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Trapezoidal Intuitionistic Fuzzy Power Heronian Aggregation Operator and Its Applications to Multiple-Attribute Group Decision-Making

Jeevaraj Selvaraj [1], Prakash Gatiyala [2] and Sarfaraz Hashemkhani Zolfani [3,*]

[1] Department of Applied Sciences (Mathematics), Atal Bihari Vajpayee Indian Institute of Information Technology and Management, Gwalior 474015, India
[2] Department of Information Technology, Atal Bihari Vajpayee Indian Institute of Information Technology and Management, Gwalior 474015, India
[3] School of Engineering, Universidad Catolica del Norte, Coquimbo Campus, Coquimbo 1281, Chile
* Correspondence: sarfaraz.hashemkhani@ucn.cl

Abstract: Decision-making problems involve imprecise and incomplete information that can be modelled well using intuitionistic fuzzy numbers (IFNs). Various IFNs are available in the literature for modelling such problems. However, trapezoidal intuitionistic fuzzy numbers (TrIFNs) are widely used. It is mainly because of the flexibility in capturing the incompleteness that occurs in the data. Aggregation operators play a vital role in real-life decision-making problems (modelled under an intuitionistic fuzzy environment). Different aggregation operators are available in the literature for better decision-making. Various aggregation operators are introduced in the literature as a generalization to the conventional aggregation functions defined on the set of real numbers. Each aggregation operator has a specific purpose in solving the problems effectively. In recent years, the power average (PA) operator has been used to reduce the effect of biased data provided by decision-makers. Similarly, the Heronian mean (HM) operator has a property that considers the inter-relationship among various criteria available in the decision-making problem. In this paper, we have considered both the operators (HM, PA) to introduce a new aggregation operator (on the set of TrIFNs), which takes advantage of both properties of these operators. In this study, firstly, we propose the idea of a trapezoidal intuitionistic fuzzy power Heronian aggregation (TrIFPHA) operator and a trapezoidal intuitionistic fuzzy power weighted Heronian aggregation (TrIFPWHA) operator by combining the idea of the Heronian mean operator and power average operator in real numbers. Secondly, we study different mathematical properties of the proposed aggregation operators by establishing a few essential theorems. Thirdly, we discuss a group decision-making algorithm for solving problems modelled under a trapezoidal intuitionistic fuzzy environment. Finally, we show the applicability of the group decision-making algorithm by solving a numerical case problem, and we compare the proposed method's results with existing methods.

Keywords: trapezoidal intuitionistic fuzzy set; Heronian mean; power average operator; trapezoidal intuitionistic fuzzy power weighted Heronian aggregation (TrIFPWHA) operator; multiple-attribute group decision-making

MSC: 03B52; 03E72; 94D05

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1. Introduction

Real-life problems mainly deal with either imprecise data or the combination of various types of data. Solving such problems with imprecise information is not an easy task. If a problem consists of precise (real number) data, it would be easy to solve such a problem using conventional decision-making algorithms. However, the problem with imprecise information and the problems with incomplete or adequate information cannot be solvable by using various conventional decision-making algorithms. Fuzzy numbers can represent decision-making problems involving imprecise information; hence, they can be solved using various fuzzy decision-making techniques. However, problems with imprecise and incomplete information can be modelled better using intuitionistic fuzzy numbers ([1–3]) than fuzzy numbers or real numbers. Further, trapezoidal intuitionistic fuzzy numbers (TrIFNs) were widely used to model problems with imprecise, adequate and qualitative information. Many decision-making algorithms are available to solve these problems modelled under an intuitionistic fuzzy environment. If the problem is modelled using TrIFNs, it is necessary to study the ranking principle to compare arbitrary TrIFNs. The ranking of TrIFNs [4,5] plays a vital role in solving problems modelled using trapezoidal intuitionistic fuzzy numbers. Researchers worldwide have introduced various ranking principles for comparing two arbitrary trapezoidal intuitionistic fuzzy numbers. However, none yield a total ordering on the class of trapezoidal intuitionistic fuzzy numbers. In 2016, Nayagam et al. [6] introduced eight different score functions in the class of TrIFNs and defined a total ordering principle by using those eight score functions. The total ordering principle on the class of TrIFNs makes the decision-making algorithm more efficient. Similarly, the aggregation operators will be used to find the aggregated performance of any alternatives concerning multiple attributes, which plays another important role in any decision-making algorithm. The same decision-making algorithm may give different results based on numerous aggregation operators. Many intuitionistic fuzzy aggregation operators developed, such as the intuitionistic fuzzy weighted averaging operator, intuitionistic fuzzy ordered weighted averaging operator, intuitionistic fuzzy hybrid aggregation operator, Heronian mean, Bonferroni mean, Dombi, trigonometric, Frank and power aggregation operator [7]. Each of these aggregation operators has its specific purposes, some of which can mitigate the specific influences of irrational data generated by biased decision-makers, such as the power aggregation operator, by allocating the weighted vector based on the degree of support between the input arguments to aggregate the input data and accomplish this purpose. The interrelationship of the aggregated arguments, such as the Heronian mean and Bonferroni mean, can also be considered by certain aggregation operators. In this paper, our main aim is to introduce a new aggregation operator on the set of trapezoidal intuitionistic fuzzy numbers by considering both power aggregation and the Heronian mean operator.

Vojinovic et al. [8] have developed the novel integrated Improved Fuzzy Stepwise Weight Assessment Ratio Analysis (IMF SWARA) method, Fuzzy Dombi weighted geometric averaging (FDWGA) operator and PESTEL. They have considered five decision-makers for evaluating six main elements of the PESTEL analysis and 30 elements more (five for each group). In total, they have created 35 models based on the developed model. Additionally, the usefulness of the developed integrated model has been demonstrated using a case example.

Riaz et al. [9] proposed two aggregation operators, namely picture fuzzy hybrid weighted arithmetic geometric aggregation (PFHWAGA) operator and picture fuzzy hybrid ordered weighted arithmetic geometric aggregation (PFHOWAGA) operator, and studied their mathematical properties. The proposed operators outperform the current PFN-defined operators. Further, they have shown the applicability of the proposed aggregation operators by solving an MCDM problem on third-party logistic provider selection. Sahu et al. [10] proposed two hybridization approaches based on the Hausdorff and Hamming distance measures. They demonstrated two case studies to validate the applicability of the proposed idea.

Zhou et al. [11] have used the hesitant fuzzy sets (HFSs) to depict the uncertainty in risk evaluation. Then, an improved HFWA (hesitant fuzzy weighted averaging) operator was adopted to fuse the risk evaluation for FMEA experts. Additionally, they have developed the novel HFWGA (hesitant fuzzy weighted geometric averaging) operator. Finally, they have solved a real example of the risk priority evaluation of power transformer parts to show the applicability and feasibility of the proposed hybrid FMEA framework.

Ali et al. [12] have proposed Einstein Geometric Aggregation Operators by using a Novel Complex Interval-valued Pythagorean Fuzzy Setting. They have applied the proposed model for solving the problem in Green Supplier Chain Management. Deveci et al. [13] proposed a novel extension of CoCoSo with the logarithmic method and the power Heronian function. Additionally, they have applied the proposed model to real-time traffic management problems. Deveci et al. [14] introduced an Ordinal Priority Approach (OPA) method for determining the criteria weights and application of a fuzzy Dombi Bonferroni (DOBI) methodology for the evaluation of alternatives.

Erdogan et al. [15] proposed hybrid power Heronian functions in which the linear normalization method is improved by applying the inverse sorting algorithm for rational and objective decision-making. Additionally, they have developed a new multi-criteria decision-making model to determine the best smart charging scheduling that meets electric vehicle (EV) user considerations at the work-places. Jeevaraj [16] has introduced the idea of interval-valued Fermatean fuzzy sets which is a generalization to many different generalized classes of fuzzy sets [17] and a total ordering principle on the class of IVFFNs by presenting four different score functions. Pratibha et al. [18] proposed a new score function for comparing arbitrary interval-valued Fermatean fuzzy numbers. Further, they have introduced a new interval-valued Fermatean fuzzy Einstein aggregation operator to combine various IVFFNs. Finally, an illustrative case study was discussed to assess the performance quality of the developed methodology. In addition, as the complexity of decision-making problems is increasing in the real world, we need to synchronously consider the following conditions in one decision-making problem to choose an optimal alternative. To alleviate these influences, we can select the PA operator to achieve this purpose by assigning the different weights generated by the support measures. We also consider the objective interrelationships between input values in certain cases, and then this function can be completed by the Heronian mean or Bonferroni mean ([19]). Since HM has some advantages over BM, however, we may expand HM to account for interactions.

The purpose of this paper is, therefore, to combine the PA operator and HM and extend them to trapezoidal intuitionist fuzzy environments and to, propose some of the power Heronian aggregation operators for trapezoidal intuitionistic fuzzy numbers (TrIFNs) and apply them to solve MAGDM problems to meet the two needs as mentioned earlier. The remainder of this paper is shown as follows to do:

1. We briefly study some basic concepts of the TrIFS, PA operator and HM in Section 2.
2. Section 3 suggests some of the power Heronian aggregation operators for TrIFNs and addresses some of these operators' useful properties and special cases.
3. We establish a Multi-attribute Group Decision-Making (MAGDM) algorithm in Section 4 based on the proposed operators.
4. To illustrate the validity of the proposed method, Section 5 gives a numerical example.
5. We give the concluding remarks in Section 6.

2. Preliminaries

Some basic definitions are given in this section. Here, we give a brief review of some preliminaries.

Definition 1 (Atanassov, [20])**.** *Consider A to be a set that is not empty. An intuitionistic fuzzy set (IFS) In in A is represented with $In = \{\langle a, y_{In}(a), z_{In}(a)\rangle | a \in A\}$, wheresoever $y_{In}(a) : A \to [0,1]$ and $z_{In}(a) : A \to [0,1], a \in A$ including the constraints $0 \leq y_{In}(a) + z_{In}(a) \leq 1, \forall a \in A$. The values $y_{In}(a)$ and $z_{In}(a)$ in the range $[0,1]$ signify the degree of membership and non-membership of a in In, correspondingly. The hesitation degree of a to lie in In is defined as $\pi_{In}(a) = 1 - y_{In}(a) - z_{In}(a)$ for any intuitionistic fuzzy subset In in A.*

Definition 2 (Grzegorzewski, [21]). *In the set of real numbers R, an intuitionistic fuzzy number* $In = (y_{In}, z_{In})$ *is described by*

$$y_{In}(a) = \begin{cases} k_{In}(a) & \text{when } p \leq a \leq q_1 \\ 1 & \text{when } q_1 \leq a \leq q_2 \\ l_{In}(a) & \text{when } q_2 \leq a \leq r \\ 0 & \text{for rest of the cases} \end{cases}$$

and

$$y_{In}(a) = \begin{cases} m_{In}(a) & \text{when } s \leq a \leq u_1 \\ 0 & \text{when } u_1 \leq a \leq u_2 \\ n_{In}(a) & \text{when } u_2 \leq a \leq v \\ 1 & \text{for rest of tha cases} \end{cases}$$

$0 \leq y_{In}(a) + z_{In}(a) \leq 1$ *is such that* $p, q_1, q_2, r, s, u_1, u_2, v \in \Re$, *and* $k_{In}, l_{In}, m_{In}, n_{In} : \Re \to [0,1]$ *is the legs of the membership function* y_{In} *and the nonmembership function* z_{In}. *Non-decreasing continuous functions* k_{In} *and* n_{In}, *as well as non-increasing continuous functions* m_{In} *and* l_{In}, *exist.*

An intuitionistic fuzzy number $\{(p, q_1, q_2, r), (s, u_1, u_2, v)\}$ *with* $(s, u_1, u_2, v) \leq (p, q_1, q_2, r)^c$ *is shown in Figure 1.*

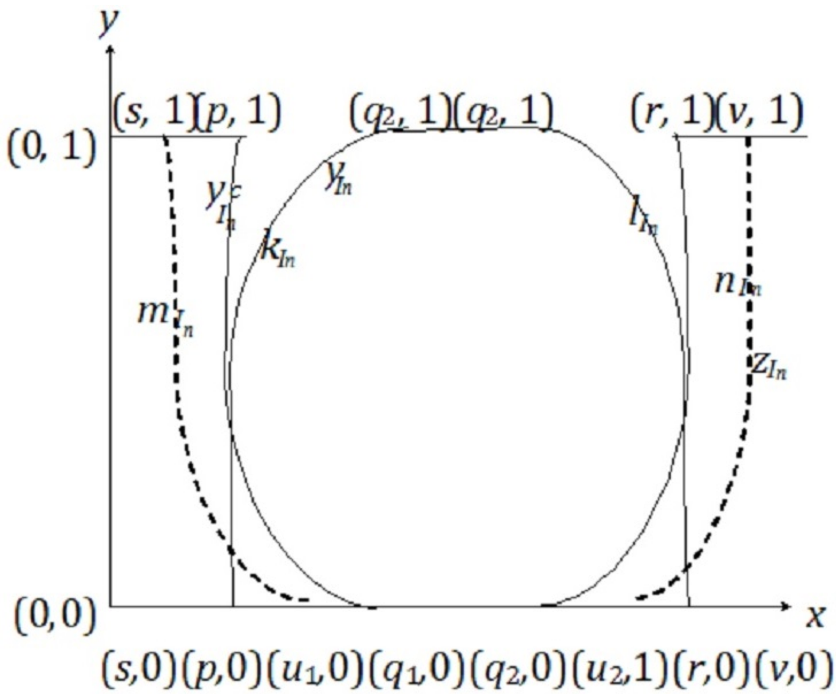

Figure 1. Intuitionistic fuzzy number.

Definition 3 (Nehi and Maleki, [22]). *In the set of real numbers* $\Re$, $Tr = \{(p, q_1, q_2, r), (s, u_1, u_2, v)\}$ *is an intuitionistic fuzzy set that is trapezoidal type Tr, which holds the* $s \leq p, u_1 \leq q_1 \leq q_2 \leq u_2, r \leq v$ *conditions. Below is its membership, and non-membership functions are given.*

$$y_{Tr}(a) = \begin{cases} \frac{a - tr_{11}}{tr_{12} - tr_{11}} & \text{whenever } tr_{11} \leq a \leq tr_{12} \\ 1 & \text{whenever } tr_{12} \leq a \leq tr_{13} \\ \frac{tr_{14} - a}{tr_{14} - tr_{13}} & \text{whenever } tr_{13} \leq a \leq tr_{14} \\ 0 & \text{for rest of the cases} \end{cases}$$

$$z_{Tr}(a) = \begin{cases} \frac{a - tr_{22}}{tr_{21} - tr_{22}} & \textit{whenever } tr_{21} \leq a \leq tr_{22} \\ 0 & \textit{whenever } tr_{22} \leq a \leq tr_{23} \\ \frac{a - tr_{23}}{tr_{24} - tr_{23}} & \textit{whenever } tr_{23} \leq a \leq tr_{24} \\ 1 & \textit{for rest of the cases} \end{cases}$$

The triangular intuitionistic fuzzy numbers are a special case of the trapezoidal intuitionistic fuzzy numbers if $tr_{12} = tr_{13}$ *(and* $tr_{22} = tr_{23}$*) in a trapezoidal intuitionistic fuzzy number Tr.*

In Figure 2, $Tr = \{(p, q_1, q_2, r), (s, u_1, u_2, v)\}$ *is an intuitionistic fuzzy set which is a trapezoidal type, which holds the* $u_1 \leq q_1$, $u_2 \geq q_2$, $s \leq p$, *and* $v \geq r$ *conditions.*

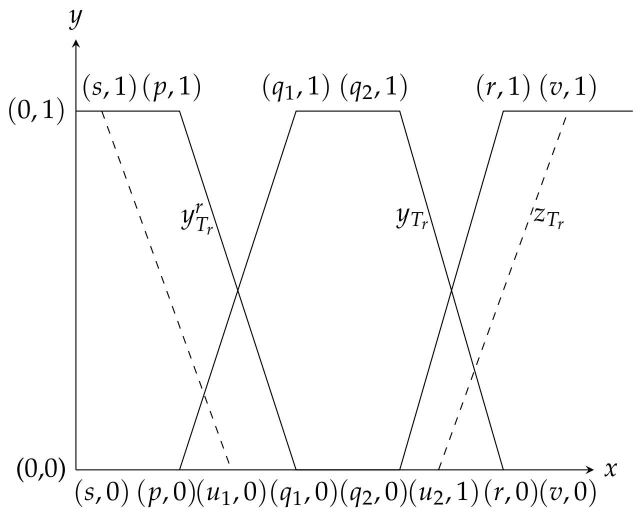

Figure 2. Intuitionistic fuzzy set of trapezoidal type.

We note that the condition $(s, u_1, u_2, v) \leq (p, q_1, q_2, r)^c$ of the trapezoidal intuitionistic fuzzy number $Tr = \{(p, q_1, q_2, r), (s, u_1, u_2, v)\}$ whose membership and nonmembership fuzzy numbers of Tr are (p, q_1, q_2, r) and (s, u_1, u_2, v) implies $u_1 \leq q_1$, $u_2 \geq q_2$, $s \leq p$, and $v \geq r$ on the legs of trapezoidal intuitionistic fuzzy number.

Definition 4 (Atanassov & Gargov, [23])**.** *Consider* $S[0,1]$ *to be the set among all closed subintervals of* $[0,1]$*. An interval valued intuitionistic fuzzy set on a set* $A \neq \phi$ *is provided by* $IV = \{\langle a, y_{IV}(a), z_{IV}(a)\rangle : a \in A\}$*, where* $y_{IV} : A \rightarrow S[0,1], z_{IV} : A \rightarrow S[0,1]$*, where* $0 < sup_a y_{IV}(a) + sup_a z_{IV}(a) \leq 1$ *is the condition.*

The $y_{IV}(a)$ and $z_{IV}(a)$ intervals express the degree of belongingness and non-belongingness of the element a to the set IV, respectively. $a \in A$, $y_{IV}(a)$ and $z_{IV}(a)$ are therefore closed intervals, with $y_{IV_L}(a)$, $z_{IV_U}(a)$ and $y_{IV_L}(x)$, $z_{IV_U}(a)$ denoting the lower and upper end points, respectively. We express $IV = \{\langle a, [y_{IV_L}(a), z_{IV_U}(a)], [y_{IV_L}(a), z_{IV_U}(a)]\rangle : a \in A\}$ wherever $0 < y_{IV}(a) + z_{IV}(a) \leq 1$.

We can calculate the unknown degree (hesitance degree) of belongingness $\pi_{IV}(a)$ to IV as $\pi_{IV}(a) = 1 - y_{IV}(a) - z_{IV}(a) = [1 - y_{IV_U}(a) - z_{IV_U}(a), 1 - y_{IV_L}(a) - z_{IV_L}(a)]$ for each element $a \in A$. For simplicity, an intuitionistic fuzzy interval number (IFIN) is indicated as $IV = ([p, q], [r, s])$.

Definition 5. *Assume* $\tilde{tr}_1 = ([p_1, q_{11}, q_{12}, r_1], [s_1, u_{11}, u_{12}, v_1])$*, and* $\tilde{tr}_2 = ([p_2, q_{21}, q_{22}, r_2], [s_2, u_{21}, u_{22}, v_2])$ *and* $\gamma \geq 0$*. Below, the TrIFNs operations are listed (Atanassov and Gargov [23], Jun Ye [24])*

$$\begin{aligned} \tilde{tr}_1 \bigoplus \tilde{tr}_2 = ([1 - (1 - p_1)(1 - p_2), 1 - (1 - q_{11})(1 - q_{21}), 1 - (1 - q_{12})(1 - q_{22}), \\ 1 - (1 - r_1)(1 - r_2)], [s_1 s_2, u_{11} u_{21}, u_{12} u_{22}, v_1 v_2]) \end{aligned} \tag{1}$$

$$\tilde{tr}_1 \otimes \tilde{tr}_2 = ([p_1p_2, q_{11}q_{21}, q_{12}q_{22}, r_1r_2], [1-(1-s_1)(1-s_2), 1-(1-u_{11})(1-u_{21}), 1-(1-u_{12})(1-u_{22}), 1-(1-v_1)(1-v_2)])) \tag{2}$$

$$\gamma\tilde{tr}_1 = ([1-(1-p_1)^\gamma, 1-(1-q_{11})^\gamma, 1-(1-q_{12})^\gamma, 1-(1-r_1)^\gamma], [s_1^\gamma, u_{11}^\gamma, u_{12}^\gamma, v_1^\gamma]) \tag{3}$$

$$\tilde{tr}_1{}^\gamma = ([p_1^\gamma, q_{11}^\gamma, q_{12}^\gamma, r_1^\gamma], [1-(1-s_1)^\gamma, 1-(1-u_{11})^\gamma, 1-(1-u_{12})^\gamma, 1-(1-v_1)^\gamma]) \tag{4}$$

Definition 6 (Nayagam et al., [6]). *Consider $\tilde{Tr}_I = ([p, q_1, q_2, r], [s, u_1, u_2, v])$ to be a TrIFN. Then, the membership (L), non-membership (LG), vague (P), imprecise (IP), widespread (WS), complete (J_6), comprehensive (J_7), and exact (J_8) score functions for TrIFN Tr_I are defined as follows:*

$$L(Tr_I) = \frac{(2(p+q_1+q_2+r) - 2(s+u_1+u_2+v) + (p+q_1)(s+u_1) + (q_2+r)(u_2+v))}{8}$$

$$LG(Tr_I) = \frac{(-2(p+q_1+q_2+r) + 2(s+u_1+u_2+v) + (p+q_1)(s+u_1) + (q_2+r)(u_2+v))}{8}$$

$$P(Tr_I) = \frac{(2(p+q_1) - 2(q_2+r) - 2(s+u_1) + 2(u_2+v) + (p+q_1)(s+u_1) + (q_2+r)(u_2+v))}{8}$$

$$IP(Tr_I) = \frac{(-2(p+q_1) + 2(q_2+r) - 2(s+u_1) + 2(u_2+v) - (p+q_1)(s+u_1) + (q_2+r)(u_2+v))}{8}$$

$$WS(Tr_I) = \frac{((p+q_2) - (q_1+r) + (s+u_2) - (u_1+v) + (p+q_2)(s+u_2) - (q_1+r)(u_1+v))}{8}$$

$$J_6(Tr_I) = \frac{((p+q_2) - (q_1+r) - (s+u_2) + (u_1+v) + (p+q_2)(u_1+v) - (q_1+r)(s+u_1))}{8}$$

$$J_7(Tr_I) = \frac{((q_2-p) + (q_1-r) - (u_2-s) - (u_1-v) - (p+r)(u_1+u_2) + (q_1+q_2)(s+v))}{8}$$

$$J_8(Tr_I) = \frac{((q_2-p) + (q_1-r) + (u_2-s) + (u_1-v) - (p+r)(s+v) + (q_1+q_2)(u_1+u_2))}{8}$$

Definition 7 (Nayagam et al., [6]). *(Ordering principle in the class of TrIFNs). Let $\tilde{Tr}_{1I} = ([p_1, q_{11}, q_{12}, r_1], [s_1, u_{11}, u_{12}, v_1])$ and $\tilde{Tr}_{2I} = ([p_2, q_{21}, q_{22}, r_2], [s_2, u_{21}, u_{22}, v_2])$ be two TrIFN. A relation 'Less than' ('<') denoted by $Tr_{1I} < Tr_{2I}$ on the entire class of TrIFNs is defined as follows:*

if $L(Tr_{1I}) < L(Tr_{2I})$ then $Tr_{1I} < Tr_{2I}$ or
if $L(Tr_{1I}) = L(Tr_{2I})$ and $LG(Tr_{1I}) > LG(Tr_{2I})$ then $Tr_{1I} < Tr_{2I}$ or
if $L(Tr_{1I}) = L(Tr_{2I})$, $LG(Tr_{1I}) = LG(Tr_{2I})$ and $P(Tr_{1I}) < P(Tr_{2I})$ then $Tr_{1I} < Tr_{2I}$ or
if $L(Tr_{1I}) = L(Tr_{2I})$, $LG(Tr_{1I}) = LG(Tr_{2I})$, $P(Tr_{1I}) = P(Tr_{2I})$ and $IP(Tr_{1I}) > IP(Tr_{2I})$ then $Tr_{1I} < Tr_{2I}$ or
if $L(Tr_{1I}) = L(Tr_{2I})$, $LG(Tr_{1I}) = LG(Tr_{2I})$, $P(Tr_{1I}) = P(Tr_{2I})$, $IP(Tr_{1I}) = IP(Tr_{2I})$ and $WS(Tr_{1I}) > WS(Tr_{2I})$ then $Tr_{1I} < Tr_{2I}$ or
if $L(Tr_{1I}) = L(Tr_{2I})$, $LG(Tr_{1I}) = LG(Tr_{2I})$, $P(Tr_{1I}) = P(Tr_{2I})$, $IP(Tr_{1I}) = IP(Tr_{2I})$, $WS(Tr_{1I}) = WS(Tr_{2I})$ and $J_6(Tr_{1I}) < J_6(Tr_{2I})$ then $Tr_{1I} < Tr_{2I}$ or
if $L(Tr_{1I}) = L(Tr_{2I})$, $LG(Tr_{1I}) = LG(Tr_{2I})$, $P(Tr_{1I}) = P(Tr_{2I})$, $IP(Tr_{1I}) = IP(Tr_{2I})$, $WS(Tr_{1I}) = WS(Tr_{2I})$, $J_6(Tr_{1I}) = J_6(Tr_{2I})$ and $J_7(Tr_{1I}) > J_7(Tr_{2I})$ then $Tr_{1I} < Tr_{2I}$ or
if $L(Tr_{1I}) = L(Tr_{2I})$, $LG(Tr_{1I}) = LG(Tr_{2I})$, $P(Tr_{1I}) = P(Tr_{2I})$, $IP(Tr_{1I}) = IP(Tr_{2I})$, $WS(Tr_{1I}) = WS(Tr_{2I})$, $J_6(Tr_{1I}) = J_6(Tr_{2I})$, $J_7(Tr_{1I}) = J_7(Tr_{2I})$ and $J_8(Tr_{1I}) < J_8(Tr_{2I})$ then $Tr_{1I} < Tr_{2I}$ or
if $L(Tr_{1I}) = L(Tr_{2I})$, $LG(Tr_{1I}) = LG(Tr_{2I})$, $P(Tr_{1I}) = P(Tr_{2I})$, $IP(Tr_{1I}) = IP(Tr_{2I})$, $WS(Tr_{1I}) = WS(Tr_{2I})$, $J_6(Tr_{1I}) = J_6(Tr_{2I})$, $J_7(Tr_{1I}) = J_7(Tr_{2I})$, $J_8(Tr_{1I}) = J_8(Tr_{2I})$ then $Tr_{1I} = Tr_{2I}$.

2.1. The Power Average Operator

The power average (PA), first proposed by Yager [25], is a useful aggregation operator that can mitigate some of the negative consequences of decision makers' overly large or small arguments. The classic PA, which is described as follows, may aggregate a collection of crisp integers where the weighting vectors solely depend on the input data.

Definition 8 (Yager [25])**.** *Consider $Tr = \{tr_a | a = 1, 2, \ldots, h\}$ to be a set of non-negative real numbers, and the power average (PA) operator is defined as*

$$PA(tr_1, tr_2, \ldots, tr_h) = \frac{\sum_{a=1}^{h}(1 + T(tr_a))tr_a}{\sum_{a=1}^{h}(1 + T(tr_a))} \tag{5}$$

where

$$T(tr_a) = \sum_{b=1, b\neq a} Sup(tr_a, tr_b) \tag{6}$$

and the support degree for tr_1 from tr_2 is $Sup(tr_1, tr_2)$. It has the properties listed below. (1) $Sup(tr_1, tr_2) \in [0,1]$; (2) $Sup(tr_1, tr_2) = Sup(tr_2, tr_1)$; (3) $Sup(tr_1, tr_2) \geq Sup(tr_3, tr_4)$, if $|tr_1 - tr_2| < |tr_3 - tr_4|$.

2.2. Heronian Mean (HM) Operator

The Heronian mean (HM) is a useful aggregation operator for capturing the interrelationships between the input parameters (Liu and Pei [26]). It can be defined as follows:

Definition 9 (Liu and Pei [26])**.** *Consider $I = [0,1]$, $e,f \geq 0$, $H^{e,f} : I^h \rightarrow I$, if $H^{e,f}$ satisfies:*

$$H^{e,f}(tr_1, tr_2, \ldots, tr_h) = \left(\frac{2}{h(h+1)} \sum_{a=1}^{h} \sum_{b=a}^{h} tr_a^e tr_b^f \right)^{\frac{1}{e+f}} \tag{7}$$

The Heronian mean (HM) operator with parameter is therefore defined as $H^{e,f}$.

The HM operator has been shown to have the properties of idempotency, monotonicity, and boundedness (Liu and Pei [26]).

3. The Trapezoidal Intuitionistic Fuzzy Power Heronian Aggregation Operators

According to operation rules defined for TrIFNs, We introduce the trapezoidal intuitionistic fuzzy power Heronian aggregation (TrIFPHA) operator and trapezoidal intuitionistic fuzzy power weighted Heronian aggregation (TrIFPWHA) operator in this section.

Definition 10. *Let $Tr = \{\tilde{tr}_b | b = 1, 2, \ldots, h\}$ (where $\tilde{tr}_b = ([p_b, q_{1b}, q_{2b}, r_b], [s_b, u_{1b}, u_{2b}, v_b])$) be the set of TrIFNs and $e, f \geq 0$, and $TrIFPHA : \theta^h \rightarrow \theta$, if*

$$TrIFPHA^{e,f}(\tilde{tr}_1, \tilde{tr}_2, \ldots, \tilde{tr}_h) = \left(\frac{2}{h(h+1)} \sum_{a=1}^{h} \sum_{b=a}^{h} \left(h \frac{(1+I(\tilde{tr}_a))}{\sum_{c=1}^{h}(1+I(\tilde{tr}_c))} \tilde{tr}_a \right)^e \otimes_H \left(h \frac{(1+I(\tilde{tr}_b))}{\sum_{c=1}^{h}(1+I(\tilde{tr}_c))} \tilde{tr}_b \right)^f \right)^{\frac{1}{e+f}} \tag{8}$$

where θ is the collection of each TrIFNs, $\overline{\theta_c} = \frac{(1+I(\tilde{tr}_1))}{\sum_{c=1}^{h}(1+I(\tilde{tr}_c))}$ and $\sum_{c=1}^{h} \overline{\theta_c} = 1$.

$I(\tilde{tr}_c) = \sum_{a=1, i\neq k}^{h} Sup(\tilde{tr}_c, \tilde{tr}_a)$, and the support degree for $\tilde{tr}_c$ from $\tilde{tr}_a$ is $Sup(\tilde{tr}_c, \tilde{tr}_a)$, which consist of the resulting properties. (1) $Sup(\tilde{tr}_c, \tilde{tr}_a) \in [0, 1]$; (2) $Sup(\tilde{tr}_c, \tilde{tr}_a) = Sup(\tilde{tr}_a, \tilde{tr}_c)$; (3) $Sup(\tilde{tr}_1, \tilde{tr}_2) \geq Sup(\tilde{a}, \tilde{b})$, if $d(\tilde{tr}_1, \tilde{tr}_2) < d(\tilde{a}, \tilde{b})$ in which $d(\tilde{tr}_1, \tilde{tr}_2)$ is the distance among TrIFNs $\tilde{tr}_1$ and $\tilde{tr}_2$. Therefore, TrIFPHA is called the Trapezoidal intuitionistic fuzzy power Heronian aggregation operator.

The expression (8) can be simplified. For that, we can determine

$$\overline{\theta_c} = \frac{(1 + I(\tilde{tr}_1))}{\sum_{c=1}^{h}(1 + I(\tilde{tr}_c))} \tag{9}$$

and call $(\overline{\theta_1}, \overline{\theta_2}, \ldots, \overline{\theta_h})$ a power weighting vector. Certainly, we hold $\overline{\theta_c} \geq 0, \sum_{c=1}^{h} \overline{\theta_c} = 1$. Thus, The expression (8) can be written as:

$$TrIFPHA^{e,f}(\tilde{tr}_1, \tilde{tr}_2, \ldots, \tilde{tr}_h) = \left(\frac{2}{h(h+1)} \sum_{a=1}^{h} \sum_{b=a}^{h} (h\overline{\theta_a}\tilde{tr}_a)^e \otimes_H (h\overline{\theta_b}\tilde{tr}_b)^f \right)^{\frac{1}{e+f}} \tag{10}$$

According to operation rules defined for TrINFS in Equations (1)–(4), Theorem 1's result is driven as shown below.

Theorem 1. *Let* $Tr = \{\tilde{tr}_b | b = 1, 2, \ldots, h\}$ *(where* $\tilde{tr}_b = ([p_b, q_{1b}, q_{2b}, r_b], [s_b, u_{1b}, u_{2b}, v_b])$*) be the set of TrIFNs and* $e, f \geq 0$*. Then, the trapezoidal intuitionistic fuzzy power Heronian aggregation operator (TrIFPHA) obtained by using Equation (10) is a TrIFN, and also*

$$
\begin{aligned}
TrIFPHA^{e,f}(\tilde{tr}_1, \tilde{tr}_2, \ldots, \tilde{tr}_h) = \Bigg(\Bigg[& \left(1 - \left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-(1-p_a)^{h\overline{\theta}_a})^e(1-(1-p_b)^{h\overline{\theta}_b})^f)\right)^{\frac{2}{h(h+1)}}\right)^{\frac{1}{e+f}}, \\
& \left(1 - \left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-(1-q_{1a})^{h\overline{\theta}_a})^e(1-(1-q_{1b})^{h\overline{\theta}_b})^f)\right)^{\frac{2}{h(h+1)}}\right)^{\frac{1}{e+f}}, \\
& \left(1 - \left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-(1-q_{2a})^{h\overline{\theta}_a})^e(1-(1-q_{2b})^{h\overline{\theta}_b})^f)\right)^{\frac{2}{h(h+1)}}\right)^{\frac{1}{e+f}}, \\
& \left(1 - \left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-(1-r_a)^{h\overline{\theta}_a})^e(1-(1-r_b)^{h\overline{\theta}_b})^f)\right)^{\frac{2}{h(h+1)}}\right)^{\frac{1}{e+f}} \Bigg], \\
& \Bigg[1 - \left(1 - \left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-s_a^{h\overline{\theta}_a})^e(1-s_b^{h\overline{\theta}_b})^f)\right)^{\frac{2}{h(h+1)}}\right)^{\frac{1}{e+f}}, \\
& 1 - \left(1 - \left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-u_{1a}^{h\overline{\theta}_a})^e(1-u_{1b}^{h\overline{\theta}_b})^f)\right)^{\frac{2}{h(h+1)}}\right)^{\frac{1}{e+f}}, \\
& 1 - \left(1 - \left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-u_{2a}^{h\overline{\theta}_a})^e(1-u_{2b}^{h\overline{\theta}_b})^f)\right)^{\frac{2}{h(h+1)}}\right)^{\frac{1}{e+f}}, \\
& 1 - \left(1 - \left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-v_a^{h\overline{\theta}_a})^e(1-v_b^{h\overline{\theta}_b})^f)\right)^{\frac{2}{h(h+1)}}\right)^{\frac{1}{e+f}} \Bigg] \Bigg)
\end{aligned} \tag{11}
$$

Proof. Let $Tr = \{\tilde{tr}_b | b = 1, 2, \ldots, h\}$ (where $\tilde{tr}_b = ([p_b, q_{1b}, q_{2b}, r_b], [s_b, u_{1b}, u_{2b}, v_b])$) be the set of TrIFNs and $e, f \geq 0$. By using Equations (1)–(4), we obtain

$$h\overline{\theta}_a\tilde{tr}_a = ([1-(1-p_a)^{h\overline{\theta}_a}, 1-(1-q_{1a})^{h\overline{\theta}_a}, 1-(1-q_{2a})^{h\overline{\theta}_a}, 1-(1-r_a)^{h\overline{\theta}_a}], [s_a^{h\overline{\theta}_a}, u_{1a}^{h\overline{\theta}_a}, u_{2a}^{h\overline{\theta}_a}, v_a^{h\overline{\theta}_a}])$$

So, $(h\overline{\theta}_a\tilde{tr}_a)^e = ([(1-(1-p_a)^{h\overline{\theta}_a})^e, (1-(1-q_{1a})^{h\overline{\theta}_a})^e, (1-(1-q_{2a})^{h\overline{\theta}_a})^e, (1-(1-r_a)^{h\overline{\theta}_a})^e], [1-(1-s_a^{h\overline{\theta}_a})^e, 1-(1-u_{1a}^{h\overline{\theta}_a})^e, 1-(1-u_{2a}^{h\overline{\theta}_a})^e, 1-(1-v_a^{h\overline{\theta}_a})^e])$.

Furthermore, we hold, $(h\overline{\theta}_b\tilde{tr}_b)^f = ([(1-(1-p_b)^{h\overline{\theta}_b})^f, (1-(1-q_{1b})^{h\overline{\theta}_b})^f, (1-(1-q_{2b})^{h\overline{\theta}_b})^f, (1-(1-r_b)^{h\overline{\theta}_b})^f], [1-(1-s_b^{h\overline{\theta}_b})^f, 1-(1-u_{1b}^{h\overline{\theta}_b})^f, 1-(1-u_{2b}^{h\overline{\theta}_b})^f, 1-(1-v_b^{h\overline{\theta}_b})^f])$

Thus, we have $(h\overline{\theta}_a\tilde{tr}_a)^e \otimes_H (h\overline{\theta}_b\tilde{tr}_b)^f = ([(1-(1-p_a)^{h\overline{\theta}_a})^e(1-(1-p_b)^{h\overline{\theta}_b})^f, (1-(1-q_{1a})^{h\overline{\theta}_a})^e(1-(1-q_{1b})^{h\overline{\theta}_b})^f, (1-(1-q_{2a})^{h\overline{\theta}_a})^e(1-(1-q_{2b})^{h\overline{\theta}_b})^f, (1-(1-r_a)^{h\overline{\theta}_a})^e(1-(1-r_b)^{h\overline{\theta}_b})^f], [1-(1-s_a^{h\overline{\theta}_a})^e(1-s_b^{h\overline{\theta}_b})^f, 1-(1-u_{1a}^{h\overline{\theta}_a})^e(1-u_{1b}^{h\overline{\theta}_b})^f, 1-(1-u_{2a}^{h\overline{\theta}_a})^e(1-u_{2b}^{h\overline{\theta}_b})^f, 1-(1-v_a^{h\overline{\theta}_a})^e(1-v_b^{h\overline{\theta}_b})^f])$

also

$$\sum_{a=1}^{h}\sum_{b=a}^{h}((h\bar{\theta}_a t\tilde{r}_a)^e \otimes_H (h\bar{\theta}_b t\tilde{r}_b)^f) =$$
$$\Bigg(\Bigg[1-\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-(1-p_a)^{h\bar{\theta}_a})^e(1-(1-p_b)^{h\bar{\theta}_b})^f),$$
$$1-\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-(1-q_{1a})^{h\bar{\theta}_a})^e(1-(1-q_{1b})^{h\bar{\theta}_b})^f),$$
$$1-\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-(1-q_{2a})^{h\bar{\theta}_a})^e(1-(1-q_{2b})^{h\bar{\theta}_b})^f),$$
$$1-\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-(1-r_a)^{h\bar{\theta}_a})^e(1-(1-r_b)^{h\bar{\theta}_b})^f)\Bigg],$$
$$\Bigg[\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-s_a^{h\bar{\theta}_a})^e(1-s_b^{h\bar{\theta}_b})^f),\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-u_{1a}^{h\bar{\theta}_a})^e(1-u_{1b}^{h\bar{\theta}_b})^f),$$
$$\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-u_{2a}^{h\bar{\theta}_a})^e(1-u_{2b}^{h\bar{\theta}_b})^f),\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-v_a^{h\bar{\theta}_a})^e(1-v_b^{h\bar{\theta}_b})^f)\Bigg]\Bigg)$$
$$\Rightarrow \frac{2}{h(h+1)}\sum_{a=1}^{h}\sum_{b=a}^{h}((h\bar{\theta}_a t\tilde{r}_a)^e \otimes_H (h\bar{\theta}_b t\tilde{r}_b)^f) =$$
$$\Bigg(\Bigg[1-\left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-(1-p_a)^{h\bar{\theta}_a})^e(1-(1-p_b)^{h\bar{\theta}_b})^f)\right)^{\frac{2}{h(h+1)}},$$
$$1-\left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-(1-q_{1a})^{h\bar{\theta}_a})^e(1-(1-q_{1b})^{h\bar{\theta}_b})^f)\right)^{\frac{2}{h(h+1)}},$$
$$1-\left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-(1-q_{2a})^{h\bar{\theta}_a})^e(1-(1-q_{2b})^{h\bar{\theta}_b})^f)\right)^{\frac{2}{h(h+1)}},$$
$$1-\left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-(1-r_a)^{h\bar{\theta}_a})^e(1-(1-r_b)^{h\bar{\theta}_b})^f)\right)^{\frac{2}{h(h+1)}}\Bigg],$$
$$\Bigg[\left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-s_a^{h\bar{\theta}_a})^e(1-s_b^{h\bar{\theta}_b})^f)\right)^{\frac{2}{h(h+1)}},$$
$$\left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-u_{1a}^{h\bar{\theta}_a})^e(1-u_{1b}^{h\bar{\theta}_b})^f)\right)^{\frac{2}{h(h+1)}},$$
$$\left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-u_{2a}^{h\bar{\theta}_a})^e(1-u_{2b}^{h\bar{\theta}_b})^f)\right)^{\frac{2}{h(h+1)}},$$
$$\left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-v_a^{h\bar{\theta}_a})^e(1-v_b^{h\bar{\theta}_b})^f)\right)^{\frac{2}{h(h+1)}}\Bigg]\Bigg)$$

Therefore,

$$\left(\frac{2}{h(h+1)}\sum_{a=1}^{h}\sum_{b=a}^{h}((h\bar{\theta}_a\tilde{tr}_a)^e \otimes_h (h\bar{\theta}_b\tilde{tr}_b)^f)\right)^{\frac{1}{e+f}} =$$

$$\left(\left[\left(1-\left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-(1-p_a)^{h\bar{\theta}_a})^e(1-(1-p_b)^{h\bar{\theta}_b})^f)\right)^{\frac{2}{h(h+1)}}\right)^{\frac{1}{e+f}},\right.\right.$$

$$\left(1-\left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-(1-q_{1a})^{h\bar{\theta}_a})^e(1-(1-q_{1b})^{h\bar{\theta}_b})^f)\right)^{\frac{2}{h(h+1)}}\right)^{\frac{1}{e+f}},$$

$$\left(1-\left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-(1-q_{2a})^{h\bar{\theta}_a})^e(1-(1-q_{2b})^{h\bar{\theta}_b})^f)\right)^{\frac{2}{h(h+1)}}\right)^{\frac{1}{e+f}},$$

$$\left.\left(1-\left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-(1-r_a)^{h\bar{\theta}_a})^e(1-(1-r_b)^{h\bar{\theta}_b})^f)\right)^{\frac{2}{h(h+1)}}\right)^{\frac{1}{e+f}}\right],$$

$$\left[1-\left(1-\left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-s_a^{h\bar{\theta}_a})^e(1-s_b^{h\bar{\theta}_b})^f)\right)^{\frac{2}{h(h+1)}}\right)^{\frac{1}{e+f}},\right.$$

$$1-\left(1-\left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-u_{1a}^{h\bar{\theta}_a})^e(1-u_{1b}^{h\bar{\theta}_b})^f)\right)^{\frac{2}{h(h+1)}}\right)^{\frac{1}{e+f}},$$

$$1-\left(1-\left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-u_{2a}^{h\bar{\theta}_a})^e(1-u_{2b}^{h\bar{\theta}_b})^f)\right)^{\frac{2}{h(h+1)}}\right)^{\frac{1}{e+f}},$$

$$\left.\left.1-\left(1-\left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-v_a^{h\bar{\theta}_a})^e(1-v_b^{h\bar{\theta}_b})^f)\right)^{\frac{2}{h(h+1)}}\right)^{\frac{1}{e+f}}\right]\right)$$

□

We have to calculate the support degree $Sup(\tilde{tr}_c, \tilde{tr}_a)$ to determine the power weighted vector $\bar{\theta}$. Usually, the similarity degree among $\tilde{tr}_c$ and $\tilde{tr}_a$ can equivalently represented as the support degree $Sup(\tilde{tr}_c, \tilde{tr}_a)$. Various similarity measures are available on the class of intuitionistic fuzzy numbers and IVIFNs [27].

Let us consider two IFNs as $\tilde{tr}_1 = (y_1, z_1)$ *and* $\tilde{tr}_2 = (y_2, z_2)$. Different similarity measures between the two IFNs are given below.

(1) Chen's [28] similarity definition is stated as mentioned below.

$$S_0 = 1 - \frac{1}{2}(|y_1 - y_2| + |z_1 - z_2|) \tag{12}$$

(2) Song et al.'s [29] similarity definition is stated as mentioned below.

$$S_1 = \frac{1}{2}\left(\sqrt{y_1 y_2} + 2\sqrt{z_1 z_2} + \sqrt{\pi_1 \pi_2} + \sqrt{(1-z_1)(1-z_2)}\right) \tag{13}$$

(3) Nguyen's [30] similarity definition is stated as mentioned below.

$$S_2 = 1 - |K_F(\tilde{tr}_1) - K_F(\tilde{tr}_2)| \tag{14}$$

where, the knowledge measures of $\tilde{tr}_1$ and $\tilde{tr}_2$ are $K_F(\tilde{tr}_1)$ and $K_F(\tilde{tr}_2)$, each.

$$K_F(\tilde{tr}_1) = \sqrt{\frac{y_1^2 + z_1^2 + (1-\pi_1)^2}{2}}, K_F(\tilde{tr}_2) = \sqrt{\frac{y_2^2 + z_2^2 + (1-\pi_2)^2}{2}} \tag{15}$$

The above-mentioned similarity definitions can be extended to TrIFNs using the fuzzy extension principle. Below are the definitions mentioned for them.

Consider $\tilde{tr}_1 = ([p_1, q_{11}, q_{12}, r_1], [s_1, u_{11}, u_{12}, v_1])$ and $\tilde{tr}_2 = ([p_2, q_{21}, q_{22}, r_2], [s_2, u_{21}, u_{22}, v_2])$ be any two trapezoidal fuzzy (intuitionistic type) numbers, thus the definitions are as follows

$$S_0 = 1 - \frac{1}{4}\Big(|p_1 - p_2| + |q_{11} - q_{21}| + |q_{12} - q_{22}| + |r_1 - r_2| + |s_1 - s_2| + |u_{11} - u_{21}| + |u_{12} - u_{22}| + |v_1 - v_2|\Big) \tag{16}$$

$$\begin{aligned} S_1 = \frac{1}{4}\Big(&\sqrt{p_1 p_2} + \sqrt{q_{11} q_{21}} + \sqrt{q_{12} q_{22}} + \sqrt{r_1 r_2} + 2\sqrt{s_1 s_2} + 2\sqrt{u_{11} u_{21}} + 2\sqrt{u_{12} u_{22}} + 2\sqrt{v_1 v_2} \\ &+ \sqrt{(1 - p_1 - s_1)(1 - p_2 - s_2)} + \sqrt{(1 - q_{11} - u_{11})(1 - q_{21} - u_{21})} \\ &+ \sqrt{(1 - q_{12} - u_{12})(1 - q_{22} - u_{22})} + \sqrt{(1 - r_1 - v_1)(1 - r_2 - v_2)}\Big) \end{aligned} \tag{17}$$

$$S_2 = 1 - |K_f(\tilde{tr}_1) - K_f(\tilde{tr}_2)| \tag{18}$$

where, the knowledge measures of $\tilde{tr}_1$ and $\tilde{tr}_2$ are $K_F(\tilde{tr}_1)$ and $K_F(\tilde{tr}_2)$, each.

$$K_f(\tilde{tr}_1) = \sqrt{\frac{p_1^2 + q_{11}^2 + q_{12}^2 + r_1^2 + s_1^2 + u_{11}^2 + u_{12}^2 + v_1^2 + (p_1 + s_1)^2 + (q_{11} + u_{11})^2 + (q_{12} + u_{12})^2 + (r_1 + v_1)^2}{4}} \tag{19}$$

$$K_f(\tilde{tr}_2) = \sqrt{\frac{p_2^2 + q_{21}^2 + q_{22}^2 + r_2^2 + s_2^2 + u_{21}^2 + u_{22}^2 + v_2^2 + (p_2 + s_2)^2 + (q_{21} + u_{21})^2 + (q_{22} + u_{22})^2 + (r_2 + v_2)^2}{4}} \tag{20}$$

In this paper, we use a similarity measure between $\tilde{tr}_c$, and $\tilde{tr}_a$ for finding $Sup(\tilde{tr}_c, \tilde{tr}_a)$. Further, Some TrIFPHA operator's properties will be discussed.

Theorem 2 (Idempotency). *Let* $\tilde{Tr} = \{\tilde{tr}_b | \, for\ all\ b = 1, \ldots, h\}$ *be a collection of TrIFNs, and* $\tilde{tr}_b = \tilde{tr}$ *for each* $b = 1, 2, \ldots, h$ *where* $\tilde{tr} = ([p, q_1, q_2, r], [s, u_1, u_2, v])$. *Then,* $TrIFPHA^{e,f}(\tilde{tr}_1, \tilde{tr}_2, \ldots, \tilde{tr}_h) = \tilde{tr}$.

Proof. From our assumption, we have, $\tilde{tr}_b = \tilde{tr} = ([p, q_1, q_2, r], [s, u_1, u_2, v])$ for all $b = 1, 2, \ldots, h$.

$$Sup(\tilde{tr}_c, \tilde{tr}_a) = 1 \text{ for all } a, c = 1, 2, \ldots, h \tag{21}$$

(Since $Sup(\tilde{tr}_c, \tilde{tr}_a)$ is replaced with a similarity measure between $\tilde{tr}_c$ and $\tilde{tr}_a$).
Equation (21) and Definition 10 imply that, $\overline{\theta_c} = \frac{1}{h}$ for all $c = 1, 2, \ldots, h$
Then, from Definition 10, we obtain

$$TrIFPHA^{e,f}(\tilde{tr}_1, \tilde{tr}_2, \ldots, \tilde{tr}_h) = TrIFPHA^{e,f}(\tilde{tr}, \tilde{tr}, \ldots, \tilde{tr})$$

$$\begin{aligned} = \Bigg(\Bigg[&\left(1 - \left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1 - (1 - (1 - p)^{h\frac{1}{h}})^e (1 - (1 - p)^{h\frac{1}{h}})^f)\right)^{\frac{2}{h(h+1)}}\right)^{\frac{1}{e+f}}, \\ &\left(1 - \left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1 - (1 - (1 - q_1)^{h\frac{1}{h}})^e (1 - (1 - q_1)^{h\frac{1}{h}})^f)\right)^{\frac{2}{h(h+1)}}\right)^{\frac{1}{e+f}}, \\ &\left(1 - \left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1 - (1 - (1 - q_2)^{h\frac{1}{h}})^e (1 - (1 - q_2)^{h\frac{1}{h}})^f)\right)^{\frac{2}{h(h+1)}}\right)^{\frac{1}{e+f}}, \\ &\left(1 - \left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1 - (1 - (1 - r)^{h\frac{1}{h}})^e (1 - (1 - r)^{h\frac{1}{h}})^f)\right)^{\frac{2}{h(h+1)}}\right)^{\frac{1}{e+f}}\Bigg], \end{aligned}$$

$$\left[1-\left(1-\left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-s^{h\frac{1}{h}})^{e}(1-s^{h\frac{1}{h}})^{f})\right)^{\frac{2}{h(h+1)}}\right)^{\frac{1}{e+f}},\right.$$
$$1-\left(1-\left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-u_1^{h\frac{1}{h}})^{e}(1-u_1^{h\frac{1}{h}})^{f})\right)^{\frac{2}{h(h+1)}}\right)^{\frac{1}{e+f}},$$
$$1-\left(1-\left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-u_2^{h\frac{1}{h}})^{e}(1-u_2^{h\frac{1}{h}})^{f})\right)^{\frac{2}{h(h+1)}}\right)^{\frac{1}{e+f}},$$
$$\left.1-\left(1-\left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-v^{h\frac{1}{h}})^{e}(1-v^{h\frac{1}{h}})^{f})\right)^{\frac{2}{h(h+1)}}\right)^{\frac{1}{e+f}}\right]\Bigg)$$
$$=\Bigg(\Bigg[\left(1-\left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-p^{e+f})\right)^{\frac{2}{h(h+1)}}\right)^{\frac{1}{e+f}},\left(1-\left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-q_1^{e+f})\right)^{\frac{2}{h(h+1)}}\right)^{\frac{1}{e+f}},$$
$$\left.\left(1-\left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-q_2^{e+f})\right)^{\frac{2}{h(h+1)}}\right)^{\frac{1}{e+f}},\left(1-\left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-r^{e+f})\right)^{\frac{2}{h(h+1)}}\right)^{\frac{1}{e+f}}\right],$$
$$\left[1-\left(1-\left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-s)^{e+f})\right)^{\frac{2}{h(h+1)}}\right)^{\frac{1}{e+f}},1-\left(1-\left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-\right.\right.\right.$$
$$\left.\left.u_1)^{e+f})\right)^{\frac{2}{h(h+1)}}\right)^{\frac{1}{e+f}},1-\left(1-\left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-u_2)^{e+f})\right)^{\frac{2}{h(h+1)}}\right)^{\frac{1}{e+f}},$$
$$\left.1-\left(1-\left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-v)^{e+f})\right)^{\frac{2}{h(h+1)}}\right)^{\frac{1}{e+f}}\right]\Bigg)$$
$$=\Bigg(\Bigg[\left(1-\left((1-p^{e+f})\right)\right)^{\frac{1}{e+f}},\left(1-\left((1-q_1^{e+f})\right)\right)^{\frac{1}{e+f}},$$
$$\left.\left(1-\left((1-q_2^{e+f})\right)^{\frac{2}{h(h+1)}}\right),\left(1-\left((1-r^{e+f})\right)\right)^{\frac{1}{e+f}}\right],$$
$$\left[1-\left(1-\left((1-(1-s)^{e+f})\right)\right)^{\frac{1}{e+f}},1-\left(1-\left((1-(1-u_1)^{e+f})\right)\right)^{\frac{1}{e+f}},\right.$$
$$\left.1-\left(1-\left((1-(1-u_2)^{e+f})\right)\right)^{\frac{1}{e+f}},1-\left(1-\left((1-(1-v)^{e+f})\right)\right)^{\frac{1}{e+f}}\right]\Bigg)$$
$=([p,q_1,q_2,r],[s,u_1,u_2,v])$.

Hence the proof. □

Theorem 3 (Boundedness)**.** *Let* $\tilde{Tr}=\{\tilde{tr}_1,\tilde{tr}_2,\ldots,\tilde{tr}_h\}$ *be a set of TrIFNs, and* $\tilde{tr}_l=min(\tilde{tr}_1,\tilde{tr}_2,\ldots,\tilde{tr}_h)=([p,q_1,q_2,r],[s,u_1,u_2,v])$, $\tilde{tr}_m=max(\tilde{tr}_1,\tilde{tr}_2,\ldots,\tilde{tr}_h)=([\overline{p},\overline{q_1},\overline{q_2},\overline{r}],[\overline{s},\overline{u_1},\overline{u_2},\overline{v}])$.

Then, the following condition holds:

$\tilde{l}\leq TrIFPHA^{e,f}(\tilde{tr}_1,\tilde{tr}_2,\ldots,\tilde{tr}_h)\leq\tilde{m}$

In which $\tilde{l}=\Bigg(\Bigg[\left(1-\left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-(1-p)^{h\overline{\theta_a}})^{e}(1-(1-p)^{h\overline{\theta_b}})^{f})\right)^{\frac{2}{h(h+1)}}\right)^{\frac{1}{e+f}},$
$$\left(1-\left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-(1-q_1)^{h\overline{\theta_a}})^{e}(1-(1-q_1)^{h\overline{\theta_b}})^{f})\right)^{\frac{2}{h(h+1)}}\right)^{\frac{1}{e+f}},$$
$$\left(1-\left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-(1-q_2)^{h\overline{\theta_a}})^{e}(1-(1-q_2)^{h\overline{\theta_b}})^{f})\right)^{\frac{2}{h(h+1)}}\right)^{\frac{1}{e+f}},$$

$$\left(1-\left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-(1-r)^{h\overline{\theta_a}})^e(1-(1-r)^{h\overline{\theta_b}})^f)\right)^{\frac{2}{h(h+1)}}\right)^{\frac{1}{e+f}}\Bigg],$$

$$\Bigg[1-\left(1-\left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-s^{h\overline{\theta_a}})^e(1-s^{h\overline{\theta_b}})^f)\right)^{\frac{2}{h(h+1)}}\right)^{\frac{1}{e+f}},$$

$$1-\left(1-\left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-u_1^{h\overline{\theta_a}})^e(1-u_1^{h\overline{\theta_b}})^f)\right)^{\frac{2}{h(h+1)}}\right)^{\frac{1}{e+f}},$$

$$1-\left(1-\left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-u_2^{h\overline{\theta_a}})^e(1-u_2^{h\overline{\theta_b}})^f)\right)^{\frac{2}{h(h+1)}}\right)^{\frac{1}{e+f}},$$

$$1-\left(1-\left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-v^{h\overline{\theta_a}})^e(1-v^{h\overline{\theta_b}})^f)\right)^{\frac{2}{h(h+1)}}\right)^{\frac{1}{e+f}}\Bigg]\Bigg)$$

$$\text{and}\quad \tilde{m}=\Bigg(\Bigg[\left(1-\left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-(1-\overline{p})^{h\overline{\theta_a}})^e(1-(1-\overline{p})^{h\overline{\theta_b}})^f)\right)^{\frac{2}{h(h+1)}}\right)^{\frac{1}{e+f}},$$

$$\left(1-\left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-(1-\overline{q_1})^{h\overline{\theta_a}})^e(1-(1-\overline{q_1})^{h\overline{\theta_b}})^f)\right)^{\frac{2}{h(h+1)}}\right)^{\frac{1}{e+f}},$$

$$\left(1-\left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-(1-\overline{q_2})^{h\overline{\theta_a}})^e(1-(1-\overline{q_2})^{h\overline{\theta_b}})^f)\right)^{\frac{2}{h(h+1)}}\right)^{\frac{1}{e+f}},$$

$$\left(1-\left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-(1-\overline{r})^{h\overline{\theta_a}})^e(1-(1-\overline{r})^{h\overline{\theta_b}})^f)\right)^{\frac{2}{h(h+1)}}\right)^{\frac{1}{e+f}}\Bigg],$$

$$\Bigg[1-\left(1-\left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-\overline{s}^{h\overline{\theta_a}})^e(1-\overline{s}^{h\overline{\theta_b}})^f)\right)^{\frac{2}{h(h+1)}}\right)^{\frac{1}{e+f}},$$

$$1-\left(1-\left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-\overline{u_1}^{h\overline{\theta_a}})^e(1-\overline{u_1}^{h\overline{\theta_b}})^f)\right)^{\frac{2}{h(h+1)}}\right)^{\frac{1}{e+f}},$$

$$1-\left(1-\left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-\overline{u_2}^{h\overline{\theta_a}})^e(1-\overline{u_2}^{h\overline{\theta_b}})^f)\right)^{\frac{2}{h(h+1)}}\right)^{\frac{1}{e+f}},$$

$$1-\left(1-\left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-\overline{v}^{h\overline{\theta_a}})^e(1-\overline{v}^{h\overline{\theta_b}})^f)\right)^{\frac{2}{h(h+1)}}\right)^{\frac{1}{e+f}}\Bigg]\Bigg)$$

Proof. From the Definition 10, we obtain

$$\begin{aligned} h\overline{\theta_a}t\tilde{r}_a &= ([1-(1-p_a)^{h\overline{\theta_a}},1-(1-q_{1a})^{h\overline{\theta_a}},1-(1-q_{2a})^{h\overline{\theta_a}},1-(1-r_a)^{h\overline{\theta_a}}],[s_a^{h\overline{\theta_a}},u_{1a}^{h\overline{\theta_a}},u_{2a}^{h\overline{\theta_a}},v_a^{h\overline{\theta_a}}]) \\ &\geq ([1-(1-p)^{h\overline{\theta_a}},1-(1-q_1)^{h\overline{\theta_a}},1-(1-q_2)^{h\overline{\theta_a}},1-(1-r)^{h\overline{\theta_a}}],[s^{h\overline{\theta_a}},u_1^{h\overline{\theta_a}},u_2^{h\overline{\theta_a}},v^{h\overline{\theta_a}}]) \end{aligned}$$

$\Rightarrow (h\overline{\theta_a}t\tilde{r}_a)^e = ([(1-(1-p_a)^{h\overline{\theta_a}})^e,(1-(1-q_{1a})^{h\overline{\theta_a}})^e,(1-(1-q_{2a})^{h\overline{\theta_a}})^e,(1-(1-r_a)^{h\overline{\theta_a}})^e],[1-(1-s_a^{h\overline{\theta_a}})^e,1-(1-u_{1a}^{h\overline{\theta_a}})^e,1-(1-u_{2a}^{h\overline{\theta_a}})^e,1-(1-v_a^{h\overline{\theta_a}})^e])$
$\geq ([(1-(1-p)^{h\overline{\theta_a}})^e,(1-(1-q_1)^{h\overline{\theta_a}})^e,(1-(1-q_2)^{h\overline{\theta_a}})^e,(1-(1-r)^{h\overline{\theta_a}})^e],[1-(1-s^{h\overline{\theta_a}})^e,1-(1-u_1^{h\overline{\theta_a}})^e,1-(1-u_2^{h\overline{\theta_a}})^e,1-(1-v^{h\overline{\theta_a}})^e])$

Likewise, we obtain

$(h\overline{\theta_b}t\tilde{r}_b)^f = ([(1-(1-p_b)^{h\overline{\theta_b}})^f,(1-(1-q_{1b})^{h\overline{\theta_b}})^f,(1-(1-q_{2b})^{h\overline{\theta_b}})^f,(1-(1-r_b)^{h\overline{\theta_b}})^f],$
$[1-(1-s_b^{h\overline{\theta_b}})^f,1-(1-u_{1b}^{h\overline{\theta_b}})^f,1-(1-u_{2b}^{h\overline{\theta_b}})^f,1-(1-v_b^{h\overline{\theta_b}})^f])$
$\geq ([(1-(1-p)^{h\overline{\theta_b}})^f,(1-(1-q_1)^{h\overline{\theta_b}})^f,(1-(1-q_2)^{h\overline{\theta_b}})^f,(1-(1-r)^{h\overline{\theta_b}})^f],[1-(1-s^{h\overline{\theta_b}})^f,1-(1-u_1^{h\overline{\theta_b}})^f,1-(1-u_2^{h\overline{\theta_b}})^f,1-(1-v^{h\overline{\theta_b}})^f])$

Thus,

$(h\overline{\theta_a}t\tilde{r}_a)^e \otimes_H (h\overline{\theta_b}t\tilde{r}_b)^f = \Big([(1-(1-p_a)^{h\overline{\theta_a}})^e(1-(1-p_b)^{h\overline{\theta_b}})^f, (1-(1-q_{1a})^{h\overline{\theta_a}})^e(1-(1-q_{1b})^{h\overline{\theta_b}})^f, (1-(1-q_{2a})^{h\overline{\theta_a}})^e(1-(1-q_{2b})^{h\overline{\theta_b}})^f, (1-(1-r_a)^{h\overline{\theta_a}})^e(1-(1-r_b)^{h\overline{\theta_b}})^f],$
$[1-(1-s_a^{h\overline{\theta_a}})^e(1-s_b^{h\overline{\theta_b}})^f, 1-(1-u_{1a}^{h\overline{\theta_a}})^e(1-u_{1b}^{h\overline{\theta_b}})^f, 1-(1-u_{2a}^{h\overline{\theta_a}})^e(1-u_{2b}^{h\overline{\theta_b}})^f, 1-(1-v_a^{h\overline{\theta_a}})^e(1-v_b^{h\overline{\theta_b}})^f] \Big) \geq \Big([(1-(1-p)^{h\overline{\theta_a}})^e(1-(1-p)^{h\overline{\theta_b}})^f, (1-(1-q_1)^{h\overline{\theta_a}})^e(1-(1-q_1)^{h\overline{\theta_b}})^f, (1-(1-q_2)^{h\overline{\theta_a}})^e(1-(1-q_2)^{h\overline{\theta_b}})^f, (1-(1-r)^{h\overline{\theta_a}})^e(1-(1-r)^{h\overline{\theta_b}})^f], [1-(1-s^{h\overline{\theta_a}})^e(1-s^{h\overline{\theta_b}})^f, 1-(1-u_1^{h\overline{\theta_a}})^e(1-u_1^{h\overline{\theta_b}})^f, 1-(1-u_2^{h\overline{\theta_a}})^e(1-u_2^{h\overline{\theta_b}})^f, 1-(1-v^{h\overline{\theta_a}})^e(1-v^{h\overline{\theta_b}})^f] \Big)$

Additionally, we obtain

$$
\begin{aligned}
&\textstyle\sum_{a=1}^{h}\sum_{b=a}^{h}((\overline{\theta_a}t\tilde{r}_a)^e \otimes_H (\overline{\theta_b}t\tilde{r}_b)^f) \\
&= \Bigg(\Bigg[1-\textstyle\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-(1-p_a)^{h\overline{\theta_a}})^e(1-(1-p_b)^{h\overline{\theta_b}})^f), \\
&\quad 1-\textstyle\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-(1-q_{1a})^{h\overline{\theta_a}})^e(1-(1-q_{1b})^{h\overline{\theta_b}})^f), \\
&\quad 1-\textstyle\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-(1-q_{2a})^{h\overline{\theta_a}})^e(1-(1-q_{2b})^{h\overline{\theta_b}})^f), \\
&\quad 1-\textstyle\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-(1-r_a)^{h\overline{\theta_a}})^e(1-(1-r_b)^{h\overline{\theta_b}})^f) \Bigg], \\
&\Bigg[\textstyle\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-s_a^{h\overline{\theta_a}})^e(1-s_b^{h\overline{\theta_b}})^f), \prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-u_{1a}^{h\overline{\theta_a}})^e(1-u_{1b}^{h\overline{\theta_b}})^f), \\
&\textstyle\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-u_{2a}^{h\overline{\theta_a}})^e(1-u_{2b}^{h\overline{\theta_b}})^f), \prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-v_a^{h\overline{\theta_a}})^e(1-v_b^{h\overline{\theta_b}})^f) \Bigg] \Bigg) \\
&\geq \Bigg(\Bigg[1-\textstyle\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-(1-p)^{h\overline{\theta_a}})^e(1-(1-p)^{h\overline{\theta_b}})^f), \\
&\quad 1-\textstyle\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-(1-q_1)^{h\overline{\theta_a}})^e(1-(1-q_1)^{h\overline{\theta_b}})^f), \\
&\quad 1-\textstyle\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-(1-q_2)^{h\overline{\theta_a}})^e(1-(1-q_2)^{h\overline{\theta_b}})^f), \\
&\quad 1-\textstyle\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-(1-r)^{h\overline{\theta_a}})^e(1-(1-r)^{h\overline{\theta_b}})^f) \Bigg], \\
&\Bigg[\textstyle\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-s^{h\overline{\theta_a}})^e(1-s^{h\overline{\theta_b}})^f), \prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-u_1^{h\overline{\theta_a}})^e(1-u_1^{h\overline{\theta_b}})^f), \\
&\textstyle\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-u_2^{h\overline{\theta_a}})^e(1-u_2^{h\overline{\theta_b}})^f), \prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-v^{h\overline{\theta_a}})^e(1-v^{h\overline{\theta_b}})^f) \Bigg] \Bigg)
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&\tfrac{2}{h(h+1)}\textstyle\sum_{a=1}^{h}\sum_{b=a}^{h}((h\overline{\theta_a}t\tilde{r}_a)^e \otimes_H (h\overline{\theta_b}t\tilde{r}_b)^f) \\
&= \Bigg(\Bigg[1-\left(\textstyle\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-(1-p_a)^{h\overline{\theta_a}})^e(1-(1-p_b)^{h\overline{\theta_b}})^f) \right)^{\frac{2}{h(h+1)}}, \\
&\quad 1-\left(\textstyle\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-(1-q_{1a})^{h\overline{\theta_a}})^e(1-(1-q_{1b})^{h\overline{\theta_b}})^f) \right)^{\frac{2}{h(h+1)}}, \\
&\quad 1-\left(\textstyle\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-(1-q_{2a})^{h\overline{\theta_a}})^e(1-(1-q_{2b})^{h\overline{\theta_b}})^f) \right)^{\frac{2}{h(h+1)}}, \\
&\quad 1-\left(\textstyle\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-(1-r_a)^{h\overline{\theta_a}})^e(1-(1-r_b)^{h\overline{\theta_b}})^f) \right)^{\frac{2}{h(h+1)}} \Bigg],
\end{aligned}
$$

$$\left[\left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-s_a^{h\overline{\theta_a}})^e(1-s_b^{h\overline{\theta_b}})^f)\right)^{\frac{2}{h(h+1)}},\left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-u_{1a}^{h\overline{\theta_a}})^e(1-u_{1b}^{h\overline{\theta_b}})^f)\right)^{\frac{2}{h(h+1)}},\right.$$

$$\left.\left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-u_{2a}^{h\overline{\theta_a}})^e(1-u_{2b}^{h\overline{\theta_b}})^f)\right)^{\frac{2}{h(h+1)}},\left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-v_a^{h\overline{\theta_a}})^e(1-v_b^{h\overline{\theta_b}})^f)\right)^{\frac{2}{h(h+1)}}\right]\Bigg)$$

$$\geq \Bigg(\Bigg[1-\left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-(1-p)^{h\overline{\theta_a}})^e(1-(1-p)^{h\overline{\theta_b}})^f)\right)^{\frac{2}{h(h+1)}},$$

$$1-\left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-(1-q_1)^{h\overline{\theta_a}})^e(1-(1-q_1)^{h\overline{\theta_b}})^f)\right)^{\frac{2}{h(h+1)}},$$

$$1-\left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-(1-q_2)^{h\overline{\theta_a}})^e(1-(1-q_2)^{h\overline{\theta_b}})^f)\right)^{\frac{2}{h(h+1)}},$$

$$1-\left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-(1-r)^{h\overline{\theta_a}})^e(1-(1-r)^{h\overline{\theta_b}})^f)\right)^{\frac{2}{h(h+1)}}\Bigg],$$

$$\Bigg[\left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-s^{h\overline{\theta_a}})^e(1-s^{h\overline{\theta_b}})^f)\right)^{\frac{2}{h(h+1)}},$$

$$\left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-u_1^{h\overline{\theta_a}})^e(1-u_1^{h\overline{\theta_b}})^f)\right)^{\frac{2}{h(h+1)}},$$

$$\left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-u_2^{h\overline{\theta_a}})^e(1-u_2^{h\overline{\theta_b}})^f)\right)^{\frac{2}{h(h+1)}},\left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-v^{h\overline{\theta_a}})^e(1-v^{h\overline{\theta_b}})^f)\right)^{\frac{2}{h(h+1)}}\Bigg]\Bigg)$$

Therefore,

$$TrIFPHA^{e,f}(t\tilde{r}_1,t\tilde{r}_2,\ldots,t\tilde{r}_h)=\left(\frac{2}{h(h+1)}\sum_{a=1}^{h}\sum_{b=a}^{h}((h\overline{\theta_a}t\tilde{r}_a)^e\otimes_H(h\overline{\theta_b}t\tilde{r}_b)^f)\right)^{\frac{1}{e+f}}$$

$$=\Bigg(\Bigg[\left(1-\left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-(1-p_a)^{h\overline{\theta_a}})^e(1-(1-p_b)^{h\overline{\theta_b}})^f)\right)^{\frac{2}{h(h+1)}}\right)^{\frac{1}{e+f}},$$

$$\left(1-\left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-(1-q_{1a})^{h\overline{\theta_a}})^e(1-(1-q_{1b})^{h\overline{\theta_b}})^f)\right)^{\frac{2}{h(h+1)}}\right)^{\frac{1}{e+f}},$$

$$\left(1-\left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-(1-q_{2a})^{h\overline{\theta_a}})^e(1-(1-q_{2b})^{h\overline{\theta_b}})^f)\right)^{\frac{2}{h(h+1)}}\right)^{\frac{1}{e+f}},$$

$$\left(1-\left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-(1-r_a)^{h\overline{\theta_a}})^e(1-(1-r_b)^{h\overline{\theta_b}})^f)\right)^{\frac{2}{h(h+1)}}\right)^{\frac{1}{e+f}}\Bigg],$$

$$\Bigg[1-\left(1-\left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-s_a^{h\overline{\theta_a}})^e(1-s_b^{h\overline{\theta_b}})^f)\right)^{\frac{2}{h(h+1)}}\right)^{\frac{1}{e+f}},$$

$$1-\left(1-\left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-u_{1a}^{h\overline{\theta_a}})^e(1-u_{1b}^{h\overline{\theta_b}})^f)\right)^{\frac{2}{h(h+1)}}\right)^{\frac{1}{e+f}},$$

$$1-\left(1-\left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-u_{2a}^{h\overline{\theta_a}})^e(1-u_{2b}^{h\overline{\theta_b}})^f)\right)^{\frac{2}{h(h+1)}}\right)^{\frac{1}{e+f}},$$

$$1-\left(1-\left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-v_a^{h\overline{\theta_a}})^e(1-v_b^{h\overline{\theta_b}})^f)\right)^{\frac{2}{h(h+1)}}\right)^{\frac{1}{e+f}}\Bigg]\Bigg)$$

$$\geq \Bigg(\Bigg[\Bigg[\Bigg(1-\Bigg(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-(1-p)^{h\overline{\theta_a}})^{e}(1-(1-p)^{h\overline{\theta_b}})^{f})\Bigg)^{\frac{2}{h(h+1)}}\Bigg)^{\frac{1}{e+f}},$$
$$\Bigg(1-\Bigg(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-(1-q_1)^{h\overline{\theta_a}})^{e}(1-(1-q_1)^{h\overline{\theta_b}})^{f})\Bigg)^{\frac{2}{h(h+1)}}\Bigg)^{\frac{1}{e+f}},$$
$$\Bigg(1-\Bigg(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-(1-q_2)^{h\overline{\theta_a}})^{e}(1-(1-q_2)^{h\overline{\theta_b}})^{f})\Bigg)^{\frac{2}{h(h+1)}}\Bigg)^{\frac{1}{e+f}},$$
$$\Bigg(1-\Bigg(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-(1-r)^{h\overline{\theta_a}})^{e}(1-(1-r)^{h\overline{\theta_b}})^{f})\Bigg)^{\frac{2}{h(h+1)}}\Bigg)^{\frac{1}{e+f}}\Bigg],$$
$$\Bigg[1-\Bigg(1-\Bigg(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-s^{h\overline{\theta_a}})^{e}(1-s^{h\overline{\theta_b}})^{f})\Bigg)^{\frac{2}{h(h+1)}}\Bigg)^{\frac{1}{e+f}},$$
$$1-\Bigg(1-\Bigg(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-u_1^{h\overline{\theta_a}})^{e}(1-u_1^{h\overline{\theta_b}})^{f})\Bigg)^{\frac{2}{h(h+1)}}\Bigg)^{\frac{1}{e+f}},$$
$$1-\Bigg(1-\Bigg(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-u_2^{h\overline{\theta_a}})^{e}(1-u_2^{h\overline{\theta_b}})^{f})\Bigg)^{\frac{2}{h(h+1)}}\Bigg)^{\frac{1}{e+f}},$$
$$1-\Bigg(1-\Bigg(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-v^{h\overline{\theta_a}})^{e}(1-v^{h\overline{\theta_b}})^{f})\Bigg)^{\frac{2}{h(h+1)}}\Bigg)^{\frac{1}{e+f}}\Bigg]\Bigg)$$
$$=\tilde{l}$$

Likewise, we can also demonstrate that $TrIFPHA^{e,f}(\tilde{tr}_1,\tilde{tr}_2,\ldots,\tilde{tr}_h)\leq\tilde{m}$
Therefore, we are able to obtain

$$\tilde{l}\leq TrIFPHA^{e,f}(\tilde{tr}_1,\tilde{tr}_2,\ldots,\tilde{tr}_h)\leq\tilde{m}$$

□

Though, $TrIFPHA^{e,f}(\tilde{tr}_1,\tilde{tr}_2,\ldots,\tilde{tr}_h)$ monotonicity cannot be proven by that.
This is how we can describe it:
Consider $\tilde{Tr}=\{\tilde{tr}_1,\tilde{tr}_2,\ldots,\tilde{tr}_h\}$ and $\tilde{Tr}'=\left\{\tilde{tr}'_1,\tilde{tr}'_2,\ldots,\tilde{tr}'_h\right\}$ are two sets of TrIFNs. If $\tilde{tr}_b\leq\tilde{tr}'_b$ for each $b=1,2,\ldots,h$, then $h\overline{\theta_a}\tilde{tr}_a=([1-(1-p_a)^{h\overline{\theta_a}},1-(1-q_{1a})^{h\overline{\theta_a}},1-(1-q_{2a})^{h\overline{\theta_a}},1-(1-r_a)^{h\overline{\theta_a}}],[s_a^{h\overline{\theta_a}},u_{1a}^{h\overline{\theta_a}},u_{2a}^{h\overline{\theta_a}},v_a^{h\overline{\theta_a}}])$, and $h\overline{\theta'_a}\tilde{tr}'_a=([1-(1-p'_a)^{h\overline{\theta'_a}},1-(1-q'_{1a})^{h\overline{\theta'_a}},1-(1-q'_{2a})^{h\overline{\theta'_a}},1-(1-r'_a)^{h\overline{\theta'_a}}],[s_a'^{h\overline{\theta'_a}},u_{1a'}^{h\overline{\theta'_a}},u_{2a'}^{h\overline{\theta'_a}},v_a'^{h\overline{\theta'_a}}])$

$(\tilde{tr}_1,\tilde{tr}_2,\ldots,\tilde{tr}_h)$ and $(\tilde{tr}'_1,\tilde{tr}'_2,\ldots,\tilde{tr}'_h)$'s support degrees represented as $\overline{o_a}$ and $\overline{o'_a}$, respectively, and which do not have any inequality relationship among them. This implies that we cannot obtain $h\overline{\theta'_a}\tilde{tr}'_a\leq h\overline{\theta_a}\tilde{tr}_a$ and hence it is not possible to obtain

$$TrIFPHA^{e,f}(\tilde{tr}'_1,\tilde{tr}'_2,\ldots,\tilde{tr}'_h)\leq TrIFPHA^{e,f}(\tilde{tr}_1,\tilde{tr}_2,\ldots,\tilde{tr}_h)$$

According to the parameter e and f, here we discuss four special cases of the proposed operator ($TrIFPHA^{e,f}$).

(1) A trapezoidal intuitionistic fuzzy power generalized linear descending weighted operator can be generated from Theorem 1 Formula (1) by letting $f\rightarrow 0$, and it is shown below.

$$
\begin{aligned}
TrIFPHA^{e,f}(\tilde{tr}_1,\tilde{tr}_2,\ldots,\tilde{tr}_h) = &\Bigg(\Bigg[\Bigg(1-\Bigg(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-(1-p_a)^{h\theta_a})^e(1-(1-p_b)^{h\theta_b})^f)\Bigg)^{\frac{2}{h(h+1)}}\Bigg)^{\frac{1}{e+f}},\\
&\Bigg(1-\Bigg(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-(1-q_{1a})^{h\theta_a})^e(1-(1-q_{1b})^{h\theta_b})^f)\Bigg)^{\frac{2}{h(h+1)}}\Bigg)^{\frac{1}{e+f}},\\
&\Bigg(1-\Bigg(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-(1-q_{2a})^{h\theta_a})^e(1-(1-q_{2b})^{h\theta_b})^f)\Bigg)^{\frac{2}{h(h+1)}}\Bigg)^{\frac{1}{e+f}},\\
&\Bigg(1-\Bigg(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-(1-r_a)^{h\theta_a})^e(1-(1-r_b)^{h\theta_b})^f)\Bigg)^{\frac{2}{h(h+1)}}\Bigg)^{\frac{1}{e+f}}\Bigg],\\
&\Bigg[1-\Bigg(1-\Bigg(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-s_a^{h\theta_a})^e(1-s_b^{h\theta_b})^f)\Bigg)^{\frac{2}{h(h+1)}}\Bigg)^{\frac{1}{e+f}},\\
&1-\Bigg(1-\Bigg(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-u_{1a}^{h\theta_a})^e(1-u_{1b}^{h\theta_b})^f)\Bigg)^{\frac{2}{h(h+1)}}\Bigg)^{\frac{1}{e+f}},\\
&1-\Bigg(1-\Bigg(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-u_{2a}^{h\theta_a})^e(1-u_{2b}^{h\theta_b})^f)\Bigg)^{\frac{2}{h(h+1)}}\Bigg)^{\frac{1}{e+f}},\\
&1-\Bigg(1-\Bigg(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-v_a^{h\theta_a})^e(1-v_b^{h\theta_b})^f)\Bigg)^{\frac{2}{h(h+1)}}\Bigg)^{\frac{1}{e+f}}\Bigg]\Bigg)
\end{aligned}
\tag{22}
$$

$$
\begin{aligned}
TrIFPHA^{e,0}(\tilde{tr}_1,\tilde{tr}_2,\ldots,\tilde{tr}_h) = &\Bigg(\Bigg[\Bigg(1-\Bigg(\prod_{a=1}^{h}(1-(1-(1-p_a)^{h\theta_a})^e)^{h+1-a}\Bigg)^{\frac{2}{h(h+1)}}\Bigg)^{\frac{1}{e}},\\
&\Bigg(1-\Bigg(\prod_{a=1}^{h}(1-(1-(1-q_{1a})^{h\theta_a})^e)^{h+1-a}\Bigg)^{\frac{2}{h(h+1)}}\Bigg)^{\frac{1}{e}},\\
&\Bigg(1-\Bigg(\prod_{a=1}^{h}(1-(1-(1-q_{2a})^{h\theta_a})^e)^{h+1-a}\Bigg)^{\frac{2}{h(h+1)}}\Bigg)^{\frac{1}{e}},\\
&\Bigg(1-\Bigg(\prod_{a=1}^{h}(1-(1-(1-r_a)^{h\theta_a})^e)^{h+1-a}\Bigg)^{\frac{2}{h(h+1)}}\Bigg)^{\frac{1}{e}}\Bigg],\\
&\Bigg[1-\Bigg(1-\Bigg(\prod_{a=1}^{h}(1-(1-s_a^{h\theta_a})^e)^{h+1-a}\Bigg)^{\frac{2}{h(h+1)}}\Bigg)^{\frac{1}{e}},\\
&1-\Bigg(1-\Bigg(\prod_{a=1}^{h}(1-(1-u_{1a}^{h\theta_a})^e)^{h+1-a}\Bigg)^{\frac{2}{h(h+1)}}\Bigg)^{\frac{1}{e}},\\
&1-\Bigg(1-\Bigg(\prod_{a=1}^{h}(1-(1-u_{2a}^{h\theta_a})^e)^{h+1-a}\Bigg)^{\frac{2}{h(h+1)}}\Bigg)^{\frac{1}{e}},\\
&1-\Bigg(1-\Bigg(\prod_{a=1}^{h}(1-(1-v_a^{h\theta_a})^e)^{h+1-a}\Bigg)^{\frac{2}{h(h+1)}}\Bigg)^{\frac{1}{e}}\Bigg]\Bigg)
\end{aligned}
$$

By Equation (22), we understand that $TrIFPHA^{e,0}(\tilde{tr}_1,\tilde{tr}_2,\ldots,\tilde{tr}_h)$ can use a heavy weight vector $(h,h-1,\ldots,1)$ to measure the information $((h\overline{\theta_1}\tilde{tr}_1)^e,(h\overline{\theta_2}\tilde{tr}_2)^e,\ldots,(h\overline{\theta_h}\tilde{tr}_h)^e)$.

(2) A trapezoidal intuitionistic fuzzy power generalized linear ascending weighted operator can be generated from Theorem 1 Formula (1) by letting $e \to 0$, and it is shown below.

$$\begin{aligned} TrIFPHA^{0,f}(\tilde{tr}_1,\tilde{tr}_2,\ldots,\tilde{tr}_h) = \Bigg(\Bigg[&\left(1-\left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-(1-p_b)^{h\theta_b})^f)\right)^{\frac{2}{h(h+1)}}\right)^{\frac{1}{f}}, \\ &\left(1-\left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-(1-q_{1b})^{h\theta_b})^f)\right)^{\frac{2}{h(h+1)}}\right)^{\frac{1}{f}}, \\ &\left(1-\left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-(1-q_{2b})^{h\theta_b})^f)\right)^{\frac{2}{h(h+1)}}\right)^{\frac{1}{f}}, \\ &\left(1-\left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-(1-r_b)^{h\theta_b})^f)\right)^{\frac{2}{h(h+1)}}\right)^{\frac{1}{f}}\Bigg], \\ \Bigg[&1-\left(1-\left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-s_b^{h\theta_b})^f)\right)^{\frac{2}{h(h+1)}}\right)^{\frac{1}{f}}, \\ &1-\left(1-\left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-u_{1b}^{h\theta_b})^f)\right)^{\frac{2}{h(h+1)}}\right)^{\frac{1}{f}}, \\ &1-\left(1-\left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-u_{2b}^{h\theta_b})^f)\right)^{\frac{2}{h(h+1)}}\right)^{\frac{1}{f}}, \\ &1-\left(1-\left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-v_b^{h\theta_b})^f)\right)^{\frac{2}{h(h+1)}}\right)^{\frac{1}{f}}\Bigg]\Bigg) \end{aligned} \tag{23}$$

By Equation (23), we understand that $TrIFPHA^{0,f}(\tilde{tr}_1,\tilde{tr}_2,\ldots,\tilde{tr}_h)$ can use a heavy weight vector $(1,2,\ldots,h)$ to measure the information $((h\overline{\theta_1}\tilde{tr}_1)^f,(h\overline{\theta_2}\tilde{tr}_2)^f,\ldots,(h\overline{\theta_h}\tilde{tr}_h)^f)$.

$TrIFPHA^{e,f}(\tilde{tr}_1,\tilde{tr}_2,\ldots,\tilde{tr}_h)$ operator can act as linear weighted function if we consider $e=0$ or $f=0$. Further, we also understand that the parameters e and f cannot be substituted for one another according to the Equations (22) and (23).

(3) A trapezoidal intuitionistic fuzzy power basic Heronian (TrIFPBH) operator can be generated from Theorem 1 Formula (1) by keeping $e=f=\frac{1}{2}$. The TrIFPBH operator is shown below.

$$\begin{aligned} TrIFPHA^{\frac{1}{2},\frac{1}{2}}(\tilde{tr}_1,\tilde{tr}_2,\ldots,\tilde{tr}_h) = \Bigg(\Bigg[&\left(1-\left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-\sqrt{(1-(1-p_a)^{h\theta_a})(1-(1-p_b)^{h\theta_b})})\right)^{\frac{2}{h(h+1)}}\right), \\ &\left(1-\left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-\sqrt{(1-(1-q_{1a})^{h\theta_a})(1-(1-q_{1b})^{h\theta_b})})\right)^{\frac{2}{h(h+1)}}\right), \\ &\left(1-\left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-\sqrt{(1-(1-q_{2a})^{h\theta_a})(1-(1-q_{2b})^{h\theta_b})})\right)^{\frac{2}{h(h+1)}}\right), \\ &\left(1-\left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-\sqrt{(1-(1-r_a)^{h\theta_a})(1-(1-r_b)^{h\theta_b})})\right)^{\frac{2}{h(h+1)}}\right)\Bigg], \\ \Bigg[&\left(\left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-\sqrt{(1-s_a^{h\theta_a})(1-s_b^{h\theta_b})})\right)^{\frac{2}{h(h+1)}}\right), \left(\left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-\sqrt{(1-u_{1a}^{h\theta_a})(1-u_{1b}^{h\theta_b})})\right)^{\frac{2}{h(h+1)}}\right), \\ &\left(\left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-\sqrt{(1-u_{2a}^{h\theta_a})(1-u_{2b}^{h\theta_b})})\right)^{\frac{2}{h(h+1)}}\right), \left(\left(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-\sqrt{(1-v_a^{h\theta_a})(1-v_b^{h\theta_b})})\right)^{\frac{2}{h(h+1)}}\right)\Bigg]\Bigg) \end{aligned} \tag{24}$$

(4) A trapezoidal intuitionistic fuzzy power line Heronian mean operator can be generated from Theorem 1 Formula (1) by assigning $e = f = 1$ and it is shown below.

$$\begin{aligned} TrIFPHA^{1,1}(\tilde{tr}_1, \tilde{tr}_2, \ldots, \tilde{tr}_h) = &\Bigg(\Bigg[\Bigg(1 - \Bigg(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-(1-p_a)^{h\theta_a})(1-(1-p_b)^{h\theta_b}))\Bigg)^{\frac{2}{h(h+1)}}\Bigg)^{\frac{1}{2}}, \\ &\Bigg(1 - \Bigg(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-(1-q_{1a})^{h\theta_a})(1-(1-q_{1b})^{h\theta_b}))\Bigg)^{\frac{2}{h(h+1)}}\Bigg)^{\frac{1}{2}}, \\ &\Bigg(1 - \Bigg(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-(1-q_{2a})^{h\theta_a})(1-(1-q_{2b})^{h\theta_b}))\Bigg)^{\frac{2}{h(h+1)}}\Bigg)^{\frac{1}{2}}, \\ &\Bigg(1 - \Bigg(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-(1-r_a)^{h\theta_a})(1-(1-r_b)^{h\theta_b}))\Bigg)^{\frac{2}{h(h+1)}}\Bigg)^{\frac{1}{2}}\Bigg], \\ &\Bigg[1 - \Bigg(1 - \Bigg(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-s_a^{h\theta_a})(1-s_b^{h\theta_b}))\Bigg)^{\frac{2}{h(h+1)}}\Bigg)^{\frac{1}{2}}, \\ &1 - \Bigg(1 - \Bigg(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-u_{1a}^{h\theta_a})(1-u_{1b}^{h\theta_b}))\Bigg)^{\frac{2}{h(h+1)}}\Bigg)^{\frac{1}{2}}, \\ &1 - \Bigg(1 - \Bigg(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-u_{2a}^{h\theta_a})(1-u_{2b}^{h\theta_b}))\Bigg)^{\frac{2}{h(h+1)}}\Bigg)^{\frac{1}{2}}, \\ &1 - \Bigg(1 - \Bigg(\prod_{a=1}^{h}\prod_{b=a}^{h}(1-(1-v_a^{h\theta_a})(1-v_b^{h\theta_b}))\Bigg)^{\frac{2}{h(h+1)}}\Bigg)^{\frac{1}{2}}\Bigg]\Bigg) \end{aligned} \tag{25}$$

We just take into account the weight vector dependent on the power operator and the interrelationships of input TrIFNs in $TrIFPHA^{e,f}(\tilde{tr}_1, \tilde{tr}_2, \ldots, \tilde{tr}_h)$ operator, and we ignore the value of each individual input TrIFN. This weight, however, is a critical parameter in many real-life decision-making situations. In a MADM problem, for example, it will be the attribute weight, which will have a significant impact on ranking the alternatives. As a result, we will define a trapezoidal intuitionistic fuzzy power weighted Heronian aggregation (TrIFPWHA) operator in the following definition.

Definition 11. *Let $\tilde{tr}_b = ([p_b, q_{1b}, q_{2b}, r_b], [s_b, u_{1b}, u_{2b}, v_b])$ be a set of TrIFNs, further $e, f \geq 0$, also TrIFPWHA: $\Theta^h \to \Theta$, when*

$$TrIFPWHA^{e,f}(\tilde{tr}_1, \tilde{tr}_2, \ldots, \tilde{tr}_h) = \left(\frac{2}{h(h+1)}\sum_{a=1}^{h}\sum_{b=a}^{h}\left(\left(\frac{h\overline{\theta_a}o_a}{\sum_{c=1}^{h}\overline{\theta_c}o_c}\tilde{tr}_a\right)^{e} \otimes_H \left(\frac{h\overline{\theta_b}o_b}{\sum_{c=1}^{h}\overline{\theta_c}o_c}\tilde{tr}_b\right)^{f}\right)\right)^{\frac{1}{e+f}} \tag{26}$$

where, each TrIFN is a collection for Θ, $\overline{\theta_c} = \frac{(1+I(\tilde{tr}_c))}{\sum_{c=1}^{h}(1+I(\tilde{tr}_c))}$ and $\sum_{c=1}^{h}\overline{\theta_c} = 1$

$I(\tilde{tr}_c) = \sum_{a=1, a\neq c}^{h} Sup(\tilde{tr}_c, \tilde{tr}_a)$, and support degree for $\tilde{tr}_c$ from $\tilde{tr}_a$ is $Sup(\tilde{tr}_c, \tilde{tr}_a)$ that has the properties mentioned below. (1) $Sup(\tilde{tr}_c, \tilde{tr}_a) \in [0,1]$; (2) $Sup(\tilde{tr}_c, \tilde{tr}_a) = Sup(\tilde{tr}_a, \tilde{tr}_c)$; (3) $Sup(\tilde{tr}_1, \tilde{tr}_2) \geq Sup(\tilde{a}, \tilde{b})$, if $d(\tilde{tr}_1, \tilde{tr}_2) < d(\tilde{a}, \tilde{b})$, and $d(\tilde{tr}_1, \tilde{tr}_2)$ is the distance among TrIFNs $\tilde{tr}_1$ and $\tilde{tr}_2$. Additionally, $o = (o_a, o_2, \ldots, o_h)^T$ is the weight vector of ($\tilde{tr}_1, \tilde{tr}_2, \ldots, \tilde{tr}_h$), $\theta_b \in [0,1], \sum_{b=1}^{h} o_b = 1$, also h is a balance parameter. Thus, the trapezoidal intuitionistic fuzzy power weighted Heronian aggregation operator is then known as TrIFPWHA.

Theorem 4 is based on the operational laws of the TrIFNs described in Equations (1)–(4).

Theorem 4. *Let $\tilde{Tr} = \{\tilde{tr}_b | \tilde{tr}_b = ([p_b, q_{1b}, q_{2b}, r_b], [s_b, u_{1b}, u_{2b}, v_b]),$ where $b = 1, 2, \ldots, h\}$ be a set of TrIFNs, and $e, f \geq 0$. The product of combining the results of Definitions 11 is still a TrIFN, and*

$$
\begin{aligned}
& TrIFPWHA^{e,f}(\tilde{tr}_1, \tilde{tr}_2, \ldots, \tilde{tr}_h) \\
& = \Bigg(\Bigg[\Bigg(1 - \Bigg(\prod_{a=1}^{h}\prod_{b=a}^{h}(1 - (1 - (1 - p_a)^{\frac{h\overline{\theta_a}o_a}{\sum_{c=1}^{h}\overline{\theta_c}o_c}})^e(1 - (1 - p_b)^{\frac{h\overline{\theta_b}o_b}{\sum_{c=1}^{h}\overline{\theta_c}o_c}})^f)\Bigg)^{\frac{2}{h(h+1)}}\Bigg)^{\frac{1}{e+f}}, \\
& \Bigg(1 - \Bigg(\prod_{a=1}^{h}\prod_{b=a}^{h}(1 - (1 - (1 - q_{1a})^{\frac{h\overline{\theta_a}o_a}{\sum_{c=1}^{h}\overline{\theta_c}o_c}})^e(1 - (1 - q_{1b})^{\frac{h\overline{\theta_b}o_b}{\sum_{c=1}^{h}\overline{\theta_c}o_c}})^f)\Bigg)^{\frac{2}{h(h+1)}}\Bigg)^{\frac{1}{e+f}}, \\
& \Bigg(1 - \Bigg(\prod_{a=1}^{h}\prod_{b=a}^{h}(1 - (1 - (1 - q_{2a})^{\frac{h\overline{\theta_a}o_a}{\sum_{c=1}^{h}\overline{\theta_c}o_c}})^e(1 - (1 - q_{2b})^{\frac{h\overline{\theta_b}o_b}{\sum_{c=1}^{h}\overline{\theta_c}o_c}})^f)\Bigg)^{\frac{2}{h(h+1)}}\Bigg)^{\frac{1}{e+f}}, \\
& \Bigg(1 - \Bigg(\prod_{a=1}^{h}\prod_{b=a}^{h}(1 - (1 - (1 - r_a)^{\frac{h\overline{\theta_a}o_a}{\sum_{c=1}^{h}\overline{\theta_c}o_c}})^e(1 - (1 - r_b)^{\frac{h\overline{\theta_b}o_b}{\sum_{c=1}^{h}\overline{\theta_c}o_c}})^f)\Bigg)^{\frac{2}{h(h+1)}}\Bigg)^{\frac{1}{e+f}}\Bigg], \\
& \Bigg[1 - \Bigg(1 - \Bigg(\prod_{a=1}^{h}\prod_{b=a}^{h}(1 - (1 - s_a^{\frac{h\overline{\theta_a}o_a}{\sum_{c=1}^{h}\overline{\theta_c}o_c}})^e(1 - s_b^{\frac{h\overline{\theta_b}o_b}{\sum_{c=1}^{h}\overline{\theta_c}o_c}})^f)\Bigg)^{\frac{2}{h(h+1)}}\Bigg)^{\frac{1}{e+f}}, \\
& 1 - \Bigg(1 - \Bigg(\prod_{a=1}^{h}\prod_{b=a}^{h}(1 - (1 - u_{1a}^{\frac{h\overline{\theta_a}o_a}{\sum_{c=1}^{h}\overline{\theta_c}o_c}})^e(1 - u_{1b}^{\frac{h\overline{\theta_b}o_b}{\sum_{c=1}^{h}\overline{\theta_c}o_c}})^f)\Bigg)^{\frac{2}{h(h+1)}}\Bigg)^{\frac{1}{e+f}}, \\
& 1 - \Bigg(1 - \Bigg(\prod_{a=1}^{h}\prod_{b=a}^{h}(1 - (1 - u_{2a}^{\frac{h\overline{\theta_a}o_a}{\sum_{c=1}^{h}\overline{\theta_c}o_c}})^e(1 - u_{2b}^{\frac{h\overline{\theta_b}o_b}{\sum_{c=1}^{h}\overline{\theta_c}o_c}})^f)\Bigg)^{\frac{2}{h(h+1)}}\Bigg)^{\frac{1}{e+f}}, \\
& 1 - \Bigg(1 - \Bigg(\prod_{a=1}^{h}\prod_{b=a}^{h}(1 - (1 - v_a^{\frac{h\overline{\theta_a}o_a}{\sum_{c=1}^{h}\overline{\theta_c}o_c}})^e(1 - v_b^{\frac{h\overline{\theta_b}o_b}{\sum_{c=1}^{h}\overline{\theta_c}o_c}})^f)\Bigg)^{\frac{2}{h(h+1)}}\Bigg)^{\frac{1}{e+f}}\Bigg]\Bigg)
\end{aligned}
\tag{27}
$$

Proof. Proof of this theorem is similar to Theorem 1, and hence it is omitted. □

If $o = \left(\frac{1}{h}, \frac{1}{h}, \ldots, \frac{1}{h}\right)^T$, then $TrIFPWHA^{e,f}(\tilde{tr}_1, \tilde{tr}_2, \ldots, \tilde{tr}_h)$ operator is obviously reduced to the $TrIFPHA^{e,f}(\tilde{tr}_1, \tilde{tr}_2, \ldots, \tilde{tr}_h)$ operator.

Note: It is simple to demonstrate that the $TrIFPWHA^{e,f}(\tilde{tr}_1, \tilde{tr}_2, \ldots, \tilde{tr}_h)$ operator just has the property of boundedness and lacks the properties of monotonicity and idempotency.

4. A Group Decision-Making Method Based on the Trapezoidal Intuitionistic Fuzzy Power Heronian Aggregation Operator and Trapezoidal Intuitionistic Fuzzy Power Weighted Heronian Aggregation Operator

The implementations of the proposed trapezoidal intuitionistic fuzzy power Heronian aggregation operator which generalizes interval-valued intuitionistic fuzzy power Heronian aggregation operator [31] and trapezoidal intuitionistic fuzzy power weighted Heronian aggregation operator in solving the Multi-attribute Group Decision-Making (MAGDM) problem will be discussed in this section.

The data which we have considered in this paper has been taken from the literature [32] and modified based on the need of a problem. Initially, the linguistic terms were given by the experts for evaluating the performance of alternatives. Then, the linguistic terms were converted into a trapezoidal intuitionistic fuzzy number. These TrIFN equivalents for various linguistic terms were identified from the literature and modified based on the case study problem.

Assume there are h alternatives $p_1, p_2, \ldots, p_h$ and g attributes $(x_1, x_2, \ldots, x_g)$, and $o = (o_1, o_2, \ldots, o_g)$ is the weight vector of the attribute $x_a(a = 1, 2, \ldots, g)$, here $o_a \geq 0, a = 1, 2, 3, \ldots, g, \sum_{a=1}^{g} o_a = 1$, in a MAGDM issue with TrIFNs. Assume there are c decision makers $(r_1, r_2, \ldots, r_c)$ and $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_c)$ is the weight vector of them with $\gamma_p \geq 0 (p = 1, 2, \ldots, c), \sum_{p=1}^{c} \gamma_p = 1$. Assume that $\tilde{D}^p = \left[\tilde{d_{ba}}^p\right]_{h \times g}$ is the decision matrix, where $\tilde{d_{ba}}^p = \left(\left[p^p_{ba}, q^p_{ba}, r^p_{ba}, s^p_{ba}\right], \left[t^p_{ba}, u^p_{ba}, v^p_{ba}, w^p_{ba}\right]\right)$ has the form of a TrIFN, the decision maker's statement d_p, for alternative p_b with respect to attribute x_a. After that, it is time to rank the alternative.

The TrIFPHA and TrIFPwHA operators are used to solve MAGDM problems in the following sections, and the following are the thorough decision-making steps:

Step 1 The attribute values should be normalised.

Usually, there are two kinds of attribute values: benefit and cost. We should transform them to the same form, with the benefit being translated to cost. If the a-th attribute x_a is cost type, the attribute values $\tilde{d_{ba}}^p = \left(\left[p^p_{ba}, q^p_{ba}, r^p_{ba}, s^p_{ba}\right], \left[t^p_{ba}, u^p_{ba}, v^p_{ba}, w^p_{ba}\right]\right) b = 1, 2, \ldots, h;$ $p = 1, 2, \ldots, c$, The following formula can be used to translate it to a beneficial one. ($\tilde{d_{ba}}^p$ continues to express transformed attribute values.)

$$\tilde{d_{ba}}^p = \left(\left[t^p_{ba}, u^p_{ba}, v^p_{ba}, w^p_{ba}\right], \left[p^p_{ba}, q^p_{ba}, r^p_{ba}, s^p_{ba}\right]\right) \tag{28}$$

Step 2 Compute the supports.

$$Sup(\tilde{d_{ba}}^p, \tilde{d_{ba}}^t) = 1 - |K_F(\tilde{d_{ba}}^p) - K_F(\tilde{d_{ba}}^t)|, p, t = 1, 2, \ldots, c; b = 1, 2, \ldots, h; a = 1, 2, \ldots, g \tag{29}$$

here,

$$K_F(\tilde{d_{ba}}^p) = \sqrt{\frac{\left(x1 + x2\right)}{4}} \text{ Where, } x1 = (p^p_{ba})^2 + (q^p_{ba})^2 + (r^p_{ba})^2 + (s^p_{ba})^2 + (t^p_{ba})^2 + (u^p_{ba})^2 + (v^p_{ba})^2 + (w^p_{ba})^2$$
$$x2 = (p^p_{ba} + t^p_{ba})^2 + (q^p_{ba} + u^p_{ba})^2 + (r^p_{ba} + v^p_{ba})^2 + (s^p_{ba} + w^p_{ba})^2 \tag{30}$$

$$K_F(\tilde{d_{ba}}^t) = \sqrt{\frac{\left(x1 + x2\right)}{4}} \text{ Where, } x1 = (p^t_{ba})^2 + (q^t_{ba})^2 + (d^t_{ba})^2 + (s^t_{ba})^2 + (t^t_{ba})^2 + (u^t_{ba})^2 + (v^t_{ba})^2 + (w^t_{ba})^2$$
$$x2 = (p^t_{ba} + t^t_{ba})^2 + (q^t_{ba} + u^t_{ba})^2 + (d^t_{ba} + v^t_{ba})^2 + (s^t_{ba} + w^t_{ba})^2 \tag{31}$$

Step 3 Compute $I(\tilde{d_{ba}}^p)$

$$I(\tilde{d_{ba}}^p) = \sum_{t=1, t \neq p}^{c} Sup(\tilde{d_{ba}}^p, \tilde{d_{ba}}^t); p = 1, 2, \ldots, c; b = 1, 2, \ldots, h; a = 1, 2, \ldots, g \tag{32}$$

Step 4 Compute the power operator's weight vector $\overline{\theta_{ba}}^p$ linked with the TrIFN $\tilde{d_{ba}}^p$

$$\overline{\theta_{ba}}^p = \frac{(1 + I(\tilde{d_{ba}}^p))}{\sum_{p=1}^{c}(1 + I(\tilde{d_{ba}}^p))}; p = 1, 2, \ldots, c; b = 1, 2, \ldots m; a = 1, 2, \ldots n \tag{33}$$

Step 5 Aggregate the results of each expert's evaluations into a single report.

$$\tilde{d_{ba}} = TrIFPWHA^{e,f}(\tilde{d_{ba}}^1, \tilde{d_{ba}}^2, \ldots, \tilde{d_{ba}}^l) \tag{34}$$

to gather information in a group. Here, $b = 1, 2, \ldots, h; a = 1, 2, \ldots, g$.

Step 6 Compute $I(\tilde{d_{ba}})$

$$I(\tilde{d_{ba}}) = \sum_{t=1, t \neq i}^{g} Sup(\tilde{d_{ba}}, \tilde{d_{bt}}); b = 1, 2, \ldots, h; a = 1, 2, \ldots, g \tag{35}$$

Step 7 Compute the power operator's weight vector $\overline{\theta_{ba}}$ linked with the TrIFN $\tilde{d_{ba}}$

$$\overline{\theta_{ba}} = \frac{(1 + I(\tilde{d_{ba}}))}{\sum_{a=1}^{g}(1 + I(\tilde{d_{ba}}))}; b = 1,2,\ldots,h; a = 1,2,\ldots,g \quad (36)$$

Step 8 Alternative's total evaluation value:

From the following formula, we can determine each alternative's total evaluation value.

$$\tilde{y_b} = TrIFPWHA^{e,f}(\tilde{d_{b1}}, \tilde{d_{b2}}, \ldots, \tilde{d_{bg}}) \quad (37)$$

where $b = 1,2,\ldots,h$;
Step 9 Using Definitions 6 and 7, we rank $\tilde{y_b}(b = 1,2,\ldots,h)$ in descending order.
Step 10 In accordance with the order of $\tilde{y_b}(b = 1,2,\ldots,h)$, rank all the alternatives and choose the best one(s).
Step 11 End.

5. An Illustrative Example

In this section, we show the applicability of the proposed decision-making method in solving a problem discussed in Zhang et al. [32].

Example 1 ([32])**.** *Assume that a company's data centre needed to upgrade the management information system in order to boost productivity. Four alternatives (software system)* $tr_b(a = 1,2,3,4)$ *may be examined after preliminary sampling, and there were four assessment attributes taken into consideration. They are* x_1*: System costs, which include both hardware and software.* x_2*: The dependability of outsourcing companies' software growth.* x_3 *: The contribution to the success of the company.* x_4*: The effort to migrate from old systems to modern systems. Where* x_1 *is an expense attribute and the others are value attributes, and the attributes' weight vector is* $o = (0.5, 0.3, 0.1, 0.1)$*. Three experts* $r_c(c = 1,2,3)$ *with a weight vector of* $\gamma = (0.2, 0.5, 0.3)$ *have been requested to join a panel to evaluate these alternatives on each attribute. The assessment values provided by the requested experts are represented by TrIFNs, and given in Tables 1–3.*

Table 1. Expert r_1's decision matrix $\tilde{D}^1$.

	x_1	x_2	x_3	x_4
tr_1	([0.2,0.4,0.6,0.8], [0.15,0.3,0.7,0.9])	([0.15,0.25,0.45,0.6], [0.1,0.2,0.6,0.7])	([0.15,0.35,0.5,0.65], [0.1,0.25,0.55,0.7])	([0.1,0.3,0.45,0.6], [0,0.25,0.55,0.65])
tr_2	([0.3,0.35,0.55,0.7], [0.1,0.25,0.7,0.85])	([0.1,0.25,0.45,0.65], [0,0.15,0.5,0.7])	([0.5,0.65,0.8,0.9], [0.3,0.45,0.85,0.9])	([0.5,0.6,0.7,0.8], [0.4,0.55,0.75,0.85])
tr_3	([0.4,0.55,0.75,0.9], [0.2,0.35,0.8,0.9])	([0.1,0.2,0.3,0.4], [0,0.15,0.35,0.45])	([0.15,0.3,0.4,0.6], [0.1,0.25,0.45,0.65])	([0.1,0.2,0.3,0.4], [0,0.15,0.35,0.5])
tr_4	([0.4,0.5,0.8,0.9], [0.2,0.4,0.8,0.9])	([0.5,0.65,0.8,0.9], [0.3,0.45,0.85,0.9])	([0,0.25,0.45,0.6], [0,0.15,0.5,0.65])	([0.2,0.35,0.5,0.65], [0.1,0.25,0.55,0.7])

Table 2. Expert r_2's decision matrix $\tilde{D}^2$.

	x_1	x_2	x_3	x_4
tr_1	([0.15,0.35,0.5,0.65], [0.1,0.25,0.55,0.7])	([0.1,0.25,0.45,0.55], [0.1,0.15,0.5,0.65])	([0.25,0.4,0.55,0.7], [0.15,0.3,0.6,0.75])	([0.2,0.3,0.45,0.6], [0.2,0.3,0.6,0.7])
tr_2	([0.25,0.35,0.5,0.65], [0.15,0.25,0.55,0.65])	([0.2,0.3,0.5,0.6], [0.1,0.2,0.6,0.7])	([0.45,0.6,0.75,0.9], [0.3,0.45,0.8,0.9])	([0.4,0.55,0.75,0.85], [0.25,0.4,0.8,0.9])
tr_3	([0.45,0.55,0.65,0.75], [0.25,0.45,0.7,0.8])	([0.25,0.45,0.6,0.75], [0.15,0.35,0.65,0.85])	([0.1,0.25,0.45,0.6], [0,0.15,0.5,0.7])	([0.15,0.35,0.45,0.6], [0,0.3,0.5,0.7])
tr_4	([0.25,0.35,0.55,0.75], [0.15,0.3,0.6,0.8])	([0.4,0.6,0.75,0.9], [0.2,0.45,0.8,0.9])	([0.2,0.4,0.5,0.6], [0.1,0.3,0.6,0.7])	([0,0.25,0.4,0.65], [0,0.15,0.45,0.7])

Table 3. Expert r_3's decision matrix $\tilde{D}^3$.

	x_1	x_2	x_3	x_4
tr_1	([0.25,0.45,0.65,0.8], [0.15,0.3,0.7,0.85])	([0.1,0.2,0.3,0.4], [0,0.1,0.4,0.5])	([0,0.2,0.35,0.5], [0,0.1,0.4,0.5])	([0.45,0.55,0.6,0.65], [0.4,0.55,0.65,0.75])
tr_2	([0.18,0.29,0.34,0.47], [0.1,0.2,0.4,0.55])	([0.2,0.4,0.55,0.7], [0.15,0.25,0.6,0.75])	([0.4,0.6,0.7,0.8], [0.3,0.5,0.7,0.8])	([0.4,0.55,0.75,0.85], [0.35,0.4,0.8,0.9])
tr_3	([0.3,0.45,0.65,0.7], [0.2,0.4,0.75,0.9])	([0.3,0.4,0.5,0.6], [0.1,0.3,0.6,0.7])	([0.2,0.4,0.6,0.75], [0.15,0.35,0.65,0.75])	([0,0.25,0.5,0.6], [0,0.15,0.6,0.7])
tr_4	([0.3,0.45,0.65,0.75], [0.2,0.35,0.7,0.8])	([0.4,0.7,0.8,0.9], [0.3,0.6,0.8,0.9])	([0.1,0.3,0.5,0.7], [0,0.2,0.6,0.8])	([0,0.2,0.4,0.6], [0,0.15,0.5,0.65])

5.1. Steps in Making a Decision

The following measures are involved in obtaining the right alternative(s):

(1) The attribute values should be normalised.

Since x_1 is a cost form, all x_1 attribute values must be converted (28).

(2) Compute the supports $Sup(\tilde{d_{ba}}^p, \tilde{d_{ba}}^t)$ $p, t = 1, 2, 3$. $b, a = 1, 2, 3, 4$ from Formulas (29)–(31).

(For the sake of clarity, we'll use the abbreviation) $Sup(\tilde{d_{ba}}^p, \tilde{d_{ba}}^t)$ with S_{ba}^{pt}, and obtain
$S_{11}^{12} = S_{11}^{21} = 0.7245$, $S_{11}^{13} = S_{11}^{31} = 0.9897$, $S_{11}^{23} = S_{11}^{32} = 0.7143$,
$S_{12}^{12} = S_{12}^{21} = 0.9049$, $S_{12}^{13} = S_{12}^{31} = 0.6550$, $S_{12}^{23} = S_{12}^{32} = 0.7501$,
$S_{13}^{12} = S_{13}^{21} = 0.8839$, $S_{13}^{13} = S_{13}^{31} = 0.6668$, $S_{13}^{23} = S_{13}^{32} = 0.5508$,
$S_{14}^{12} = S_{14}^{21} = 0.9235$, $S_{14}^{13} = S_{14}^{31} = 0.6089$, $S_{14}^{23} = S_{14}^{32} = 0.6854$,
$S_{21}^{12} = S_{21}^{21} = 0.8144$, $S_{21}^{13} = S_{21}^{31} = 0.5459$, $S_{21}^{23} = S_{21}^{32} = 0.7316$,
$S_{22}^{12} = S_{22}^{21} = 0.9389$, $S_{22}^{13} = S_{22}^{31} = 0.8218$, $S_{22}^{23} = S_{22}^{32} = 0.8829$,
$S_{23}^{12} = S_{23}^{21} = 0.9425$, $S_{23}^{13} = S_{23}^{31} = 0.8134$, $S_{23}^{23} = S_{23}^{32} = 0.8709$,
$S_{24}^{12} = S_{24}^{21} = 0.9817$, $S_{24}^{13} = S_{24}^{31} = 0.9973$, $S_{24}^{23} = S_{24}^{32} = 0.9844$,
$S_{31}^{12} = S_{31}^{21} = 0.8647$, $S_{31}^{13} = S_{31}^{31} = 0.8363$, $S_{31}^{23} = S_{31}^{32} = 0.9716$,
$S_{32}^{12} = S_{32}^{21} = 0.3336$, $S_{32}^{13} = S_{32}^{31} = 0.5361$, $S_{32}^{23} = S_{32}^{32} = 0.7974$,
$S_{33}^{12} = S_{33}^{21} = 0.9777$, $S_{33}^{13} = S_{33}^{31} = 0.6985$, $S_{33}^{23} = S_{33}^{32} = 0.7208$,
$S_{34}^{12} = S_{34}^{21} = 0.6482$, $S_{34}^{13} = S_{34}^{31} = 0.6447$, $S_{34}^{23} = S_{34}^{32} = 0.9965$,
$S_{41}^{12} = S_{41}^{21} = 0.6532$, $S_{41}^{13} = S_{41}^{31} = 0.7674$, $S_{41}^{23} = S_{41}^{32} = 0.8857$,
$S_{42}^{12} = S_{42}^{21} = 0.9201$, $S_{42}^{13} = S_{42}^{31} = 0.9840$, $S_{42}^{23} = S_{42}^{32} = 0.9042$,
$S_{43}^{12} = S_{43}^{21} = 0.8485$, $S_{43}^{13} = S_{43}^{31} = 0.8099$, $S_{43}^{23} = S_{43}^{32} = 0.9613$,
$S_{44}^{12} = S_{44}^{21} = 0.8817$, $S_{44}^{13} = S_{44}^{31} = 0.8419$, $S_{44}^{23} = S_{44}^{32} = 0.9602$,

(3) Compute $I(\tilde{d_{ba}}^p)$ $b, a = 1, 2, 3, 4$, $p = 1, 2, 3$. (For the sake of clarity, we'll use the abbreviation $I(\tilde{d_{ba}}^p)$ with I_{ba}^p from Equation (32), and obtain
$I_{11}^1 = 1.7143$, $I_{11}^2 = 1.4388$, $I_{11}^3 = 1.7040$, $I_{12}^1 = 1.5600$, $I_{12}^2 = 1.6550$, $I_{12}^3 = 1.4052$,
$I_{13}^1 = 1.5508$, $I_{13}^2 = 1.4347$, $I_{13}^3 = 1.2176$, $I_{14}^1 = 1.5324$, $I_{14}^2 = 1.6089$, $I_{14}^3 = 1.2944$,
$I_{21}^1 = 1.3603$, $I_{21}^2 = 1.5459$, $I_{21}^3 = 1.2775$, $I_{22}^1 = 1.7606$, $I_{22}^2 = 1.8218$, $I_{22}^3 = 1.7046$,
$I_{23}^1 = 1.7560$, $I_{23}^2 = 1.8134$, $I_{23}^3 = 1.6843$, $I_{24}^1 = 1.9790$, $I_{24}^2 = 1.9662$, $I_{24}^3 = 1.9817$,
$I_{31}^1 = 1.7010$, $I_{31}^2 = 1.8363$, $I_{31}^3 = 1.8079$, $I_{32}^1 = 0.8697$, $I_{32}^2 = 1.1310$, $I_{32}^3 = 1.3336$,
$I_{33}^1 = 1.6762$, $I_{33}^2 = 1.6985$, $I_{33}^3 = 1.4192$, $I_{34}^1 = 1.2928$, $I_{34}^2 = 1.6447$, $I_{34}^3 = 1.6412$,
$I_{41}^1 = 1.4206$, $I_{41}^2 = 1.5389$, $I_{41}^3 = 1.6532$, $I_{42}^1 = 1.9042$, $I_{42}^2 = 1.8243$, $I_{42}^3 = 1.8882$,
$I_{43}^1 = 1.6584$, $I_{43}^2 = 1.8099$, $I_{43}^3 = 1.7712$, $I_{44}^1 = 1.7236$, $I_{44}^2 = 1.8419$, $I_{44}^3 = 1.8020$,

(4) Using Formula (33), compute the power weights $\overline{\theta_{ba}}^p$ $(b, a = 1, 2, 3, 4.\ p = 1, 2, 3)$ and obtain
$\overline{\theta_{11}}^1 = 0.3455$, $\overline{\theta_{11}}^2 = 0.3104$, $\overline{\theta_{11}}^3 = 0.3442$, $\overline{\theta_{12}}^1 = 0.3359$, $\overline{\theta_{12}}^2 = 0.3484$, $\overline{\theta_{12}}^3 = 0.3156$,
$\overline{\theta_{13}}^1 = 0.3541$, $\overline{\theta_{13}}^2 = 0.3380$, $\overline{\theta_{13}}^3 = 0.3079$, $\overline{\theta_{14}}^1 = 0.3406$, $\overline{\theta_{14}}^2 = 0.3509$, $\overline{\theta_{14}}^3 = 0.3086$,
$\overline{\theta_{21}}^1 = 0.3286$, $\overline{\theta_{21}}^2 = 0.3544$, $\overline{\theta_{21}}^3 = 0.3170$, $\overline{\theta_{22}}^1 = 0.3331$, $\overline{\theta_{22}}^2 = 0.3405$, $\overline{\theta_{22}}^3 = 0.3264$,
$\overline{\theta_{23}}^1 = 0.3339$, $\overline{\theta_{23}}^2 = 0.3409$, $\overline{\theta_{23}}^3 = 0.3252$, $\overline{\theta_{24}}^1 = 0.3337$, $\overline{\theta_{24}}^2 = 0.3323$, $\overline{\theta_{24}}^3 = 0.3340$,
$\overline{\theta_{31}}^1 = 0.3237$, $\overline{\theta_{31}}^2 = 0.3399$, $\overline{\theta_{31}}^3 = 0.3365$, $\overline{\theta_{32}}^1 = 0.2952$, $\overline{\theta_{32}}^2 = 0.3364$, $\overline{\theta_{32}}^3 = 0.3684$,
$\overline{\theta_{33}}^1 = 0.3434$, $\overline{\theta_{33}}^2 = 0.3462$, $\overline{\theta_{33}}^3 = 0.3104$, $\overline{\theta_{34}}^1 = 0.3025$, $\overline{\theta_{34}}^2 = 0.3490$, $\overline{\theta_{34}}^3 = 0.3485$,
$\overline{\theta_{41}}^1 = 0.3180$, $\overline{\theta_{41}}^2 = 0.3335$, $\overline{\theta_{41}}^3 = 0.3485$, $\overline{\theta_{42}}^1 = 0.3370$, $\overline{\theta_{42}}^2 = 0.3278$, $\overline{\theta_{42}}^3 = 0.3352$,

$\overline{\theta_{43}}^1 = 0.3226$, $\overline{\theta_{43}}^2 = 0.3410$, $\overline{\theta_{43}}^3 = 0.3363$, $\overline{\theta_{44}}^1 = 0.3255$, $\overline{\theta_{44}}^2 = 0.3396$, $\overline{\theta_{44}}^3 = 0.3349$,

(5) Aggregate the results of each expert's evaluations into a single report from Formula (34) (consider e = f = 2).

To make it easier, we will start by calculating the $\xi_{ba}^s = \frac{l\overline{\theta_{ba}}^s \gamma_s}{\sum_{p=1}^{c} \overline{\theta_{ba}}^p \gamma_p}$

$\xi_{11}^1 = 0.6328$, $\xi_{11}^2 = 1.4215$, $\xi_{11}^3 = 0.9457$, $\xi_{12}^1 = 0.5997$, $\xi_{12}^2 = 1.555$, $\xi_{12}^3 = 0.8452$,
$\xi_{13}^1 = 0.6396$, $\xi_{13}^2 = 1.5263$, $\xi_{13}^3 = 0.8341$, $\xi_{14}^1 = 0.608$, $\xi_{14}^2 = 1.5658$, $\xi_{14}^3 = 0.8262$,
$\xi_{21}^1 = 0.5832$, $\xi_{21}^2 = 1.5727$, $\xi_{21}^3 = 0.8441$, $\xi_{22}^1 = 0.597$, $\xi_{22}^2 = 1.5256$, $\xi_{22}^3 = 0.8774$,
$\xi_{23}^1 = 0.5984$, $\xi_{23}^2 = 1.5273$, $\xi_{23}^3 = 0.8743$, $\xi_{24}^1 = 0.6011$, $\xi_{24}^2 = 1.4963$, $\xi_{24}^3 = 0.9025$,
$\xi_{31}^1 = 0.5786$, $\xi_{31}^2 = 1.5191$, $\xi_{31}^3 = 0.9023$, $\xi_{32}^1 = 0.5243$, $\xi_{32}^2 = 1.494$, $\xi_{32}^3 = 0.9816$,
$\xi_{33}^1 = 0.6152$, $\xi_{33}^2 = 1.5507$, $\xi_{33}^3 = 0.8341$, $\xi_{34}^1 = 0.5346$, $\xi_{34}^2 = 1.5416$, $\xi_{34}^3 = 0.9238$,
$\xi_{41}^1 = 0.5697$, $\xi_{41}^2 = 1.4937$, $\xi_{41}^3 = 0.9366$, $\xi_{42}^1 = 0.6094$, $\xi_{42}^2 = 1.4816$, $\xi_{42}^3 = 0.9091$,
$\xi_{43}^1 = 0.5763$, $\xi_{43}^2 = 1.5227$, $\xi_{43}^3 = 0.901$, $\xi_{44}^1 = 0.5823$, $\xi_{44}^2 = 1.519$, $\xi_{44}^3 = 0.8986$,

After that, by $\tilde{d_{ba}} = TrIFPWHA^{2,2}(\tilde{d_{ba}}^1, \tilde{d_{ba}}^2, \tilde{d_{ba}}^3)$, and obtain

$$D = \begin{bmatrix}
([0.1288,0.2827,0.6375,0.8084],[0.1964,0.384,0.542,0.6862]), & ([0.1157,0.2575,0.4366,0.5378],[0,0.1507,0.471,0.5951]), & ([0.232,0.3762,0.5132,0.6486],[0,0.2148,0.5184,0.6472]), & ([0.2943,0.3971,0.5069,0.6197],[0,0.3362,0.5867,0.6851]), \\
([0.1523,0.2586,0.56,0.6828],[0.239,0.3294,0.4442,0.5831]), & ([0.207,0.3394,0.5189,0.6449],[0,0.2083,0.5701,0.6945]), & ([0.4591,0.6145,0.7439,0.8628],[0.3002,0.4535,0.7612,0.8588]), & ([0.4291,0.5663,0.7384,0.8322],[0.3028,0.4127,0.781,0.8856]), \\
([0.2505,0.4407,0.7359,0.8545],[0.3842,0.5064,0.6431,0.7303]), & ([0.2741,0.4338,0.5556,0.674],[0,0.2874,0.5672,0.7087]), & ([0.1436,0.3138,0.5013,0.6532],[0,0.2238,0.5083,0.6859]), & ([0.1439,0.3358,0.4682,0.5893],[0,0.2159,0.4864,0.6535]), \\
([0.1811,0.3422,0.6796,0.8203],[0.2881,0.3923,0.581,0.7447]), & ([0.4288,0.6459,0.7739,0.8908],[0.2512,0.4765,0.7922,0.8923]), & ([0.189,0.38,0.5056,0.6384],[0,0.2338,0.5706,0.6971]), & ([0.0779,0.2686,0.4294,0.6397],[0,0.1751,0.466,0.6707]),
\end{bmatrix}$$

(6) From Formula (35), compute $I(\tilde{d_{ba}})$ $(b, a = 1, 2, 3, 4)$ (For the sake of clarity, we'll use the abbreviation $I(\tilde{d_{ba}})$ with I_{ba}), obtain

$I_{11} = 2.414$, $I_{12} = 2.2855$, $I_{13} = 2.6109$, $I_{14} = 2.6109$,
$I_{21} = 1.9169$, $I_{22} = 1.9928$, $I_{23} = 1.9472$, $I_{24} = 1.9928$,
$I_{31} = 1.9041$, $I_{32} = 2.4479$, $I_{33} = 2.4479$, $I_{34} = 2.3277$,
$I_{41} = 2.1308$, $I_{42} = 1.4902$, $I_{43} = 2.1308$, $I_{44} = 1.9031$,

(7) From Formula (36), compute the power weights $\overline{\theta_{ba}}$ $(b, a = 1, 2, 3, 4)$, and obtain

$\overline{\theta_{11}} = 0.2452$, $\overline{\theta_{12}} = 0.236$, $\overline{\theta_{13}} = 0.2594$, $\overline{\theta_{14}} = 0.2594$,
$\overline{\theta_{21}} = 0.2462$, $\overline{\theta_{22}} = 0.2526$, $\overline{\theta_{23}} = 0.2487$, $\overline{\theta_{24}} = 0.2526$,
$\overline{\theta_{31}} = 0.2212$, $\overline{\theta_{32}} = 0.2626$, $\overline{\theta_{33}} = 0.2626$, $\overline{\theta_{34}} = 0.2535$,
$\overline{\theta_{41}} = 0.2686$, $\overline{\theta_{42}} = 0.2137$, $\overline{\theta_{43}} = 0.2686$, $\overline{\theta_{44}} = 0.2491$,

(8) From the Formula (37), we can determine each alternative's total evaluation value (let take $e = f = 2$), and obtain

$\tilde{y_1} = ([0.1663, 0.3238, 0.5849, 0.7061], [0, 0.2731, 0.4932, 0.621])$
$\tilde{y_2} = ([0.239, 0.3697, 0.5993, 0.7114], [0, 0.2941, 0.4776, 0.5919])$
$\tilde{y_3} = ([0.2839, 0.4568, 0.6526, 0.7524], [0, 0.3501, 0.5668, 0.6852])$
$\tilde{y_4} = ([0.297, 0.4786, 0.6882, 0.801], [0, 0.36, 0.5699, 0.7178])$

(9) Using Definition 6, compute the necessary score functions of TrIFN $\tilde{y_b}(b = 1,2,3,4)$, and obtain

L($\tilde{y_1}$) = 0.2949, L($\tilde{y_2}$) = 0.3366, L($\tilde{y_3}$) = 0.3882, and L($\tilde{y_4}$) = 0.4289.

Since all the values of score function L for all TrIFNs are unique, we can rank them according to their respective score values.

(10) Rank the alternatives

By using Definition 7, we obtain the ranking as follows,

$$Tr_4 > Tr_3 > Tr_2 > Tr_1$$

Note: Here the ranking is based on the membership score (L). We have used only the membership score function, since it differentiates all the TrIFNs given in the problem effectively. In general, we may need other score functions for proper ranking of alternatives.

5.2. Discussion

Further, in Table 4, we have shown a ranking outcome for the above discussed example for different values of e and f. We have considered e and f values in steps (5) and (8) to rank the alternatives.

Table 4. Ranking outcome for different values of e and f.

e and f	Score Function $\tilde{y_b}(b = 1,2,3,4)$	Ranking
$e = f = \frac{1}{2}$	L($\tilde{y_1}$) = 0.1476, L($\tilde{y_2}$) = 0.2327, L($\tilde{y_3}$) = 0.2017, L($\tilde{y_4}$) = 0.2563	$Tr_4 > Tr_2 > Tr_3 > Tr_1$
$e = \frac{1}{2}, f = 0$	L($\tilde{y_1}$) = 0.3373, L($\tilde{y_2}$) = 0.3664, L($\tilde{y_3}$) = 0.3934, L($\tilde{y_4}$) = 0.4723	$Tr_4 > Tr_3 > Tr_2 > Tr_1$
$e = 0, f = \frac{1}{2}$	L($\tilde{y_1}$) = 0.0555, L($\tilde{y_2}$) = 0.1918, L($\tilde{y_3}$) = 0.1174, L($\tilde{y_4}$) = 0.1474	$Tr_2 > Tr_4 > Tr_3 > Tr_1$
$e = 1, f = 0$	L($\tilde{y_1}$) = 0.3623, L($\tilde{y_2}$) = 0.3879, L($\tilde{y_3}$) = 0.4328, L($\tilde{y_4}$) = 0.5019	$Tr_4 > Tr_3 > Tr_2 > Tr_1$
$e = 0, f = 1$	L($\tilde{y_1}$) = 0.0929, L($\tilde{y_2}$) = 0.2132, L($\tilde{y_3}$) = 0.1658, L($\tilde{y_4}$) = 0.2052	$Tr_2 > Tr_4 > Tr_3 > Tr_1$
$e = 1, f = 1$	L($\tilde{y_1}$) = 0.2004, L($\tilde{y_2}$) = 0.2696, L($\tilde{y_3}$) = 0.2715, L($\tilde{y_4}$) = 0.3258	$Tr_4 > Tr_3 > Tr_2 > Tr_1$
$e = 1, f = 2$	L($\tilde{y_1}$) = 0.2313, L($\tilde{y_2}$) = 0.2883, L($\tilde{y_3}$) = 0.3156, L($\tilde{y_4}$) = 0.3692	$Tr_4 > Tr_3 > Tr_2 > Tr_1$
$e = 2, f = 1$	L($\tilde{y_1}$) = 0.2869, L($\tilde{y_2}$) = 0.3354, L($\tilde{y_3}$) = 0.3731, L($\tilde{y_4}$) = 0.4136	$Tr_4 > Tr_3 > Tr_2 > Tr_1$
$e = 1, f = 5$	L($\tilde{y_1}$) = 0.3615, L($\tilde{y_2}$) = 0.3781, L($\tilde{y_3}$) = 0.4614, L($\tilde{y_4}$) = 0.4996	$Tr_4 > Tr_3 > Tr_2 > Tr_1$
$e = 5, f = 1$	L($\tilde{y_1}$) = 0.4327, L($\tilde{y_2}$) = 0.461, L($\tilde{y_3}$) = 0.542, L($\tilde{y_4}$) = 0.5507	$Tr_4 > Tr_3 > Tr_2 > Tr_1$
$e = 5, f = 5$	L($\tilde{y_1}$) = 0.4539, L($\tilde{y_2}$) = 0.4695, L($\tilde{y_3}$) = 0.5691, L($\tilde{y_4}$) = 0.5781	$Tr_4 > Tr_3 > Tr_2 > Tr_1$

Table 4, Using the TrIFPWHA operator, we can see that the ranking outcomes for alternatives are different if we consider different parameters e and f for them. From the above result, we can say the best alternative is Tr_4 or Tr_2. By choosing the high values of e and f, there is more emphasis on the interactions of attribute values. For the linear weighting, we consider simple values like we can choose $e = f = 0$, or for simple computation, we can consider values like $e = f = 1$ or $e = f = \frac{1}{2}$. We can conclude that parameters e and f in the TrIFPWHA operator provide flexibility for decision-making. This flexibility will help choose the most suitable alternative in decision-making problems, as we can consider the best possible values of e and f for the decision-making problems.

5.3. Comparison of Proposed Method with the Existing Methods

The Advantages of the Proposed Method

(1) Comparison with the method proposed by Peide Liu and Ying Liu [33]

The Peide Liu and Ying Liu [33] method is built on PA operators, but the proposed method is based on PA operators as well as it considers relationships of the aggregate arguments. The PA operators help in minimizing the influence of irrational data. Below, we will show the result of the intuitionistic trapezoidal fuzzy power generalized weighted average (ITFPGWA) operator for different values of λ. Further, we will deliver the advantage (as discussed above) of the proposed method using the above-discussed example.

5.4. Results and Discussion

Using Table 5, we can see that the ITFPGWA operator does not have the impact of Λ. However, our proposed method has the impact of e and f values. Tr_2 as best alternative produce by ITFPGWA operator. As we know, the ITFPGWA operator did not consider the relationships of the aggregated attributes. Using Table 4, we can see that in the proposed methods, by choosing the high values of e and f, there is more emphasis on the interactions of attribute values. The proposed method that gives the best alternative is Tr_4 or Tr_2. Additionally, the ITFPGWA operator has only four parameters in its trapezoidal sets, i.e., it considers more generalized trapezoidal fuzzy numbers. It is denoted as

$$\tilde{p} = ([p, q, r, s]; u_{\tilde{p}}, v_{\tilde{p}})$$

However, the proposed method considers a more specific representation of the trapezoidal sets, and it considers the 8 parameter for its representation and is denoted as

$$\tilde{p} = ([p, q_1, q_2, r], [s, u_1, u_2, v])$$

So, it is evident that the proposed has more advantages in handling problems more precisely.

Table 5. The impact of the parameters Λ on the ranking outcome of example 1 for ITFPGWA operator.

Λ	**Expected Values $\tilde{y_b}(b = 1, 2, 3, 4)$**	**Ranking**
Λ = 0.5	I($\tilde{y_1}$) = 0.421, I($\tilde{y_2}$) = 0.563, I($\tilde{y_3}$) = 0.385, I($\tilde{y_4}$) = 0.501	$Tr_2 > Tr_4 > Tr_1 > Tr_3$
Λ = 1	I($\tilde{y_1}$) = 0.441, I($\tilde{y_2}$) = 0.58, I($\tilde{y_3}$) = 0.405, I($\tilde{y_4}$) = 0.527	$Tr_2 > Tr_4 > Tr_1 > Tr_3$
Λ = 2	I($\tilde{y_1}$) = 0.471, I($\tilde{y_2}$) = 0.608, I($\tilde{y_3}$) = 0.432, I($\tilde{y_4}$) = 0.562	$Tr_2 > Tr_4 > Tr_1 > Tr_3$
Λ = 3	I($\tilde{y_1}$) = 0.497, I($\tilde{y_2}$) = 0.63, I($\tilde{y_3}$) = 0.45, I($\tilde{y_4}$) = 0.588	$Tr_2 > Tr_4 > Tr_1 > Tr_3$
Λ = 4	I($\tilde{y_1}$) = 0.521, I($\tilde{y_2}$) = 0.648, I($\tilde{y_3}$) = 0.463, I($\tilde{y_4}$) = 0.608	$Tr_2 > Tr_4 > Tr_1 > Tr_3$
Λ = 5	I($\tilde{y_1}$) = 0.543, I($\tilde{y_2}$) = 0.661, I($\tilde{y_3}$) = 0.473, I($\tilde{y_4}$) = 0.624	$Tr_2 > Tr_4 > Tr_1 > Tr_3$
Λ = 6	I($\tilde{y_1}$) = 0.561, I($\tilde{y_2}$) = 0.672, I($\tilde{y_3}$) = 0.482, I($\tilde{y_4}$) = 0.637	$Tr_2 > Tr_4 > Tr_1 > Tr_3$
Λ = 7	I($\tilde{y_1}$) = 0.578, I($\tilde{y_2}$) = 0.682, I($\tilde{y_3}$) = 0.489, I($\tilde{y_4}$) = 0.648	$Tr_2 > Tr_4 > Tr_1 > Tr_3$
Λ = 8	I($\tilde{y_1}$) = 0.592, I($\tilde{y_2}$) = 0.689, I($\tilde{y_3}$) = 0.495, I($\tilde{y_4}$) = 0.657	$Tr_2 > Tr_4 > Tr_1 > Tr_3$
Λ = 10	I($\tilde{y_1}$) = 0.614, I($\tilde{y_2}$) = 0.701, I($\tilde{y_3}$) = 0.505, I($\tilde{y_4}$) = 0.671	$Tr_2 > Tr_4 > Tr_1 > Tr_3$
Λ = 5000	I($\tilde{y_1}$) = 0.25, I($\tilde{y_2}$) = 0.472, I($\tilde{y_3}$) = 0.234, I($\tilde{y_4}$) = 0.469	$Tr_2 > Tr_4 > Tr_1 > Tr_3$

Our proposed method is more compatible with real-world problems because, in MADM problems, there is usually a relationship among the attributes.

In a term, the method proposed can take advantage of the operators of the PA and BM, i.e., it can take into account the relationships of the aggregated claims and remove the power weighting effect of the unreasonable data, and it can provide a more logical ranking results than some current methods. Of course, the estimation of the suggested approach is somewhat complicated due to the simultaneous consideration of the PA and BM operators.

Depending upon the problem, aggregation operators have specific purposes. Some aggregation operators can mitigate the specific influences of incomplete and inadequate data generated by the decision-makers with different confidence levels. Aggregation operators, such as the PA operator, allocate the weighted vector based on the degree of support between the input arguments to aggregate the input data and accomplish this purpose. Certain aggregation operators consider the interrelationship among various aggregated arguments. More specifically, based on the complexity of many decision-making problems, we shall choose our aggregation operators, which provide more accurate, intuitive/believable results. In this paper, we selected the PA operator by assigning different weights to alleviate the influences of various parameters. Additionally, we considered the interrelationships between input values in a few cases, and then this function was completed by the Heronian mean/Bonferroni mean. The proposed aggregation operator/group decision-making algorithm can be used for many real-life problems. For example, the proposed group decision-making method can be applied to Supply Chain Management problems in manufacturing industries. Especially in recent years, many manufacturing organizations have thrived in the post-COVID-19 scenario dealing with incomplete and imprecise information. They are trying to perform well in the market by introducing/utilizing various strategies to tackle the present disruption. The supplier selection problem can be considered as a multi-criteria decision-making problem. If we consider the example defined here, then the selection of the best supplier will be based on different criteria such as cost, quality and social sustainability dimensions, etc. Here, a few qualitative criteria can be modelled better using TrIFNs. The traditional MCDM models or fuzzy MCDM models cannot solve problems with incomplete information. The proposed group decision-making model presents an effective way to select resilient suppliers under an intuitionistic fuzzy environment. To select the best supplier, we shall utilize the proposed group decision-making model, which utilizes TrIFPHA and TrIFWPHA operators.

6. Conclusions and Future Scope

In this paper, we merged the power average operator with the Heronian mean operator and introduced the trapezoidal intuitionistic fuzzy power Heronian aggregation (TrIFPHA) operator, and trapezoidal intuitionistic fuzzy power weighted Heronian aggregation (TrIFPWHA) operator. These operators can make full use of the benefits of the PA operator and the Heronian mean, i.e., they can take into account the relationships of the aggregated arguments and remove the power weighting influences of the unreasonable data. In addition, we studied some of the properties of these new aggregation operators, addressed some special cases, and developed a new approach based on these operators for solving MAGDM problems with TrIFNs. Finally, to demonstrate the efficacy of the developed method, we gave a numerical example and explained its benefits by contrasting it with the existing methods. In future studies, we may extend the idea of the proposed model to the class of interval-valued Fermatean fuzzy sets and Q-rung orthopair fuzzy sets, which are generalizations of interval-valued intuitionistic fuzzy numbers. Additionally, we shall define a power aggregation operator on several new classes of fuzzy numbers, such as neutrosophic numbers (Liu et al. [33,34]), interval type-2 fuzzy sets (Celik et al. [35]), and hesitant fuzzy numbers (Hu et al. [36]). In this framework, we have used the existing similarity measures. The available similarity measures are defined by using distance measures, and many of the available distance measures are not ultra-metric. So, one can think of introducing a new similarity measure based on ultra-metric, and they can change the existing similarity measures with the ultra-metric similarity measure and study the performance of a decision-making algorithm. We will also explore some applications in supply chain management, and energy and environment evaluations (Huang and Yu, [37]; Liu et al. [38]; Shaw and Roy, [39]; Zha and Kavuri [40]) simultaneously.

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