \vartheta^c, t) \\ &\quad * \mu^\diamond(\varrho \cap v^c \cap \vartheta, t) * \mu^\diamond(\varrho \cap v^c \cap \vartheta^c, t). \end{aligned} \tag{6}$$

Since $(v \cup \vartheta) = (v \cap \vartheta) \cup (v \cap \vartheta^c) \cup (v^c \cap \vartheta)$ and sup-additivity, we derive

$$\mu^\diamond(\varrho \cap ((v \cap \vartheta) \cup (v \cap \vartheta^c) \cup (v^c \cap \vartheta)), t) \ge \mu^\diamond(\varrho \cap v \cap \vartheta, t) * \mu^\diamond(\varrho \cap v \cap \vartheta^c, t) * \mu^\diamond(\varrho \cap v^c \cap \vartheta, t).$$

Using (6) implies that

$$\mu^{\diamond}(\varrho, t) \leq \mu^{\diamond}(\varrho \cap (v \cup \vartheta), t) * \mu^{\diamond}(\varrho \cap (v \cup \vartheta)^c, t).$$

It follows that $v \cup \vartheta \in \mathcal{M}$, i.e., $\mathcal{M}$ is an algebra. Moreover, when $v, \vartheta \in \mathcal{M}$ and $v \cap \vartheta = \emptyset$, we have

$$\mu^{\diamond}(v \cup \vartheta, t) = \mu^{\diamond}((v \cup \vartheta) \cap v, t) * \mu^{\diamond}((v \cup \vartheta) \cap v^c, t) = \mu^{\diamond}(v, t) * \mu^{\diamond}(\vartheta, t),$$

which implies that $\mu^{\diamond}$ is finitely additive on $\mathcal{M}$.

Consider a sequence of disjoint sets in $\mathcal{M}$ i.e., $\{v_\ell\}_{\ell=1}^{+\infty}$, and $\vartheta_n = \bigcup_{\ell=1}^{n} v_\ell$ and $\vartheta = \bigcup_{\ell=1}^{+\infty} v_\ell$. Then for any $\varrho \subset X$, we have

$$\begin{aligned} \mu^{\diamond}(\varrho \cap \vartheta_n, t) &= \mu^{\diamond}(\varrho \cap \vartheta_n \cap v_n, t) * \mu^{\diamond}(\varrho \cap \vartheta_n \cap v_n^c, t) \\ &= \mu^{\diamond}(\varrho \cap v_n, t) * \mu^{\diamond}(\varrho \cap \vartheta_{n-1}, t). \end{aligned}$$

Now, a simple induction shows that $\mu^{\diamond}(\varrho \cap \vartheta_n, t) = *_{\ell=1}^{n} \mu^{\diamond}(\varrho \cap v_\ell, t)$. Thus,

$$\begin{aligned} \mu^{\diamond}(\varrho, t) &= \mu^{\diamond}(\varrho \cap \vartheta_n, t) * \mu^{\diamond}(\varrho \cap \vartheta_n^c, t) \\ &= *_{\ell=1}^{n} \mu^{\diamond}(\varrho \cap v_\ell, t) * \mu^{\diamond}(\varrho \cap \vartheta_n^c, t) \\ &\leq *_{\ell=1}^{n} \mu^{\diamond}(\varrho \cap v_\ell, t) * \mu^{\diamond}(\varrho \cap \vartheta^c, t), \end{aligned}$$

and letting $n \to +\infty$ we obtain

$$\begin{aligned} \mu^{\diamond}(\varrho, t) &\leq *_{\ell=1}^{+\infty} \mu^{\diamond}(\varrho \cap v_\ell, t) * \mu^{\diamond}(\varrho \cap \vartheta^c, t) \\ &\leq \mu^{\diamond}\left(\bigcup_{\ell=1}^{+\infty} (\varrho \cap v_\ell), t \right) * \mu^{\diamond}(\varrho \cap \vartheta^c, t) \\ &= \mu^{\diamond}(\varrho \cap \vartheta, t) * \mu^{\diamond}(\varrho \cap \vartheta^c, t) \\ &\leq \mu^{\diamond}(\varrho, t). \end{aligned}$$

Thus $\mu^{\diamond}(\varrho, t) = \mu^{\diamond}(\varrho \cap \vartheta, t) * \mu^{\diamond}(\varrho \cap \vartheta^c, t)$. From $\vartheta \in \mathcal{M}$ and taking $\varrho = \vartheta$, we get $\mu^{\diamond}(\vartheta, t) = *_{\ell=1}^{+\infty} \mu^{\diamond}(\varrho \cap v_\ell, t)$; thus $\mu^{\diamond}$ is countably additive on $\mathcal{M}$. Finally, if $\mu^{\diamond}(\varrho, t) = 1$ for any $\varrho \subset X$ we have

$$\mu^{\diamond}(\varrho, t) \geq \mu^{\diamond}(\varrho \cap v, t) * \mu^{\diamond}(\varrho \cap v^c, t) = \mu^{\diamond}(\varrho \cap v^c, t) \geq \mu^{\diamond}(\varrho, t),$$

because $v \in \mathcal{M}$. Hence $\mu^*|_{\mathcal{M}}$ is a complete $*$–FM. □

Definition 8. *Consider the algebra $\mathcal{A}$ of $\mathcal{P}(X)$; we say $\mu_\diamond : \mathcal{A} \times J \to I$ is a $*$-fuzzy premeasure ($*$-FPM), when*

(i) $\mu_\diamond(\emptyset, t) = 1$, *and*

(ii) *if $\{v_\ell\}_{\ell=1}^{+\infty}$ is a sequence of disjoint sets in $\mathcal{A}$ such that $\bigcup_{\ell=1}^{+\infty} v_\ell \in \mathcal{A}$, then* $\mu_\diamond\left(\bigcup_{\ell=1}^{+\infty} v_\ell, t \right) = *_{\ell=1}^{+\infty} \mu_\diamond(v_\ell, t)$.

In particular, any $*$-FPM is finitely additive because $v_\ell = \emptyset$ for $\ell \geq n$.

Let $\mu_\diamond$ be a $*$–FPM on $\mathcal{A} \subset \mathcal{P}(X)$, Theorem 3, implies that

$$\mu^{\diamond}(\varrho, t) = \sup \left\{ *_{\ell=1}^{+\infty} \mu_\diamond(v_\ell, t) : v_\ell \in \mathcal{A}, \varrho \subseteq \bigcup_{\ell=1}^{+\infty} v_\ell \right\}. \tag{7}$$

Let S be the set of intervals $(a,b]$ and $* = *_H$. Let $\mathcal{A}$ be the collection of sets $A \subset \mathbb{R}$ representable as finite unions of disjoint intervals,

$$A = \bigcup_{i=1}^{k} (a_i, b_i],$$

one may check that $\mathcal{A}$ is an algebra. We define

$$\mu_\diamond(A,t) = \frac{t}{t + \sum_{i=1}^{k}(b_i - a_i)}.$$

It is easy to show that $\mu_\diamond$ is a $*$-FPM on $\mathcal{A}$.

Theorem 5. *Consider $*$-FPM $\mu_\diamond$ on $\mathcal{A}$ then*

(i) $\mu^\diamond|_{\mathcal{A}} = \mu_\diamond$,
(ii) *elements of $\mathcal{A}$ are $\mu^\diamond$-$*$-fuzzy measurable.*

Proof.

(i) Suppose $\varrho \in \mathcal{A}$. Let $\varrho \subset \bigcup_{\ell=1}^{+\infty} v_\ell$ with $v_\ell \in \mathcal{A}$ and $\vartheta_n = \varrho \cap \left(v_n - \bigcup_{\ell=1}^{n-1} v_\ell\right)$. Then the ϑ_n's are disjoint members of $\mathcal{A}$ whose $\bigcup_{n=1}^{+\infty} \vartheta_n = \varrho$, thus

$$\begin{aligned}\mu_\diamond(\varrho,t) &= \mu_\diamond\left(\bigcup_{n=1}^{+\infty} \vartheta_n, t\right) \\ &= *_{n=1}^{+\infty}\mu_\diamond(\vartheta_n,t) \geq *_{n=1}^{+\infty}\mu_\diamond(v_n,t),\end{aligned}$$

and so

$$\sup \mu_\diamond(\varrho,t) \geq \sup\left\{ *_{n=1}^{+\infty}\, \mu_\diamond(v_n,t) : \varrho \subseteq \bigcup_{n=1}^{+\infty} v_n \right\},$$

hence

$$\mu_\diamond(\varrho,t) \geq \mu^\diamond(\varrho,t), \tag{8}$$

also $\varrho \subseteq \varrho$, thus

$$\mu^\diamond(\varrho,t) \geq \mu_\diamond(\varrho,t). \tag{9}$$

From (8) and (9) we have

$$\mu^\diamond(\varrho,t) = \mu_\diamond(\varrho,t).$$

(ii) If $v \in \mathcal{A}$, $\varrho \subset X$, and $0 < \varepsilon < 1$, there is a sequence $\{\vartheta_\ell\}_{\ell=1}^{+\infty} \subset v$ with $\varrho \subset \bigcup_{\ell=1}^{+\infty} \vartheta_\ell$ and $\mu^\diamond(\varrho,t) - \varepsilon < *_{\ell=1}^{+\infty}\mu_\diamond(\vartheta_\ell,t)$. Since $\mu_\diamond$ is $*$-additive on $\mathcal{A}$, we have

$$\begin{aligned}\mu^\diamond(\varrho,t) - \varepsilon &< *_{\ell=1}^{+\infty}\mu_\diamond(\vartheta_\ell,t) \\ &= *_{\ell=1}^{+\infty}\mu_\diamond(\vartheta_\ell \cap (v \cup v^c),t) \\ &= *_{\ell=1}^{+\infty}\mu_\diamond((\vartheta_\ell \cap v) \cup (\vartheta_\ell \cap v^c),t) \\ &= *_{\ell=1}^{+\infty}[\mu_\diamond(\vartheta_\ell \cap v,t) * (\vartheta_\ell \cap v^c,t)] \\ &= [*_{\ell=1}^{+\infty}\mu_\diamond(\vartheta_\ell \cap v,t)] * [*_{\ell=1}^{+\infty}\mu_\diamond(\vartheta_\ell \cap v^c,t)] \\ &\leq \mu^\diamond(\varrho \cap v,t) * \mu^\diamond(\varrho \cap v^c,t).\end{aligned}$$

Since $0 < \varepsilon < 1$ is arbitrary, we come to

$$\mu^\diamond(\varrho,t) \leq \mu^\diamond(\varrho \cap v,t) * \mu^\diamond(\varrho \cap v^c,t).$$

Thus v is $\mu^{\diamond}$-$*$-fuzzy measurable.
□

Theorem 6. *Consider the algebra $\mathcal{A}$ of $\mathcal{P}(X)$, $*$-FPM $\mu_{\diamond}$ on $\mathcal{A}$, and the generated σ-algebra $\mathcal{M}$ by $\mathcal{A}$. Then we can find a $*$–FM μ on $\mathcal{M}$ that $\mu = \mu^{\diamond}|_{\mathcal{M}}$ where $\mu^{\diamond}$. Let ν be a different $*$–FM on $\mathcal{M}$ that extends $\mu_{\diamond}$, then $\nu(\varrho, t) \geq \mu(\varrho, t)$ for each $t > 0$ and $\varrho \in \mathcal{M}$, with equality when $\mu(\varrho, t) > 0$. If $\mu_{\diamond}$ is σ-bounded, then μ is the unique extension of $\mu_{\diamond}$ to a $*$–FM on $\mathcal{M}$.*

Proof. Let $\varrho \in \mathcal{M}$ and $\varrho \subset \bigcup_{\ell=1}^{+\infty} v_{\ell}$ such that $v_{\ell} \in \mathcal{A}$, then

$$\nu(\varrho, t) \geq *_{\ell=1}^{+\infty} \nu(v_{\ell}, t) = *_{\ell=1}^{+\infty} \mu_{\diamond}(v_{\ell}, t)$$

and so

$$\begin{aligned} \sup\{\nu(\varrho, t)\} &\geq \sup\{*_{\ell=1}^{+\infty} \mu_{\diamond}(v_{\ell}, t) : \varrho \subseteq \cup v_{\ell}\} \\ \nu(\varrho, t) &\geq \mu(\varrho, t). \end{aligned} \tag{10}$$

Further, if we set $v = \bigcup_{\ell=1}^{+\infty} v_{\ell}$, we get

$$\begin{aligned} \nu(v, t) &= \nu\left(\bigcup_{\ell=1}^{+\infty} v_{\ell}, t\right) = \lim_{n \to +\infty} \nu\left(\bigcup_{\ell=1}^{n} v_{\ell}, t\right) \\ &= \lim_{n \to +\infty} \mu\left(\bigcup_{\ell=1}^{n} v_{\ell}, t\right) = \mu\left(\bigcup_{\ell=1}^{+\infty} v_{\ell}, t\right) = \mu(v, t). \end{aligned}$$

If $\mu(\varrho, t) > 0$, there are v_{ℓ}'s such that

$$\mu(\varrho, t) - \varepsilon < \mu(v, t). \tag{11}$$

On the other hand

$$\begin{aligned} \mu(v, t) &= \mu(\varrho \cup (v \setminus \varrho), t) \\ &\geq \mu(\varrho, t) * \mu(v \setminus \varrho, t) \\ &\geq \max\{\mu(\varrho, t) + \mu(v \setminus \varrho, t) - 1, 0\} \\ &= \mu(\varrho, t) + \mu(v \setminus \varrho, t) - 1, \end{aligned}$$

so

$$\mu(\varrho, t) - \mu(v, t) < 1 - \mu(v \setminus \varrho, t). \tag{12}$$

From (11) and (12) we conclude that

$$1 - \mu(v \setminus \varrho, t) > \varepsilon,$$

or

$$\mu(v \setminus \varrho, t) < 1 - \varepsilon.$$

Thus,

$$\begin{aligned} \mu(\varrho, t) &\geq \mu(v, t) = \nu(v, t) \\ &= \nu(\varrho \cup (v \setminus \varrho), t) = \nu(\varrho, t) * \nu(v \setminus \varrho, t) \\ &\geq \nu(\varrho, t) * \mu(v \setminus \varrho, t) \geq \nu(\varrho, t) * (1 - \varepsilon). \end{aligned}$$

Since $0 < \varepsilon < 1$ is arbitrary we have

$$\mu(\varrho, t) \geq \nu(\varrho, t). \tag{13}$$

From (10) and (13) we get

$$\mu(\varrho, t) = \nu(\varrho, t).$$

Finally, suppose $X = \bigcup_{\ell=1}^{+\infty} \upsilon_\ell$ with $\mu_\diamond(\upsilon_\ell, t) > 0$, such that $\upsilon_\ell \cap \upsilon_k = \emptyset$, $\ell \neq k$, then for each $\varrho \in \mathcal{M}$, we have

$$\mu(\varrho, t) = \mu\left(\bigcup_{\ell=1}^{+\infty} (\varrho \cap \upsilon_\ell), t\right) = *_{\ell=1}^{+\infty} \mu(\varrho \cap \upsilon_\ell, t)$$
$$= *_{\ell=1}^{+\infty} \nu(\varrho \cap \upsilon_\ell, t) = \nu\left(\bigcup_{\ell=1}^{+\infty} (\varrho \cap \upsilon_\ell), t\right) = \nu(\varrho, t),$$

which implies

$$\nu = \mu.$$

□

4. Conclusions

We considered an uncertain measure based on the concept of fuzzy sets and triangular norms named by $*$-fuzzy measure. Next, we have extended $*$-FPM $\mu_\diamond$ on $\mathcal{A}$ to a $*$–FM μ on $\mathcal{M}$ (the σ-algebra generated by $\mathcal{A}$) such that $\mu|_{\mathcal{A}} = \mu_\diamond$ based on Caratheodory's Theorem. In addition, we showed that μ is the unique extension of $\mu_\diamond$ to a $*$-FM on $\mathcal{M}$ if the outer $*$-FM generated by (1) satisfying $\mu^\diamond|_{\mathcal{M}} = \mu$ and $\mu_\diamond$ is σ-bounded. We expect applications of our results in several domains dealing with modeling of time-dependent situations, such as quantum physics or filtering in image processing.

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