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Backward Stochastic Differential Equations (BSDEs) Using Infinite-Dimensional Martingales with Subdifferential Operator

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Abstract: In this paper, we focus on a family of backward stochastic differential equations (BSDEs) with subdifferential operators that are driven by infinite-dimensional martingales. We shall show that the solution to such infinite-dimensional BSDEs exists and is unique. The existence and uniqueness of the solution are established using Yosida approximations. Furthermore, as an application of the main result, we shall show that the backward stochastic partial differential equation driven by infinite-dimensional martingales with a continuous linear operator has a unique solution under the special condition that the $\mathcal{F}_t$-progressively measurable generator F of the model we proposed in this paper equals zero.

Keywords: backward stochastic differential equations (BSDEs); variational inequalities; martingales; subdifferential operators

MSC: 60H15; 60H30

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1. Introduction

In 1990, Pardoux and Peng [1] initially proposed the general nonlinear case of backward stochastic differential equations (BSDEs): let (ξ, f) include a square-integrable random variable ξ and a progressively measurable process f, and let $W_{t(0\leq t\leq T)}$ be a k-dimensional Brownian process. It can be proven that there exists a unique solution of an adapted process (Y, Z) of the following type of BSDEs:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\,\mathrm{d}s - \int_t^T Z_s\,\mathrm{d}W_s,\ \ 0 \leq t \leq T.$$

Since then, many scholars have begun to carry out more in-depth research on BSDEs. As a result, BSDEs have developed rapidly, whether in their own development or in many other related fields such as financial mathematics, stochastic control, biology, the financial futures market, the theory of partial differential equations, and stochastic games. Reference can be made to Karoui et al. [2], Hamadene and Lepeltial [3], Peng [4,5], Ren and Xia [6], and Luo et al. [7], among others. Among the BSDEs, Pardoux and Răşcanu [8] considered BSDEs involving a subdifferential operator, which are also dubbed Backward Stochastic Variational Inequalities (BSVIs), and also utilized them with the Feymann–Kac formula to represent a solution of the multivalued parabolic partial differential equations (PDEs). Pardoux and Răşcanu [9] demonstrated that the result could be easily expanded to a spatial setting Hilbert by giving examples of backward stochastic partial differential equations with solutions. Diomande and Maticiuc [10] used a mixed Euler–Yosida scheme to prove the existence of the solution of the multivalued BSDEs with time-delayed generators; Maticiuc and Rotenstein [11] provided the numerical results of the multivalued BSDEs. Boufoussi [12] showed that there is an existing and unique solution to a type of generalized backward doubly stochastic differential equation with a symmetric backward stochastic Itô integral. Wang and Yu [13] explored this problem with an anticipated type of generalized backward doubly stochastic differential equation. Instead of normal Brownian motion as the interference source, Yang et al. [14] showed the existence and uniqueness of the solution for a type of BSDE driven by a finite G-Brownian process with the subdifferential operator by using the Method of Approximation of Moreau–Yosida.

Some authors have also obtained results in the type spaces of L^p, among which Briand et al. [15] obtained an a priori estimate and demonstrated the existence and uniqueness of solutions in $L^p, p > 1$. Under normal conditions, Fan et al. [16] studied bounded solutions, $L^p(p > 1)$ solutions, and L^1 solutions of one-dimensional equations.

Instead of focusing on one-dimensional BSDEs ($Y \in \mathbb{R}$), it is possible to extend to multi-dimensional settings. Bahlali [17] had proven the existence, uniqueness, and stability of the solution for multi-dimensional BSDEs with a local monotonous coefficient. Maticiuc and Răşcanu [18] extended the existence and uniqueness results of the previous work of Pardoux and Răşcanu [9] by supposing a weaker boundedness condition for the generator and by considering the random time interval $[0, T]$, the Lebesgue–Stieltjes integral terms, where a fixed convex boundary is induced by the subdifferential of an appropriate lower semicontinuous convex function. Răşcanu [19] proved that in the case of $p \geq 2$, the variational solution is a strong one since they have certified the uniqueness of that solution.

Moreover, the martingale has a broader range of applications than Brownian motion. The properties of the martingale described may not hold true, and one generally needs to enter more martingale into the response. Hamaguchi [20] proposed an endless dimensional BSDE driven by a barrel-shaped martingale, demonstrated the presence and uniqueness of the arrangement of such boundless dimensional BSDEs, and showed the grouping of arrangements of related BSDEs. El Karoui and Huang [2] studied BSDEs driven by finite-dimension martingales. Al-Hussein [21] demonstrated an aftereffect of the presence and uniqueness of the solution of a BSDE which is driven by a limitless dimensional martingale and applied the outcome to track down a special answer for a regressive stochastic fractional differential condition in boundless measurements. Because the case of $p = 2$ is more common and $p > 2$ is more complex in L^p space, it is necessary to study BSDEs with the subdifferential operator, whose drives are infinite-dimensional martingales in L^2 space.

By considering the subdifferential operator and martingale simultaneously, Nie [22] concentrated on the existence and uniqueness of the solution to a multi-dimensional forward-backward stochastic differential equation (FBSDE) with the subdifferential operator in the backward condition where the backward equation is reflected on the boundary of a closed convex area. However, as far as we know, research on infinite dimensional martingale has not been done before.

The purpose of this paper is to consider a class of BSDEs driven by infinite dimensional martingales with the subdifferential operator of the following type:

$$\begin{cases} \mathrm{d}Y_t + F(t, Y_t, Z_t \mathcal{Q}_t^{1/2})\,\mathrm{d}t \in \partial\varphi(Y_t)\,\mathrm{d}t + Z_t\,\mathrm{d}M_t + \mathrm{d}N_t,\ 0 \leq t \leq T, \\ Y(T) = \xi. \end{cases} \quad (1)$$

Equation (1) is written in the context of a completion probability space $(\Omega, \mathcal{F}, P)$ with a continuous filter $\{\mathcal{F}_t\}_{t \geq 0}$ on the right side. Here ξ is a random variable, given as a final value; the function F is a mapping from $\Omega \times [0, \infty) \times H \times L^2(H)$ to H; M is a continuous martingale in the space of H; and $\mathcal{Q}_M$ is a predictable process that captures values from the space $L_2(H)$ of nuclear operators on H, that was introduced by Al-Hussein [23], and will be explained in the next section.

The main aim of this paper is to find an adapted process (Y, Z, U, N) in a proper space such that the BSDE in Equation (1) holds. Then, it allows us to establish the uniqueness of the viscosity solution of a certain type of non-local variational inequality. The following is a list of how this paper is organized. Section 2 introduces certain fundamental notations, assumptions, and preliminaries, as well as the a priori estimation of a series of penalized approximations to the equations. In Section 3, we verify the existence and uniqueness of the BSDE solution using the Yosida approximation approach. In Section 4, an example is provided for illustration of the proposed methodology.

2. Preliminaries

Al-Hussein [23] established the concepts of space and martingales as follows: Denote $\mathcal{M}^2_{[0,T]}(H)$ as the vector space of the cadlag square-integrable martingales $\{M(t), 0 \le t \le T\}$, that take values in the space of H; moreover $\mathbb{E}[|M(t)|_H{}^2] < \infty$ for each $t \in [0,T]$. A Hilbert space with respect to the inner product $(M,N) \mapsto \mathbb{E}[\langle M(T), N(T)\rangle_H]$ if $\mathbb{P}$-equivalence classes have been established. Let $\mathcal{M}^{2,c}_{[0,T]}(H)$ be a Hilbert subspace containing continuous square integrable martingale in H. These are *very strongly orthogonal* for $M, N \in \mathcal{M}^2_{[0,T]}(H)$, for all $[0,T]$-valued stopping times u, if we can satisfy $\mathbb{E}[M(u) \otimes N(u)] = \mathbb{E}[M(0) \otimes N(0)]$. In particular, if $N(0) = 0$, $\mathbb{E}[M(u) \otimes N(u)] = 0$, then M and N are very strongly orthogonal.

Let $M \in \mathcal{M}^2_{[0,T]}(H)$, and let the process $\langle M\rangle$ represent the predictable quadratic variation of M; let $\mathcal{Q}_M$ represent a predicted process that takes values from the set of positive symmetric elements that is linked to a Doléans measure of $M \otimes N$. We define $\langle\langle M\rangle\rangle_t = \int_0^t \mathcal{Q}_M(s)\, \mathrm{d}\langle M\rangle_s$, and assume there exists a predictable process $\mathcal{Q}(t,\cdot)$ which is a symmetric positive definite nuclear operator on H and satisfies $\langle\langle M\rangle\rangle_t = \int_0^t \mathcal{Q}(s)\, \mathrm{d}s$.

Under the space $L^*(H;\mathcal{P},M)$ of processes Φ, we first consider $\mathcal{E}(L(H))$ to be the space of predictable simple processes, and let $\Lambda^2(H;\mathcal{P},M)$ be the closure of $\mathcal{E}(L(H))$ in $L^*(H;\mathcal{P},M)$. Hence, the space $\Lambda^2(H;\mathcal{P},M)$ is one Hilbert subspace of $L^*(H;\mathcal{P},M)$. Additionally, the stochastic integral $\int \Phi\, \mathrm{d}M$ is defined for an element $\Phi \in \Lambda^2(H;\mathcal{P},M)$ which belongs to $\mathcal{M}^2_{[0,T]}(H)$, and also fulfills the condition

$$\mathbb{E}\left[\left|\int_0^T \Phi(t)\, \mathrm{d}M(t)\right|_H^2\right] = \mathbb{E}\left[\int_0^T |\Phi(t) \circ \mathcal{Q}_M^{1/2}(t)|^2_{L_2(H)}\, \mathrm{d}\langle M\rangle_t\right] < \infty.$$

Consider the following spaces [21]:

$$L^2_{\mathcal{F}}(0,T;H) := \left\{\phi : [0,T] \times \Omega \to H, \phi \text{ is predictable and satisfies } \mathbb{E}\left[\int_0^T |\Phi(t)|^2_H\, \mathrm{d}\langle M\rangle_t\right] < \infty\right\};$$

$$\mathcal{S}^2(H) := \left\{\phi : [0,T] \times \Omega \to H, \phi \text{ is continuous, adaptable and satisfies } \mathbb{E}\left[\sup_{0\le t\le T} |\phi(t)|^2_H\right] < \infty\right\}.$$

As stated in Al-Hussein [21], $\mathcal{S}^2(H)$ is a separable Banach space which conforms to the norm

$$\|\phi\|^2_{\mathcal{S}}(H) = \left(\mathbb{E}\left[\sup_{0\le t\le T} |\phi(t)|^2_H\right]\right)^{1/2}.$$

Let $M \in \mathcal{M}^2_{[0,T]}(H)$ be $M(0) = 0$ and consider the following assumptions:

(H1) The function $F : \Omega \times [0,\infty) \times H \times L^2(H) \to H$ fulfills the requirement $\alpha \in \mathbb{R}, \beta, \gamma \ge 0$, and also let η be one $\mathcal{F}_t-$progressively measurable process.

(H2) (i) $F(\cdot,\cdot,y,z)$ is $\mathcal{F}_t-$ progressively measurable,
(ii) $y \mapsto F(t,y,z)$ is continuous, $dp \times dt$ a.e. ,
(iii) $\forall y, y' \in H$ and $\forall z, z' \in L_2(H)$
$$(F(t,y,z) - F(t,y',z), y - y') \le \alpha|y - y'|^2,$$
(iv) $|F(t,y,z) - F(t,y,z')| \le \beta\|z - z'\|$,
(v) $|F(t,y,0)| \le \eta_t + \gamma|y|$,

(vi) $\mathbb{E}[\int_0^T |F(t,0,0)|_H^2\,\mathrm{d}t] < \infty$.

(H3) (i) φ is just a valid convex function,

(ii) $\varphi(y) \geq \varphi(0) = 0$.

(H4) (i) $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; H)$,

(ii) $\mathbb{E}[e^{\lambda t}(|\xi|^2 + |\varphi(\xi)|)] < \infty$,

(iii) $\mathbb{E}[\int_0^t e^{\lambda t}|\eta(s)^2|\,\mathrm{d}s] < \infty$, here $\lambda > 2\alpha + \beta^2$.

(H5) Every H-valued square integrable martingale with filtering $\{\mathcal{F}_t, 0 \leq t \leq T\}$ has a continuous version.

We introduce φ, which is a subdifferential of the l.s.c. convex function from the space H to $\mathbb{R}$. $\partial\varphi$ is a multivalued function from the space H to H, which was given by Pardoux and Răşcanu [1].

For any $u \in H$,

$$\partial\varphi(u) = \{h \in H : (h, v-u) + \varphi(u) \leq \varphi(v), \forall v \in H\}.$$

Let $\mathrm{Dom}(\partial\varphi)$ be the set of $u \in H$ such that $\partial\varphi(u)$ is not empty, and define $(u,v) \in \partial\varphi$ to imply that $u \in \mathrm{Dom}(\partial\varphi)$ and $v \in \partial\varphi(u)$.

The function φ is then approximated by the convex C^1-function φ_ε, $\varepsilon > 0$ which was defined by Pardoux and Răşcanu [8] as

$$\varphi_\varepsilon(u) = \inf\left\{\frac{1}{2}|u-v|^2 + \varepsilon\varphi(v) : v \in H\right\} = \frac{1}{2}|u - J_\varepsilon u|^2 + \varepsilon\varphi(J_\varepsilon u),$$

where $J_\varepsilon u = (I + \varepsilon\partial\varphi)^{-1}(u)$. For all $u, v \in H, \varepsilon > 0$, the properties of the approximation presented by Barbu [24] are given by

$$\begin{aligned} &\tfrac{1}{\varepsilon}D\varphi_\varepsilon(u) = \tfrac{1}{\varepsilon}\partial\varphi_\varepsilon(u) = \tfrac{1}{\varepsilon}(u - J_\varepsilon u) \in \partial\varphi(J_\varepsilon u), \\ &|\varphi(J_\varepsilon u) - \varphi(J_\varepsilon v)| \leq |u - v|, \quad \text{and} \quad \lim_{\varepsilon\to 0} J_\varepsilon u = Pr_{\overline{Dom\varphi}}(u). \end{aligned}$$

Hence, for all $u, v \in H, \varepsilon > 0, \varepsilon' > 0$, we have $0 \leq \varphi_\varepsilon \leq (D\varphi_\varepsilon(u), u)$ where

$$\left(\frac{1}{\varepsilon}D\varphi_\varepsilon(u) - \frac{1}{\varepsilon'}D\varphi_{\varepsilon'}(v)\right) \geq -\left(\frac{1}{\varepsilon} + \frac{1}{\varepsilon'}\right)|D\varphi_\varepsilon(u)| \times |D\varphi_{\varepsilon'}(v)|. \tag{2}$$

Consider the approximating equation

$$Y_t^\varepsilon + \frac{1}{\varepsilon}\int_t^T D\varphi_\varepsilon(Y_s^\varepsilon)\,\mathrm{d}s = \xi + \int_t^T F(t, Y_s^\varepsilon, Z_s^\varepsilon Q_s^{1/2})\,\mathrm{d}s - \int_t^T Z_s^\varepsilon\,\mathrm{d}M_s - \int_t^T \mathrm{d}N_s^\varepsilon. \tag{3}$$

As a result of the conclusion of Al-Hussein [21], for this Equation (3) there exists a unique solution $(Y^\varepsilon, Z^\varepsilon, N^\varepsilon) \in S^2_{[0,T]}(H) \times M^2_{[0,T]}(L^2(H)) \times M^2_{[0,T]}(H)$.

Lemma 1. *Let the assumptions (H1)-(H5) be satisfied, then for all* $0 \leq a \leq T$,

$$\mathbb{E}\left[\sup_{a\leq t\leq T} \mathrm{e}^{\lambda t}|Y_t^\varepsilon|^2 + \int_a^T \mathrm{e}^{\lambda t}(|Y_s^\varepsilon|^2 + \|Z_s^\varepsilon Q_s^{1/2}\|^2)\,\mathrm{d}s + \int_a^T \mathrm{e}^{\lambda s}\,\mathrm{d}\langle N\rangle_s\right] \leq C\Gamma_1(a,T), \tag{4}$$

where

$$\Gamma_1(a,T) = \mathbb{E}\left[\mathrm{e}^{\lambda T}|\xi|^2 + \int_a^T \mathrm{e}^{\lambda s}|F(s,0,0)|^2\,\mathrm{d}s\right].$$

Proof. Firstly, Itô's formula for $e^{\lambda t}|Y_t^\varepsilon|^2$ yields

$$e^{\lambda t}|Y_t^\varepsilon|^2 + \int_t^T e^{\lambda s}(\lambda|Y_s^\varepsilon|^2 + \|Z_s^\varepsilon \mathcal{Q}_s^{1/2}\|^2)\,ds + \int_t^T e^{\lambda s}\,d\langle N\rangle_s + \frac{2}{\varepsilon}\int_t^T e^{\lambda s}(Y_s^\varepsilon, D\varphi_\varepsilon(Y_s^\varepsilon)\,ds$$
$$= e^{\lambda T}|\xi|^2 + 2\int_t^T e^{\lambda s}(Y_s^\varepsilon, F(t, Y_s^\varepsilon, Z_s^\varepsilon \mathcal{Q}_s^{1/2})\,ds - 2\int_t^T e^{\lambda s}(Y_s^\varepsilon, Z_s^\varepsilon\,dM_s) - 2\int_t^T e^{\lambda s}(Y_s^\varepsilon, dN_s^\varepsilon).$$

Then applying Schwarz's inequalites and considering $\left(\frac{1}{\varepsilon}D\varphi_\varepsilon(y), y\right) \geq 0$,

$$\begin{aligned} 2(y, F(s,y,z)) &= 2(y, F(s,y,z) - F(s,y,0) + F(s,y,0) - F(s,0,0) + F(s,0,0)) \\ &\leq 2\beta(y,z) + 2\alpha|y|^2 + 2(y, F(s,0,0)) \\ &\leq (2\alpha + (1+r)\beta^2 + r)|y|^2 + \frac{1}{1+r}\|z\|^2 + \frac{1}{r}|F(s,0,0)|^2, \end{aligned}$$

where $\lambda > 2\alpha + \beta^2, 0 \leq r \leq \frac{\lambda - (2\alpha+\beta^2)}{1+\beta^2} \wedge 1$. Hence,

$$e^{\lambda t}|Y_t^\varepsilon|^2 + \int_t^T e^{\lambda s}[(\lambda - 2\alpha - \beta^2 - r(1+\beta^2))|Y_s^\varepsilon|^2 + \frac{r}{r+1}\|Z_s^\varepsilon \mathcal{Q}_s^{1/2}\|^2]\,ds + \int_t^T e^{\lambda s}\,d\langle N\rangle_s$$
$$\leq e^{\lambda T}|\xi|^2 + \frac{1}{r}\int_t^T e^{\lambda s}|F(s,0,0)|^2\,ds - 2\int_t^T e^{\lambda s}(Y_s^\varepsilon, Z_s^\varepsilon\,dM_s) - 2\int_t^T e^{\lambda s}(Y_s^\varepsilon, dN_s).$$

It can be shown that

$$\begin{aligned} \sup_{a\leq t\leq T} e^{\lambda t}|Y_t^\varepsilon|^2 &\leq e^{\lambda T}|\xi|^2 + \frac{1}{r}\int_t^T e^{\lambda s}|F(s,0,0)|^2\,ds + 2\sup_{a\leq t\leq T}\left|\int_t^T e^{\lambda s}(Y_s^\varepsilon, Z_s^\varepsilon\,dM_s)\right| \\ &\quad +2\sup_{a\leq t\leq T}\left|\int_t^T e^{\lambda s}(Y_s^\varepsilon, dN_s)\right|. \end{aligned}$$

Then, taking the expectation in the above inequality using Burkholder–Davise–Gundy's inequality,

$$\begin{aligned} \mathbb{E}\left[\sup_{a\leq t\leq T} e^{\lambda t}|Y_t^\varepsilon|^2\right] &\leq C_1 + 2\mathbb{E}\left[\sup_{a\leq t\leq T}\left|\int_t^T e^{\lambda s}(Y_s^\varepsilon, Z_s^\varepsilon\,dM_s)\right|\right] \\ &\quad +2\mathbb{E}\left[\sup_{a\leq t\leq T}\left|\int_t^T e^{\lambda s}(Y_s^\varepsilon, dN_s)\right|\right] \\ &\leq C_1 + \frac{1}{2}\mathbb{E}\left[\sup_{a\leq t\leq T} e^{\lambda t}|Y_t^\varepsilon|^2\right] + C_2\mathbb{E}\left[\int_a^T e^{\lambda t}\|Z_s^\varepsilon \mathcal{Q}_s^{1/2}\|^2\,ds\right] \\ &\quad +\frac{1}{2}\mathbb{E}\left[\sup_{a\leq t\leq T} e^{\lambda t}|Y_t^\varepsilon|^2\right] + C_3\mathbb{E}\left[\int_a^T e^{\lambda t}\,d\langle N\rangle_s\right]. \end{aligned}$$

Hence, the proof is completed. □

Lemma 2. *Let the assumptions (H1)–(H5) be satisfied, then there exists a positive constant C such that for* $0 \leq a \leq T$,

$$\mathbb{E}\left[\int_a^T e^{\lambda s}\left(\frac{1}{\varepsilon}D\varphi_\varepsilon(Y_s^\varepsilon)\right)^2 ds\right] \leq C\Gamma_2(a,T), \tag{5}$$

$$\mathbb{E}\left[e^{\lambda a}\varphi(J_\varepsilon Y_s^\varepsilon)\right] + \mathbb{E}\left[\int_a^T e^{\lambda s}\varphi(J_\varepsilon Y_s^\varepsilon)\,ds\right] \leq C\Gamma_2(a,T), \tag{6}$$

$$\mathbb{E}\left[e^{\lambda a}|Y_s^\varepsilon - J_\varepsilon Y_s^\varepsilon|^2\right] \leq \varepsilon^2 C\Gamma_2(a,T), \tag{7}$$

where

$$\Gamma_2(a,T) = \mathbb{E}\left[e^{\lambda T}(|\xi|^2 + \varphi(\xi)) + \int_0^T e^{\lambda s}|\eta(s)|^2\,ds \right].$$

Proof. Consider the subdifferential inequality below:

$$e^{\lambda s}\varphi_\varepsilon(Y_s^\varepsilon) \geq (e^{\lambda s} - e^{\lambda r})\varphi_\varepsilon(Y_s^\varepsilon) + e^{\lambda r}\varphi_\varepsilon(Y_r^\varepsilon) + e^{\lambda r}(D\varphi_\varepsilon(Y_r^\varepsilon), Y_s^\varepsilon - Y_r^\varepsilon),$$

for $s = t_{i+1} \wedge T$,$r = t_i \wedge T$, where $t = t_0 < t_1 < t_2 < \cdots$ and $t_{i+1} - t_i = 1/n$. By summing up over i, and going to the limit as $n \to \infty$, $\forall t \in [0,T]$, we can deduce that

$$\begin{aligned}
&e^{\lambda t}\varphi_\varepsilon(Y_t^\varepsilon) + \int_t^T \lambda e^{\lambda s}\varphi_\varepsilon(Y_s^\varepsilon)\,ds + \frac{1}{\varepsilon}\int_t^T e^{\lambda s}|D\varphi_\varepsilon(Y_s^\varepsilon)|^2\,ds \\
&\leq e^{\lambda T}\varphi_\varepsilon(\xi) + \int_t^T (D\varphi_\varepsilon(Y_s^\varepsilon), F(s, Y_s^\varepsilon, Z_s^\varepsilon))\,ds - \int_t^T e^{\lambda s}(D\varphi_\varepsilon(Y_s^\varepsilon), Z_s^\varepsilon\,dM_s).
\end{aligned}$$

As a consequence, we obtain the result by combining Equation (4) with the inequalities in Proposition 2.2 from Pardoux and Răşcanu [8]:

$$\begin{aligned}
&\frac{1}{2}|D\varphi_\varepsilon(y)|^2 + \varepsilon\varphi(J_\varepsilon y) = \varphi_\varepsilon(y), \;\; \varepsilon\varphi(J_\varepsilon y) \leq \varphi_\varepsilon(y), \\
&-\lambda\varphi_\varepsilon(y) \leq |\lambda|\varphi_\varepsilon(y) \leq |\lambda|(D\varphi_\varepsilon(y), y), \\
&\varphi_\varepsilon(\xi) \leq \varepsilon\varphi(\xi), \\
&0 \leq \varphi_\varepsilon(u) \leq (D\varphi_\varepsilon(u), u),
\end{aligned}$$

$$\begin{aligned}
(D\varphi_\varepsilon(y), |\lambda|y + F(s,y,z)) &\leq \frac{1}{2\varepsilon}|D\varphi_\varepsilon(y)|^2 + \frac{\varepsilon}{2}(|\lambda||y| + |F(s,y,z)|)^2 \\
&\leq \frac{1}{2\varepsilon}|D\varphi_\varepsilon(y)|^2 + \varepsilon(|\lambda|^2|y|^2 + |F(s,y,z)|^2) \\
&\leq \frac{1}{2\varepsilon}|D\varphi_\varepsilon(y)|^2 + \varepsilon[|\lambda|^2|y|^2 \\
&\quad +4(\beta^2\|z\|^2 + \gamma^2|y|^2 + \eta^2(s))].
\end{aligned}$$

Hence, the end result is obtained. □

Lemma 3. *Assuming that assumptions (H1)–(H5) are satisfied, then for any* $\varepsilon, \varepsilon' > 0$,

$$\mathbb{E}\left[\int_0^T e^{\lambda s}(|Y_s^\varepsilon - Y_s^{\varepsilon'}|^2 + \|Z_s^\varepsilon Q_s^{1/2} - Z_s^{\varepsilon'} Q_s^{1/2}\|^2)\,ds \right] \leq (\varepsilon + \varepsilon')C\Gamma(T), \tag{8}$$

$$\mathbb{E}\left[\sup_{0\leq t\leq T} e^{\lambda t}|Y_s^\varepsilon - Y_s^{\varepsilon'}|^2 \right] \leq (\varepsilon + \varepsilon')C\Gamma(T), \tag{9}$$

where

$$\Gamma(T) = \mathbb{E}\left[e^{\lambda T}(|\xi|^2 + \varphi(\xi)) + \int_0^T |F(s,0,0)|^2\,ds \right].$$

.

Proof. Firstly, Itô's formula for $e^{\lambda t}|Y_s^\varepsilon - Y_s^{\varepsilon'}|^2$ yields

$$\begin{aligned}
&e^{\lambda t}|Y_t^\varepsilon - Y_s^{\varepsilon'}|^2 + \int_t^T e^{\lambda s}(\lambda|Y_s^\varepsilon - Y_s^{\varepsilon'}|^2 + \|Z_s^\varepsilon Q_s^{1/2} - Z_s^{\varepsilon'} Q_s^{1/2}\|^2)\,ds \\
&+ \int_t^T e^{\lambda s}\,d\langle N^\varepsilon - N^{\varepsilon'}\rangle_s + 2\int_t^T e^{\lambda s}(Y_s^\varepsilon - Y_s^{\varepsilon'}, \frac{1}{\varepsilon}D\varphi_\varepsilon(Y_s^\varepsilon) - \frac{1}{\varepsilon'}D\varphi_{\varepsilon'}(Y_s^{\varepsilon'}))\,ds \\
&= e^{\lambda T}|\xi|^2 + 2\int_t^T e^{\lambda s}(Y_s^\varepsilon - Y_s^{\varepsilon'}, F(t, Y_s^\varepsilon, Z_s^\varepsilon Q_s^{1/2}) - F(t, Y_s^\varepsilon, Z_s^\varepsilon Q_s^{1/2}))\,ds \\
&\quad - 2\int_t^T e^{\lambda s}(Y_s^\varepsilon - Y_s^{\varepsilon'}, (Z_s^\varepsilon - Z_s^{\varepsilon'})\,dM_s) - 2\int_t^T e^{\lambda s}(Y_s^\varepsilon - Y_s^{\varepsilon'}, dN_s^\varepsilon - dN_s^{\varepsilon'}).
\end{aligned}$$

Moreover,

$$\begin{aligned}
&2(Y_s^\varepsilon - Y_s^{\varepsilon'}, F(t, Y_s^\varepsilon, Z_s^\varepsilon Q_s^{1/2}) - F(t, Y_s^\varepsilon, Z_s^\varepsilon Q_s^{1/2})) \\
&\quad \le (2\alpha + (1+r)\beta^2)|Y_t^\varepsilon - Y_s^{\varepsilon'}|^2 + \frac{1}{1+r}\|Z_s^\varepsilon Q_s^{1/2} - Z_s^{\varepsilon'} Q_s^{1/2}\|^2.
\end{aligned}$$

Based on Equation (2), hence,

$$\begin{aligned}
&e^{\lambda t}|Y_t^\varepsilon - Y_s^{\varepsilon'}|^2 + \int_t^T e^{\lambda s}\,d\langle N^\varepsilon - N^{\varepsilon'}\rangle_s \\
&+ \int_t^T e^{\lambda s}\left[(\lambda - 2\alpha - \beta^2 - r\beta^2)|Y_s^\varepsilon - Y_s^{\varepsilon'}|^2 + \frac{r}{r+1}\|Z_s^\varepsilon Q_s^{1/2} - Z_s^{\varepsilon'} Q_s^{1/2}\|^2\right] ds \\
&\le 2\left(\frac{1}{\varepsilon} + \frac{1}{\varepsilon'}\right)\int_t^T e^{\lambda s}|D\varphi_\varepsilon Y_t^\varepsilon| \times |D\varphi_{\varepsilon'} Y_s^{\varepsilon'}|\,ds - 2\int_t^T e^{\lambda s}(Y_s^\varepsilon - Y_s^{\varepsilon'}, Z_s^\varepsilon - Z_s^{\varepsilon'}\,dM_s) \\
&- 2\int_t^T e^{\lambda s}(Y_s^\varepsilon - Y_s^{\varepsilon'}, dN_s^\varepsilon - dN_s^{\varepsilon'}).
\end{aligned}$$

Taking the expectations on both sides of the above inequation, and combining it with the inequation below from Lemma 2,

$$2\left(\frac{1}{\varepsilon} + \frac{1}{\varepsilon'}\right)\mathbb{E}\left[\int_t^T e^{\lambda s}|D\varphi_\varepsilon Y_t^\varepsilon| \times |D\varphi_{\varepsilon'} Y_s^{\varepsilon'}|\,ds\right] \le C(\varepsilon + \varepsilon')\Gamma(T),$$

we can obtain

$$\mathbb{E}\left[\int_0^T e^{\lambda s}(|Y_s^\varepsilon - Y_s^{\varepsilon'}|^2)ds\right] \le C(\varepsilon + \varepsilon')\Gamma(T).$$

On the other hand, on the basis of Equation (10), we obtain

$$\mathbb{E}\left[e^{\lambda s}\|Z_s^\varepsilon Q_s^{1/2} - Z_s^{\varepsilon'} Q_s^{1/2}\|^2\,ds\right] \le C(\varepsilon + \varepsilon')\mathbb{E}\left[\int_t^T e^{\lambda s}|D\varphi_\varepsilon Y_t^\varepsilon| \times |D\varphi_{\varepsilon'} Y_s^{\varepsilon'}|\,ds\right].$$

Indeed, it follows from Burkholder–Davis–Gundy's inequality that the result below can be obtained:

$$\mathbb{E}\left[\sup_{0\le t\le T} e^{\lambda t}|Y_s^\varepsilon - Y_s^{\varepsilon'}|^2\right] \le (\varepsilon + \varepsilon')C\Gamma(T).$$

We then complete the proof. □

3. The Existence and Uniqueness of the Solution

Lemma 4. *Let the assumptions (H1)–(H5) be satisfied, and let (Y, Z, U, N) be a solution to the BSDE in Equation (1) and (Y', Z', U', N') likewise be another solution to this type of BSDE. Denote $(\delta Y, \delta Z, \delta U, \delta N) \triangleq (Y - Y', Z - Z', U - U', N - N')$, and let λ be a real number, hence,*

$$\mathbb{E}\left[\int_t^T e^{\lambda s}(|\delta Y_s|^2 + \|\delta Z_s \mathcal{Q}_s^{1/2}\|^2)\,ds\right] = 0, \tag{10}$$

$$\mathbb{E}\left[\sup_{0\le t\le T} e^{\lambda t}|\delta Y_t|^2\right] = 0. \tag{11}$$

Proof. Itô's formula for $e^{\lambda t}|\delta Y_t|^2$ yields

$$\begin{aligned} e^{\lambda T}|\delta Y_T|^2 - e^{\lambda t}|\delta Y_t|^2 \quad = \quad & \int_t^T e^{\lambda s}(\lambda|\delta Y_s|^2 + \|\delta Z_s \mathcal{Q}_s^{1/2}\|^2)\,ds + \int_t^T e^{\lambda s}\,d\langle \delta N\rangle_s \\ & +2\int_t^T e^{\lambda s}(\delta Y_s, [F(s, Y_s, Z_s \mathcal{Q}_s^{1/2}) - F(s, \delta Y_s, \delta Z_s \mathcal{Q}_s^{1/2})]\,ds \\ & \qquad - |\delta U_s|\,ds - |\delta Z_s|\,dM_s - \,d\delta N_s). \end{aligned}$$

Taking the expectation of the above equation, we obtain

$$\begin{aligned} & \mathbb{E}\left[e^{\lambda t}|\delta Y_t|^2\right] + \mathbb{E}\left[\int_t^T e^{\lambda s}(\lambda|\delta Y_s|^2 + \|\delta Z_s \mathcal{Q}_s^{1/2}\|^2)\,ds\right] \\ & + \mathbb{E}\left[\int_t^T e^{\lambda s}\,d\langle \delta N\rangle_s\right] + 2\mathbb{E}\left[\int_t^T e^{\lambda s}(\delta Y_t, \delta U_s)\,ds\right] \\ & = 2\mathbb{E}\left[\int_t^T e^{\lambda s}(\delta Y_t, [F(s, Y_s, Z_s \mathcal{Q}_s^{1/2}) - F(s, \delta Y_s, \delta Z_s \mathcal{Q}_s^{1/2})])\,ds\right]. \end{aligned}$$

However, consider the following,

$$2(\delta Y_t, \delta U_s) \ge 0,$$

$$\begin{aligned} (\delta Y_t, F(s, Y_s, Z_s \mathcal{Q}_s^{1/2}) - F(s, \delta Y_s, \delta Z_s \mathcal{Q}_s^{1/2})) \quad & \le \quad 2\alpha|\delta Y_t|^2 + \beta(|\delta Y_s|^2 + \|\delta Z_s \mathcal{Q}_s^{1/2}\|^2) \\ & \le \quad (2\alpha + \beta^2 + r\beta^2)|\delta Y_t|^2 + \frac{1}{1+r}\|\delta Z_s \mathcal{Q}_s^{1/2}\|^2. \end{aligned}$$

Hence, we can obtain Equation (10). By the use of Burkholder–Davis–Gundy's inequality, Equation (11) can also be obtained. □

Theorem 1. *Suppose that the conditions (H1)–(H5) hold, then there will exist a unique quadruple (Y, Z, U, N) so that*

$$Y \in S^2_{[0,T]}(H),\ Z \in M^2_{[0,T]}(L^2(H)),\ U \in M^2_{[0,T]}(H),\ N \in M^2_{[0,T]}(H),$$

$$\mathbb{E}\left[\int_t^T e^{\lambda s}\varphi(Y_s)\,ds\right] \le \infty, \tag{12}$$

$$\begin{cases} Y_t + \displaystyle\int_t^T U_s\,ds = \xi + \int_t^T F(t, Y_s, Z_s \mathcal{Q}_s^{1/2})\,ds - \int_t^T Z_s\,dM_s - \int_t^T dN_s, \\ Y_t \in \operatorname{Dom}(\partial\varphi), \text{ and } U_t \in \partial\varphi(Y_t). \end{cases} \tag{13}$$

Proof. Uniqueness can be proven by using Lemma 4 above. As a limit of the quadruple $(Y_s^\varepsilon, Z_s^\varepsilon, \frac{1}{\varepsilon}D\varphi_\varepsilon(Y_s^\varepsilon), N_s^\varepsilon)$, the existence of the solution (Y, Z, U, N) is established. The following results come from Lemma 3,

$$\exists Y \in S^2_{[0,T]}(H), Z \in M^2_{[0,T]}(L^2(H)),$$

$$\lim_{\varepsilon\to 0} Y^\varepsilon = Y \text{ in } S^2_{[0,T]}(H),$$

$$\lim_{\varepsilon\to 0} Z^\varepsilon = Z \text{ in } M^2_{[0,T]}(L^2(H)),$$

and from Equations (5) and (7), for $\forall t \in [0, T]$, we have

$$\lim_{\varepsilon\to 0} J_\varepsilon(Y^\varepsilon) = Y \text{ in } S^2_{[0,T]}(H),$$

$$\lim_{\varepsilon\to 0} \mathbb{E}\left[\mathrm{e}^{\lambda t}|J_\varepsilon(Y_t^\varepsilon) - Y_t|^2\right] = 0.$$

Equation (12) follows from Equations (6) and (10).

For all $\varepsilon > 0$, let $U_t^\varepsilon = \frac{1}{\varepsilon}D\varphi_\varepsilon(Y_t^\varepsilon)$, and $\widehat{U}_t^\varepsilon = \int_t^T U_s^\varepsilon\, \mathrm{d}s$. As a result of the convergence result which was presented by Pardoux and Rascanu [8] and Equation (13), there exists a progressively measurable process $\{\widehat{U}_t, 0 \le t \le T\}$ so that for each $T > 0$,

$$\mathbb{E}\left[\sup_{0\le t\le T} |\widehat{U}_t^\varepsilon - \widehat{U}_t|^2\right] \to 0,\ \varepsilon \to 0.$$

Furthermore, from Equation (5),

$$\sup_{\varepsilon>0} \mathbb{E}\left[\int_0^T \mathrm{e}^{\lambda t}|U_t^\varepsilon|^2\, \mathrm{d}s\right] < \infty.$$

In the space $L^2(\Omega, H^1(0, T))$, $\widehat{U}^\varepsilon$ is bounded for all $T < 0$, and $\lim_{\varepsilon\to 0} \widehat{U}^\varepsilon = \widehat{U}$ in $L^2(\Omega, H^1(0, T))$; specifically, $\widehat{U}_t$ adopts the form $\widehat{U}_t = \int_t^T U_s\, \mathrm{d}s$, where $\{\widehat{U}_t, 0 \le t \le T\}$ is gradually measurable, and $\widehat{U}$ is completely continuous.

Moreover, from Gegout–Petit and Pardoux's Lemma 5.8 [25], for each $0 \le a \le b \le T$, $V \in M^2_{[a,b]}(H)$,

$$\int_a^b (U_t^\varepsilon, V_t - Y_t^\varepsilon)\, \mathrm{d}t \to \int_a^b (U_t, V_t - Y_t)\, \mathrm{d}t$$

in probability, and from Equation (5) we obtain $\int_a^b (U_t^\varepsilon, J_\varepsilon(Y_t^\varepsilon) - Y_t^\varepsilon)\, \mathrm{d}t \to 0$. Now, since $U_t^\varepsilon \in \partial\varphi(J_\varepsilon(Y_t^\varepsilon))$,

$$\int_a^b (U_t^\varepsilon, V_t - J_\varepsilon(Y_t^\varepsilon))\, \mathrm{d}t + \int_a^b \varphi(J_\varepsilon(Y_t^\varepsilon))\, \mathrm{d}t \le \int_a^b \varphi(V_t)\, \mathrm{d}t.$$

Taking the limit inferior in the probability of the above inequation, we obtain

$$\int_a^b (U_t, V_t - Y_t)\, \mathrm{d}t + \int_a^b \varphi(Y_t)\, \mathrm{d}t \le \int_a^b \varphi(V_t)\, \mathrm{d}t.$$

When the constants a, b, and the process V are random, Equation (13) can be proven. □

4. Examples

Considering Theorem 4.2 and Example 4.3 of Al-Hussein [21], the following backward stochastic partial differential equation (BSPDE) has its solution (Y, Z, N) where

$$\begin{cases} -\mathrm{d}Y_t = A\,\mathrm{d}t - Z_t\,\mathrm{d}M_t - \mathrm{d}N_t,\ 0 \le t \le T, \\ Y(T) = \xi. \end{cases} \tag{14}$$

Here, let $F(t, Y_t, Z_t \mathcal{Q}_t^{1/2}) = 0$, and assume A is a linear operator with no bounds from $\mathcal{D}(A)$ to H. If $A : V \to V'$ is a continuous linear operator, Equation (14) has an unique solution.

Now, let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a full probability space and $\mathcal{D} \subset \mathbb{R}^d$ be an open bounded subset with suitably smooth border $\partial(\mathcal{D})$. Let M_t be martingales and set $H = H_1 := L^2(\mathcal{D})$. Then, consider the BSPDE given below:

$$\begin{cases} -\mathrm{d}Y_t + \partial j(Y_t)\mathrm{d}t \ni A\,\mathrm{d}t + F(t, Y_t, Z_t\mathcal{Q}_t^{1/2})\mathrm{d}t - Z_t\,\mathrm{d}M_t - \mathrm{d}N_t,\ 0 \le t \le T, \\ Y_t = 0 \ \text{ on } \ \Omega \times [0, T] \times \partial(\mathcal{D}). \end{cases} \tag{15}$$

Firstly, let us apply Theorem 3.2 of Maticiuc and Răşcanu [18], where φ is a function from $L^2(\mathcal{D}) \to \mathbb{R}$, which is provided below:

$$\varphi(u) = \begin{cases} \dfrac{1}{2}\displaystyle\int_{\mathcal{D}} |\nabla u(x)|^2\,\mathrm{d}x + \int_{\mathcal{D}} j(u(x))\,\mathrm{d}x, & \text{if } u \in H_0^1(\mathcal{D}), j(u) \in L^1(\mathcal{D}), \\ +\infty, & \text{otherwise}. \end{cases} \tag{16}$$

Then, considering Proposition 2.8 of Barbu [26], the following properties hold:

(i) φ is a function what proper, convex as well as 1.s.c.,

(ii) $\partial\varphi(u) = \{u^* \in L^2(\mathcal{D}) : * \in \partial j(u(x)) - \triangle u(x)\ \ a.e. \text{ on } \mathcal{D}\}, \forall u \in \mathrm{Dom}(\partial\varphi)$,

(iii) $\mathrm{Dom}(\partial\varphi) = \{u \in H^1(\mathcal{D}) \bigcap H^2(\mathcal{D}) : u(x) \in \mathrm{Dom}(\partial j)\ \ a.e. \text{ on } \mathcal{D}\}$,

(iv) $\|u\| \le C\|u^*\|, \forall (u, u^*) \in \partial\varphi$.

Lastly, by applying Theorem 1 from Section 3, we decide that, under the above conditions, Equation (15) has an unique solution $(Y, Z, U, N) \in S^2(H) \times M^2(L^2(H)) \times M^2(H) \times M^2(H)$, such that $(Y_t, Z_t) = (\eta, 0)$, where η is a $H^1(\mathcal{D})$-valued random variable, $\mathcal{F}_t$-measurable and

(a) $Y_t + \int_t^T U_s\,\mathrm{d}s = \xi + \int_t^T AY_s\,\mathrm{d}\mathcal{Q}_s - \int_t^T Z_s\,\mathrm{d}M_s - \int_t^T \mathrm{d}N_s\ \ a.s.$,

(b) $Y_t \in H^1(\mathcal{D}) \cap H^2(\mathcal{D})$,

(c) $Y_t(x) \in \mathrm{Dom}(j)$,

(d) $U_t(x) \in \partial j(Y_t(x))$.

5. Conclusions

The goal of this paper is to present and study a type of BSDEs that is driven by infinite-dimensional martingales with subdifferential operators. We have shown that the adaptive solution of this BSDE exists and is unique. Additionally, we have presented a special example for a simple case. For future work, we will focus on this interesting problem and pay more attention to the simulation of numerical solutions of BSDEs of multidimensional and even infinite-dimensional types and their applications in finance and computing, such as [27–29].

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