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Refinements to Relation-Theoretic Contraction Principle

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Abstract: After the appearance of relation-theoretic contraction principle proved in a metric space equipped with an amorphous binary relation (often termed as relational metric space), various core fixed point results have been proved in the setting of different relational distance spaces by varying underlying contraction conditions. In proving such results, the notions of completeness of ambient space, continuity of involved mapping and d-self-closedness of underlying binary relation are of paramount importance. The aim of this paper is to further refine the relation-theoretic contraction principle by relaxing the conditions of completeness and continuity by replacing their respective relation-theoretic analogues. Moreover, we observe that the notion of d-self-closedness utilized in relation-theoretic contraction principle is more general than the concepts of regularity and strong regularity utilized by earlier authors.

Keywords: $\mathcal{R}$-continuous mappings; T-closed binary relations; $\mathcal{R}$-connected sets

MSC: Primary: 47H10; Secondary: 54H25

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Citation: Alam, A.; George, R.; Imdad, M. Refinements to Relation-Theoretic Contraction Principle. *Axioms* **2022**, *11*, 316. https://doi.org/10.3390/axioms11070316

Academic Editor: Sidney A. Morris

Received: 19 April 2022
Accepted: 31 May 2022
Published: 28 June 2022

1. Introduction

In 2015, the first and last author [1] of the current paper proved relation-theoretic analogue of Banach contraction principle and observed that the partial order, preorder, transitive relation, tolerance, strict order, symmetric closure etc. utilized in several earlier core metrical fixed point theorems are not optimal and can further be weakened to the extent of an amorphous binary relation. In a short span of last seven years, this result has attracted the attention of various researchers and by now this paper has already earned more than hundred citations. It will not be exaggeration to say that this result has already inspired a lot of research around it which lead to the completion of several Ph.D. theses. This result is core enough to be part of an undergraduate syllabi as mentioned in AMS mathematical review (see MR Number: 3421979).

In 2008, Jachymski [2] investigated a new variant of Banach contraction principle employing the idea of directed graph. To prove their result, Jachymski [2] hypothesis that the set of all edges of the directed graph contains all loops, which amounts to saying that the set of all edges forms a reflexive binary relation on the underlying metric space. As mentioned earlier, Alam and Imdad [1] proved Banach contraction principle under an amorphous binary relation, therefore the result due to Jachymski [2] can be deduced from the result of Alam and Imdad [1]. Thus, in all, the relational-theoretic approach remains a genuine improvement over graphical approach. To substantiate our claim, if we take strict order "$<$" as a binary relation, then the relevant fixed point theorem is obtained from relation-theoretic contraction principle. However, we can never deduce such a result from the graphical fixed point theorem due to the fact that the relation $<$ is irreflexive.

Present paper is a continuation of [1], wherein a more sharpened version of the relation-relation-theoretic contraction principle will be proved. We also highlighted that the notion of d-self-closedness utilized by Alam and Imdad [1] is better than the notion of regularity adopted by various authors.

2. *d*-Self-Closedness and Regularity

As usual, $\mathbb{N}$ and $\mathbb{N}_0$ denote the sets of positive integers and nonnegative integers, respectively. In the sequel, we assume that (X,d) is a metric space, $\mathcal{R}$ is a binary relation on X and T is a self-mapping on X. We say that two elements x and y of X are $\mathcal{R}$-comparative if either $(x,y)\in\mathcal{R}$ or $(y,x)\in\mathcal{R}$. We denote it by $[x,y]\in\mathcal{R}$. Thus, we have $(x,y)\in\mathcal{R}^s$ iff $[x,y]\in\mathcal{R}$. A sequence $\{x_n\}\subset X$ is called $\mathcal{R}$-preserving if $(x_n,x_{n+1})\in\mathcal{R}$ for all $n\in\mathbb{N}_0$.

Definition 1 ([1]). *We say that $\mathcal{R}$ is d-self-closed if for any $\mathcal{R}$-preserving sequence $\{x_n\}$ such that $x_n\xrightarrow{d}x$, there exists a subsequence $\{x_{n_k}\}$* of $\{x_n\}$ with $[x_{n_k},x]\in\mathcal{R}$ *for all $k\in\mathbb{N}_0$.*

Definition 2 ([3]). *We say that the triplet $(X,d,\mathcal{R})$ is regular if for any $\mathcal{R}$-preserving sequence $\{x_n\}$ such that $x_n\xrightarrow{d}x$, there exists a subsequence $\{x_{n_k}\}$* of $\{x_n\}$ with $(x_{n_k},x)\in\mathcal{R}$ *for all $k\in\mathbb{N}_0$.*

Definition 3 ([4]). *We say that the triplet $(X,d,\mathcal{R})$ is strongly regular if for any $\mathcal{R}$-preserving sequence $\{x_n\}$ such that $x_n\xrightarrow{d}x$, then we have $(x_n,x)\in\mathcal{R}$ for all $n\in\mathbb{N}_0$.*

Notice that Shahzad et al. [4] used the same term '*Regularity*'. However, instead we use the term '*Strongly Regularity*' to distinguish with the notion of regularity utilized by Samet and Turinici [3]. Unfortunately, many authors used the notions of regularity to prove their results, but it is better to use the concept of d-self-closedness, as it is clear from the definitions that

strongly regularity **implies** *regularity* **implies** *d-self-closedness*.

Due to this reason, we have to visit to these concepts.

3. Additional Observations

For the sake of completeness, we firstly recall the following notions.

Definition 4 ([1]). *$\mathcal{R}$ is said to be T-closed if for any $x,y\in X$,*

$$(x,y)\in\mathcal{R}\Rightarrow(Tx,Ty)\in\mathcal{R}.$$

Proposition 1 ([5]). *If $\mathcal{R}$ is T-closed, then, for all $n\in\mathbb{N}_0$, $\mathcal{R}$ is also T^n-closed, where T^n denotes nth iterate of T.*

Definition 5 ([6]). *For $x,y\in X$, a path of length k (where k is a natural number) in $\mathcal{R}$ from x to y is a finite sequence $\{z_0,z_1,z_2,\ldots,z_k\}\subset X$ satisfying the following ones:*

(i) $z_0=x$ and $z_k=y$,
(ii) $(z_i,z_{i+1})\in\mathcal{R}$ *for each* i $(0\le i\le k-1)$.

We use the following notations:

(i) $F(T)$:=the set of all fixed points of T,
(ii) $X(T,\mathcal{R}):=\{x\in X:(x,Tx)\in\mathcal{R}\}$,
(iii) $\Upsilon(x,y,\mathcal{R})$:=the class of all paths in $\mathcal{R}$ from x to y (where $x,y\in X$).

The statement of relation-theoretic contraction principle proved by Alam and Imdad [1] runs as follows:

Theorem 1 ([1]). *Let (X,d) be a metric space, $\mathcal{R}$ a binary relation on X and T a self-mapping on X. Suppose that the following conditions hold:*

(*a*) (X, d) *is complete,*
(*b*) $\mathcal{R}$ *is* T*-closed,*
(*c*) *either* T *is continuous or* $\mathcal{R}$ *is* d*-self-closed,*
(*d*) $X(T, \mathcal{R})$ *is nonempty,*
(*e*) *there exists* $\alpha \in [0, 1)$ *such that*

$$d(Tx, Ty) \leq \alpha d(x, y) \text{ for all } x, y \in X \textit{ with } (x, y) \in \mathcal{R}.$$

Then T *has a fixed point. Moreover, if* $\Upsilon(x, y, \mathcal{R}^s)$ *is nonempty, for each* $x, y \in X$*, then* T *has a unique fixed point.*

In Theorem 1, it is clear that the contractivity condition (e) is compatible with given binary relation $\mathcal{R}$. In view of further improvement making compatible (with given binary relation $\mathcal{R}$) another involved metrical notions "completeness" and "continuity" Alam and Imdad [7] introduced the following notions:

Definition 6 ([7])**.** *We say that* (X, d) *is* $\mathcal{R}$*-complete if every* $\mathcal{R}$*-preserving Cauchy sequence in* X *converges.*

Clearly, every complete metric space is $\mathcal{R}$-complete, for any binary relation $\mathcal{R}$. Particularly, under the universal relation the notion of $\mathcal{R}$-completeness coincides with usual completeness.

Definition 7 ([7])**.** *We say that* T *is* $\mathcal{R}$*-continuous at* $x \in X$ *if for any* $\mathcal{R}$*-preserving sequence* $\{x_n\}$ *such that* $x_n \xrightarrow{d} x$*, we have* $T(x_n) \xrightarrow{d} T(x)$*. Moreover,* T *is called* $\mathcal{R}$*-continuous if it is* $\mathcal{R}$*-continuous at each point of* X*.*

Clearly, every continuous mapping is $\mathcal{R}$-continuous, for any binary relation $\mathcal{R}$. Particularly, under the universal relation the notion of $\mathcal{R}$-continuity coincides with usual continuity.

Definition 8. *We say that* T *is* $\mathcal{R}$*-preserving contraction if there exists* $\alpha \in [0, 1)$ *such that*

$$d(Tx, Ty) \leq \alpha d(x, y) \text{ for all } x, y \in X \textit{ with } (x, y) \in \mathcal{R}.$$

Using the symmetry property of metric d, the following result holds straightforward.

Proposition 2. T *is an* $\mathcal{R}$*-preserving contraction iff* T *is also* $\mathcal{R}^s$*-preserving contraction.*

Proposition 3. *Let* T *be an* $\mathcal{R}$*-preserving contraction. If* $(X, d, \mathcal{R})$ *is strongly regular, then* T *is* $\mathcal{R}$*-continuous.*

Proof. Take an arbitrary $\mathcal{R}$-preserving sequence $\{x_n\} \subset X$ such that $x_n \xrightarrow{d} x \in X$. Using strong regularity of $(X, d, \mathcal{R})$, we have $(x_n, x) \in \mathcal{R}$. As T is $\mathcal{R}$-preserving contraction, there exists $\alpha \in [0, 1)$ such that

$$d(Tx_n, Tx) \quad \leq \quad \alpha d(x_n, x) \to 0 \text{ as } n \to +\infty$$

so that $T(x_n) \xrightarrow{d} T(x)$. Hence, T is $\mathcal{R}$-continuous. □

The above result indicates that over the idea of 'strong regularity' not only 'd-self-closedness' has the superiority but also '$\mathcal{R}$-continuity'.

The following notion also is introduced by Alam and Imdad utilized instead of using the hypothesis "$\Upsilon(x, y, \mathcal{R}^s) \neq \emptyset$."

Definition 9 ([7])**.** *Let X be a nonempty set and $\mathcal{R}$ a binary relation on X. A subset E of X is called $\mathcal{R}$-connected if there exists a path in $\mathcal{R}$ between each pair of elements of E.*

Clearly the condition "$\Upsilon(x, y, \mathcal{R}^s)$ is nonempty, for each $x, y \in X$" is equivalent to saying that "X is called $\mathcal{R}^s$-connected".

4. Main Results

We slightly modify Theorem 1 in the following respects:

- The notions of completeness and continuity are replaced by their respective $\mathcal{R}$-analogues.
- "$\Upsilon(x, y, \mathcal{R}^s) \neq \emptyset$" is replaced alternately by more weaker condition "$\Upsilon(Tx, Ty, \mathcal{R}^s) \neq \emptyset$", which is equivalent to saying that "$T(X)$ is called $\mathcal{R}^s$-connected".

Theorem 2. *Let (X, d) be a metric space, $\mathcal{R}$ a binary relation on X and T a self-mapping on X. Suppose that the following conditions hold:*

(a) *(X, d) is $\mathcal{R}$-complete;*
(b) *$\mathcal{R}$ is T-closed;*
(c) *either T is $\mathcal{R}$-continuous or $\mathcal{R}$ is d-self-closed;*
(d) *$X(T, \mathcal{R})$ is nonempty;*
(e) *T is $\mathcal{R}$-preserving contraction.*

Then T has a fixed point. Moreover, if $T(X)$ is $\mathcal{R}^s$-connected, then T has a unique fixed point.

Proof. In view of assumption (d), take arbitrarily $x_0 \in X(T, \mathcal{R})$. Construct the sequence $\{x_n\}$ of Picard iteration based at the initial point x_0, i.e,

$$x_n = T^n(x_0) = T(x_{n-1}) \text{ for all } n \in \mathbb{N}. \tag{1}$$

As $(x_0, Tx_0) \in \mathcal{R}$, using T-closedness of $\mathcal{R}$ and Proposition 1, we obtain

$$(T^n x_0, T^{n+1} x_0) \in \mathcal{R}$$

which in lieu of (1) becomes

$$(x_n, x_{n+1}) \in \mathcal{R} \text{ for all } n \in \mathbb{N}_0. \tag{2}$$

Thus, the sequence $\{x_n\}$ is $\mathcal{R}$-preserving. Applying the contractivity condition (e) to (2), we deduce, for some $\alpha \in [0, 1)$ and for all $n \in \mathbb{N}_0$ that

$$d(x_{n+1}, x_{n+2}) \leq \alpha d(x_n, x_{n+1}),$$

which by induction yields that

$$d(x_{n+1}, x_{n+2}) \leq \alpha^{n+1} d(x_0, Tx_0) \text{ for all } n \in \mathbb{N}_0. \tag{3}$$

For all $m, n \in \mathbb{N}$ with $m < n$, using (3) and triangular inequality, we get

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \cdots + d(x_{n-1}, x_n) \\ &\leq (\alpha^m + \alpha^{m+1} + \cdots + \alpha^{n-1}) d(x_0, Tx_0) \\ &= \alpha^m (1 + \alpha + \alpha^2 + \cdots + \alpha^{n-m-1}) d(x_0, Tx_0) \\ &\leq \frac{\alpha^m}{1 - \alpha} d(x_0, Tx_0), (0 \leq \alpha < 1) \\ &\to 0 \text{ as } m \text{ (and hence } n) \to +\infty, \end{aligned}$$

which implies that the sequence $\{x_n\}$ is Cauchy in X. Hence, $\{x_n\}$ is an $\mathcal{R}$-preserving Cauchy sequence. By $\mathcal{R}$-completeness of X, there exists $x \in X$ such that $x_n \xrightarrow{d} x$.

Finally, we use assumption (c) to show that x is a fixed point of T. Suppose that T is $\mathcal{R}$-continuous. As $\{x_n\}$ is an $\mathcal{R}$-preserving with $x_n \xrightarrow{d} x$, $\mathcal{R}$-continuity of T implies that $x_{n+1} = T(x_n) \xrightarrow{d} T(x)$. Using the uniqueness of limit, we obtain $T(x) = x$, i.e, x is a fixed point of T.

Alternately, assume that $\mathcal{R}$ is d-self-closed. Again as $\{x_n\}$ is a $\mathcal{R}$-preserving sequence and $x_n \xrightarrow{d} x$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $[x_{n_k}, x] \in \mathcal{R}$ for all $k \in \mathbb{N}_0$. On using assumption (e), Proposition 2, $[x_{n_k}, x] \in \mathcal{R}$ and $x_{n_k} \xrightarrow{d} x$, we obtain

$$\begin{aligned} d(x_{n_k+1}, Tx) &= d(Tx_{n_k}, Tx) \leq \alpha d(x_{n_k}, x) \\ &\to 0 \text{ as } k \to +\infty \end{aligned}$$

so that $x_{n_k+1} \xrightarrow{d} T(x)$. Again, owing to the uniqueness of limit, we obtain $T(x) = x$ so that x is a fixed point of T.

To prove uniqueness, take $x, y \in F(T)$, we have

$$T^n(x) = x \text{ and } T^n(y) = y. \tag{4}$$

As $x, y \in T(X)$ and $T(X)$ is $\mathcal{R}^s$-connected, there exists a path (say $\{z_0, z_1, z_2, \ldots, z_k\}$) of some finite length k in $\mathcal{R}^s$ from x to y so that

$$z_0 = x,\ z_k = y \text{ and } [z_i, z_{i+1}] \in \mathcal{R} \text{ for each } i\ (0 \leq i \leq k-1). \tag{5}$$

As $\mathcal{R}$ is T-closed, by using Proposition 1, we have

$$[T^n z_i, T^n z_{i+1}] \in \mathcal{R} \text{ for each } i\ (0 \leq i \leq k-1) \text{ and for each } n \in \mathbb{N}_0. \tag{6}$$

Making use of (4)–(6), triangular inequality, assumption (e) and Proposition 2, we obtain

$$\begin{aligned} d(x,y) &= d(T^n z_0, T^n z_k) \leq \sum_{i=0}^{k-1} d(T^n z_i, T^n z_{i+1}) \leq \alpha \sum_{i=0}^{k-1} d(T^{n-1} z_i, T^{n-1} z_{i+1}) \\ &\leq \alpha^2 \sum_{i=0}^{k-1} d(T^{n-2} z_i, T^{n-2} z_{i+1}) \leq \cdots \leq \alpha^n \sum_{i=0}^{k-1} d(z_i, z_{i+1}) \\ &\to 0 \text{ as } n \to +\infty \end{aligned}$$

so that $x = y$. Hence T has a unique fixed point. □

However, Theorem 2 is also available in [7], wherein authors deduce it from their newly proved coincidence theorem as a consequence. However, for the sake of completeness, its independent proof is given.

Now, we furnish an illustrative example in support of Theorem 2, which does not satisfy the hypotheses of Theorem 1.

Example 1. *Consider $X = (0,1]$ equipped with usual metric $d(x,y) = |x-y|$. On X, define a binary relation $\mathcal{R} = \{(x,y) : \frac{1}{4} \leq x \leq y \leq \frac{1}{3}$ or $\frac{1}{2} \leq x \leq y \leq 1\}$. Then (X,d) is an $\mathcal{R}$-complete metric space, although it is not complete. Consider the mapping $T : X \to X$ defined by*

$$T(x) = \begin{cases} \frac{1}{4} & \text{if } 0 \leq x < \frac{1}{2} \\ 1 & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

Clearly, T is $\mathcal{R}$-continuous but not continuous. Furthermore, $\mathcal{R}$ is T-closed. We can easily verify assumption (e) of Theorem 2 for any arbitrary $\alpha \in [0,1)$. Thus, all the conditions $(a) - (e)$ of Theorems 2 are satisfied and T has a fixed point in X.

Moreover, $T(X)$ is not $\mathcal{R}^s$-connected, as there is no chain between $\frac{1}{4}$ and 1. Hence, we have not guarantee about uniqueness of fixed point. Notice that there are two fixed points of T (namely: $x = \frac{1}{4}$ and $x = 1$).

Author Contributions: All the authors have contributed equally in all parts. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Conflicts of Interest: The authors declare no conflict of interest.

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