Abstract: The goal of this paper is to propose and investigate new iterative methods for examining an approximate solution of a fixed-point problem, an equilibrium problem, and a finite collection of variational inclusions in the Hadamard manifold’s structure. Operating under some assumptions, we extend the proximal point algorithm to estimate the common solution of stated problems and obtain a strong convergence theorem for the common solution. We also present several consequences of the proposed iterative methods and their convergence results.

Keywords: proximal point method; equilibrium problem; fixed-point problem; variational inclusion problem; Hadamard manifold

MSC: 47J20; 47H10; 49J40

1. Introduction

Many nonlinear problems, such as equilibrium, optimization, variational inequality, and fixed-point problems, have recently been transformed from linear spaces to Hadamard manifolds; see [1–15]. Fan [16] initiated the equilibrium problem (EP), which was later developed by Blum and Oettli [17] in real Hilbert space. It was Colao [5] who studied the equilibrium problem for the first time on the Hadamard manifold. For a bi-function $F: K \times K \to \mathbb{R}$, such that $F(u, u) = 0, \forall u \in K$, $K$ is a nonempty subset of the Hadamard manifold $X$. The equilibrium problem is to locate a point $u^* \in K$, such that

$$F(u^*, u) \geq 0, \forall u \in K. \quad (1)$$

They studied the existence of equilibrium point of equilibrium problem (1), and utilized their results to find the solution of mixed variational inequality problems, fixed-point problems and Nash equilibrium problems in Hadamard manifolds. They also introduced the Picard iterative method to approximate a solution of the equilibrium problem (1). Recently, Kamphaewong et al. [10,18] studied the splitting type algorithms for equilibrium and inclusion problems on Hadamard manifolds. We denote by $EP(F)$ the set of equilibrium points of the equilibrium problem (1).

The variational inclusion problem in Hilbert space $\mathbb{H}$ is to locate a point $u \in D$, such that

$$0 \in V(u) + G(u), \quad (2)$$
where \( V : D \to \mathbb{H} \) and \( G : D \to 2^\mathbb{H} \) are single valued and set-valued mappings, respectively, defined on a nonempty subset \( D \) of Hilbert space \( \mathbb{H} \). The solution set of the problem (2) is denoted by \( (V + G)^{-1}(0) \).

Due to its application-oriented nature, the problem (2) has been investigated extensively by a number of researchers in diverse directions. The proximal point method due to Martinet [19] is a fundamental approach for solving the inclusion problem \( u \in G^{-1}(0) \), and Rockafellar [15] generalized this strategy to solve the variational inclusion problem (2). Li et al. [11] introduced the proximal point method for the inclusion problem in Hadamard manifold. Ansari et al. [2] examined Korpelevich’s method to find the solution of the variational inclusion problem (2) in the structure of the Hadamard manifold.

Several practical problems can be formulated as a fixed-point problem:

\[
S(u) = u, \tag{4}
\]

where \( S \) is a nonlinear mapping. The solutions of this equation are called fixed points of \( S \), which is denoted by \( \text{Fix}(S) \). Li et al. [13] extended the Mann and Halpern iteration scheme to find the fixed point of nonexpansive mappings from Hilbert spaces to Hadamard manifolds. Recently, Al-Homidan et al. [1] proposed and analyzed the Halpern and Mann-type iterative methods to find the solution of a variational inclusion problem (2) and fixed-point problem (4) of self nonexpansive mapping \( S \) in the Hadamard manifold, which is to locate \( u \in K \), such that

\[
u \in \text{Fix}(S) \cap (V + G)^{-1}(0). \tag{5}\]

Most of the problems originating in nonlinear science, such as signal processing, image recovery, signal processing, optimization, machine learning, etc., are switchable to either variational inclusion, an equilibrium problem or a fixed-point problem. Therefore, many mathematicians have recently transformed and studied the inclusion problems, equilibrium recovery, signal processing, optimization, machine learning, etc., are switchable to either

As zero of the sum of monotone mapping \( V + G \) is the fixed point of resolvent \( J^G_\lambda (\exp (\lambda V(x))) \), \( \lambda > 0 \), following the work of Ansari et al. [2], and Al-Homidan et al. [1], Chang et al. [4] investigated the problem:

\[
\text{Find } u \in K \text{ such that } u \in \text{Fix}(S) \bigcap_{i=1}^{N} (V_i + G)^{-1}(0) \bigcap_{i=1}^{N} \text{EP}(F), \tag{6}\]

where \( \text{Fix}(S) \) and \( \text{EP}(F) \) represent the set of fixed points of the mapping \( S \) and equilibrium points of equilibrium function \( F \), respectively, and \( \bigcap_{i=1}^{N} (V_i + G)^{-1}(0) \) is the set of common singularities of \( N \) variational inclusion problems, defined as:

\[
\text{Find } u \in K \text{ such that } 0 \in V_i(u) + G(u), \ \forall i \in \{1, 2 \cdots N\}. \tag{7}\]

If \( V_i = V \), for all \( i = 1, 2, \cdots N \), we have

\[
\text{Find } u \in K \text{ such that } u \in \text{Fix}(S) \bigcap (V + G)^{-1}(0) \bigcap \text{EP}(F). \tag{7}\]
Inspired by the works of Ansari and Babu [3], Al-Homidan et al. [1] and following contemporary research work, our motive in this article is to propose new iterative algorithms to solve problems (5)-(7) in the setting of Hadamard manifold. We also bring out some consequences of proposed iterative algorithms. The following section contains some definitions, symbols, and useful results on Riemannian manifolds. Section 3 contains the main results describing the iterative algorithms for the problems (5)-(7). In the last section, we discuss some of the consequences of the suggested algorithms and their convergence results for solving variational inequality problems with equilibrium and fixed-point problems.

2. Preliminaries

We consider $\mathcal{X}$ to be a differentiable manifold of finite dimension. Let $T_u\mathcal{X}$ indicate the tangent space of $\mathcal{X}$ at $u$, and the tangent bundle of $\mathcal{X}$ is indicated by $T\mathcal{X} = \bigcup_{u \in \mathcal{X}} T_u\mathcal{X}$, which is obviously a manifold. An inner product $\langle \cdot, \cdot \rangle_u$ on $T_u\mathcal{X}$ is termed as a Riemannian metric on $T_u\mathcal{X}$. A tensor $\langle \cdot, \cdot \rangle : u \rightarrow \langle \cdot, \cdot \rangle_u$ is said to be the Riemannian metric on $T_u\mathcal{X}$, if $\langle \cdot, \cdot \rangle_u$ is a Riemannian metric on $T_u\mathcal{X}$ for each $u \in \mathcal{X}$. We denote the Riemannian metric on $\mathcal{X}$ by $\langle \cdot, \cdot \rangle$ and corresponding norm by $\| \cdot \|_u$, which is given by $\| w \|_u = \sqrt{\langle w, w \rangle_u}$, for all $w \in T_u\mathcal{X}$. We assume that $\mathcal{X}$ is equipped with the Riemannian metric $\langle \cdot, \cdot \rangle_u$ and its corresponding norm is $\| \cdot \|_u$. For simplicity, we omit the subscript.

The length of a piecewise smooth curve joining $u$ to $v$ (i.e., $\gamma(u) = a$ and $\gamma(v) = b$) is defined as $L(\gamma) = \int_a^b \| \gamma'(x) \| dx$. The Riemannian distance $d(u, v)$ yields the original topology on $\mathcal{X}$, which minimizes the length over the set of all such curves which connect $u$ and $v$.

We denote the Levi-Civita connection associated to $\mathcal{X}$ by $\nabla$. We know that if $\nabla_XY = 0$, the vector field $F$ is parallel along a smooth curve $\gamma$. If $\gamma'$ is parallel along $\gamma$, then $\gamma$ is said to be geodesic and in this case $\| \gamma' \|$ is constant. $\gamma$ is called a normalized geodesic, if $\| \gamma' \| = 1$. A minimal geodesic is a geodesic connecting $u$ to $v$ in $\mathcal{X}$ with the length equal to $d(u, v)$. A complete Riemannian manifold is one in which for any $u \in \mathcal{X}$, all geodesics that originate from $u$ are defined for all real numbers $\kappa \in (-\infty, \infty)$. Due to Hopf–Rinow Theorem [23], it is known to us that in a complete Riemannian manifold $\mathcal{X}$, any $u, v \in \mathcal{X}$ can be attached through a minimal geodesic.

Moreover, the exponential map $\exp_u : T_u\mathcal{X} \rightarrow \mathcal{X}$ at $u$ is defined by $\exp_u(w) = \gamma_w(1, u)$ for each $w \in T_u\mathcal{X}$, where $\gamma_w(\cdot, u)$ is the geodesic starting from $u$ with velocity $w$ (that is, $\gamma_w(0, u) = u$ and $\gamma_w'(0, u) = w$). We know that $\exp_u(tw) = \gamma_w(t, u)$ for each real number $t$ and $\exp_u0 = \gamma_w(0; u) = u$. It is known to us that for any $u \in \mathcal{X}$, the exponential map $\exp_u$ is differentiable on $T_u\mathcal{X}$ and the derivative of $\exp_u(0)$ is the identity vector of $T_u\mathcal{X}$. Hence, using inverse mapping theorem, there is an inverse exponential map $\exp^{-1}_u : \mathcal{X} \rightarrow T_u\mathcal{X}$. Moreover, for any $u, v \in \mathcal{X}$, we have $d(u, v) = \| \exp^{-1}_u v \| = \| \exp^{-1}_u v \|$, where $\| \exp^{-1}_u v \| = \sqrt{\langle \exp^{-1}_u v, \exp^{-1}_u v \rangle}$. In particular, if $\mathcal{X} = \mathbb{R}^n$ the Euclidian space, then $\exp^{-1}_u v = v - u$ for all $u, v \in \mathbb{R}^n$.

A Hadamard manifold is a Riemannian manifold with nonpositive sectional curvature which is complete and simply connected.

Lemma 1 ([23]). Let $\mathcal{X}$ be a finite dimensional manifold and $\gamma : [0, 1] \rightarrow \mathcal{X}$ be a geodesic joining $u$ to $v$. Then,

$$d(\gamma(\kappa_1), \gamma(\kappa_2)) = |\kappa_1 - \kappa_2| d(u, v), \quad \text{for all } \kappa_1, \kappa_2 \in [0, 1].$$

(8)

Proposition 1 ([23]). Let $\mathcal{X}$ be a Hadamard manifold. Then

(i) The exponential map $\exp_u : T_u\mathcal{X} \rightarrow \mathcal{X}$ is a diffeomorphism for all $u \in \mathcal{X}$.

(ii) For any pair of point $u, v \in \mathcal{X}$, there exists a unique normalized geodesic $\gamma : [0, 1] \rightarrow \mathcal{X}$ joining $u = \gamma(0)$ to $v = \gamma(1)$, which is in fact a minimal geodesic defined by

$$\gamma(\kappa) = \exp_u \kappa \exp^{-1}_u v, \quad \text{for all } \kappa \in [0, 1].$$
A subset $K$ of Hadamard manifold $\mathbb{X}$ is called a convex set if, for any $u, v \in K$, any geodesic joining $u$ and $v$ must be in $K$. In other words, if $\gamma : [a, b] \to \mathbb{X}$ is a geodesic, such that $u = \gamma(a)$ and $v = \gamma(b)$, then $\gamma((1 - \kappa)a + \kappa b) \in K$, for all $\kappa \in [0, 1]$.

A function $h : K \to (-\infty, \infty]$ is called a geodesic convex function, if any geodesic $\gamma : [a, b] \to \mathbb{X}$, the composition function $h \circ \gamma : [a, b] \to \mathbb{R}$ is convex; that is,

$$(h \circ \gamma)(t_1 + (1 - \kappa)t_2) \leq h(t_1) + (1 - \kappa)(h \circ \gamma)(t_2), \quad \forall \kappa \in [0, 1] \text{ and } \forall t_1, t_2 \in \mathbb{R}.$$

**Proposition 2** ([23]). The Riemannian distance $d : \mathbb{X} \times \mathbb{X} \to \mathbb{R}$ is a convex function with respect to the product Riemannian metric, i.e., given any pair of geodesics $\gamma_1 : [0, 1] \to \mathbb{X}$ and $\gamma_2 : [0, 1] \to \mathbb{X}$, the following inequality holds for all $\kappa \in [0, 1]$:

$$d(\gamma_1(\kappa), \gamma_2(\kappa)) \leq (1 - \kappa)d(\gamma_1(0), \gamma_2(0)) + \kappa d(\gamma_1(1), \gamma_2(1)). \quad (9)$$

In particular, for each $u \in \mathbb{X}$, the function $d(\cdot, u) : \mathbb{X} \to \mathbb{R}$ is a convex function.

For $n$-dimensional manifold $\mathbb{X}$, we conclude by Proposition 1 that $\mathbb{X}$ is diffeomorphic to the Euclidean space $\mathbb{R}^n$; hence, $\mathbb{X}$ and $\mathbb{R}^n$ have the same differential structure and topology. Moreover, Euclidean spaces and Hadamard manifold have certain identical geometric prospects. Some of these are stated in the following results.

In a Riemannian manifold $\mathbb{X}$, geodesic triangle $\Delta(r_1, r_2, r_3)$ is a collection of three points $r_1, r_2$ and $r_3$ and the three minimal geodesics $\gamma_k$ joining $\phi_k$ to $\phi_{k+1}$, where $k = 1, 2, 3$ mod (3).

**Lemma 2** ([13]). Let $\Delta(r_1, r_2, r_3)$ be a geodesic triangle in Hadamard manifold $\mathbb{X}$. Then, $r_1', r_2', r_3' \in \mathbb{R}^2$, such that

$$d(r_1, r_2) = ||r_1' - r_2'||, \quad d(r_2, r_3) = ||r_2' - r_3'||, \quad \text{and} \quad d(r_3, r_1) = ||r_3' - r_1'||.$$

The points $r_1', r_2', r_3'$ are called the comparison points to $r_1, r_2, r_3$, respectively. The triangle $\Delta(r_1', r_2', r_3')$ is called the comparison triangle of the geodesic triangle $\Delta(r_1, r_2, r_3)$, which is unique to the isometry of $\mathbb{X}$.

**Lemma 3** ([13]). Let $\Delta(r_1, r_2, r_3)$ be a geodesic triangle in Hadamard manifold $\mathbb{X}$ and $\Delta(r_1', r_2', r_3') \in \mathbb{R}^2$ be its comparison triangle.

(i) Let $\theta_1, \theta_2, \theta_3$ (respectively, $\theta_1', \theta_2', \theta_3'$) be the angles of $\Delta(r_1, r_2, r_3)$ (respectively, $\Delta(r_1', r_2', r_3')$) at the vertices $(r_1, r_2, r_3)$ (respectively, $r_1', r_2', r_3'$). Then, the following inequalities hold:

$$\theta_1' \geq \theta_1, \quad \theta_2' \geq \theta_2, \quad \theta_3' \geq \theta_3.$$

(ii) Let $v$ be a point on the geodesic joining $r_1$ to $r_2$ and $v'$ be its comparison point in the interval $[r_1', r_2']$. Suppose that $d(v, r_1) = ||v - r_1'||$ and $d(v, r_2) = ||v' - r_2'||$. Then,

$$d(v, r_3) \leq ||v' - r_3'||.$$

**Proposition 3** ([23]). (Comparison Theorem for Triangle) Let $\Delta(r_1, r_2, r_3)$ be a geodesic triangle. Denote, for each $k = 1, 2, 3$ mod (3), by $\gamma_k : [0, l_k] \to \mathbb{X}$ geodesic joining $r_k$ to $r_{k+1}$ and set $l_k = L(\gamma_k), \theta_k = \angle(\gamma_k'(0), \gamma_k'(-l_{k-1}))$. Then,

$$\theta_1 + \theta_2 + \theta_3 \leq \pi, \quad (10)$$

$$l_k^2 + l_{k+1}^2 - 2l_k l_{k+1} \cos \theta_{k+1} \leq l_{k-1}^2. \quad (11)$$

In terms of $d$ and exp, (11) can be expressed as

$$d^2(r_k, r_{k+1}) + d^2(r_{k+1}, r_{k+2}) - 2\langle \exp_{r_{k+1}}^{-1} r_k, \exp_{r_{k+1}}^{-1} r_{k+2} \rangle \leq d^2(r_{k-1}, r_k), \quad (12)$$
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...Maximal monotone if $G$ is monotone and for $u$

Monotone if for all $u$

Strongly monotone if there exists a constant

Definition 1 ([24]). A single-valued vector field $V \in \Omega(X)$ is said to be

Monotone if

\[
\langle V(u), \exp_{u}^{-1} v \rangle \leq \langle V(v), -\exp_{v}^{-1} u \rangle, \quad \forall u, v \in X.
\]

Strongly monotone if there exists a constant $\eta > 0$ such that

\[
\langle V(u), \exp_{u}^{-1} v \rangle + \langle V(v), -\exp_{v}^{-1} u \rangle \leq -\eta d^2(u, v), \quad \forall u, v \in X.
\]

$\varphi$-Lipschitz continuous if there exists a constant $\varphi > 0$, such that

\[
\|P_{u,v}V(u) - V(v)\| \leq \varphi d(u, v), \quad \forall u, v \in X.
\]

Definition 2 ([25]). A set-valued vector field $G \in \chi(X)$ is said to be

Monotone if for all $u, v \in D(X)$,

\[
\langle w, \exp_{u}^{-1} v \rangle \leq \langle z, -\exp_{v}^{-1} u \rangle, \quad \forall w \in G(u), \forall z \in G(v).
\]

Maximal monotone if $G$ is monotone and for $u \in D(G)$ and $w \in T_u(X)$, the condition

\[
\langle w, \exp_{u}^{-1} v \rangle \leq \langle z, -\exp_{v}^{-1} u \rangle,
\]

implies $w \in G(u)$.

Definition 3 ([25]). A set-valued vector field $G \in \chi(X)$ is called upper Kuratowski semicontinuous at $u \in D(G)$ if, for any sequence $\{u_k\} \subseteq D(G)$ and $\{v_k\} \subseteq T_X$ with $v_k \in G(u_k)$, the relation $\lim_{k \to \infty} v_k = v$ and $\lim_{k \to \infty} u_k = u$ imply $v \in G(u)$. Moreover, $G$ is called upper Kuratowski semicontinuous on $X$ if it is Kuratowski semicontinuous at each $u \in D(G)$. 

since

\[
\langle \exp_{r_k}^{-1} r_k, \exp_{r_k}^{-1} r_{k+2} \rangle = d(r_k, r_{k+1})d(r_{k+1}, r_{k+2}) \cos \theta_{k+1}.
\]
**Definition 4.** Let \((X,d)\) be a complete metric space and \(K \subseteq X\) be a nonempty set. A sequence \(\{u_n\}\) in \(X\) is called Fejér convergent to \(K\) if, for all \(u \in K\) and \(k \geq 0\),

\[
d(u_{k+1}, u) \leq d(u_k, u).
\]

**Lemma 5 ([8]).** Let \((X,d)\) be a complete metric space. If \(u_k \subseteq X\) is a Fejér convergent to a nonempty set \(K \subseteq X\), then \(\{u_k\}\) is bounded. Moreover, if cluster point \(u\) of \(\{u_k\}\) belongs to \(K\), then \(\{u_k\}\) converges to \(u\).

Let \(K \subseteq X\) and \(F : K \times K \to \mathbb{R}\) be a bifunction satisfying the following conditions:

(A) \(F(u, u) \geq 0, \forall u \in K\);
(B) \(F\) is monotone; that is, for all \(u, v \in K\), \(F(u, v) + F(v, u) \leq 0\);
(C) For every \(v \in K\), \(u \to F(u, v)\) is upper semicontinuous;
(D) For all \(u \in K\), \(v \to F(u, v)\) is geodesic convex and lower semicontinuous;
(E) There exists a compact set \(C \subseteq X\) and a point \(u \in C \cap K\), such that

\[
F(u, v) < 0, \quad \forall v \in K \setminus C.
\]

The resolvent \(T^F_t : X \ni K\) of a bifunction \(F\), a set-valued operator introduced by Colao [5] in the setting of the Hadamard manifold, is defined by

\[
T^F_t(u) = \{w \in K : F(w, v) - \frac{1}{t} \langle \exp^{-1}_w u, \exp^{-1}_w v \rangle \geq 0, \forall v \in K\}, \quad \forall u \in X.
\]

**Lemma 6 ([5]).** Let \(K \subseteq X\) and \(F : K \times K \to \mathbb{R}\) be a bifunction satisfying (A)–(E). Then, for \(t > 0\),

(a) The resolvent \(T^F_t\) of \(F\) is nonempty and single valued;
(b) The resolvent \(T^F_t\) of \(F\) is firmly nonexpansive;
(c) The fixed point of \(T^F_t\) is the equilibrium point set of \(F\);
(d) The equilibrium point set \(EP(F)\) is closed and geodesic convex.

3. Main Results

The solution to problem (6) is assumed to be consistent, and it is denoted by \(\Gamma\). We propose the following iterative procedure to solve the problem (6) in \(X\), based on the proximal point method (3).

**Algorithm 1.** Suppose that \(V_i \in \Omega(X), (i = 1, \cdots, N), G \in \chi((X), F, S\text{ and }T^F_t\text{ are the same as described above.}\)

Choose arbitrary \(z_0 \in K\), to define the sequences \(\{u_k^i\}, i \in \{1, 2, \cdots, N\}, \{v_k\}\) and \(\{z_k\}\) as follows:

\[
0 \in P_{u_k^i \subseteq z_k} V_i(z_k) + G(u_k^i) - \frac{1}{\lambda_k} \exp^{-1}_k z_k,
\]

\[
v_k = \exp_{z_k}(1 - \alpha_k) \exp^{-1}_k T^F_t(u_k^i),
\]

\[
z_{k+1} = \exp_{z_k}(1 - \beta_k) \exp^{-1}_k S(v_k),
\]

where \(k_i \in \{1, 2, \cdots, N\}\) such that \(d(u_k^i, z_k) = \max_{i=1, \cdots, N} \{d(u_k^i, z_k)\}, \alpha, \beta, \epsilon \in (0, 1), 0 < \epsilon < 1\) and \(\lambda_k > 0\).

If \(V_i = V\) for all \(i \in \{1, 2, \cdots, N\}\), we have the following iterative algorithm to solve the problem (7).
Algorithm 2. For arbitrary $z_0 \in K$, obtain the sequences $\{u_k\}$, $\{v_k\}$ and $\{z_k\}$ as follows:

$$
0 \in P_{u_k,z_k} V(z_k) + G(u_k) - \frac{1}{\lambda_k} \exp_{u_k}^{-1} z_k,
$$

$$
v_k = \exp_{z_k} (1 - \alpha_k) \exp_{z_k}^{-1} T_k^f (u_k),
$$

$$
z_{k+1} = \exp_{z_k} (1 - \rho_k) \exp_{z_k}^{-1} S(v_k),
$$

where $\alpha_k, \beta_k \in (0, 1), 0 < \eta < 1$ and $\lambda_k > 0$.

If $V_i = V$ for all $i \in \{1, 2, \cdots, N\}$ and $F = 0$, then we have the following iterative algorithm to solve the problem (5).

Algorithm 3. For arbitrary $z_0 \in K$, obtain the sequences $\{u_k\}$ and $\{z_k\}$ as follows:

$$
0 \in P_{u_k,z_k} V(z_k) + G(u_k) - \frac{1}{\lambda_k} \exp_{u_k}^{-1} z_k,
$$

$$
z_{k+1} = \exp_{z_k} (1 - \rho_k) \exp_{z_k}^{-1} S(u_k),
$$

where $\beta_k \in (0, 1), 0 < \eta < 1$ and $\lambda_k > 0$.

Theorem 1. Let $K$ be the nonempty, closed and geodesic convex subset of $X$. Suppose that for every $i \in \{1, 2, \cdots, N\}$, vector field $V_i \in \Omega(X)$ is $\eta_i$-strongly monotone and $\varphi_i$-Lipschitz continuous and $G \in \chi(X)$ is maximally monotone. Let $F: K \times K \to \mathbb{R}$ be a bifunction enjoying the conditions (A) - (E) and $T^f_k$ be the resolvent of $F$, $S: K \to K$ as a nonexpansive mapping. If $\Gamma \neq \emptyset$ and

$\eta = \min \{ \eta_i \}, \varphi = \max \{ \varphi_i \}, \alpha_n, \beta_n \in (0, 1), 0 < \eta < 1$ and $\lambda_k > 0$ satisfy the following conditions:

(H1) $0 < \bar{\lambda} < \lambda_k < \lambda < \frac{1}{2 \eta},$ and $\varphi < 2 \eta$.

(H2) $\sum_{k=0}^{\infty} \beta_k = \infty$.

(H3) $0 < b < \alpha_k, \beta_k < c < 1$.

Then, the sequence $\{z_k\}$ obtained from Algorithm 1 converges to an element in $\Gamma$.

Proof. The proof is divided into the following three steps:

Step I. First, we justify that the sequence $\{z_k\}$ is Fejér monotone with respect to $\Gamma$.

Let $z^* \in \Gamma$, then $-V_i(z^*) \in G(z^*)$ for each $i \in \{1, 2, \cdots, N\}$. For any arbitrary $z_0 \in K$, from Algorithm 1, we have

$$
-P_{u_k,z_k} V(z_k) + \frac{1}{\lambda_k} \exp_{u_k}^{-1} z_k \in G(u_k),
$$

with monotonicity of $G$, which implies that

$$
\langle -P_{u_k,z_k} V(z_k) + \frac{1}{\lambda_k} \exp_{u_k}^{-1} z_k, \exp_{u_k}^{-1} z^* \rangle \leq \langle -V_i(z^*), \exp_{u_k}^{-1} z^* \rangle.
$$

(14)

Since $V_i$ is $\eta_i$-strongly monotone vector field for each $i \in \{1, 2, \cdots, N\}$, then

$$
\langle V_i(z^*), \exp_{u_k}^{-1} u_k \rangle \leq \langle -V_i(u_k), \exp_{u_k}^{-1} z^* \rangle - \eta_i d^2(u_k, z^*).
$$

(15)

Combining (14) and (15), we get

$$
\langle -P_{u_k,z_k} V(z_k) + \frac{1}{\lambda_k} \exp_{u_k}^{-1} z_k, \exp_{u_k}^{-1} z^* \rangle \leq \langle -V_i(z^*), \exp_{u_k}^{-1} z^* \rangle - \eta_i d^2(u_k, z^*),
$$

where $\alpha_k, \beta_k \in (0, 1), 0 < \eta < 1$ and $\lambda_k > 0$. 
or,

\[ \langle \exp^{-1}z_k, \exp^{-1}z^* \rangle \leq \lambda_k \langle P_{\exp^{-1}z_i} V_i(z_k) - V_i(u_k^i), \exp^{-1}z^* \rangle - \eta d^2(u_k^i, z^*). \]  

(16)

Since \( V_i \) is \( \varphi_i \)-Lipschitz continuous monotone vector field for each \( i \in \{1, 2 \cdots N\} \) and \( \varphi = \max_{i=1,2 \cdots N} \{ \varphi_i \} \), using Cauchy–Schwartz inequality, we get

\[
\langle P_{\exp^{-1}z_i} V_i(z_k) - V_i(u_k^i), \exp^{-1}z^* \rangle \leq \| P_{\exp^{-1}z_i} V_i(z_k) - V_i(u_k^i) \| \| \exp^{-1}z^* \| \\
\leq \varphi_i \| \exp^{-1}z_k \| \| \exp^{-1}z^* \|
\]

Thus, inequality (16) becomes

\[
2 \langle \exp^{-1}z_k, \exp^{-1}z^* \rangle \leq \varphi \lambda_k \left\{ d^2(z_k, u_k^i) + d^2(u_k^i, z^*) \right\} - 2\eta d^2(u_k^i, z^*).
\]

(17)

For fixed \( k \in \mathbb{N} \) and \( i \in \{1, 2 \cdots N\} \), let \( \Delta(z_k, u_k^i, z^*) \subseteq \mathbb{R} \). Then, using (12), we get

\[
d^2(z_k, u_k^i) + d^2(u_k^i, z^*) - 2 \langle \exp^{-1}z_k, \exp^{-1}z^* \rangle \leq d^2(z_k, z^*).
\]

(18)

From inequalities (17) and (18), and using \( \eta = \min_{i=1,2 \cdots N} \{ \eta_i \} \), we have

\[
d^2(z_k, u_k^i) + d^2(u_k^i, z^*) \leq \varphi \lambda_k d^2(z_k, u_k^i) + \varphi \lambda_k d^2(u_k^i, z^*) + d^2(z_k, z^*) - 2\eta \lambda_k d^2(u_k^i, z^*).
\]

Since, \( 0 < \lambda < \lambda_k < \lambda < \frac{1}{2\eta} \) and \( \varphi < 2\eta \), we have

\[
d^2(z_k, u_k^i) + d^2(u_k^i, z^*) \leq \varphi \lambda d^2(z_k, u_k^i) + d^2(z_k, z^*),
\]

or

\[
d^2(u_k^i, z^*) \leq d^2(z_k, z^*) - (1 - \varphi \lambda) d^2(z_k, u_k^i).
\]

(19)

Since \( \lambda < \frac{1}{2\eta} \) and \( \varphi < 2\eta \), implies that \( \varphi \lambda < 1 \), we get

\[
d^2(u_k^i, z^*) \leq d^2(z_k, z^*), \quad i \in \{1, 2 \cdots N\}, \quad k \in \mathbb{N}.
\]

(20)

Let \( k_i \in \{1, 2 \cdots N\} \) such that \( d(u_k^{k_i}, z^*) = \max_{k \in \{1,2 \cdots N\}} \{ d(u_k^i, z^*) \} \leq d(z_k, z^*) \). From (20), Algorithm 1, we have

\[
d(u_k^i, z^*) = d(\gamma_k(1 - a_k), z^*) \leq (1 - a_k) d(\gamma_k(0), z^*) + a_n d(\gamma_k(1), z^*) \leq (1 - a_k) d(z_k, z^*) + a_n d(T^k (u_k^i), z^*) \leq (1 - a_k) d(z_k, z^*) + a_n d(u_k^i, z^*) \leq (1 - a_k) d(z_k, z^*) + a_n d(z_k, z^*) = d(z_k, z^*).
\]

(21)
Step II. Next, we show that 
\[ z \rightarrow z^* \]
that is, \( \{z_k\} \) is Fejér monotone and hence bounded by Lemma 5, and therefore the sequence \( \{u_k\}, \{v_k\} \) all are bounded and \( \lim_{k \to \infty} d(z_k, z^*) \) exists.

For fixed \( k \in \mathbb{N} \), let \( p_k = S(v_k) \) and \( \triangle(z_k, p_k, z^*) \) be the geodesic triangle and \( \triangle(x_k, \bar{x}_k, \bar{z}^*) \subseteq X \) be the comparison triangle. Then, we have
\[
d^2(z_{k+1}, z^*) \leq \|x_{k+1} - \bar{x}\|^2 \\
= \| (1 - \bar{\beta}_k) \bar{x} + \bar{\beta}_k \bar{p}_k - \bar{x} \|^2 \\
= \| (1 - \bar{\beta}_k) (x_k - \bar{x}) + \bar{\beta}_k |\bar{x} - \bar{p}_k| \|^2 \\
= (1 - \bar{\beta}_k) \|x_k - \bar{x}\|^2 + \bar{\beta}_k \|\bar{p}_k - \bar{x}\|^2 - \bar{\beta}_k (1 - \bar{\beta}_k) \|p_k - x_k\|^2 \\
\leq (1 - \bar{\beta}_k) d^2(z_k, z^*) + \bar{\beta}_k d^2(p_k, z^*) - \bar{\beta}_k (1 - \bar{\beta}_k) d^2(p_k, z_k) \\
\leq (1 - \bar{\beta}_k) d^2(z_k, z^*) + \bar{\beta}_k d^2(S(v_k), z^*) - \bar{\beta}_k (1 - \bar{\beta}_k) d^2(S(v_k), z_k) \\
\leq (1 - \bar{\beta}_k) d^2(z_k, z^*) + \bar{\beta}_k d^2(S(v_k), z^*) - \bar{\beta}_k (1 - \bar{\beta}_k) d^2(S(v_k), z_k) \\
\leq d^2(S(v_k), z^*) - \bar{\beta}_k (1 - \bar{\beta}_k) d^2(S(v_k), z_k)
\]
or,
\[
d^2(S(v_k), z_k) \leq \frac{1}{\bar{\beta}_k (1 - \bar{\beta}_k)} \left\{ d^2(z_k, z^*) - d^2(z_{k+1}, z^*) \right\}. \\
\] (24)

Further, using condition (H3), we have
\[
d^2(z_k, S(v_k)) \leq \frac{1}{\bar{\beta} (1 - \bar{\beta})} \left\{ d^2(z_k, z^*) - d^2(z_{k+1}, z^*) \right\}. \\
\] (25)

Since \( \{z_k\} \) is Fejér monotone with respect to \( \Gamma \), \( \lim_{k \to \infty} d(z_k, z^*) \) exists; hence, we get
\[
d(z_k, S(v_k)) = 0, k \to \infty. \\
\] (26)

Using (23) and (26), we obtain
\[
d(z_{k+1}, z_k) = 0, k \to \infty. \\
\] (27)
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Since \{z_k\}, \{v_k\}, are bounded, there exists \(N_1 > 0\) with \(d(z_k, z^*) \leq N_1\), and for each \(k \in \mathbb{N}\), we have

\[
\begin{align*}
d(z_k, z^*) & \leq d(\gamma_{k-1}(1 - \varrho \beta_k), z^*) \\
& \leq (1 - \varrho \beta_k) d(\gamma_{k-1}(1), z^*) + \varrho \beta_k d(z_k, z^*) \\
& = (1 - \varrho \beta_k) d(z_k, z^*) + \varrho \beta_k d(S(v_{k-1}), z^*) \\
& \leq (1 - \varrho \beta_k) d(z_k, z^*) + \varrho \beta_k d(v_{k-1}, z^*). 
\end{align*}
\]

(28)

Furthermore, for each \(i\),

\[
\begin{align*}
d(z_k, u^i_k) &= d(z_k, z_{k+1}) + d(z_{k+1}, z^*) + d(u^i_k, z^*) \\
& = d(z_k, z_{k+1}) + d(z_{k+1}, z^*) + d(z_k, z^*) \\
& \leq d(z_k, z_{k+1}) + d(z_{k+1}, z^*) + d(z_k, z^*) \\
& = d(z_k, z_{k+1}) + 2d(z_k, z^*) \\
& \leq d(z_k, z_{k+1}) + 2N_1 \sum_{j=m}^{k-1} \left\{ (1 - \varrho \beta_j) \prod_{n=j+1}^{k} \varrho \beta_n \right\} + 2N_1 \prod_{j=m}^{k-1} \varrho \beta_j.
\end{align*}
\]

Using condition (H2), we have

\[
\lim_{m \to \infty} \sum_{j=m}^{\infty} \left\{ (1 - \varrho \beta_j) \prod_{n=j+1}^{\infty} \varrho \beta_n \right\} = 0, \quad \lim_{m \to \infty} \prod_{j=m}^{\infty} \varrho \beta_j = 0.
\]

Thus, using (27), we get

\[
\lim_{k \to \infty} d(z_k, u^i_k) = 0, \quad k \to \infty \quad \text{for each} \quad i \in \{1, \cdots, N\}.
\]

(30)

Furthermore, for each \(i \in \{1, 2, \cdots, N\}\),

\[
\begin{align*}
d(z_k, T^i (u^i_k)) &= d(z_k, z_{k+1}) + d(z_{k+1}, z^*) + d(T^i (u^i_k), z^*) \\
& \leq d(z_k, z_{k+1}) + d(z_{k+1}, z^*) + d(u^i_k, z^*) \\
& \leq d(z_k, z_{k+1}) + d(z_{k+1}, z^*) + d(z_k, z^*) \to 0, \quad k \to \infty,
\end{align*}
\]

(31)

and

\[
\begin{align*}
d(z_k, v_k) &= d(z_k, z_{k+1}) + d(z_{k+1}, z^*) + d(v_k, z^*) \\
& \leq d(z_k, z_{k+1}) + d(z_{k+1}, z^*) + d(z_k, z^*) \to 0, \quad k \to \infty.
\end{align*}
\]

(32)

Step III. Finally, we show that the limit of a sequence \(\{z_k\}\) belongs in \(\Gamma\).

From Step I, we know that the sequence \(\{z_k\}\) is bounded, so there is a subsequence \(\{z_{k_n}\}\) of \(\{z_k\}\) converging to a cluster point \(w^*\) of \(\{z_k\}\). From (26), we have \(v_{k_n} \to w^*\), and (32) implies \(d(z_{k_n}, S(v_{k_n})) \to 0\) as \(n \to \infty\); thus, due to the nonexpansiveness of \(S\), we get

\[
\begin{align*}
d(S(w^*), w^*) &= d(S(w^*), S(v_{k_n})) + d(S(v_{k_n}), z_{k_n}) + d(z_{k_n}, w^*) \\
& \leq d(w^*, v_{k_n}) + d(S(v_{k_n}), z_{k_n}) + d(z_{k_n}, w^*) \to 0, \quad n \to \infty,
\end{align*}
\]

Thus, we obtain \(w^* \in \text{Fix}(S)\).
Since $T^F_i$ is also nonexpansive, using (31), we get
\[ d(T^F_i(w^*), w^*) = d(T^F_i(w^*), T^F_i(u^i_{k_n}^i)) + d(T^F_i(u^i_{k_n}^i), z_{k_n}) + d(z_{k_n}, w^*) \leq d(w^*, u^i_{k_n}^i) + d(T^F_i(u^i_{k_n}^i), z_{k_n}) + d(z_{k_n}, w^*) \to 0, \ n \to \infty, \] (33)
which amounts to $w^* \in \text{Fix}(T^F_i)$.

From Algorithm 1, we have
\[ \psi_{k_n+1} = -P_{u^i_{k_n}^i \cap \Omega} V_i(z_{k_n}) + \frac{1}{\lambda_{k_n}} \exp^{-1} u^i_{k_n} \in G(u^i_{k_n}). \] (34)

From (30), we have $\lim_{k \to \infty} d(z_k, u^i_k) = 0$ and since $0 < \bar{\lambda} < \lambda_k < \lambda < 1$, and we deduce that $\lim_{k \to \infty} d(z_k, u^i_k) = 0$, for every $i \in \{1, 2, \ldots, N\}$. Thus, we have
\[ \lim_{n \to \infty} \frac{1}{\lambda_{k_n}} \| \exp^{-1} u^i_{k_n} \| = \lim_{n \to \infty} \frac{1}{\lambda_{k_n}} d(u^i_{k_n}, z_{k_n}) = 0, \] (35)
and so,
\[ \lim_{n \to \infty} \frac{1}{\lambda_{k_n}} \exp^{-1} u^i_{k_n} = 0. \] (36)

Since $V_i$ is the Lipschitz continuous vector field and $z_{k_n} \to w^*$ as $n \to \infty$, taking into account (34) and (36), we get
\[ \lim_{n \to \infty} \psi_{k_n+1} = -V_i(w^*), \ i \in \{1, 2, \ldots, N\}. \] (37)

$G$ is upper Kuratowski semiconcave, as it is maximally monotone; then, we have $-V_i(w^*) \in G(w^*)$, for every $i \in \{1, 2, \ldots, N\}$, that is $w^* \in \bigcap_{i=1}^{N} (V_i + G)^{-1}(0)$. Hence, $w^* \in \Gamma$. This completes the proof by appealing to Lemma 5. \qed

If $V_i = V$, then we have the following convergence result for Algorithm 2.

**Corollary 1.** Let $K$ be nonempty, closed and geodesic convex subset of $\mathbb{X}$. Let vector field $V \in \Omega(\mathbb{X})$ be a $\eta$-strongly monotone and $\varphi$-Lipschitz continuous; $G \in \chi(\mathbb{X})$ is maximally monotone. Let $F : K \times K \to \mathbb{R}$ be a bifunction enjoying the conditions (A) – (E), and $T^F_i$ be the resolvent of $F$, $S : K \to K$ be a nonexpansive mapping. If $\text{Fix}(S) \cap (V + G)^{-1}(0) \cap \text{EP}(F) \neq \emptyset$ and $\alpha_n, \beta_n \in (0, 1)$, $0 < \varphi < 1$ and $\lambda_k > 0$ satisfy the following conditions (H1)–(H3), then the sequence $\{z_k\}$ obtained by Algorithm 2 converges to the solution of problem (7).

For Algorithm 3, we have the following result to solve $\text{Fix}(S) \cap (V + G)^{-1}(0)$.

**Corollary 2.** Let $K$ be a nonempty, closed and geodesic convex subset of $\mathbb{X}$ and vector field $V \in \Omega(\mathbb{X})$ be a $\eta$-strongly monotone and $\varphi$-Lipschitz continuous. $G \in \chi(\mathbb{X})$ is maximally monotone and $S : K \to K$ is a nonexpansive mapping. If $\text{Fix}(S) \cap (V + G)^{-1}(0) \neq \emptyset$ and $\beta_n \in (0, 1)$, $0 < \varphi < 1$ and $\lambda_k > 0$ satisfy the following conditions (H1)–(H3), then the sequence $\{z_k\}$ obtained by Algorithm 3 converges to the solution of problem (5).

**4. Consequences**

Németh [14], introduced and studied the following variational inequality problem $VI(V, K)$: Find $u \in K$, such that
\[ \langle V(u), \exp^{-1} v \rangle \geq 0, \ \text{for all} \ v \in K, \] (38)
where $V : K \to T\mathbb{X}$ is a single-valued vector field defined on $K \subseteq \mathbb{X}$.
We know that \( u \in K \) is a solution of \( \text{VI}(V, K) \) if and only if \( u \) satisfies
\[
0 \in V(u) + N_K(u),
\]
where \( N_K(u) \) is the normal cone to \( K \) at \( u \in K \), defined by
\[
N_K(u) = \{ p \in T_u X : \langle p, \exp^{-1}_u v \rangle \leq 0, \text{ for all } v \in K \}.
\]
The indicator function \( \gamma_K \) of \( K \) is defined by
\[
\gamma_K(u) = \begin{cases} 0, & \text{if } u \in K, \\ +\infty, & \text{if } u \notin K. \end{cases}
\]
Since \( \gamma_K \) is proper, lower semicontinuous, then the differential \( \partial \gamma_K(u) \) of \( \gamma_K \) is maximally monotone, which is defined by
\[
\partial \gamma_K(u) = \{ p \in T_u X : \langle p, \exp^{-1}_u v \rangle \leq \gamma_K(v) - \gamma_K(u) = 0 \}.
\]
Thus, we have
\[
\partial I_K(u) = \{ p \in T_u X : \langle p, \exp^{-1}_u v \rangle \leq 0 \} = N_K(u).
\]
For \( \lambda > 0 \), the resolvent of \( \partial \gamma_K \), defined by
\[
\mathcal{R}_K(u) = \{ q \in X : u \in \exp_q \lambda \partial \gamma_K(q) \} = P_K(u), \text{ for all } u \in X.
\]
Thus, for \( V : K \rightarrow X \) and for all \( u \in K \), we have
\[
u \in (V + \partial \gamma_K)^{-1}(0) = 0 \in V(u) + \partial \gamma_K(u) = -V(u) \in \partial \gamma_K(u) \iff \langle -V(u), \exp^{-1}_u v \rangle \leq 0, \text{ for all } v \in K \]
\[
\iff u \in \text{VI}(V, K).
\]
Let \( V_i : K \rightarrow TX, i \in \{1, 2, \cdots, N\} \) be a finite collection of monotone mappings, then the variational inequality problem for \( V_i \) is defined as
\[
\langle V_i(u^*_i), \exp^{-1}_u v \rangle \geq 0, \forall v \in K \text{ and } i \in \{1, 2, \cdots, N\},
\]
and the solution set is denoted by \( \bigcap_{i=1}^{N} \text{VI}(V_i, K) \).

**Algorithm 4.** For an arbitrary \( z_0 \in K \), compute the sequences \( \{u_k^i\}, i \in \{1, 2, \cdots\}, \{v_k\} \) and \( \{z_k\} \) as follows:
\[
0 \in P_{u_k} V_i(z_k) + \partial \gamma_K(u_k^i) - \frac{1}{\lambda_k} \exp^{-1}_u z_k,
\]
\[
v_k = \exp_{z_k} (1 - \alpha_k) \exp^{-1}_{z_k} T_i^k(u_k^i),
\]
\[
z_{k+1} = \exp_{z_k} (1 - \beta_k) \exp^{-1}_{z_k} S(v_k),
\]
where \( k_i \in \{1, 2, \cdots, N\} \) such that \( d(u_k^i, z_k) = \max_{i=1, \cdots, N} \{ d(u_k^i, z_k) \} \), \( \alpha_k, \beta_k \in (0, 1), 0 < \phi < 1 \) and \( \lambda_k > 0 \).

If \( V_i = V \) for all \( i \in \{1, 2, \cdots, N\} \), then we give the following algorithm to solve \( \text{Fix}(S) \cap \text{VI}(V, K) \cap \text{EP}(F) \).
Algorithm 5. For arbitrary $z_0 \in K$, compute the sequences $\{u_k\}$, $\{v_k\}$ and $\{z_k\}$ as follows:

\[
0 \in P_{u_k-z_k} V(z_k) + \partial \gamma_k(u_k) - \frac{1}{\lambda_k} \exp_{u_k}^{-1} z_k,
\]

\[
v_k = \exp_{z_k} (1 - a_k) \exp_{z_k}^{-1} T^f_k(u_k),
\]

\[
z_{k+1} = \exp_{z_k} (1 - \rho \beta_k) \exp_{z_k}^{-1} S(v_k),
\]

where $a_k, \beta_k \in (0, 1)$, $0 < \rho < 1$ and $\lambda_k > 0$.

If $V_i = V$ for all $i \in \{1, 2, \cdots, N\}$ and $F = 0$, then we propose the following algorithm to solve $\text{Fix}(S) \cap VI(V, K)$.

Algorithm 6. For arbitrary $z_0 \in K$, compute the sequences $\{u_k\}$ and $\{z_k\}$ as follows:

\[
0 \in P_{u_k-z_k} V(z_k) + \partial \gamma_k(u_k) - \frac{1}{\lambda_k} \exp_{u_k}^{-1} z_k,
\]

\[
z_{k+1} = \exp_{z_k} (1 - \rho \beta_k) \exp_{z_k}^{-1} S(u_k),
\]

where $\beta_k \in (0, 1)$, $0 < \rho < 1$ and $\lambda_k > 0$.

Corollary 3. Let $V_i \in \Omega(\mathbb{X})$ be $\eta_i$-strongly monotone and $\varphi_i$-Lipschitz continuous monotone vector fields for each $i \in \{1, 2, \cdots, N\}$. Let $F : K \times K \to \mathbb{R}$ be a bifunction satisfying the conditions (A)–(E) and $T^f_k$ be the resolvent of $F$, $S : K \to K$, which is a nonexpansive mapping. If $\text{Fix}(S) \cap VI(V_r, K) \cap \text{EP}(F) \neq \emptyset$ and $\eta = \min_{i=1,2,\cdots,N} \{\eta_i\}$, $\varphi = \max_{i=1,2,\cdots,N} \{\varphi_i\}$, $a_k, \beta_k \in (0, 1)$, $0 < \rho < 1$ and $\lambda_k > 0$ satisfy the conditions given in Theorem 1, then the sequence $\{z_k\}$ obtained by Algorithm 4 converges to an element in $\text{Fix}(S) \cap VI(V_r, K) \cap \text{EP}(F)$.

Corollary 4. Let $V \in \Omega(\mathbb{X})$ be $\eta$-strongly monotone and $\varphi$-Lipschitz continuous monotone vector field. Let $F : K \times K \to \mathbb{R}$ be a bifunction satisfying the conditions (A)–(E) and $T^f_k$ be the resolvent of $F$, $S : K \to K$, which is a nonexpansive mapping. If $\text{Fix}(S) \cap VI(V, K) \cap \text{EP}(F) \neq \emptyset$, $a_k, \beta_k \in (0, 1)$, $0 < \rho < 1$ and $\lambda_k > 0$ satisfy the conditions given in Theorem 1, then the sequence $\{z_k\}$ obtained by Algorithm 5 converges to an element in $\text{Fix}(S) \cap VI(V, K) \cap \text{EP}(F)$.

Corollary 5. Let $V \in \Omega(\mathbb{X})$ be $\eta$-strongly monotone and $\varphi$-Lipschitz continuous monotone vector field. Let $S : K \to K$, which is a nonexpansive mapping. If $\text{Fix}(S) \cap VI(V, K) \neq \emptyset$, $\beta_k \in (0, 1)$, $0 < \rho < 1$ and $\lambda_k > 0$ satisfy the conditions given in Theorem 1, then the sequence $\{z_k\}$ obtained by Algorithm 6 converges to an element in $\text{Fix}(S) \cap VI(V, K)$.

5. Conclusions

This work is concerned with the investigation of the common solution of a fixed-point problem, an equilibrium problem and a finite collection of variational inclusion problems. Our proposed algorithms are advanced and can be considered improvements to the methods discussed in the paper [3]. Several consequences of the suggested algorithms are discussed for variational inequalities, equilibrium and fixed-point problems. We anticipate that the methods presented in this paper can be extended to more general settings; for example, hyperbolic spaces, geodesic spaces and a CAT(0) space.

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