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Article

Graded Many-Valued Modal Logic and Its Graded Rough Truth

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Abstract: Much attention is focused on the relationship between rough sets and many-valued modal logic to deal with approximate reasoning. This paper discusses the graded modal logic and puts forward the graded many-valued modal logic G(S5). Secondly, by employing the graded operators that correspond to graded modal operations in G(S5), we introduce the concept of graded upper and lower rough truth degrees of a logical formula. Then, we propose the graded upper and lower conditional rough truth degrees. Several basic interesting properties are addressed. Finally, in order to make a distinction between any two rough formulas in graded many-valued modal logic, the graded upper and lower rough similarity degrees between two graded modal formulas are established in a very natural way.

Keywords: modal many-valued logic; rough set; graded rough truth; graded rough conditional truth

MSC: 03B45; 03B50

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1. Introduction

In classical logic, every proposition takes one of the two values of truth or false. Each proposition is either true or false. Many-valued logic is a non-classical logical system. The values of a proposition cannot be binary. Many-valued logic is different classical logic by the fundamental fact that they allow for a larger set of truth degrees. The theory that deals with the logical relations between such propositions is called many-valued logic. It has been used in computer science and artificial intelligence [1]. Modal logic is widely studied and a more mature non-classical logic. It provides a good balance between logical reasoning and computational complexity [2]. Necessity and possibility provide a rich context for modeling and studying concepts from many fields, including proof theory, time and cognitive concepts, workflow in software applications, and more. On the other hand, substructure logic (give up structure rules), especially produced by bounded and integral exchange surplus logic (thus usually keeping concept lattices and the existence of absolutely true and false) provides a formal framework, in the form of a kind of very universal and adaptable way to manage fuzzy sensitive information and resources.

Many-valued modal logic appears in the literature both to pursue the development of pure theories and to provide a richer framework for modeling complex environments that may require valued information and qualification operators. Although the earliest publications on this subject date back to the 1990s [3,4] (focusing on problems on finite Heyting algebras), it is only in recent years that more systematic work has developed. Nonetheless, the research may still be too narrow in a sense. The basic idea is to preserve the general notion of possible world semantics, while allowing formulas to have values in the many-valued space of each possible world. A brief study of modal system S5 over BL algebra is given in [1], but it is only in more recent work that modal system over arbitrary Kripke frames (also known in the literature as minimum modal logic) is studied [5]. Several works since have studied different aspects of this logic. Most relevant for the present paper are the works related to axiomatizability and proof theoretic questions, addressing the minimal modal system over finite MTL algebras [6]. Concerning computability, Caicedo et al. [7] proposed new semantics and used them to establish the decidability for Gödel modal logic. They also established the decidability of the extension of S5 by using the similar methodology.

Rough set theory, proposed by Pawlak in 1982 [8], is a mathematical tool for dealing with incompleteness and uncertainty (uncertain factors and incomplete information), which can effectively analyze various incomplete information, such as that which is imprecise, inconsistent, and incomplete. It can also analyze and reason about data, discover hidden knowledge and reveal underlying laws. There has been wide interest in the application of rough sets, such as data mining, data analysis, knowledge discovery, approximate reasoning, decision making, machine learning, and other related fields [9–13]. Based on Pawlakian rough set theory, there have been many extended forms of it so far. Its general extended form is the variable precision rough set, the probabilistic rough set, the decision–theoretic rough set, the generalized rough set, the tolerance rough set, the dominance rough set, the fuzzy rough set and the rough fuzzy set [14–18], etc. In some cases, only a portion of the logical research related to various rough set models is indicated in the literature from an algebraic point of view [13,19,20]. Pawlakian rough set theory is apparently related to the modal system S5. Pawlakian approximation space (X, R) of the rough set is just a Kripke frame for the modal logic S5, where R is an equivalence relation on a set X [21]. The lower and upper approximations can be seen as operations that interpret the necessity and possibility, respectively. Later, based on rough set theory, various study of logic system were proposed continuously. There are two main methods in these directions: one formula is interpreted as a set in an approximate space, and the other is interpreted as a rough set with respect to the approximate space. Nonetheless, the structure of both methods remains a Pawlakian rough set. Pawlak discussed the relationship between rough set theory and modal logic . The results showed that Pawlakian rough set is directly related to the modal logic system S5 [22]. Along with the introduction of various other upper and lower approximation operators, other modal systems have also emerged. Furthermore, the possibility of new modal systems also arises. In 2016, Ma and Chakraborty [23] pointed out that the P4 logic is exactly the modal system S5. The modal systems for the remaining logic are so far unknown. For the modal systems P2, P3, C1, C3 and CGr, one feature they have in common is that the modal property K does not hold.

Later, many literary works have been devoted to the probabilistic rough set model, variable precision rough set model, graded rough set model, rough logic, and rough algebra in [10,14,16,24,25], which were extended from rough sets. There has been extensive discussion on the logical foundation of rough sets and their relationships to non-standard logic. For example, Yao and Lin explored the relationship between rough set and modal logic and discussed the graded modal system and graded rough set model [26]. Orłowska proposed logic for reasoning about concepts using the notion of rough sets, which is essentially the modal system S5 with the modal operators interpreted using the lower and upper approximations [9,27]. A similar approach was also adopted by Chakraborty and Banerjee [19]. The modal semantics of these logic systems have been investigated in many works in the literature. In [28], the rough logic was defined, the language of which was taken to be propositional. Modal many-valued logic and fuzzy modal logic were introduced in [29,30]. Among these research, a formal logic system called the pre-rough logic was proposed with respect to the pre-rough algebra in [20]. In addition, the algorithm is sound and complete in rough set semantics and was proved in the same paper. Naturally, the study of the pre-rough logic has become an important aspect of approximate reasoning [31,32].

The established connections between rough set and modal logic have very important implications. Based on such relationships, one can enrich each theory by the results from the other theory. What has been lacking so far is the study on graded many-valued modal logic. However, some fundamental studies have been published in this direction [3,4], where some new speculations are put forward, and some new questions are also raised. In a sense, the current study helps to move in this direction. Modal logic and algebraic semantics are always closely related. In the application of computer or artificial intelligence, most of the time, we deal with vague or uncertain information. The processing method mainly applies the uncertainty reasoning method based on modal many-valued logic. In classical many-valued modal logic, the modal necessity operator $\Box$ has only one level of operation, while the possibility operator $\Diamond$ also has only one level of operation. In the process of uncertain reasoning, these are not enough and cannot be refined. In order to solve this problem, this paper introduces the hierarchical modal operators and proposes the hierarchical roughness truth by extending the hierarchical modal logic, which will help us better understand the hierarchical modal logic.

The remainder of this paper is organized as follows. In Section 2, we introduce most of the notions that we will be using throughout the paper and some preliminary definitions, including rough sets, modal logics, a pre-rough algebra, and a pre-rough logic. In Section 3, we discuss the graded many-valued modal logic system. In Section 4, we further introduce the algebraic structure corresponding to the graded many-valued modal system G(S5). In Section 5, we give the graded operators in the algebraic structure of the graded many-valued modal logic and investigate the properties of graded operators and graded rough truth degrees. We obtain some results from them. In Section 6, we propose the graded conditional rough truth degree of a rough formula in G(S5). Some properties of the graded conditional rough truth are investigated. In Section 7, we propose the graded rough similarity between any two rough formulas in G(S5).

2. Rough Sets and Modal System S5

In this section, we briefly review several basic concepts that will be used in the following sections.

As it is well known, rough set theory is based on the notion of an approximation space, which is a pair (X, R), X being a non-empty set and R an equivalence relation on it [8,12]. If $A \subseteq X$, the lower and upper approximations of (X, R) are defined as follows:

$$\underline{R}(A) = \{x \in X | [x]_R \subseteq A\}, \quad \overline{R}(A) = \{x \in X | [x]_R \cap A \neq \emptyset\} \tag{1}$$

where $[x]$ denotes the equivalence class containing the element x. The triple (X, R, A) is called a rough set. Note that X is a definable set if and only if $\underline{R}(A) = \overline{R}(A)$, and therefore, we also treat classical sets as special cases of rough sets.

We study modal logic in the context of a language of necessity and possibility as usual [2]. The language is founded on a countable set of atomic proposition $p_1, p_2, p_3, \ldots$. These are the simplest sentences. These formulas are formed using logical connectives $\neg A, A \wedge B, A \vee B, A \to B, A \leftrightarrow B, \Box A, \Diamond A$.

Let us consider the following schemas K, D, T, B, 4 and 5 in modal system and rough set in Table 1, respectively.

Table 1. Schemas K, D, T, B, 4 and 5 in modal system and rough set, respectively.

	Modal Logic	Rough Set
(K)	$\Box(A \to B) \to (\Box A \to \Box B)$	$\underline{R}(A^c \cup B) \subseteq (\underline{R}(A))^c \cup \underline{R}(B)$
(D)	$\Box A \to \Diamond A$	$\underline{R}(A) \subseteq \overline{R}(A)$
(T)	$\Box A \to A$	$\underline{R}(A) \subseteq A$
(B)	$A \to \Box\Diamond A$	$A \subseteq \underline{R}(\overline{R}(A))$
(4)	$\Box A \to \Box\Box A$	$\underline{R}(A) \subseteq \underline{R}(\underline{R}(A))$
(5)	$\Diamond A \to \Box\Diamond A$	$\overline{R}(A) \subseteq \underline{R}(\overline{R}(A))$

Where X^c denotes the complement of the set X. The usual axioms of modal system S5 are only with (K), (T), (5) or (K), (T), (4), (B). The established link in rough set shows that Pawlakian rough set model is a counterpart of modal system S5.

Banerjee and Chakraborty investigated the algebraic structure of rough sets in order to arrive at rough logic theory. They proposed the pre-rough algebra and the pre-rough logic [13,20].

Definition 1 (See [20]). *An algebraic structure $\mathcal{A} = (P, \leqslant, \sqcap, \sqcup, L, 0, 1)$ is a pre-rough algebra, if and only if the following conditions hold for any $a, b \in P$:*

(1) $(P, \leqslant, \sqcap, \sqcup, 0, 1)$ *is a bounded distributive lattice.*
(2) $\neg\neg a = a$.
(3) $\neg(a \sqcup b) = \neg a \sqcap \neg b$.
(4) $L0 = 0$, $L1 = 1$.
(5) $La \leqslant a$.
(6) $L(a \sqcap b) = La \sqcap Lb$.
(7) $La \leqslant Ma$.
(8) $La \leqslant Lb$ *and* $Ma \leqslant Mb$ *imply* $a \leqslant b$.
(9) $a \to b = (\neg La \sqcup Lb) \sqcap (\neg Ma \sqcup Mb)$.

here $\forall a \in P, Ma = \neg L \neg a$.

The language of pre-rough logic [13,20] is constructed on the set of atomic formulas $S = \{p_1, p_2, \ldots, p_m, \cdots\}$ and primitive logical connectives $\neg, \to$ and L. The set of all formulas in the pre-rough logic, denoted by $F(S)$, is a free algebra of type $(\neg, \wedge, L)$ that is generated by the set S. In the pre-rough logic, three additional connectives $\vee$, M and $\to$ are defined as follows: *for any* $A, B \in F(S)$

$$A \vee B = \neg(\neg A \wedge \neg B), \quad A \to B = (\neg LA \vee LB) \wedge (\neg MA \vee MB), \quad MA = \neg L \neg A. \tag{2}$$

Definition 2 (See [13]). *A valuation v in a pre-rough logic is a map from the set of rough formulas $F(S)$ to any pre-rough algebra $(P, \leqslant, \sqcap, \sqcup, \neg, L, 0, 1)$ satisfying $\forall A, B \in F(S)$,*

$$v(A \wedge B) = v(A) \sqcap v(B), \quad v(\Box A) = L(v(A)), \quad v(\neg A) = \neg v(A), \tag{3}$$

where L is the valuation of necessity operator $\Box$ and M is the valuation of possibility operator $\Diamond$, respectively, i.e.,

$$La = 1 \text{ if } a = 1, \quad Mb = 0 \text{ iff } b = 0.$$

Example 1 (See [13]). *Consider algebraic structure $\mathbf{3} = \left(\{0, \frac{1}{2}, 1\}, \leqslant, \wedge, \vee, \neg, 0, 1\right)$, where $\wedge$ and $\vee$ are the minimum and maximum, respectively. Give the operations of $\neg$, L and M in $\{0, \frac{1}{2}, 1\}$, as shown in Table 2.*

Table 2. The operations of $\neg$, L and M in $\{0, \frac{1}{2}, 1\}$.

	0	$\mathbf{\frac{1}{2}}$	**1**
$\neg$	1	$\frac{1}{2}$	0
L	0	0	1
M	0	1	1

*Then, we have that algebra **3** is a pre-rough algebra and is also the smallest non-trivial pre-rough algebra.*

Meanwhile, axiom schemes and rules of inference are provided. The soundness and completeness are proved in the pre-rough logic in [20].

3. Graded Many-Valued Modal System G(S5)

Some efforts have been attempted in both rough and modal logic. In this section, a graded many-valued modal system G(S5), which extends the classical modal system S5 [26,33,34] and whose language with modal operators $\Box_i$, is interpreted by employing graded operators φ_i.

Definition 3 (See [26]). *The language of the graded modal system G(S5) consists of the following:*

(1) *The set of atomic formulas* $S = \{p_1, p_2, \ldots\}$;

(2) *The propositional connectives* $\neg$ *and* $\wedge$*;*
(3) *The graded modal operators* $\Box_i$*;*
(4) *A finite set of parentheses.*

The set of all modal system formulas is denoted by $F(S)$*, which is a free algebra of type* $(\neg, \wedge, \Box_i)$ *that is generated by the set* S*.*

The remaining logical connections, $\vee, \rightarrow, \leftrightarrow$ and $\Diamond_i$, can be constructed as the following:

(1) $A \vee B = \neg(\neg A \wedge \neg B)$
(2) $A \rightarrow B = \neg A \vee B$
(3) $A \leftrightarrow B = (A \rightarrow B) \wedge (B \rightarrow A)$
(4) $\Diamond_i A = \neg \Box_i \neg A$
(5) $\Diamond!_i A = \begin{cases} \neg \Diamond_0 A, & \text{if } i = 0 \\ \Diamond_{i-1} A \wedge \neg \Diamond_i A, & \text{if } i > 0 \end{cases}$

Obviously, graded modal operators $\Box_i$ and $\Diamond_i$ are dual operators under the negation operator $\neg$:

$$\Box_i A = \neg \Diamond_i \neg A, \quad \Diamond_i A = \neg \Box_i \neg A. \tag{4}$$

If $i = 0$, then they reduce to normal operators $\Box$ and $\Diamond$, namely,

$$\Box_0 A = \Box A, \quad \Diamond_0 A = \Diamond A.$$

The axioms of G(S5) are all the instances of the following schemata [26,33,35]:
For any $A, B \in F(S)$, and $i, i_1, i_2 \in I = \{0, 1, 2, \ldots, n, \ldots\}$,

(Ax.1) $A \rightarrow \Diamond_i A$
(Ax.2) $\Diamond_{i+1} A \rightarrow \Diamond_i A$
(Ax.3) $\Diamond_i A \rightarrow \Box_i \Diamond_i A$
(Ax.4) $\Box_0 (A \rightarrow B) \rightarrow (\Box_i A \rightarrow \Box_i B)$
(Ax.5) $\Box_0 \neg(A \wedge B) \rightarrow \big((\Diamond!_{i_1} A \wedge \Diamond!_{i_2} B) \rightarrow \Diamond!_{i_1+i_2}(A \vee B)\big)$

The inference rules of G(S5) are as follows:

$$\begin{array}{llll} (MP) & \dfrac{\begin{array}{c} A \\ A \rightarrow B \end{array}}{B} & (RN) & \dfrac{A}{\Box_0 A} \\ (HS) & \dfrac{\begin{array}{c} A \rightarrow B \\ B \rightarrow C \end{array}}{A \rightarrow C} & (N) & \dfrac{A \rightarrow B}{\neg B \rightarrow \neg A} \end{array} \tag{5}$$

We write $\vdash A$ if A is a theorem of G(S5), and write $\Sigma \vDash A$ if A is a syntactic consequence of Σ.

Theorem 1. *In G(S5), for any* $A, B \in F(S), i \in \mathbb{N}^+$*,*

(1) $\vdash \Diamond_i A \leftrightarrow \neg \Box_i \neg A$
(2) $\vdash \Diamond_i \neg A \leftrightarrow \neg \Box_i A$
(3) $\vdash \neg \Diamond_i A \leftrightarrow \Box_i \neg A$

Proof. It is easily verified because $\Box_i$ and $\Diamond_i$ are the dual operators. □

Theorem 2. *In G(S5), for any* $A, B \in F(S), i \in \mathbb{N}^+$*,*

(1) *If* $\vdash A \rightarrow B$*, then* $\vdash \Box_i A \rightarrow \Box_i B$ *and* $\vdash \Diamond_i A \rightarrow \Diamond_i B$*.*
(2) *If* $\vdash A \leftrightarrow B$*, then* $\vdash \Box_i A \leftrightarrow \Box_i B$ *and* $\vdash \Diamond_i A \leftrightarrow \Diamond_i B$*.*

Proof. (1) On the one hand, due to $\vdash A \rightarrow B$ and the inference RN, it is enough to prove $\vdash \Box_0(A \rightarrow B)$. According to Ax.4, it follows that $\vdash \Box_i A \rightarrow \Box_i B$. On the another hand, since

$\vdash A \to B$, it is easy to show that $\vdash \neg B \to \neg A$. Simplifying the result gives $\vdash \Box_i \neg B \to \Box_i \neg A$. Thus, it turns out that $\vdash \Diamond_i A \to \Diamond_i B$.

(2) Thank to the definition of connectives $\leftrightarrow$ and the above conclusion , it is now obvious that the results hold. $\Box$

Remark 1. *Theorem 2 shows some derived rules of inference for G(S5),*

$$(DR1) \quad \frac{A \to B}{\Box_i A \to \Box_i B} \qquad (DR2) \quad \frac{A \to B}{\Diamond_i A \to \Diamond_i B}$$
$$(DR3) \quad \frac{A \leftrightarrow B}{\Box_i A \leftrightarrow \Box_i B} \qquad (DR4) \quad \frac{A \leftrightarrow B}{\Diamond_i A \leftrightarrow \Diamond_i B} \tag{6}$$

Theorem 3. *In G(S5), for any $A, B \in F(S), i \in \mathbb{N}^+$*

(1) *If $i \leqslant j$ then $\vdash \Diamond_j A \to \Diamond_i A$;*
(2) *If $i \leqslant j$ then $\vdash \Box_i A \to \Box_j A$.*

Proof. (1) When $i = j$, the proof is trivial. When $i < j$, by Ax. 2, we have $\vdash \Diamond_j A \to \Diamond_{j-1} A, \vdash \Diamond_{j-1} A \to \Diamond_{j-2} A, \ldots, \vdash \Diamond_{i+1} A \to \Diamond_i A$. Hence, $\vdash \Diamond_j A \to \Diamond_i A$.

(2) By substituting the rule (*N*) *and $\Box_i A = \neg \Diamond_i \neg A$ into the first case of this theorem, the proof is easily verified.*

$\Box$

4. Algebraic Structure of G(S5)

In order to investigate, quite a few algebraic properties of logical calculi over valuation domain follow L_{2n+1}, whereas $L_{2n+1} = \{0, \frac{1}{2n}, \ldots, \frac{2n-1}{2n}, 1\}$. We introduce unary operators φ_i over L_{2n+1}, where $n \in \mathbb{N}^+$.

Negative operations come in many forms [1]. In the remainder of this paper, we adopt the standard negation, i.e.,

$$\neg a = 1 - a, \text{ for } \forall a \in [0, 1].$$

Definition 4. *For any $\frac{k}{2n} \in L_{2n+1}$, the operators φ_i are defined as follows:*

$$\varphi_i\left(\frac{k}{2n}\right) = \begin{cases} 0, & i + k < 2n + 1, \\ 1, & i + k \geqslant 2n + 1. \end{cases} \tag{7}$$

These operators φ_i are called the graded operators on L_{2n+1}. These reflect the ordered structure of logical values over L_{2n+1} .

Example 2. *When $n = 2$ and $L_5 = \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$, Table 3 shows the nagation operator $\neg$ and the graded operators φ_1, φ_2 on L_5.*

Table 3. Negation operator $\neg$ and graded operators φ_1, φ_2 on L_5.

	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	**1**
$\neg$	1	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	0
φ_1	0	0	0	0	1
φ_2	0	0	0	1	1

Theorem 4. *The algebraic structure $\mathcal{L} = \left(L_{2n+1}, \wedge, \vee, \neg, \{\varphi_i\}, 0, 1\right)$ satisfies the following:*

(1) *$(L_{2n+1}, \leqslant, \wedge, \vee, 0, 1)$ is a bounded distributive lattice and $\neg$ is a dual involutive homomorphism of L_{2n+1} into itself (i.e., $\neg(a \vee b) = \neg a \wedge \neg b, \neg(a \wedge b) = \neg a \vee \neg b$ and $\neg\neg a = a$).*
(2) *$\varphi_i(a) \wedge \neg\varphi_i(a) = 0; \quad \varphi_i(a) \vee \neg\varphi_i(a) = 1$.*
(3) *$\varphi_i(\varphi_j(a)) = \varphi_j(a)$.*
(4) *If $i \leqslant j$ then $\varphi_i(a) \leqslant \varphi_j(a)$.*

(5) $\varphi_i(\neg a) = \neg\varphi_{2n+1-i}(a)$.
(6) *If* $\varphi_i(a) = \varphi_i(b)$ *then* $a = b$.

where $a \wedge b = \min\{a, b\}$, $a \vee b = \max\{a, b\}$, $\neg a = 1 - a$, $a, b \in L_{2n+1}$ *and* $i, j \in \{1, 2, \ldots, n\}$.

Proof. Clearly, (1)–(3) hold, obviously.

The proof of (4) holds by direct checking. If $i + k \geqslant 2n + 1$ for each $\frac{k}{2n} \in L_n$ and $i \leqslant j$. Hence, $\varphi_i(x) = 1$ implies $\varphi_j(x) = 1$, and thus $\varphi_i(a) \leqslant \varphi_j(a)$.

Now, only (5) needs to be verified. Let $x = \frac{k}{2n}$. We have that

$$\varphi_i(\neg x) = \varphi_i\left(1 - \frac{k}{2n}\right) = \varphi_i\left(\frac{2n-k}{2n}\right).$$

Hence, $\varphi_i(\neg x) = 0$ in case $i - k < 1$, and $\varphi_i(\neg x) = 1$ in case $i - k \geqslant 1$.

On the other hand, $\neg\varphi_i(\frac{k}{2n}) = 0$ if and only if $\varphi_{2n+1-i}(\frac{k}{2n}) = 1$, i.e., if and only if $i - k < 1$. The above two states are equivalent. Hence, (5) holds.

Finally, to prove (6), let $x = \frac{i}{2n} < y = \frac{k}{2n}$, which implies $\varphi_{2n-i}(\frac{i}{2n}) = 0$ while $\varphi_{2n-i}(\frac{k}{2n}) = 1$. The proof is thus concluded. □

Definition 5. *For any* $a \in L_{2n+1}$, *the operators* ψ_i *are defined as follows:*

$$\psi_i(a) = \neg\varphi_i(\neg a). \tag{8}$$

Those operators ψ_i *are called dual graded operators of* φ_i *on* L_{2n+1}.

Corollary 4.1. *For any* $a, b \in L_{2n+1}$ *and* $i, j \in \{1, 2, \ldots, n\}$ *we can obtain the following properties:*

(1) *If* $a \leqslant b$ *then* $\varphi_i(a) \leqslant \varphi_i(b)$.
(2) *If* $i \leqslant j$ *then* $\varphi_i(a) \leqslant \varphi_j(a)$.
(3) $\varphi_i(a \wedge b) = \varphi_i(a) \wedge \varphi_i(b)$.
(4) $\varphi_i(a \vee b) = \varphi_i(a) \vee \varphi_i(b)$.
(5) $\varphi_i(a) = \neg\varphi_{2n+1-i}(\neg a)$.

Proof. The proof can be shown similarly as that of Theorem 4. □

Following the study of Banerjee and Chakraborty, the algebra of rough sets was investigated [20] in order to arrive at a logic for the rough logic theory. An algebraic structure, called graded modal logic algebra, is proposed.

Definition 6. *The algebraic structure* $(L_{2n+1}, \wedge, \vee, \neg, \{\varphi_i\}_{i \in I}, 0, 1)$ *is called a graded modal algebra(GM-algebra), if and only if for any* $a, b \in L_{2n+1}$:

(1) $(L_{2n+1}, \leqslant, \wedge, \vee, 0, 1)$ *is a bounded distributive lattice;*
(2) $\neg\neg a = a$;
(3) $\neg(a \vee b) = \neg a \wedge \neg b$;
(4) $\varphi_i(0) = 0$, $\varphi_i(1) = 1$;
(5) $a \leqslant \psi_i(a)$;
(6) $\varphi_i(a) \leqslant \psi_i(a)$;
(7) $\varphi_i(a \wedge b) = \varphi_i(a) \wedge \varphi_i(b)$;
(8) $\varphi_i(a \vee b) = \varphi_i(a) \vee \varphi_i(b)$;

where $\psi_i(a) = \neg\varphi_i(\neg a)$, $\wedge$ *and* $\vee$ *are the minimum and maximum, respectively.*

Correspondingly, the GM-algebra is framed and observed to be sound and complete concerning semantics based on rough sets.

Example 3. *(1) If* $L_3 = \{0, \frac{1}{2}, 1\}$, *then we have the graded operator* φ *and its dual graded operator* ψ *on* L_3, *as shown in Table 4.*

Table 4. Graded operator φ and its dual graded operator ψ on L_3.

	0	$\mathbf{\frac{1}{2}}$	**1**
φ	0	0	1
ψ	0	1	1

(2) If $L_5 = \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$, then we have the graded operators φ_1, φ_2 and its dual graded operators ψ_1, ψ_2 on L_5, as shown in Table 5.

Table 5. Graded operators φ_1, φ_2 and its dual graded operators ψ_1, ψ_2 on L_5.

	0	$\mathbf{\frac{1}{4}}$	$\mathbf{\frac{1}{2}}$	$\mathbf{\frac{3}{4}}$	**1**
φ_1	0	0	0	0	1
φ_2	0	0	0	1	1
ψ_2	0	0	1	1	1
ψ_1	0	1	1	1	1

Theorem 5. *For any $a \in L_{2n+1}$, the structure $(\varphi_i(L_{2n+1}), \wedge, \vee, \neg, 0, 1)$ is a Boolean algebra, for each $i \in I$.*

Proof. The definition of Boolean algebra makes this point clear. $\square$

5. Graded Rough Truth Degree in G(S5)

In this section, the concept of graded rough truth for a formula in G(S5), whichplays an important role of quantitative logic, is introduced by using unary graded operators φ_i that are discussed in the algebraic structure $(L_{2n+1}, \wedge, \vee, \neg, \varphi_i, 0, 1)$ in the above section.

Let formula $A = A(p_1, \ldots, p_m) \in F(S)$ consist of atomic and the logic connectives, and L_{2n+1} be a valuation domain.

Definition 7. *A valuation function v is a map from the set of rough formulas $F(S)$ to any GM-algebra $(L_{2n+1}, \leqslant, \wedge, \vee, \neg, \varphi_i, 0, 1)$, i.e., $v : F(S) \to L_{2n+1}$, satisfying for any $A, B \in F(S)$:*

$$v(A \wedge B) = v(A) \wedge v(B), \quad v(\neg A) = 1 - v(A), \quad v(\Box_i(A)) = \varphi_i(v(A)). \tag{9}$$

From these three clauses, we can define the behavior of valuations for the other connectives:

$$v(A \vee B) = v(A) \vee v(B), \quad v(A \to B) = v(A) \to v(B), \quad v(\Diamond_i A) = \psi_i(v(A)). \tag{10}$$

We will also denote by Ω the set of all valuation vectors of a formula A over L_{2n+1}. With a valuation function v, we can characterize a proposition by the set of valuation vectors that are generated by the valuation function, in which the proposition is true. In other words, we can define a mapping $\pi : F(S) \to \Omega$ as follows:

$$\pi(A) = \{v \in \Omega | \models_v A\}, \qquad \text{for any } A \in F(S) \tag{11}$$

The set $\pi(A)$ can be thought of as the set of truth evaluations of logical propositions. It is also considered the occurrence domain of A. The map π can be called an associative map. An induced function can be built up by substituting $x_1, x_2, \ldots, x_m$ for atomic $p_1, p_2, \ldots, p_m$, respectively, and interpreting the logic connectives $\neg, \wedge$ and φ_i as the corresponding operators on L_{2n+1}. Then a m-ary function $f_A(x_1, x_2, \ldots, x_m) : L_{2n+1}^m \to L_{2n+1}$, called the truth function, is induced by rough formula A. Ω is denoted by the set of all truth functions. We define $\Omega_m = \{v \in \Omega : v(p_k) = 0, k > m\}$. Then, $\forall u \in \Omega$, there is a unique valuation $v \in \Omega_m$, satisfying $u(A) = v(A)$ and $u(p_i) = v(p_i)$ for all $i = 1, 2, \ldots, m$. By the construction of f_A, we have

$$v(A) = f_A(v(p_1), v(p_2), \ldots, v(p_m)),$$

where $A = A(p_1, \ldots, p_m) \in F(S), v \in \Omega$.

Definition 8. *In G(S5) for any $A = A(p_1, p_2, \ldots, p_m) \in F(S)$, we define:*

$$\underline{\pi}_i(A) = \{v \in \Omega : v(\Box_i(A)) = 1\}, \quad \overline{\pi}_i(A) = \{v \in \Omega : v(\Diamond_i(A)) = 1\}, \tag{12}$$

where $L_{2n+1} = \{0, \frac{1}{2n}, \cdots, \frac{2n-1}{2n}, 1\}$ and $i = 1, 2, \ldots, n$. $\underline{\pi}_i(A)$ and $\overline{\pi}_i(A)$ are called $i-$th lower and upper approximation valuation space with respect to A, respectively.

On the basis of the i-th lower and upper approximations, the positive, negative and boundary valuation regions of A are defined as follows:

$$POS_i(A) = \underline{\pi}_i(A), \quad NEG_i(A) = (\overline{\pi}_i(A))^c, \quad BND_i(A) = \overline{\pi}_i(A) \setminus \underline{\pi}_i(A). \tag{13}$$

The i-th lower and upper approximation valuation spaces satisfy the following properties.

Proposition 1. *In G(S5), for any $A, B \in F(S)$, we have*

(1) If $\vdash A \to B$, then $\underline{\pi}_i(A) \subseteq \underline{\pi}_i(B)$ and $\overline{\pi}_i(A) \subseteq \overline{\pi}_i(B)$.
(2) If $i \leqslant j$ then $\underline{\pi}_i(A) \subseteq \underline{\pi}_j(A)$ and $\overline{\pi}_j(A) \subseteq \overline{\pi}_i(A)$.

Proof. (1) It follows from Theorem 3 and Definition 8.

(2) If $i \leqslant j$, by Theorem 3 (2), we have $\Box_i A \to \Box_j A$. Hence, $\underline{\pi}_i(\Box_i A) \subseteq \underline{\pi}_i(\Box_j A)$. Therefore, it follows immediately from Definition 4 and Definition 8.

(3) It can be shown similarly. □

Proposition 2. *In G(S5), for any $A \in F(S)$, we have*

(1) $\underline{\pi}_1(A) \subseteq \underline{\pi}_2(A) \subseteq \ldots \subseteq \underline{\pi}_n(A)$
(2) $\overline{\pi}_1(A) \supseteq \overline{\pi}_2(A) \supseteq \ldots \supseteq \overline{\pi}_n(A)$

Proof. This proof is similar to that of Proposition 1. □

Remark 2. *According to Axiom 2 $\Diamond_{i+1} A \to \Diamond_i A$, we have $\Box_i A \to \Box_{i+1} A$. Hence, Proposition 2 is the conclusion that follows from this. Moreover, from Axiom 4, we have*

$$\underline{\pi}_i(A \vee B) \subseteq \underline{\pi}_i(A) \cup \underline{\pi}_i(B), \text{for each } i.$$

Definition 9. *In G(S5), for any $A, B \in F(S)$,*

(1) If $\underline{\pi}_i(A) \subseteq \underline{\pi}_i(B)$, then A is said to be of i-th lower roughly logical equivalence with B.
(2) If $\overline{\pi}_i(A) \subseteq \overline{\pi}_i(B)$, then A is said to be of i-th upper roughly logical equivalence with B.
(3) If $\underline{\pi}_i(A) \subseteq \underline{\pi}_i(B)$ and $\overline{\pi}_i(A) \subseteq \overline{\pi}_i(B)$, then A and B are said to be of i-th roughly logical equivalence.

Definition 10. *In G(S5), for any formula $A = A(p_1, \ldots, p_m) \in F(S)$, $i \in I$, we define*

$$\underline{\tau}_i(A) = \frac{|\underline{\pi}_i(A)|}{|\Omega_m|}, \qquad \overline{\tau}_i(A) = \frac{|\overline{\pi}_i(A)|}{|\Omega_m|}, \tag{14}$$

where $|X|$ denotes the cardinality of set X which is not null. $\underline{\tau}_i(A)$ and $\overline{\tau}_i(A)$ are called the i-th lower and upper truth degrees for a m-dimension formulas A in G(S5).

Remark 3. *Since the domain of the truth function* $f_A(x_1, x_2, \ldots, x_m) : L_{2n+1}^m \to L_{2n+1}$ *is associated to rough formula* A *is* L_{2n+1}^m*, in which there are a total of* $(2n+1)^m$ *vectors* $(x_1, x_2, \ldots, x_m)$*, for each rough formula* A*, every vector can be viewed as a valuation* $v : F(S) \to L_{2n+1}$ *satisfying* $v(p_1) = x_1, v(p_2) = x_2, \ldots, v(p_m) = x_m$*. Hence, for any rough formula* $A = A(p_1, \ldots, p_m) \in F(S)$*, we have* $|v(A)| = (2n+1)^m$*. Moreover, we can obtain the following,* $i \in \{1, 2 \ldots, n\}$*,*

$$\underline{\tau}_i(A) = \frac{|\underline{\pi}_i(A)|}{(2n+1)^m}, \quad \overline{\tau}_i(A) = \frac{|\overline{\pi}_i(A)|}{(2n+1)^m}. \tag{15}$$

We can interpret the results of rough truth as in the following example.

Example 4. *Let* $A = p_1$ *and* $B = (p_1 \wedge p_2) \vee p_3$ *be rough formulas and* $L_5 = \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$ *be the domain of valuation when* $n = 2$*. The semantics of operators* $\neg, \varphi_1, \varphi_2, \psi_1$*, and* ψ_2 *are all in Table 6.*

Table 6. The semantics for operations $\neg, \varphi_1, \varphi_2, \psi_1, \psi_2$ on L_5.

	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	**1**
$\neg$	1	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	0
φ_1	0	0	0	0	1
φ_2	0	0	0	1	1
ψ_2	0	0	1	1	1
ψ_1	0	1	1	1	1

According to Definition 8, it is easy to obtain that

$$\underline{\pi}_1(A) = \{1\}, \quad \overline{\pi}_1(A) = \{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}, \quad \underline{\pi}_2(A) = \{\frac{3}{4}, 1\}, \quad \overline{\pi}_2(A) = \{\frac{1}{2}, \frac{3}{4}, 1\}.$$

Thus,

$$\underline{\tau}_1(A) = \frac{1}{5}, \quad \overline{\tau}_1(A) = \frac{4}{5}, \quad \underline{\tau}_2(A) = \frac{2}{5}, \quad \overline{\tau}_2(A) = \frac{3}{5},$$

and

$$POS_1(A) = \{1\}, \quad NEG_1(A) = \{0\}, \quad BND_1(A) = \{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}.$$

Moreover,

$$\begin{aligned}\underline{\pi}_1(B) = \{&(0,0,1), (0, \frac{1}{4}, 1), (0, \frac{1}{2}, 1), (0, \frac{3}{4}, 1), (0,1,1), (\frac{1}{4}, 0, 1), (\frac{1}{4}, \frac{1}{4}, 1), (\frac{1}{4}, \frac{1}{2}, 1), \\ &(\frac{1}{4}, \frac{3}{4}, 1), (\frac{1}{4}, 1, 1), (\frac{1}{2}, 0, 1), (\frac{1}{2}, \frac{1}{4}, 1), (\frac{1}{2}, \frac{1}{2}, 1), (\frac{1}{2}, \frac{3}{4}, 1), (\frac{1}{2}, 1, 1), (\frac{3}{4}, 0, 1), \\ &(\frac{3}{4}, \frac{1}{4}, 1), (\frac{3}{4}, \frac{1}{2}, 1), (\frac{3}{4}, \frac{3}{4}, 1), (\frac{3}{4}, 1, 1), (1, 0, 1), (1, \frac{1}{4}, 1), (1, \frac{1}{2}, 1), (1, \frac{3}{4}, 1), \\ &(1,1,1), (1,1,0), (1,1,\frac{1}{4}), (1,1,\frac{1}{2}), (1,1,\frac{3}{4})\}.\end{aligned}$$

Similarly, we can obtain $\overline{\pi}_1(B)$*,* $\underline{\pi}_2(B)$ *and* $\overline{\pi}_2(B)$*. Hence,*

$$\underline{\tau}_1(B) = \frac{29}{125}, \quad \overline{\tau}_1(B) = \frac{116}{125}, \quad \underline{\tau}_2(B) = \frac{62}{125}, \quad \overline{\tau}_2(B) = \frac{93}{125}.$$

Theorem 6. *In G(S5), for any* $A, B \in F(S)$ *the following hold:*

(1) $0 \leqslant \underline{\tau}_i(A) \leqslant \overline{\tau}_i(A) \leqslant 1$.
(2) $\underline{\tau}_i(\neg A) = 1 - \overline{\tau}_i(A), \quad \overline{\tau}_i(\neg A) = 1 - \underline{\tau}_i(A)$.
(3) $\underline{\tau}_i(A \vee B) = \underline{\tau}_i(A) + \underline{\tau}_i(B) - \underline{\tau}_i(A \wedge B)$,
$\overline{\tau}_i(A \vee B) = \overline{\tau}_i(A) + \overline{\tau}_i(B) - \overline{\tau}_i(A \wedge B)$.

Proof. (1) According to Definitions 8 and 10, the proof is obvious.
(2) Due to Definition 8 and Theorem 1, we have

$$\begin{aligned}\underline{\pi}_i(\neg A) =& \{v \in \Omega_m : v(\Box_i \neg A) = 1\} \\ =& \{v \in \Omega_m : v(\neg \Diamond_i A) = 1\} \\ =& \{v \in \Omega_m : v(\Diamond_i A) = 0\} \\ =& \Omega_m \setminus \{v \in \Omega_m : v(\Diamond_i A) = 1\} \\ =& \Omega_m \setminus \overline{\pi}_i(A).\end{aligned}$$

It is directly available according to Definition 10.
(3) Note that

$$\begin{aligned}\underline{\pi}_i(A \vee B) &= \{v \in \Omega_m : v(\Box_i(A \vee B)) = 1\} \cup \{v \in \Omega_m : v(\Box_i A) \vee v(\Box_i B) = 1\} \\ &= \{v \in \Omega_m : v(\Box_i A) = 1\} \cup \{v \in \Omega_m : v(\Box_i B) = 1\} \\ &\quad \setminus \{v \in \Omega_m : v(\Box_i(A \wedge B)) = 1\}.\end{aligned}$$

Thus, we have $\underline{\pi}_i(A \vee B) = \underline{\pi}_i(A) + \underline{\pi}_i(B) - \underline{\pi}_i(A \wedge B)$. The other one can be proved similarly. □

Theorem 7. *In G(S5), the following properties hold for any $A, B \in F(S)$,*

(1) If $\underline{\tau}_i(A) \geqslant a, \underline{\tau}_i(A \to B) \geqslant b$, then $\underline{\tau}_i(B) \geqslant a + b - 1$.
(2) If $\overline{\tau}_i(A) \geqslant a, \overline{\tau}_i(A \to B) \geqslant b$, then $\overline{\tau}_i(B) \geqslant a + b - 1$.

Proof. Owing to Theorem 6 (2) and (3), we have

$$\begin{aligned}\underline{\tau}_i(A \to B) &= \underline{\tau}_i(\neg A \vee B) \\ &= \underline{\tau}_i(\neg A) + \underline{\tau}_i(B) - \underline{\tau}_i(\neg A \wedge B).\end{aligned}$$

Then,

$$\begin{aligned}\underline{\pi}_i(B) &= \overline{\pi}_i(A) - 1 + \underline{\pi}_i(A \to B) + \underline{\pi}_i(\neg A \vee B) \\ &\geqslant \underline{\pi}_i(A) + \underline{\pi}_i(A \to B) - 1 \\ &\geqslant a + b - 1.\end{aligned}$$

The other is similar. □

Theorem 8. *In G(S5), for any $a, b \in [0, 1]$, A, B and $C \in F(S)$, the following properties hold:*

(1) If $\underline{\tau}_i(A \to B) \geqslant a, \underline{\tau}_i(B \to C) \geqslant b$, then $\underline{\tau}_i(A \to C) \geqslant a + b - 1$.
(2) If $\overline{\tau}_i(A \to B) \geqslant a, \overline{\tau}_i(B \to C) \geqslant b$, then $\overline{\tau}_i(A \to C) \geqslant a + b - 1$.

Proof. (1) Since $\vdash \big((A \to B) \to ((B \to C) \to (A \to C))\big)$, we have

$$\underline{\tau}_i((A \to B) \to ((B \to C) \to (A \to C))) = \overline{\tau}_i((A \to B) \to ((B \to C) \to (A \to C))) = 1.$$

If $\underline{\tau}_i(A \to B) \geqslant a$ and $\underline{\tau}_i(B \to C) \geqslant b$, then $\underline{\tau}_i((B \to C) \to (A \to C)) \geqslant \underline{\tau}_i(A \to B)$ and $\underline{\tau}_i(A \to C) \geqslant a + b - 1$ by using Theorem 7 twice.
(2) It can be obtained similarly and is thus omitted. □

Theorem 9. *In G(S5), for any $A \in F(S)$, we have*

(1) $\underline{\tau}_1(A) \leqslant \underline{\tau}_2(A) \leqslant \ldots \leqslant \underline{\tau}_{n-1}(A) \leqslant \underline{\tau}_n(A)$.
(2) $\overline{\tau}_1(A) \geqslant \overline{\tau}_2(A) \geqslant \ldots \geqslant \overline{\tau}_{n-1}(A) \geqslant \overline{\tau}_n(A)$.

Proof. It follows immediately from Proposition 2 and Definition 10. □

6. Graded Rough Conditional Truth Degree in G(S5)

In this section, we propose the notion of conditional graded rough truth on the basis of conditional probability. As a result, a new research for approximation reasoning can be established in the many-valued modal system G(S5).

To begin with, the notion of conditional truth for formula A given B is defined as following:

Definition 11. *In G(S5), for any $A, B \in F(S)$, we define*

$$\underline{\tau}_i(A|B) = \frac{\underline{\tau}_i(A \wedge B)}{\underline{\tau}_i(B)}, \quad \overline{\tau}_i(A|B) = \frac{\overline{\tau}_i(A \wedge B)}{\overline{\tau}_i(B)}, \tag{16}$$

$\underline{\tau}_i(A|B)$ and $\overline{\tau}_i(A|B)$ are called the i-th graded conditional lower and upper truth degrees of A on the condition of B.

Theorem 10. *In G(S5), for any rough formula $A, B \in F(S)$, we have*

(1) $0 \leqslant \underline{\tau}_i(A|B) \leqslant \overline{\tau}_i(A|B) \leqslant 1$.

(2) *If $\vdash A$, then $\underline{\tau}_i(A|B) = \overline{\tau}_i(A|B) = 1$.*

(3) *If $\vdash B$, then $\underline{\tau}_i(A|B) = \underline{\tau}_i(A), \quad \overline{\tau}_i(A|B) = \overline{\tau}_i(A)$.*

Proof. The proofs are clear by Definition 11, so they are omitted here. □

Note that, if $\vdash B$, then $\underline{\tau}_i(A)$ and $\overline{\tau}_i(A)$ are special cases of $\underline{\tau}_i(A|B)$ and $\overline{\tau}_i(A|B)$, respectively.

Theorem 11. *In G(S5), for any rough formula A, B, C and $D \in F(S)$, we have*

(1) $\underline{\tau}_i(A \vee B|C) = \underline{\tau}_i(A|C) + \underline{\tau}_i(B|C) - \underline{\tau}_i(A \wedge B|C)$,
$\overline{\tau}_i(A \vee B|C) = \overline{\tau}_i(A|C) + \overline{\tau}_i(B|C) - \overline{\tau}_i(A \wedge B|C)$.

(2) *If $\underline{\tau}_i(A|C) \geqslant a$ and $\underline{\tau}_i(A \to B|C) \geqslant b$, then $\underline{\tau}_i(B|C) \geqslant a + b - 1$,*
If $\overline{\tau}_i(A|C) \geqslant a$ and $\overline{\tau}_i(A \to B|C) \geqslant b$, then $\overline{\tau}_i(B|C) \geqslant a + b - 1$.

(3) *If $\underline{\tau}_i(A \to B|D) \geqslant a$ and $\underline{\tau}_i(B \to C|D) \geqslant b$, then $\underline{\tau}_i(A \to C|D) \geqslant a + b - 1$,*
If $\overline{\tau}_i(A \to B|D) \geqslant a$ and $\overline{\tau}_i(B \to C|D) \geqslant b$, then $\overline{\tau}_i(A \to C|D) \geqslant a + b - 1$.

Proof. The proof is quite similar to Theorems 6–8. □

Example 5. *In G(S5), assume that $A = p_1, B = p_2$ and $L_5 = \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$. We have*

(1) $\underline{\tau}_1(A \to B) = \frac{9}{25}, \overline{\tau}_1(A \to B) = \frac{24}{25}, \underline{\tau}_2(A \to B) = \frac{16}{25}, \overline{\tau}_2(A \to B) = \frac{21}{25}$,
$\underline{\tau}_1(A|B) = \frac{1}{5}, \overline{\tau}_1(A|B) = \frac{4}{5}, \underline{\tau}_2(A|B) = \frac{2}{5}, \overline{\tau}_2(A|B) = \frac{3}{5}$.

(2) *According to Definition 11, we have*

$$\underline{\tau}_i(A \to B|A) = \underline{\tau}_i(p_1 \to p_2|p_1) = \frac{\underline{\tau}_i((p_1 \to p_2) \wedge p_1)}{\underline{\tau}_i(p_1)}.$$

From Definition 8, we obtain

$$\begin{aligned}\underline{\pi}_i\big((p_1 \to p_2) \wedge p_1\big) &= \{v \in \Omega_m : v\big(\Box_i((p_1 \to p_2) \wedge p_1)\big) = 1\} \\ &= \big\{v \in \Omega_m : \varphi_i\big(\min\{\max\{1 - v(p_1), v(p_2)\}, v(p_1)\}\big) = 1\big\},\end{aligned}$$

and

$$\begin{aligned}\overline{\pi}_i\big((p_1 \to p_2) \wedge p_1\big) &= \{v \in \Omega_m : v\big(\Diamond_i((p_1 \to p_2) \wedge p_1)\big) = 1\} \\ &= \big\{v \in \Omega_m : \psi_i\big(\min\{\max\{1 - v(p_1), v(p_2)\}, v(p_1)\}\big) = 1\big\}.\end{aligned}$$

So, it is easy to show that

$$
\begin{aligned}
\underline{\pi}_1((p_1 \to p_2) \wedge p_1) &= \{(1,1)\},\\
\overline{\pi}_1((p_1 \to p_2) \wedge p_1) &= \{(\frac{1}{4},0),(\frac{1}{4},\frac{1}{4}),(\frac{1}{4},\frac{1}{2}),(\frac{1}{4},\frac{3}{4}),(\frac{1}{4},1),(\frac{1}{2},0),(\frac{1}{2},\frac{1}{4}),\\
&\quad (\frac{1}{2},\frac{1}{2}),(\frac{1}{2},\frac{3}{4}),(\frac{1}{2},1),(\frac{3}{4},0),(\frac{3}{4},\frac{1}{4}),(\frac{3}{4},\frac{1}{2}),(\frac{3}{4},\frac{3}{4}),\\
&\quad (\frac{3}{4},1),(1,\frac{1}{4}),(1,\frac{1}{2}),(1,\frac{3}{4}),(1,1)\}
\end{aligned}
$$

and

$$
\begin{aligned}
\underline{\pi}_2((p_1 \to p_2) \wedge p_1) &= \{(\frac{3}{4},\frac{3}{4}),(\frac{3}{4},1),(1,\frac{3}{4}),(1,1)\},\\
\overline{\pi}_2((p_1 \to p_2) \wedge p_1) &= \{(\frac{1}{2},0),(\frac{1}{2},\frac{1}{4}),(\frac{1}{2},\frac{1}{2}),(\frac{1}{2},\frac{3}{4}),(\frac{1}{2},1),(\frac{3}{4},\frac{1}{2}),(\frac{3}{4},\frac{3}{4}),\\
&\quad (\frac{3}{4},1),(1,\frac{1}{2}),(1,\frac{3}{4}),(1,1)\}.
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
\underline{\tau}_1(A \to B|A) &= \frac{1}{5}, \quad \overline{\tau}_1(A \to B|A) = \frac{19}{20},\\
\underline{\tau}_2(A \to B|A) &= \frac{2}{5}, \quad \overline{\tau}_2(A \to B|A) = \frac{11}{15}.
\end{aligned}
$$

7. Graded Rough Similarity in G(S5)

The goal of the graded rough similarity in G(S5) is to find the approximate formula of the error at a different rung. Finally, approximate reasoning of logical formulas is realized.

The logical equivalence between any two formulas plays an important role in logic systems, as well as rough equality in Pawlakian rough set theory. In this section, we aim to establish the graded rough similarity degree between any two formulas.

Definition 12. *In G(S5), for any $A \in F(S), i \in I$, we define*

$$
\underline{\xi}_i(A,B) = \underline{\tau}_i(A \leftrightarrow B), \quad \overline{\xi}_i(A,B) = \overline{\tau}_i(A \leftrightarrow B), \tag{17}
$$

where $\underline{\xi}_i(A,B)$ and $\overline{\xi}_i(A,B)$ are called the graded lower and upper similarity degrees with respect to A and B, respectively.

The graded rough similarity degrees enjoy the following properties.

Theorem 12. *In G(S5), for any $A, B \in F(S)$, the following hold:*

(1) $0 \leqslant \underline{\xi}_i(A,B) \leqslant \overline{\xi}_i(A,B) \leqslant 1$.

(2) $\underline{\xi}_i(A,A) = \overline{\xi}_i(A,A) = 1$.

(3) $\underline{\xi}_i(A,B) = \underline{\xi}_i(B,A), \overline{\xi}_i(A,B) = \overline{\xi}_i(A,B)$.

Proof. The proofs can be obtained directly by Definition 10 and Definition 12. □

Example 6. *Assume that $A = p_1$ and $B = p_2$ in G(S5), $L_5 = \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$.*

(1)All valuation of formula $A \leftrightarrow B$ are given in Table 7.

Table 7. All valuations of formula $A \leftrightarrow B$.

$v(A \leftrightarrow B)$ \ $v(B)$ / $v(A)$	**0**	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	**1**
0	1	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	0
$\frac{1}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{3}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{3}{4}$
1	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1

Then, we obtain

$$\underline{\xi}_1(A,B) = \frac{2}{25}, \quad \overline{\xi}_1(A,B) = \frac{23}{25}, \quad \underline{\xi}_2(A,B) = \frac{8}{25}, \quad \overline{\xi}_2(A,B) = \frac{17}{25}.$$

(2) All valuation of formula $A \leftrightarrow \neg A$ are given in Table 8.

Table 8. All valuation of formula $A \leftrightarrow \neg A$.

	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	**1**
$v(A \leftrightarrow \neg A)$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	0

Then, we have

$$\underline{\xi}_1(A,\neg A) = 0, \quad \overline{\xi}_1(A,\neg A) = \frac{3}{5}, \quad \underline{\xi}_2(A,\neg A) = 0, \quad \overline{\xi}_2(A,\neg A) = \frac{1}{5}.$$

8. Conclusions

In this paper, firstly, we discuss the graded modal system G(S5), which is an extension of the classical modal systems. Secondly, we introduce the graded operators φ_i over the valuation domain L_{2n+1} of modal logic formulas, and propose the algebraic structure of the many-valued modal system. Some properties are investigated in detail. Thirdly, based on G(S5), we propose the graded truth degree and graded conditional truth degree of G(S5) formula by establishing the relation between rough set and graded modal logic. Finally, we introduce the graded rough similarity between any two rough formulas to make a distinction of them. As a further research topic, one may develop a kind of approximate reasoning method in the framework of the graded many-valued modal system later.

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