Article

Some New Integral Inequalities Involving Fractional Operator with Applications to Probability Density Functions and Special Means

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Abstract: Fractional calculus manages the investigation of supposed fractional derivatives and integrations over complex areas and their applications. Fractional calculus is the purported assignment of differentiations and integrations of arbitrary non-integer order. Lately, it has attracted the attention of several mathematicians because of its real-life applications. More importantly, it has turned into a valuable tool for handling elements from perplexing frameworks within different parts of the pure and applied sciences. Integral inequalities, in association with convexity, have a strong relationship with symmetry. The objective of this article is to introduce the notion of operator refined exponential type convexity. Fractional versions of the Hermite–Hadamard type inequality employing generalized R – L fractional operators are established. Additionally, some novel refinements of Hermite–Hadamard type inequalities are also discussed using some established identities. Finally, we present some applications of the probability density function and special means of real numbers.

Keywords: Hermite–Hadamard inequality; refined exponential convex function; hypergeometric function; power mean inequality; fractional integral operator; probability density function; Hölder inequality

1. Introduction

In the recent past, the theory of inequalities via different types of convexities has made significant contributions to many areas of mathematics. The theory of inequalities plays a significant role in various branches of mathematics such as optimization, differential equations, functional analysis, probability, numerical analysis, finite element method, fractional calculus, etc. Moreover, convexity implies a significant interest in mathematical inequalities because of the nature of its definition. Variants of convex functions play a critical role in several branches of sciences such as biological systems, economics, and optimization. Mathematical inequalities for various convex functions have been generalized colossally, having a critical impact on traditional investigation. Symmetry, convexity, and fractional operator have a very strong connection because of their fascinating properties. Whichever one we work on, it can be applied to the accompanying one because of the solid relationship passed on between them. One can refer to the references [1–8] for different types of convexities and related inequalities.
In the literature, inequalities of the Simpson, Fejer, Ostrowski, Oslon, Hardy, and midpoint type are studied for different classes of convex functions. The celebrated Hermite–Hadamard double inequality for convex function on an interval of the real line, discovered by C. Hermite [9] and J. Hadamard [10] has been the topic of extensive research.

The novelty of this article is that a new notion is introduced and, employing this new definition, classical and fractional inequalities are derived. This article is arranged by C. Hermite [9] and J. Hadamard [10] has been the topic of extensive research.

Definition 1. [11] For an interval $I$ in $\mathbb{R}$, a function $G : I \rightarrow \mathbb{R}$ is said to be convex on $I$ if,

$$G(ud_1 + (1-u)d_2) \leq uG(d_1) + (1-u)G(d_2),$$

holds for all $d_1, d_2 \in I$, and $u \in [0,1]$.

Let $G : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex function with $d_1 < d_2$ and $d_1, d_2 \in I$. Then, the H–H inequality is expressed as follows (see [10]): Let a function $G : I \rightarrow \mathbb{R}$ be a convex function on $I$ in $\mathbb{R}$ and $d_1, d_2 \in I$ with $d_1 < d_2$, then,

$$G \left( \frac{d_1 + d_2}{2} \right) \leq \frac{1}{d_2 - d_1} \int_{d_1}^{d_2} G(x)dx \leq \frac{G(d_1) + G(d_2)}{2}. \quad (1)$$

Tunç et al. [12] established new versions of Hermite–Hadamard inequality using both classical as well as fractional integral operators for $tgs$-convex functions as follows:

Let $G : I \rightarrow \mathbb{R}$ be a $tgs$-convex function on $I$ in $\mathbb{R}$ and $d_1, d_2 \in I$ with $d_1 < d_2$, then,

$$2G \left( \frac{d_1 + d_2}{2} \right) \leq \frac{1}{d_2 - d_1} \int_{d_1}^{d_2} G(x)dx \leq \frac{G(d_1) + G(d_2)}{6}. \quad (2)$$

The corresponding Hermite–Hadamard inequality via R-L fractional operator is expressed as:

Let $G : I \subseteq \mathbb{R}$ be a positive function with $d_1 < d_2$ and $L_1[d_1, d_2]$. If $G$ is a $tgs$-convex function on $[d_1, d_2]$, then the following inequalities for fractional integrals hold:

$$2G \left( \frac{d_1 + d_2}{2} \right) \leq \frac{\Gamma(\lambda + 1)}{2(d_2 - d_1)\lambda} \left[ t_{d_1}^\lambda G(d_2) + t_{d_2}^\lambda G(d_1) \right] \leq \frac{\lambda[G(d_1) + G(d_2)]}{2(\lambda + 1)(\lambda + 2)}. \quad (3)$$

For some recent generalizations on various forms of convex functions, we refer interested readers to see [13–16] and the references cited therein.

After reviewing the hypothesis of convexity and its generalizations in the theory of inequality, we learned about a new class of convexity called exponential convexity. Due to its applications in various fields such as big-data analysis, deep learning, and information theory, several researchers have shown their deep interest in exponential convexity. Study on big-data analysis and deep learning has recently expanded the adequacy of information theory involving exponentially convex functions (see [17,18]). Especially in the last few decades, different mathematicians worked on the idea of exponential-type convexity in various directions and contributed to the field of analysis such as Jakšetić et al. [19], Dragomir et al. [20], and Awan et al. [21] who presented Hermite–Hadamard type inequalities for exponential convex functions. Saima et al. [22] proved fractional versions of the Hermite–Hadamard inequality for exponential convex functions. Furthermore, Noor
et al. [23] generalized the exponential convex function to the exponential preinvex function and discussed some of its properties. In 2020, Kadakal et al. [24] introduced a new notion of exponential type convex function and generalized some known integral inequalities. Kadakal et al. [24], introduced the concept of exponential type convex function that generalizes convex functions.

**Definition 2.** [24] A function \( G : \mathbb{I} \rightarrow \mathbb{R} \subseteq \mathbb{R} \) is said to be an exponential type convex function, if \( G \) is non-negative, for all \( d_1, d_2 \in \mathbb{I} \) and \( u \in [0, 1] \), we have

\[
G(ud_1 + (1-u)d_2) \leq (e^u - 1)G(d_1) + (e^{1-u} - 1)G(d_2). \tag{3}
\]

Very recently Jung et al. [25], introduced the concept of refined \((\alpha, h-m)\) convex function, given as

\[
G(ud_1 + m(1-u)d_2) \leq h(u^\alpha)h(1-u^\alpha)[G(d_1) + mG(d_2)]. \tag{4}
\]

The above definition generalizes some well-known convexities such as \((\alpha, m)\)-convex function, \(m\)-convex function, \((s, m)\)-convex function, refined \(h\)-convex, refined \((h - m)\) convex, refined \((s, m)\)-convex functions, etc.

In the last decade, fractional calculus has gotten a lot of consideration. This topic has attracted the interest of many researchers because of its broad applications in different fields such as probability theory, biomathematics, image processing, fluid mechanics, material science, viscoelasticity and designing, etc. In recent times it is seen that several mathematicians utilize their notations and approaches to study a variety of definitions that fit the possibility of fractional-order integrals and derivatives. The form that is discussed most in the realm of fractional calculus is the R-L operator and its variants. It is interesting to note that the R-L meaning of a fractional derivative gives us the same outcome as that acquired by Lacroix [26].

In the modern era, fractional analysis and inequality theory have developed together. A fundamental component of applied sciences and mathematics is fractional calculus. Academics urge many students to think about applying fractional calculus to solve difficulties in the real world. The Hermite–Hadamard type integral inequalities [27], Hermite–Hadamard–Mercer inequalities [28], the Ostrowski inequality [29], and the Simpson type inequality [30] have all been used studying the Riemann–Liouville fractional integral operators. The Simpson–Mercer integral inequality was studied utilizing the Atangana–Baleanu fractional operator in [31]. The Hermite–Hadamard inequality and the Fejér type integral inequalities were investigated via Katugampola type fractional integral operators in [32]. The Hermite–Hadamard inequality and its Mercer equivalent were also examined using the Caputo–Fabrizio fractional integrals [33,34]. The data described above demonstrates the strong connection between integral inequalities and fractional operators.

To additionally encourage the conversation started in this article, we present the definition of the R-L fractional operator and \(\psi\)-R-L fractional operator.

**Definition 3.** [27] Let \([d_1, d_2] \rightarrow \mathbb{R}\). Then, R-L fractional integrals \(I_{d_1}^\lambda G(u)\) and \(I_{d_2}^\lambda G(u)\) of order \(\lambda > 0\) are defined by

\[
I_{d_1}^\lambda G(z) = \frac{1}{\Gamma(\lambda)} \int_{d_1}^{z} (z-u)^{\lambda-1}G(u)du
\]

and

\[
I_{d_2}^\lambda G(z) = \frac{1}{\Gamma(\lambda)} \int_{z}^{d_2} (u-z)^{\lambda-1}G(u)du,
\]

where, \(\Gamma(\cdot)\) is the Gamma function.
Definition 4. [35] Let \([d_1, d_2] \rightarrow \mathbb{R}\). Then, the \(\psi\)-R-L fractional integrals \(I_{d_1}^{\lambda, \psi} G(z)\) and \(I_{d_2}^{\lambda, \psi} G(z)\) of order \(\lambda > 0\) are defined by

\[
I_{d_1}^{\lambda, \psi} G(z) = \frac{1}{\Gamma(\lambda)} \int_{d_1}^z \psi(u)(\psi(z) - \psi(u))^{\lambda-1} G(u) du
\]

and

\[
I_{d_2}^{\lambda, \psi} G(z) = \frac{1}{\Gamma(\lambda)} \int_{d_2}^z \psi(u)(\psi(u) - \psi(z))^{\lambda-1} G(u) du,
\]

where, \(\Gamma(.)\) is the Gamma function.

In [36], Sariyaka and Yildirm proved the following Hadamard type inequalities for R-L fractional integrals as follows:

Theorem 1. [36] Let \(G : [d_1, d_2] \rightarrow \mathbb{R}\) be a positive mapping with \(0 \leq d_1 \leq d_2\), \(G \in L[d_1, d_2]\), and \(I_{d_1}^{\lambda} G\) and \(I_{d_2}^{\lambda} G\) be fractional operators. Then, the following inequality for fractional integral holds if \(G\) is a convex function.

\[
G \left(\frac{d_1 + d_2}{2}\right) \leq \frac{2^{\lambda+1} \Gamma(\lambda + 1)}{(d_2 - d_1)^{\lambda}} \left[ I_{d_1}^{\lambda} G(d_2) + I_{d_2}^{\lambda} G(d_1) \right] \leq \frac{G(d_1) + G(d_2)}{2}.
\]

Theorem 2. [27] Let \(G : [d_1, d_2] \rightarrow \mathbb{R}\) be a positive mapping with \(0 \leq d_1 \leq d_2, G \in L[d_1, d_2]\), and \(I_{d_1}^{\lambda} G\) and \(I_{d_2}^{\lambda} G\) be fractional operators. Then the following inequality for fractional integral holds if \(G\) is a convex function.

\[
G \left(\frac{d_1 + d_2}{2}\right) \leq \frac{\Gamma(\lambda + 1)}{2(d_2 - d_1)^{\lambda}} \left[ I_{d_1}^{\lambda} G(d_2) + I_{d_2}^{\lambda} G(d_1) \right] \leq \frac{G(d_1) + G(d_2)}{2}.
\]

To acquire some knowledge about novel refinements of the Hermite–Hadamard inequality via some new type of fractional operators, interested readers can refer to [37–42].

Confluent Hypergeometric function

For \(\Re(d_1), \Re(d_2) \geq 0\) such that \(d_1 < d_2\) and \(|k| \leq 1\), the following identities hold.

\[
_{1}F_{1}(d_1, d_2, k) = M(d_1, d_2, k) = \frac{\Gamma(d_2)}{\Gamma(d_1)\Gamma(d_2 - d_1)} \int_{0}^{1} e^{ku} u^{d_1-1} (1 - u)^{d_2-d_1-1} du
\]

Remark 1. The following identities will be used in the results to follow.

\[
_{1}F_{1} \left(\lambda, \lambda + 1, \frac{1}{2}\right) = \lambda \int_{0}^{1} e^{u} u^{\lambda-1} du.
\]

\[
_{1}F_{1} \left(1, \lambda + 1, 1\right) = \lambda \int_{0}^{1} e^{1-u} u^{\lambda-1} du.
\]

\[
\frac{\sqrt{\pi}}{\lambda} _{1}F_{1} \left(1, \lambda + 1, \frac{1}{2}\right) = \int_{0}^{1} e^{1-u} u^{\lambda-1} du.
\]

The fractional integral inequalities, the definition of refined \((a, h - m)\) convex, and exponential type convexity are the motivation of our results for this paper. In this article, a new notion of a generalized exponential convex function, i.e., refined exponential type convex function is studied. Employing this new notion, we generalized the \(H - H\) inequality via both classical integral and fractional integral operators. Some novel refinements of the \(H - H\) type inequality via generalized \(R - L\) and \(\psi - R - L\) fractional integral operators.
are discussed as well. This article also deals with applications to the probability density function and special means.

3. Refined Exponential Type Convexity and H — H Type Inequalities

Now, we introduce a new notion of convex function, i.e., a refined exponential type convex function, and establish some results based on the said convexity.

**Definition 5.** A function $G : I \rightarrow \mathbb{R}$ is said to be a refined exponential type convex function if for every $d_1, d_2 \in I$ and $u \in (0, 1)$

$$G(ud_1 + (1-u)d_2) \leq (e^u - 1)(e^{1-u} - 1)[G(d_1) + G(d_2)].$$ (5)

**Lemma 1.** The inequalities $u \leq e^u - 1$, $1 - u \leq e^{1-u} - 1$, and $u(1-u) \leq (e^u - 1)(e^{1-u} - 1)$ hold true for all $u \in (0, 1)$.

**Proposition 1.** Every non-negative tgs convex function is a refined exponential type convex function.

**Proof.** Using Lemma 1, we have $u \leq e^u - 1$ and $1 - u \leq e^{1-u} - 1$. This implies that

$$G(ud_1 + (1-u)d_2) \leq u(1-u)[G(d_1) + G(d_2)] \leq (e^u - 1)(e^{1-u} - 1)[G(d_1) + G(d_2)].$$

Next, we prove the classical version of the Hermite–Hadamard inequality corresponding to the introduced new notion of refined exponential type convex function.

**Theorem 3.** Suppose $G : [d_1, d_2] \rightarrow \mathbb{R}$ be a mapping with $0 \leq d_1 \leq d_2$ and $G \in \mathcal{L}[d_1, d_2]$. If $G$ is a refined exponential type convex function, then the following inequality holds:

$$\frac{1}{2(e^{1/2} - 1)^2}G\left(\frac{d_1 + d_2}{2}\right) \leq \frac{1}{d_2 - d_1} \int_{d_1}^{d_2} G(x)dx \leq [G(d_1) + G(d_2)]\left[3 - e^{1/2}\right].$$ (6)

**Proof.** Let $u \in [0, 1]$, consider $x, y \in [d_1, d_2], \ d_1 \geq 0$.

As $G$ is a refined exponential type convex function, we have

$$G(ux + (1-u)y) \leq (e^u - 1)(e^{1-u} - 1)[G(x) + G(y)].$$

Consequently, putting $u = \frac{1}{2}$

$$G\left(\frac{x + y}{2}\right) \leq \left(e^{1/2} - 1\right)^2 [G(x) + G(y)].$$ (7)

Putting $x = ud_1 + (1-u)d_2$ and $y = (1-u)d_1 + ud_2$ in (7), and finally integrating the consequent inequality w.r.t to $u$ over $[0, 1]$, we obtain

$$G\left(\frac{d_1 + d_2}{2}\right) \leq \left(e^{1/2} - 1\right)^2 \int_0^1 [G(ud_1 + (1-u)d_2) + G((1-u)d_1 + ud_2)]du$$

$$\leq \left(e^{1/2} - 1\right)^2 \frac{2}{d_2 - d_1} \int_{d_1}^{d_2} G(x)dx.$$ (8)
This completes the proof of the first part and to complete the second part, we use the definition of refined exponential type convex function:

$$
\mathcal{G}(ud_1 + (1-u)d_2) \leq (e^u - 1)(e^{1-u} - 1)[\mathcal{G}(d_1) + \mathcal{G}(d_2)].
$$

Using the change of the variable technique and then integrating with respect to \(u\) over \([0,1]\), we obtain

$$
\frac{1}{d_2 - d_1} \int_{d_1}^{d_2} \mathcal{G}(x)dx \leq [\mathcal{G}(d_1) + \mathcal{G}(d_2)] \int_{0}^{1} (e^u - 1)(e^{1-u} - 1)du
\leq [\mathcal{G}(d_1) + \mathcal{G}(d_2)] (3 - e).
$$

From (8) and (9), we establish the desired result (6). \(\square\)

Now, we prove some fractional versions of the Hermite–Hadamard inequality corresponding to the refined exponential type convex function via Riemann–Liouville and \(\psi\)-Riemann–Liouville fractional integral operator. The proven results show that we can apply different fractional operators for these types of convexities. The results also show the application of hypergeometric functions for inequalities as well.

**Fractional Inequalities of \(H - H\) Type**

**Theorem 4.** Let \(I_{d_1}^\lambda, \mathcal{G}\) and \(I_{d_2}^\lambda, \mathcal{G}\) be R-L fractional operators. Moreover, suppose \(\mathcal{G} : [d_1,d_2] \rightarrow \mathbb{R}\) is a mapping with \(0 \leq d_1 \leq d_2\) and \(\mathcal{G} \in L[d_1,d_2]\). If \(\mathcal{G}\) is a refined exponential type convex function, then the following inequality holds:

$$
\frac{1}{(e^\frac{x}{2} - 1)^2} \mathcal{G}\left(\frac{d_1 + d_2}{2}\right) \leq \Gamma(\lambda + 1) \left[I_{d_1}^\lambda \mathcal{G}(d_2) + I_{d_2}^\lambda \mathcal{G}(d_1)\right] \leq 2[\mathcal{G}(d_1) + \mathcal{G}(d_2)][e - \frac{1}{1}F_1(\lambda, \lambda + 1, 1) - \frac{1}{1}F_1(1, \lambda + 1, 1)].
$$

**Proof.** As \(\mathcal{G}\) is a refined exponential type convex function, one has

$$
\mathcal{G}\left(\frac{x + y}{2}\right) \leq (\sqrt{e} - 1)^2 [\mathcal{G}(x) + \mathcal{G}(y)].
$$

Putting \(x = ud_1 + (1-u)d_2\) and \(y = (1-u)d_1 + ud_2\) in (11), then multiplying both sides of the resultant inequality by \(u^{\lambda - 1}\), and finally integrating w.r.t to \(u\) over \([0,1]\), we obtain

$$
\mathcal{G}\left(\frac{d_1 + d_2}{2}\right) \int_{0}^{1} u^{\lambda - 1}du
\leq (\sqrt{e} - 1)^2 \left[\int_{0}^{1} u^{\lambda - 1}\mathcal{G}(ud_1 + (1-u)d_2)du + \int_{0}^{1} u^{\lambda - 1}\mathcal{G}(ud_2 + (1-u)d_1)du\right].
$$

Consequently,

$$
\frac{1}{(\sqrt{e} - 1)^2} \mathcal{G}\left(\frac{d_1 + d_2}{2}\right)
\leq \frac{\lambda}{\Gamma(\lambda + 1) (d_2 - d_1)} \left[I_{d_1}^\lambda \mathcal{G}(d_2) + I_{d_2}^\lambda \mathcal{G}(d_1)\right].
$$
This proves the first part of the theorem. For the second part of the inequality we use the definition of refined exponential type convexity of \( G \), i.e.,

\[
G(ud_1 + (1 - u)d_2) \leq (e^u - 1)(e^{1-u}) - 1)G(d_1) + G(d_2),
\]

and

\[
G(ud_2 + (1 - u)d_1) \leq (e^u - 1)(e^{1-u}) - 1)G(d_1) + G(d_2).
\]

Adding the last two inequalities and then following the same procedure, we obtain

\[
\int_0^1 [G(ud_1 + (1 - u)d_2) + G(ud_2 + (1 - u)d_1)]u^{\lambda-1}du
\]

\[
\leq \int_0^1 2(e^u - 1)(e^{1-u}) - 1)G(d_1) + G(d_2)]u^{\lambda-1}du.
\]

Using definition 3, we have

\[
\frac{\Gamma(\lambda + 1)}{(d_2 - d_1)^\lambda} \left[ I_{d_1}^{\lambda + 1}G(d_2) + I_{d_2}^{\lambda + 1}G(d_1) \right]
\]

\[
\leq 2[G(d_1) + G(d_2)]\left[e - 1 F_1(\lambda, \lambda + 1, 1) - 1 F_1(1, \lambda + 1, 1) + 1 \right].
\]

This completes the proof of Theorem 5. □

It is evident from articles [34,36,43–46] that many researchers are now focusing on the mid-point type inequalities corresponding to the Hermite–Hadamard inequality. Our next result is aimed toward this as well.

**Theorem 5.** Let \( I_{d_1}^{\lambda} \) and \( I_{d_2}^{\lambda} \) be the R-L fractional operator. Moreover, suppose \( G : [d_1, d_2] \rightarrow \mathbb{R} \) is a mapping with \( 0 \leq d_1 \leq d_2 \) and \( G \in \mathcal{L}(d_1, d_2) \). If \( G \) is a refined exponential type convex function, then the following inequality holds:

\[
\frac{1}{(e^\frac{x}{2} - 1)^2} G\left(\frac{d_1 + d_2}{2}\right) \leq \frac{2}{(d_2 - d_1)^\lambda} \left[ I_{d_1}^{\lambda + 1}G(d_2) + I_{d_2}^{\lambda + 1}G(d_1) \right]
\]

\[
\leq 2[G(d_1) + G(d_2)]\left[e - 1 F_1(\lambda, \lambda + 1, 1) - 1 F_1(1, \lambda + 1, 1) + 1 \right].
\]

**Proof.** Since \( G \) is a refined exponential type convex function, one has

\[
G\left(\frac{x + y}{2}\right) \leq (\sqrt{e} - 1)^2 [G(x) + G(y)].
\]

Putting \( x = \frac{d_1}{2} + \frac{1}{2}d_1 \) and \( y = \frac{d_2}{2}d_2 + \frac{1}{2}d_2 \) in (15) and then multiplying both sides by \( u^{\lambda-1} \) and finally integrating with respect to \( u \) over \([0,1] \), we have

\[
G\left(\frac{d_1 + d_2}{2}\right) \int_0^1 u^{\lambda-1}du
\]

\[
\leq (\sqrt{e} - 1)^2 \left[ \int_0^1 u^{\lambda-1}G\left(\frac{u}{2}d_1 + \frac{1}{2}d_2\right)du + \int_0^1 u^{\lambda-1}G\left(\frac{2 - u}{2}d_1 + \frac{u}{2}d_2\right)du \right].
\]
Consequently,
\[
\frac{1}{(\sqrt{e} - 1)^2} G\left(\frac{d_1 + d_2}{2}\right)
\leq \lambda 2^\lambda \left[ \int_{d}^{d_1 + d_2} \left( \frac{d_2 - z}{d_2 - d_1} \right)^{\lambda - 1} G(z) \frac{1}{d_1 - d_2} dz + \int_{d}^{d_1 + d_2} \left( \frac{w - d_1}{d_2 - d_1} \right)^{\lambda - 1} G(w) \frac{1}{d_2 - d_1} dw \right]
= \frac{2^\lambda \Gamma(\lambda + 1)}{(d_2 - d_1)^2} I^\lambda_{d_1 + d_2} G(d_2) + I^\lambda_{d_1 + d_2} G(d_1).
\]

This completes the proof of the first part. Next, to prove the second part we use the definition of refined exponential type convexity of $G$, i.e.,
\[
G\left(\frac{u}{2} d_1 + \frac{2 - u}{2} d_2\right) \leq (e^x - 1)(e^{\frac{1}{2}u} - 1)[G(d_1) + G(d_2)],
\]
and
\[
G\left(\frac{u}{2} d_2 + \frac{2 - u}{2} d_1\right) \leq (e^x - 1)(e^{\frac{1}{2}u} - 1)[G(d_1) + G(d_2)].
\]

Upon adding the last two inequalities and then following the same procedure as above, we obtain
\[
\frac{2^\lambda \Gamma(\lambda + 1)}{(d_2 - d_1)^2} I^\lambda_{d_1 + d_2} G(d_2) + I^\lambda_{d_1 + d_2} G(d_1)
\leq 2[G(d_1) + G(d_2)] \left[ e - \text{I}_1 F_1\left(\lambda, \lambda + 1, \frac{1}{2}\right) - e^\frac{1}{2} I_1 F_1\left(1, \lambda + 1, \frac{1}{2}\right) + 1 \right].
\]

Equations (16) and (17) lead to the proof of Theorem 5.  

4. Further Estimates on $H - H$ Inequalities

In the following theorems, we prove some trapezoidal type inequalities with the help of some classical inequalities such as Hölder’s inequality, Young’s inequality, and power mean inequality via Riemann–Liouville fractional operator.

**Lemma 2** ([27]). Let $G : [d_1, d_2] \to \mathbb{R}$ be a differentiable mapping on $I^0$, where $d_1, d_2 \in I^0$ with $0 \leq d_1 \leq d_2$. Then the following equality holds:
\[
\frac{G(d_1) + G(d_2)}{2} - \frac{\Gamma(\lambda + 1)}{2(d_2 - d_1)\lambda} \left[ I^\lambda_{d_1} G(d_2) + I^\lambda_{d_2} G(d_1) \right]
= \frac{d_2 - d_1}{2} \int_0^1 \left[ (1 - u)\lambda - u\lambda \right] G'(u)(d_1 + (1 - u)d_2) du.
\]

**Lemma 3** ([37]). Let $G : [d_1, d_2] \to \mathbb{R}$ be a differentiable mapping on $I^0$, where $d_1, d_2 \in I^0$ with $0 \leq d_1 \leq d_2$. Then the following equality holds:
\[
\frac{G(a) - G(d_1)}{d_1 - d_2} + \frac{\Gamma(\lambda + 1)}{(d_2 - d_1)\lambda} \left[ I^\lambda_{d_1} G(d_2) - I^\lambda_{d_2} G(d_1) \right]
= \int_0^1 u^\lambda G'(u)(d_1 + (1 - u)d_2) du + \int_0^1 (1 - u)^\lambda G'(u)(d_1 + (1 - u)d_2) du.
\]
Let \( G : [d_1, d_2] \rightarrow \mathbb{R} \) be a differentiable mapping with \( 0 \leq d_1 \leq d_2 \) and \( I_{d_1}^\lambda G, I_{d_2}^\lambda G \) be R-L fractional operators. If \( |G'| \) is a refined exponential type convex function, then the following inequality holds:

\[
\frac{G(d_1) + G(d_2)}{2} - \frac{\Gamma(\lambda + 1)}{2(d_2 - d_1)^\lambda} \left[ I_{d_1}^\lambda G(d_2) + I_{d_2}^\lambda G(d_1) \right] \\
\leq \frac{(d_2 - d_1)(|G'(d_1)| + |G'(d_2)|)}{2} \left[ e - iF_1(\lambda + 1, \lambda + 2, 1) - iF_1(1, \lambda + 1, 1) \right].
\]

**Proof.** Using Lemma 2, the refined exponential type convexity of \(|G|\), we have

\[
\frac{|G(a) + G(d_1)|}{2} - \frac{\Gamma(\lambda + 1)}{2(d_2 - d_1)^\lambda} \left[ I_{d_1}^\lambda G(d_2) + I_{d_2}^\lambda G(d_1) \right] \\
\leq \frac{d_2 - d_1}{2} \int_0^1 [(1 - u)^\lambda + u^\lambda] |G'(ud_1 + (1 - u)d_2)| \, du \\
\leq \frac{d_2 - d_1}{2} \left( \int_0^1 [(1 - u)^\lambda + u^\lambda](a^\lambda - 1)(a^\lambda - 1) [ |G'(d_1)| + |G'(d_2)| ] \, du \right) \\
\leq \frac{(d_2 - d_1)(|G'(d_1)| + |G'(d_2)|)}{2} \left[ e - iF_1(\lambda + 1, \lambda + 2, 1) - iF_1(1, \lambda + 1, 1) \right].
\]

This completes the proof. \( \square \)

**Theorem 7.** Let \( G : [d_1, d_2] \rightarrow \mathbb{R} \) be a differentiable mapping with \( 0 \leq d_1 \leq d_2 \) and \( I_{d_1}^\lambda G, I_{d_2}^\lambda G \) be R-L fractional operators. For \( q > 1 \), if \( |G'|^q \) is a refined exponential type convex function, then the following inequality holds:

\[
\frac{G(d_1) + G(d_2)}{2} - \frac{\Gamma(\lambda + 1)}{2(d_2 - d_1)^\lambda} \left[ I_{d_1}^\lambda G(d_2) + I_{d_2}^\lambda G(d_1) \right] \\
\leq (d_2 - d_1) \left[ \frac{1}{p(\lambda p + 1)} (3 - e) ||G'(d_1)||^q + ||G'(d_2)||^q \right].
\]

**Proof.** Using Lemma 2, the refined exponential type convexity of \(|G|\), we have

\[
\frac{|G(a) + G(d_1)|}{2} - \frac{\Gamma(\lambda + 1)}{2(d_2 - d_1)^\lambda} \left[ I_{d_1}^\lambda G(d_2) + I_{d_2}^\lambda G(d_1) \right] \\
\leq \frac{d_2 - d_1}{2} \int_0^1 [(1 - u)^\lambda + u^\lambda] |G'(ud_1 + (1 - u)d_2)| \, du \\
\leq \frac{d_2 - d_1}{2} \left( \int_0^1 (1 - u)^\lambda |G'(ud_1 + (1 - u)d_2)| \, du + \int_0^1 u^\lambda |G'(ud_1 + (1 - u)d_2)| \, du \right)
\]

Using Young’s inequality \( ab \leq \frac{1}{p} a^p + \frac{1}{q} b^q \), where \( \frac{1}{p} + \frac{1}{q} = 1 \)

\[
\leq \frac{d_2 - d_1}{2} \left[ \left( \frac{1}{p} \int_0^1 (1 - u)^p \, du + \frac{1}{q} |G'(ud_1 + (1 - u)d_2)|^q \, du \right) \\
+ \left( \frac{1}{p} \int_0^1 u^p \, du + \frac{1}{q} |G'(ud_1 + (1 - u)d_2)|^q \, du \right) \right] \\
\leq (d_2 - d_1) \left[ \frac{1}{p(\lambda p + 1)} (3 - e) ||G'(d_1)||^q + ||G'(d_2)||^q \right].
\]

This completes the proof. \( \square \)
Theorem 8. Let $G: [d_1, d_2] \rightarrow \mathbb{R}$ be a differentiable mapping with $0 \leq d_1 \leq d_2$ and $I^\lambda_{d_2^{-}} G$, $I^\lambda_{d_1^{+}} G$ be R-L fractional operators. For $q \geq 1$, if $|G'|^q$ is a refined exponential type convex function, then the following inequality holds:

$$
\frac{G(d_1) + G(d_2)}{2} - \frac{\Gamma(\lambda + 1)}{2(d_2 - d_1) \lambda} \left[ I^\lambda_{d_1^{+}} G(d_2) + I^\lambda_{d_2^{-}} G(d_1) \right]
\leq \frac{d_2 - d_1}{2} \left( \frac{2}{\lambda + 1} \right)^{1 - \frac{1}{q}}
\times \left( |G'(d_1)|^q + |G'(d_2)|^q \right) \left[ \left| e^{-1} F_1(\lambda + 1, \lambda + 2, 1) - 1 F_1(1, \lambda + 1, 1) + 1 \right| \right]^{\frac{1}{q}}.
$$

Proof. Using Lemma 2, the refined exponential type convexity and power-mean inequality, we have

$$
\left| \frac{G(a) + G(d_1)}{2} - \frac{\Gamma(\lambda + 1)}{2(d_2 - d_1) \lambda} \left[ I^\lambda_{d_1^{+}} G(d_2) + I^\lambda_{d_2^{-}} G(d_1) \right] \right|
\leq \frac{d_2 - d_1}{2} \int_0^1 \left| (1-u) - u^\lambda \right| |G''(ud_1 + (1-u)d_2)| du
\leq \frac{d_2 - d_1}{2} \int_0^1 \left| (1-u) - u^\lambda \right| |G''(d_1)|^q + |G''(d_2)|^q du
d^{2 - \frac{1}{q}} \left( \frac{2}{\lambda + 1} \right)^{1 - \frac{1}{q}}
\times \left( |G'(d_1)|^q + |G'(d_2)|^q \right) \left[ \left| e^{-1} F_1(\lambda + 1, \lambda + 2, 1) - 1 F_1(1, \lambda + 1, 1) + 1 \right| \right]^{\frac{1}{q}}.
$$

This completes the proof. \(\square\)

Theorem 9. Let $G: [d_1, d_2] \rightarrow \mathbb{R}$ be a positive mapping with $0 \leq d_1 \leq d_2$ and $I^\lambda_{d_2^{-}} G$, $I^\lambda_{d_1^{+}} G$ be R-L fractional operators. If $G \in \mathcal{L}[d_1, d_2]$ and is a refined exponential type convex function, then the following inequality holds:

$$
\frac{G(d_1) - G(d_2)}{d_1 - d_2} + \frac{\Gamma(\lambda + 1)}{(d_2 - d_1) \lambda} \left[ I^\lambda_{d_1^{+}} G(d_2) - I^\lambda_{d_2^{-}} G(d_1) \right]
\leq \frac{\left| G'(d_1) \right| + \left| G'(d_2) \right| \left[ \left| e^{-1} F_1(\lambda + 1, \lambda + 2, 1) - 1 F_1(1, \lambda + 1, 1) + 1 \right| \right]}{\lambda + 1}.
$$

Proof. Taking Lemma 3 and the refined exponential type convexity of $|G|$ into consideration, we have

$$
\left| \frac{G(d_1) - G(d_2)}{d_1 - d_2} + \frac{\Gamma(\lambda + 1)}{(d_2 - d_1) \lambda} \left[ I^\lambda_{d_1^{+}} G(d_2) - I^\lambda_{d_2^{-}} G(d_1) \right] \right|
\leq \int_0^1 u^\lambda |G''(ud_1 + (1-u)d_2)| du + \int_0^1 (1-u)^\lambda |G''(ud_1 + (1-u)d_2)| du.
\leq \left| G'(d_1) \right| + \left| G'(d_2) \right| \left( \int_0^1 u^\lambda (e^u - 1)(e^{1-u} - 1) du + \int_0^1 (1-u)^\lambda (e^u - 1)(e^{1-u} - 1) du \right)
= \left| G'(d_1) \right| + \left| G'(d_2) \right| \frac{\left[ \left| e^{-1} F_1(\lambda + 1, \lambda + 2, 1) - 1 F_1(1, \lambda + 1, 1) + 1 \right| \right]}{\lambda + 1}.
$$
Theorem 10. Let \( G : [d_1, d_2] \to \mathbb{R} \) be a positive mapping with \( 0 \leq d_1 \leq d_2 \) and \( I^\lambda_{d_1}, G, I^\lambda_{d_2}, G \) be R-L fractional operators. If \( G \in L[d_1, d_2] \) and \( |G|^q \) is a refined exponential type convex function, then the following inequality holds:

\[
\left| \frac{G(d_1) - G(d_2)}{d_1 - d_2} + \frac{\Gamma(\lambda + 1)}{(d_2 - d_1)} \left[ I^\lambda_{d_1} G(d_2) - I^\lambda_{d_2} G(d_1) \right] \right| \\
\leq \frac{2}{(p\lambda + 1)^2} \left( |G'(d_1)|^q + |G'(d_2)|^q \right)^{\frac{1}{q}}. 
\]

Proof. Taking Lemma 3, Hölder’s inequality, and the refined exponential type convexity of \( |G|^q \) into consideration, we have

\[
\left| \frac{G(d_1) - G(d_2)}{d_1 - d_2} + \frac{\Gamma(\lambda + 1)}{(d_2 - d_1)} \left[ I^\lambda_{d_1} G(d_2) - I^\lambda_{d_2} G(d_1) \right] \right| \\
\leq \int_0^1 u^\lambda |G'(u d_1 + (1 - u)d_2)| du + \int_0^1 (1 - u)^\lambda |G'(u d_1 + (1 - u)d_2)| du. \\
\leq \left( \int_0^1 u^p du \right)^{\frac{1}{p}} \left( |G'(d_1)|^q + |G'(d_2)|^q \right)^{\frac{1}{q}} \int_0^1 (e^u - 1)(e^{1-u} - 1) du \\
+ \left( \int_0^1 (1 - u)^p du \right)^{\frac{1}{p}} \left( |G'(d_1)|^q + |G'(d_2)|^q \right)^{\frac{1}{q}} \int_0^1 (e^u - 1)(e^{1-u} - 1) du \\
= \frac{2}{(p\lambda + 1)^2} \left( |G'(d_1)|^q + |G'(d_2)|^q \right)^{\frac{1}{q}}. 
\]

5. Inequalities via Generalized R-L Fractional Integral Operator

Here, we intend to establish new results for a generalized fractional integral operator, i.e., \( \psi \)-Riemann–Liouville fractional integral operator and inequalities such as Hölder’s inequality and power mean inequality to show the efficiency of the main results.

Theorem 11. Let \( G : [d_1, d_2] \to \mathbb{R} \) be a refined exponential type convex function with \( 0 \leq d_1 \leq d_2 \) and \( I^{\lambda \psi}_{\psi^{-1}(d_1)}, (G \circ \psi), I^{\lambda \psi}_{\psi^{-1}(d_2)}, (G \circ \psi) \) be \( \psi \)-R-L fractional operators. If \( G \in L[b, d] \), then the following inequality for fractional integral holds:

\[
\frac{1}{2(e^\frac{x}{2} - 1)^2} G \left( \frac{d_1 + d_2}{2} \right) \\
\leq \frac{\Gamma(\lambda + 1)}{2(d_2 - d_1)^\lambda} \left[ I^{\lambda \psi}_{\psi^{-1}(d_1)} (G \circ \psi) \left( \psi^{-1} (d_2) \right) + I^{\lambda \psi}_{\psi^{-1}(d_2)} (G \circ \psi) \left( \psi^{-1} (d_1) \right) \right] \\
\leq |G(d_1) + G(d_2)| \left[ e^{-1} F_1(\lambda, \lambda + 1, 1, 1) - 1 + F_1(1, \Lambda + 1, 1) + 1 \right]. 
\]

Proof. Since \( G \) is a refined exponential type convex function, one has

\[
G \left( \frac{x + y}{2} \right) \leq (\sqrt{e} - 1)^2 [G(x) + G(y)]. 
\quad (20)
\]
Putting \( x = ud_1 + (1 - u)d_2 \) and \( y = ud_2 + (1 - u)d_1 \) in (20), multiplying both sides by \( u^{\lambda - 1} \), and then integrating with respect to \( u \) over \([0, 1]\), we obtain
\[
G \left( \frac{d_1 + d_2}{2} \right) \int_0^1 u^{\lambda - 1} du \\
\leq (\sqrt{\alpha} - 1)^2 \left[ \int_0^1 u^{\lambda - 1} G(ud_1 + (1 - u)d_2) du + \int_0^1 u^{\lambda - 1} G(ud_2 + (1 - u)d_1) du \right].
\]

Consequently,
\[
\frac{1}{\lambda (\sqrt{\alpha} - 1)^2} G \left( \frac{d_1 + d_2}{2} \right) \\
\leq \int_0^1 u^{\lambda - 1} G(ud_1 + (1 - u)d_2) du + \int_0^1 u^{\lambda - 1} G(ud_2 + (1 - u)d_1) du.
\]

Consider,
\[
\frac{\Gamma(\lambda + 1)}{2(d_2 - d_1)\lambda} \left[ \Gamma_{\psi^{-1}(d_2)} (G \circ \psi) \left( \psi^{-1}(d_2) \right) + \Gamma_{\psi^{-1}(d_1)} (G \circ \psi) \left( \psi^{-1}(d_1) \right) \right] \\
= \frac{\Gamma(\lambda + 1)}{2(d_2 - d_1)\lambda} \Gamma(\lambda) \\
\times \left[ \int_{\psi^{-1}(d_1)} \psi'(v)(d_2 - \psi(v))^{\lambda - 1}(G \circ \psi)(v)dv + \int_{\psi^{-1}(d_1)} \psi'(v)(\psi(v) - d_1)^{\lambda - 1}(G \circ \psi)(v)dv \right] \\
= \frac{\lambda}{2} \left[ \int_0^1 u^{\lambda - 1} G(ud_1 + (1 - u)d_2) du + \int_0^1 u^{\lambda - 1} G(ud_2 + (1 - u)d_1) du \right] \\
\geq \frac{1}{2(\sqrt{\alpha} - 1)^2} G \left( \frac{d_1 + d_2}{2} \right).
\]

The proof of the first part is completed, next to prove the second part we use the definition of refined exponential type convexity:
\[
G(ud_1 + (1 - u)d_2) \leq (e^u - 1)(e^{(1-u)} - 1)(G(d_1) + G(d_2)),
\]
and
\[
G(ud_2 + (1 - u)d_1) \leq (e^u - 1)(e^{(1-u)} - 1)(G(d_1) + G(d_2)).
\]

Upon adding the last two inequalities and following the same procedure as above, we obtain
\[
\int_0^1 [G(ud_1 + (1 - u)d_2) + G(ud_2 + (1 - u)d_1)] u^{\lambda - 1} du \\
\leq \int_0^1 u^{\lambda - 1} 2(e^u - 1)(e^{(1-u)} - 1)[G(d_1) + G(d_2)] du.
\]

Again, from the proof of the first inequality
\[
\frac{\Gamma(\lambda)}{(d_2 - d_1)\lambda} \left[ \Gamma_{\psi^{-1}(d_2)} (G \circ \psi) \left( \psi^{-1}(d_2) \right) + \Gamma_{\psi^{-1}(d_1)} (G \circ \psi) \left( \psi^{-1}(d_1) \right) \right] \\
\leq \frac{2}{\lambda} [G(d_1) + G(d_2)](e - 1)F_1(\lambda, \lambda + 1, 1) - 1F_1(1, \lambda + 1, 1) + 1. \quad (21)
\]

This completes the proof of Theorem 11. □
Theorem 12. Let $G : [d_1, d_2] \to \mathbb{R}$ be an exponential type convex function with $0 \leq d_1 \leq d_2$ and $I_{\psi^{-1}(d_1)}^\lambda (G \circ \psi)$, $I_{\psi^{-1}(d_2)}^\lambda (G \circ \psi)$ be $\psi$-R-L fractional operators. If $G \in L[b, d]$, then the following inequality for fractional integral holds

$$\frac{1}{2(\varepsilon^2 - 1)} G \left( \frac{d_1 + d_2}{2} \right) \leq \frac{\Gamma(\lambda + 1)}{2(d_2 - d_1)^\lambda} \left[ I_{\psi^{-1}(d_1)}^\lambda (G \circ \psi) \left( \psi^{-1}(d_2) \right) + I_{\psi^{-1}(d_2)}^\lambda (G \circ \psi) \left( \psi^{-1}(d_1) \right) \right] \leq \frac{G(d_1) + G(d_2)}{2} [ \lambda F_1(\lambda, \lambda + 1, 1) ] + [ \lambda F_1(1, \lambda + 1, 1) - 2].$$

(22)

Proof. Since $G$ is an exponential type convex function, one has

$$G \left( \frac{x + y}{2} \right) \leq (\sqrt{\varepsilon} - 1) |G(x) + G(y)|.$$  

(23)

Putting $x = ud_1 + (1 - u)d_2$ and $y = ud_2 + (1 - u)d_1$ in (23), multiplying the resultant inequality by $u^{\lambda - 1}$, and then integrating with respect to $u$ over $[0, 1]$, we obtain

$$G \left( \frac{d_1 + d_2}{2} \right) \int_0^1 u^{\lambda - 1} du \leq (\sqrt{\varepsilon} - 1) \left[ \int_0^1 u^{\lambda - 1} G(ud_1 + (1 - u)d_2) du + \int_0^1 u^{\lambda - 1} G(ud_2 + (1 - u)d_1) du \right].$$

Consequently,

$$\frac{1}{\lambda (\sqrt{\varepsilon} - 1)} G \left( \frac{d_1 + d_2}{2} \right) \leq \int_0^1 u^{\lambda - 1} G(ud_1 + (1 - u)d_2) du + \int_0^1 u^{\lambda - 1} G(ud_2 + (1 - u)d_1) du.$$

Consider,

$$\frac{\Gamma(\lambda + 1)}{2(d_2 - d_1)^\lambda} \left[ I_{\psi^{-1}(d_1)}^\lambda (G \circ \psi) \left( \psi^{-1}(d_2) \right) + I_{\psi^{-1}(d_2)}^\lambda (G \circ \psi) \left( \psi^{-1}(d_1) \right) \right] = \frac{\Gamma(\lambda + 1)}{2(d_2 - d_1)^\lambda} \left[ \int_{\psi^{-1}(d_1)}^{\psi^{-1}(d_2)} \psi'(v) (d_2 - \psi(v))^{\lambda - 1} (G \circ \psi)(v) dv + \int_{\psi^{-1}(d_2)}^{\psi^{-1}(d_1)} \psi'(v) (\psi(v) - d_1)^{\lambda - 1} (G \circ \psi)(v) dv \right]$$

$$= \frac{\lambda}{2} \left[ \int_0^1 u^{\lambda - 1} G(ud_1 + (1 - u)d_2) du + \int_0^1 u^{\lambda - 1} G(ud_2 + (1 - u)d_1) du \right] \geq \frac{1}{2(\sqrt{\varepsilon} - 1)} G \left( \frac{d_1 + d_2}{2} \right).$$

This proves the first part of the theorem; next to prove the second part, we use the definition of refined exponential type convexity, i.e.,

$$G(ud_1 + (1 - u)d_2) \leq (e^u - 1) G(d_1) + (e^{(1-u)} - 1) G(d_2),$$

and

$$G(ud_2 + (1 - u)d_1) \leq (e^u - 1) G(d_2) + (e^{(1-u)} - 1) G(d_1).$$
Upon adding the last two inequalities and then following the same procedure as above, we have
\[
\int_0^1 \left| \mathcal{G}(ud_1 + (1 - u)d_2) + \mathcal{G}(ud_2 + (1 - u)d_1) \right| u^{\lambda - 1} du \\
\leq \int_0^1 u^{\lambda - 1} |e^u + e^{1-u} - 2| [\mathcal{G}(d_1) + \mathcal{G}(d_2)] du.
\]
Further computations give
\[
\frac{\Gamma(\lambda)}{(d_2 - d_1)\lambda} \left[ \frac{1}{\psi' - 1}(d_1) \right] \left[ \mathcal{G}(d_1) + \mathcal{G}(d_2) \right] F_1(\lambda, \lambda + 1, 1) + F_1(\lambda, 1, \lambda + 1, 1) - 2.
\]
Multiplying by \( \frac{1}{4} \), we obtain the desired result. □

**Lemma 4** ([35], Lemma 3.1). Let \( I_{\psi' - 1}(d_1) \), \( \mathcal{G} \), and \( \psi' \) be \( \psi \)-R-L fractional operators. Moreover, consider \( \mathcal{G}' \in \mathcal{L}(d_1, d_2) \), \( \psi(v) \), a positive monotonically increasing function on \( (d_1, d_2) \) with \( \psi'(v) \) being continuously differentiable on \( (d_1, d_2) \) and \( \lambda \in (0, 1) \). Then, the following equality holds:
\[
\frac{\mathcal{G}(d_1) + \mathcal{G}(d_2)}{2} - \frac{\Gamma(\lambda + 1)}{(d_2 - d_1)\lambda} \left[ I_{\psi' - 1}(d_1) \right] \left[ \mathcal{G}(d_1) + \mathcal{G}(d_2) \right] \left[ (\psi'(d_1))^\lambda - (d_2 - \psi(v))^\lambda \right] (\mathcal{G} \circ \psi)(\psi'(v)) dv = \frac{1}{2(d_2 - d_1)^{\lambda}} \left[ \left( (\mathcal{G}'(d_1))^{\alpha} + |\mathcal{G}'(d_2)| (3 - \alpha) \right) \right] \frac{1}{\lambda^2 + 1}.
\]

**Theorem 13.** Let \( I_{\psi' - 1}(d_1) \), \( \mathcal{G} \), and \( \psi(v) \) be \( \psi \)-R-L fractional operators and \( \psi(v) \) be a positive increasing function on \( (d_1, d_2) \) with \( \psi'(v) \) being continuously differentiable on \( (d_1, d_2) \) and \( \lambda \in (0, 1) \). If \( \mathcal{G} : (d_1, d_2) \rightarrow \mathbb{R} \) is a differentiable mapping on \( (d_1, d_2) \) and \( |\mathcal{G}'|^{\alpha} \) is a refined exponential type convex function, then the following inequality holds:
\[
\frac{\mathcal{G}(d_1) + \mathcal{G}(d_2)}{2} - \frac{\Gamma(\lambda + 1)}{(d_2 - d_1)\lambda} \left[ I_{\psi' - 1}(d_1) \right] \left[ \mathcal{G}(d_1) + \mathcal{G}(d_2) \right] \left[ (\psi'(d_1))^\lambda - (d_2 - \psi(v))^\lambda \right] (\mathcal{G} \circ \psi)(\psi'(v)) dv \leq \frac{d_2 - d_1}{2(\lambda^2 + 1)^{\frac{1}{\lambda}}},
\]
where \( \frac{1}{\lambda^2 + 1} = 1 \)

**Proof.** Using Lemma 4 and the refined exponential type convexity,
\[
\frac{\mathcal{G}(d_1) + \mathcal{G}(d_2)}{2} - \Gamma(\lambda + 1) \left[ I_{\psi' - 1}(d_1) \right] \left[ \mathcal{G}(d_1) + \mathcal{G}(d_2) \right] \left[ (\psi'(d_1))^\lambda - (d_2 - \psi(v))^\lambda \right] (\mathcal{G} \circ \psi)(\psi'(v)) dv \leq \frac{d_2 - d_1}{2(\lambda^2 + 1)^{\frac{1}{\lambda}}} \left[ \int_0^1 (1 - u)^{\lambda - 1} |G'(ud_1 + (1 - u)d_2)| du + \int_0^1 u^{\lambda} |G'(ud_1 + (1 - u)d_2)| du \right]
\]
Since, \( v \in (\psi'(d_1), \psi'(d_2)) \), one has \( d_1 \leq \psi(v) \leq d_2 \).
Considering \( u \in (d_2 - \psi(v), d_1) \), then \( \psi(v) = ud_1 + (1 - u)d_2 \)
\[
= -\frac{d_2 - d_1}{2} \int_0^1 (1 - u)^{\lambda - 1} - u^{\lambda} |G'(ud_1 + (1 - u)d_2)| du
\leq \frac{d_2 - d_1}{2} \left[ \int_0^1 (1 - u)^{\lambda} |G'(ud_1 + (1 - u)d_2)| du + \int_0^1 u^{\lambda} |G'(ud_1 + (1 - u)d_2)| du \right]
\]
where \( \frac{1}{\lambda^2 + 1} = 1 \).
Now, using Hölder’s inequality
\[
\leq \frac{d_2 - d_1}{2} \left[ \left( \int_0^1 (1 - u)^{\lambda p} \right)^{\frac{1}{p}} \left( \int_0^1 |G'(ud_1 + (1 - u)d_2)|^q du \right)^{\frac{1}{q}} + \left( \int_0^1 u^{\lambda p} \right)^{\frac{1}{p}} \left( \int_0^1 |G'(ud_1 + (1 - u)d_2)|^q du \right)^{\frac{1}{q}} \right] 
\leq \frac{d_2 - d_1}{2} \left[ \left( \frac{1}{\lambda p + 1} \right)^{\frac{1}{p}} \left([|G'(d_1)|^q + |G'(d_2)|^q]\int_0^1 (e^u - 1)(e^{1-u} - 1) du \right)^{\frac{1}{q}} + \left( \frac{1}{\lambda p + 1} \right)^{\frac{1}{p}} \left([|G'(d_1)|^q + |G'(d_2)|^q]\int_0^1 (e^u - 1)(e^{1-u} - 1) du \right)^{\frac{1}{q}} \right] 
= \frac{d_2 - d_1}{(\lambda p + 1)^{\frac{1}{p}}} \left([|G'(d_1)|^q + |G'(d_2)|^q](3 - e)\right)^{\frac{1}{q}}.
\]
This completes the proof. □

**Theorem 14.** Let \(J_{\lambda}^{\psi,\lambda_{-1}(d_1)}(G \circ \psi), J_{\lambda}^{\psi,\lambda_{-1}(d_2)}(G \circ \psi)\) be \(\psi\)-R-L fractional operators and \(\psi(v)\) be a positive increasing function on \((d_1, d_2)\) with \(\psi'(v)\) being continuously differentiable on \((d_1, d_2)\) and \(\lambda \in (0, 1)\). If \(G : [d_1, d_2] \to \mathbb{R}\) is a differentiable mapping on \((d_1, d_2)\), \(q \geq 1\), and \(|G'|^q\) is a refined exponential type convex function, then the following inequality holds:
\[
\frac{|G(d_1) + G(d_2)|}{2} \leq \frac{\Gamma(\lambda + 1)}{(d_2 - d_1)^{\lambda}} \left[ J_{\lambda}^{\psi,\lambda_{-1}(d_1)}(G \circ \psi)\left(\psi'(d_2)\right) + J_{\lambda}^{\psi,\lambda_{-1}(d_2)}(G \circ \psi)\left(\psi'(d_1)\right) \right] 
\leq \frac{d_2 - d_1}{2} \left( \frac{2}{\lambda + 1} \left( \frac{1}{2} \right)^{\lambda} \right)^{1 - \frac{1}{q}} \times \left([|G'(d_1)|^q + |G'(d_2)|^q]\left[\frac{C_1 - F_1(\lambda + 1, \lambda + 2, 1) - F_1(1, \lambda + 1, 1) + 1}{\lambda + 1}\right]\right)^{\frac{1}{q}}.
\]

**Proof.** Using Lemma 4 and the refined exponential type convexity,
\[
\frac{|G(d_1) + G(d_2)|}{2} \leq \frac{\Gamma(\lambda + 1)}{(d_2 - d_1)^{\lambda}} \left[ J_{\lambda}^{\psi,\lambda_{-1}(d_1)}(G \circ \psi)\left(\psi'(d_2)\right) + J_{\lambda}^{\psi,\lambda_{-1}(d_2)}(G \circ \psi)\left(\psi'(d_1)\right) \right] 
\leq \frac{1}{2(d_2 - d_1)^{\lambda}} \int_{\psi^{-1}(d_1)}^{\psi^{-1}(d_2)} \left[ ((\psi'(v) - d_1)^{\lambda} - (d_2 - \psi'(v))^{\lambda}) \right] (G'(v) \psi'(v) v) dv.
\]
Since \(v \in (\psi^{-1}(d_1), \psi^{-1}(d_2))\), one has \(d_1 \leq \psi(v) \leq d_2\).
Considering \(u \in \frac{d_2 - \psi(v)}{d_2 - d_1}\), then \(\psi(v) = ud_1 + (1 - u)d_2\)
\[
= \frac{d_2 - d_1}{2} \int_0^1 [(1 - u)\lambda - u\lambda] |G'(ud_1 + (1 - u)d_2)| du 
\leq \frac{d_2 - d_1}{2} \int_0^1 [(1 - u)\lambda - u\lambda] \left[ (e^u - 1)(e^{1-u} - 1) \right] |G'(d_1)| + |G'(d_2)|] du.
\]
Now, using the power-mean inequality
\[ \frac{d_2 - d_1}{2} \int_0^1 ((1-u)^\lambda + u^\lambda) du \]
\[ \leq \left( \int_0^1 ((1-u)^\lambda + u^\lambda)(e^u - 1)(e^{1-u} - 1) \left[ |G'(d_1)|^\|G'(d_2)|^\| d\right] u \right)^{\frac{1}{\|}} \]
\[ \leq \frac{d_2 - d_1}{2} \left( \frac{2}{\lambda + 1} \left( 1 - \frac{1}{\lambda + 1} \right) \right)^{1 - \frac{1}{\lambda}} \]
\[ \times \left( \left[ |G'(d_1)|^\|G'(d_2)|^\right] \left[ e - \frac{1}{\Gamma_1} \left( \lambda + 1, \lambda + 2, 1 - \frac{1}{\Gamma_1} \left( \lambda + 1, 1, 1 + 1 \right) \right) \right] \right)^\| . \]

This completes the proof. \( \Box \)

**Theorem 15.** Let \( \psi^1 \in \psi^1(d_1), (G \circ \psi), \| \psi^1 \in \psi^1(d_2), (G \circ \psi) \) be \( \psi \)-R-L fractional operators and \( \psi(v) \), a positive increasing function on \( (d_1, d_2) \) with \( \psi'(v) \) being continuously differentiable on \( (d_1, d_2) \) and \( \lambda \in (0, 1) \). If \( G : (d_1, d_2) \rightarrow \mathbb{R} \) is a differentiable mapping on \( (d_1, d_2) \), \( q \geq 1 \), and \( |G'|^q \) is a refined exponential type convex function, then the following inequality holds:

\[ \frac{G(d_1) + G(d_2)}{2} - \frac{\Gamma(\lambda + 1)}{(d_2 - d_1)^\lambda} \left[ \psi^1(d_1), (G \circ \psi) \left( \psi^{-1}(d_2) \right) \right] \]
\[ \leq \frac{1}{2(d_2 - d_1)^\lambda} \left( \int_{\psi^{-1}(d_1)}^{\psi^{-1}(d_2)} \left[ (\psi(v) - d_1)^\lambda - (d_2 - \psi(v))^\lambda \right] \left[ G' \circ \psi \right](v)\psi'(v)dv \right]. \]

**Proof.** Using Lemma 4 and the refined exponential type convexity,

\[ \frac{G(d_1) + G(d_2)}{2} - \frac{\Gamma(\lambda + 1)}{(d_2 - d_1)^\lambda} \left[ \psi^1(d_1), (G \circ \psi) \left( \psi^{-1}(d_2) \right) \right] \]
\[ \leq \frac{1}{2(d_2 - d_1)^\lambda} \left( \int_{\psi^{-1}(d_1)}^{\psi^{-1}(d_2)} \left[ (\psi(v) - d_1)^\lambda - (d_2 - \psi(v))^\lambda \right] \left[ G' \circ \psi \right](v)\psi'(v)dv \right]. \]

Since \( v \in (\psi^{-1}(d_1), \psi^{-1}(d_2)) \), one has \( d_1 \leq \psi(v) \leq d_2 \).

Considering \( u \in \frac{d_2 - \psi(v)}{d_2 - d_1} \), then \( \psi(v) = ud_1 + (1 - u)d_2 \).

\[ = \frac{d_2 - d_1}{2} \int_0^1 ((1-u)^\lambda + u^\lambda) |G'(ud_1 + (1-u)d_2)| du \]
\[ \leq \frac{d_2 - d_1}{2} \int_0^1 ((1-u)^\lambda + u^\lambda) \left[ (e^u - 1)(e^{1-u} - 1) \right] \left[ |G'(d_1)| + |G'(d_2)| \right] du. \]

Now, using the power-mean inequality

\[ \frac{d_2 - d_1}{2} \int_0^1 ((1-u)^\lambda + u^\lambda) du \]
\[ \leq \left( \int_0^1 ((1-u)^\lambda + u^\lambda)(e^u - 1)(e^{1-u} - 1) \left[ |G'(d_1)|^\|G'(d_2)|^\right] du \right)^{\frac{1}{\lambda}} \]
\[ \leq \frac{d_2 - d_1}{2} \left( \frac{2}{\lambda + 1} \right)^{1 - \frac{1}{\lambda}} \]
\[ \times \left( \left[ |G'(d_1)|^\|G'(d_2)|^\right] \left[ e - \frac{1}{\Gamma_1} \left( \lambda + 1, \lambda + 2, 1 - \frac{1}{\Gamma_1} \left( \lambda + 1, 1, 1 + 1 \right) \right) \right] \right)^\| . \]
This completes the proof. □

**Theorem 16.** Let $\Gamma^{\lambda\psi}_{\psi^{-1}(d_1)}(G \circ \psi), \Gamma^{\lambda\psi}_{\psi^{-1}(d_2)}(G \circ \psi)$ be $\psi$-$R$-$L$ fractional operators and $\psi(v)$, a positive increasing function on $(d_1, d_2)$ with $\psi'(v)$ being continuously differentiable on $(d_1, d_2)$ and $\lambda \in (0, 1)$. If $G : [d_1, d_2] \rightarrow \mathbb{R}$ is a differentiable mapping on $(d_1, d_2)$, then the following inequality holds:

$$\left| \frac{G(d_1) + G(d_2)}{2} - \frac{\Gamma(\lambda + 1)}{(d_2 - d_1)^{\lambda}} \left[ \Gamma^{\lambda\psi}_{\psi^{-1}(d_1)}(G \circ \psi) \left( \psi^{-1}(d_2) \right) + \Gamma^{\lambda\psi}_{\psi^{-1}(d_2)}(G \circ \psi) \left( \psi^{-1}(d_1) \right) \right] \right| \leq \frac{(d_2 - d_1)[|G'(d_1)| + |G'(d_2)|][F_1(1, \lambda + 2, 1) + 1]F_1(1, 1, 1)}{2(\lambda + 1)}.$$  

(28)

**Proof.** Using Lemma 4 and the exponential type convexity,

$$\left| \frac{G(d_1) + G(d_2)}{2} - \frac{\Gamma(\lambda + 1)}{(d_2 - d_1)^{\lambda}} \left[ \Gamma^{\lambda\psi}_{\psi^{-1}(d_1)}(G \circ \psi) \left( \psi^{-1}(d_2) \right) + \Gamma^{\lambda\psi}_{\psi^{-1}(d_2)}(G \circ \psi) \left( \psi^{-1}(d_1) \right) \right] \right| \leq \frac{1}{2(d_2 - d_1)^{\lambda}} \left[ \int_{\psi^{-1}(d_1)}^{\psi^{-1}(d_2)} \left[ \left| (\psi(v) - d_1)^{\lambda} - (d_2 - \psi(v))^{\lambda} \right| \right] (G' \circ \psi)(v) \psi'(v) dv \right].$$

Since $v \in (\psi^{-1}(d_1), \psi^{-1}(d_2))$, one has $d_1 \leq \psi(v) \leq d_2$.

Considering $u \in \frac{d_2 - d_1}{d_2 - d_1}$, then $\psi(v) = u d_1 + (1 - u) d_2$.

$$= \frac{d_2 - d_1}{2} \int_0^1 \left| (1 - u)^{\lambda} - u^{\lambda} \right| |G'(ud_1 + (1 - u)d_2)| du$$

$$\leq \frac{d_2 - d_1}{2} \int_0^1 \left| (1 - u)^{\lambda} - u^{\lambda} \right| \left[ (e^{u^{\lambda}} - 1)G'(d_1) + (e^{1-u^{\lambda}} - 1)G'(d_2) \right] du$$

$$\leq \frac{d_2 - d_1}{2} \int_0^1 \left| (1 - u)^{\lambda} + u^{\lambda} \right| \left[ (e^{u^{\lambda}} - 1)G'(d_1) + (e^{1-u^{\lambda}} - 1)G'(d_2) \right] du$$

$$= \frac{(d_2 - d_1)[|G'(d_1)| + |G'(d_2)|][F_1(1, \lambda + 2, 1) + 1]F_1(1, 1, 1) - 2}{2(\lambda + 1)}.$$

This completes the proof. □

**Theorem 17.** Let $\Gamma^{\lambda\psi}_{\psi^{-1}(d_1)}(G \circ \psi), \Gamma^{\lambda\psi}_{\psi^{-1}(d_2)}(G \circ \psi)$ be $\psi$-$R$-$L$ fractional operators and $\psi(v)$ be a positive increasing function on $(d_1, d_2)$ with $\psi'(v)$ being continuously differentiable on $(d_1, d_2)$ and $\lambda \in (0, 1)$. If $G : [d_1, d_2] \rightarrow \mathbb{R}$ is a differentiable mapping on $(d_1, d_2)$ and $G'^{\varphi}$ is an exponential type convex function, then the following inequality holds:

$$\left| \frac{G(d_1) + G(d_2)}{2} - \frac{\Gamma(\lambda + 1)}{(d_2 - d_1)^{\lambda}} \left[ \Gamma^{\lambda\psi}_{\psi^{-1}(d_1)}(G \circ \psi) \left( \psi^{-1}(d_2) \right) + \Gamma^{\lambda\psi}_{\psi^{-1}(d_2)}(G \circ \psi) \left( \psi^{-1}(d_1) \right) \right] \right| \leq \frac{d_2 - d_1}{\frac{\lambda}{\varphi} + \frac{1}{\varphi}} \left[ \left| G'(d_1) \right|^{\varphi} + \left| G'(d_2) \right|^{\varphi} (e - 2)^{\frac{1}{\varphi}} \right],$$

(29)

where $\frac{1}{\varphi} + \frac{1}{\varphi} = 1$

**Proof.** Using Lemma 4 and the exponential type convexity,

$$\left| \frac{G(d_1) + G(d_2)}{2} - \frac{\Gamma(\lambda + 1)}{(d_2 - d_1)^{\lambda}} \left[ \Gamma^{\lambda\psi}_{\psi^{-1}(d_1)}(G \circ \psi) \left( \psi^{-1}(d_2) \right) + \Gamma^{\lambda\psi}_{\psi^{-1}(d_2)}(G \circ \psi) \left( \psi^{-1}(d_1) \right) \right] \right| \leq \frac{1}{2(d_2 - d_1)^{\lambda}} \left[ \int_{\psi^{-1}(d_1)}^{\psi^{-1}(d_2)} \left[ \left| (\psi(v) - d_1)^{\lambda} - (d_2 - \psi(v))^{\lambda} \right| \right] (G' \circ \psi)(v) \psi'(v) dv \right].$$

g, v \in (\psi^{-1}(d_1), \psi^{-1}(d_2))$, one has $d_1 \leq \psi(v) \leq d_2$. 

Considering \( u \in \frac{d_2 - \psi(v)}{d_2 - d_1} \), then \( \psi(v) = ud_1 + (1 - u)d_2 \)
\[
\frac{d_2 - d_1}{2} \int_0^1 |(1 - u)^\lambda - u^\lambda||G'(ud_1 + (1 - u)d_2)|du
\leq \frac{d_2 - d_1}{2} \left[ \int_0^1 (1 - u)^\lambda |G'(ud_1 + (1 - u)d_2)|du + \int_0^1 u^\lambda |G'(ud_1 + (1 - u)d_2)|du \right]
\]
Now, using Hölder’s inequality
\[
\leq \frac{d_2 - d_1}{2} \left[ \left( \int_0^1 (1 - u)^{\lambda p} \right)^{\frac{1}{p}} \left( \int_0^1 |G'(ud_1 + (1 - u)d_2)|^{q}du \right)^{\frac{1}{q}} + \left( \int_0^1 u^{\lambda q} \right)^{\frac{1}{q}} \left( \int_0^1 |G'(ud_1 + (1 - u)d_2)|^{q}du \right)^{\frac{1}{q}} \right]
\]
\[
= \frac{d_2 - d_1}{2} \left[ \int |G'(d_1)|^q + |G'(d_2)|^q(e - 2) \right].
\]
This completes the proof. \( \square \)

6. Applications

6.1. Application to Probability Density Function

Let \( X \) be a random variable in the finite interval \([d_1, d_2]\) with the probability density function \( G : [d_1, d_2] \to [0, 1] \) with the cumulative distribution function
\[
\Omega(x) = Pr(X \leq x) = \int_{d_1}^{x} G(u)du.
\]

Proposition 2. Taking the assumptions given in Theorem 3 into consideration, we have
\[
\frac{1}{2(e^{1/2} - 1)^2} \left( \frac{d_1 + d_2}{2} \right) \leq \frac{d_2 - E(x)}{d_2 - d_1} \leq [\Omega(d_1) + \Omega(d_2)][3 - e].
\]

Proof. Let \( G = \Omega \) in Theorem 3 and considering the following into account
\[
E(u) = \int_{d_1}^{d_2} u\Omega(u) = d_2 - \int_{d_1}^{d_2} \Omega(u)du.
\]
\( \square \)

6.2. Application to Special Means

In this section we apply our results to establish several new inequalities for special means.

Arithmetic mean :
\[
A(d_1, d_2) = \frac{d_1 + d_2}{2}.
\]

Generalized logarithmic mean :
\[
L_n(d_1, d_2) = \left( \frac{d_2^{n+1} - d_1^{n+1}}{(d_2 - d_1)(n+1)} \right)^{\frac{1}{n}}, d_1 \neq d_2, \ d_1, d_2 > 0.
\]

Proposition 3. Let \( d_1, d_2 \in (0, \infty) \) with \( d_1 < d_2 \). Then,
\[
\frac{A^2(d_1, d_2)}{(e^2 - 1)} \leq L_2^2(d_1, d_2) \leq A(d_1^2, d_2^2)[2, f_1(1, 2, 1) - 2].
\]
Proof. Proposition 3 follows directly from Theorem 12 for the function $G(x) = x^2, \psi(x) = x$ and $\alpha = 1$.

Let $G(x) = x^n, \psi(x) = x$, and $\alpha = 1$, then we have a general result.

$$\frac{A^n(d_1, d_2)}{\left(e^{\frac{d_1}{d_2}} - 1\right)} \leq L_n^n(d_1, d_2) \leq A(d_1^n, d_2^n)[2_1F_1(1, 2, 1) - 2].$$

Proposition 4. Let $d_1, d_2 \in (0, \infty)$ with $d_1 < d_2$. Then,

$$\frac{A^{-1}(d_1, d_2)}{\left(e^{\frac{d_1}{d_2}} - 1\right)} \leq L^{-1}(d_1, d_2) \leq H^{-1}(d_1, d_2)[2_1F_1(1, 2, 1) - 2].$$

Proof. Proposition 4 follows directly from Theorem 12 for the function $G(x) = \frac{1}{x}, \psi(x) = x$ and $\alpha = 1$.

Proposition 5. Let $d_1, d_2 \in (0, \infty)$ with $d_1 < d_2$. Then,

$$|A(d_1^n, d_2^n) - L^{-1}(d_1, d_2)| \leq \frac{n(d_2 - d_1)}{2} A(d_1^{n-1}, d_2^{-n-1})[1F_1(2, 3, 1) + 1F_1(1, 3, 1) - 2].$$

Proof. Proposition 5 follows directly from Theorem 16 for the function $G(x) = x^n, \psi(x) = x$ and $\alpha = 1$.

7. Conclusions

In recent years, the use of fractional calculus to prove various integral inequalities using different convex functions has surged. Recent developments in the areas of differential equations, modeling, and mathematical inequalities all benefit from fractional operators. In this article, we focused on introducing the notion of refined exponential type convexity and presenting the fractional Hermite–Hadamard inequality and its refinements. The ability to demonstrate inequalities of the $H - H$ type on coordinates, quantum calculus, and interval-valued analysis using the ideas discussed in this article will be an intriguing test of their viability.


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Abbreviations

The following abbreviations are used in this manuscript:

H – H Hermite–Hadamard
R – L Riemann–Liouville
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References


