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Fixed Point Results for a Family of Interpolative F-Contractions in b-Metric Spaces

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Abstract: In this paper, we introduce a new generalized concept, namely, extended interpolative Cirić–Reich–Rus-type F-contraction in b-metric space. In addition, we put forward the notion of interpolative Kannan-type F-contractions. Fixed point results for these new interpolative contraction mappings are established, and non-trivial examples involving finite and infinite sets are provided to validate the results.

Keywords: Cirić–Reich–Rus type mapping; Kannan-type mapping; fixed point; contraction; b-metric space

MSC: 47H10; 54H25; 54E50

1. Introduction

Following the most celebrated fixed point results of Banach [1] in 1922, fixed point theory has witnessed breakthrough developments in different directions. One such direction is the replacement of the contraction condition with extended versions; another direction is the development of the metric space itself by inducing advanced properties. In the current research, we follow the first direction, i.e., the enhancement of the contraction condition.

For our work, we consider the F-contraction introduced by Wardowski [2] in 2012. Major improvements to Banach’s contraction principle were accomplished by Boyd and Wong [3], Chatterjea [4], Cirić [5], Kannan [6], and Meir and Keeler [7]. For developments in interpolative contractions, we refer to [8–11].

Bakhtin [12] and Czerwik [13] introduced the b-metric space, which non-trivially extended the class of metric spaces. Subsequently, tremendous improvement in fixed point theory in the framework of b-metric space have taken place [14–23].

In this paper, we put forward the concept of extended interpolative Cirić–Reich–Rus type (CRR-type) F-contraction and interpolative Kannan-type F-contraction in a b-metric space (bMS). These new interpolative results provide a new direction in the area of integral equations to find new solutions. We establish a number of important results while investigating this connection, and provide examples to validate our results. Finally, we present an application of the newly established results towards the solution of a particular type of integral equations.

2. Preliminaries

Several important results in the present context are listed below:

Definition 1 ([12,13]). Consider a mapping $Y : M \times M \rightarrow [0, \infty)$ where $M \neq \phi$ is a set. When $Y$ satisfies the following conditions:

(bM1) $Y(t_1, t_2) = 0$ if and only if $t_1 = t_2$;
(bM2) $Y(t_1, t_2) = Y(t_2, t_1)$ for all $t_1, t_2 \in X$;
(bM3) There exists a real number $s \geq 1$ such that $Y(t_1, t_3) \leq s[Y(t_1, t_2) + Y(t_2, t_3)]$ for all $t_1, t_2, t_3 \in X$.
Definition 2 ([24]). Consider a sequence \( \{u_n\} \) in \( M \), where \((M, Y)\) is a b-metric space and \( u \in M \). Then:

(a) \( \{u_n\} \) is known as a convergent sequence in \((M, Y)\), and converges to \( u \) if, for every \( \varepsilon > 0 \exists n_0 \in \mathbb{N} \) such that \( Y(u_n, u) < \varepsilon \ \forall \ n > n_0 \), which can be written as \( \lim_{n \to \infty} u_n = u \) or \( u_n \to u \) as \( n \to \infty \).

(b) \( \{u_n\} \) is known as a Cauchy sequence in \((M, Y)\) if, for every \( \varepsilon > 0 \), \( \exists n_0 \in \mathbb{N} \) such that \( Y(u_n, u_{n+p}) < \varepsilon \ \forall \ n > n_0, p > 0 \).

(c) If every Cauchy sequence in \( M \) converges to some \( u \in M \), then \((M, Y)\) is known as a complete b-metric space.

Definition 3 ([2]). Consider a real mapping \( F : (0, \infty) \to (-\infty, +\infty) \) satisfying the following conditions:

(F1) \( F \) follows the strictly increasing property;

(F2) For a sequence \( \{t_n\}_{n \in \mathbb{N}} \subset (0, \infty) \), for every \( \{t_n\}_{n \in \mathbb{N}} \), \( \lim_{n \to \infty} t_n = 0 \) iff \( \lim_{n \to \infty} F(t_n) = -\infty \);

(F3) \( \exists s \in (0, 1) \) such that \( \lim_{t \to 0} t F(t) = 0 \).

Suppose \( F \) is the collection of all mappings \( F \). If \((M, \rho)\) is a metric space, then a function \( S : M \to M \) is called an \( F \)-contraction if \( \exists \tau > 0, F \in F \) such that \( \forall p, q \in M \). Then, we have

\[
\rho(S(p), S(q)) > 0 \Rightarrow \tau + F(\rho(S(p), S(q))) \leq F(\rho(p, q)).
\]

3. Extended Interpolative \( F \)-Contraction

Here, we present our main results. We first introduce the definition of an extended interpolative CRR-type \( F \)-contraction, then establish a fixed point theorem.

Definition 4. Consider the bMS \((M, \xi, s)\) and a self-map \( S : M \to M \). Then, the function \( S \) is known as an extended interpolative modified CRR-type \( F \)-contraction if, for \( \tau > 0 \exists c_1, c_2 \in [0, 1) \) with \( c_1 + c_2 < 1 \) and \( F \in F \) such that

\[
\forall \xi, q \in M \setminus \text{Fix}(S), \text{ where Fix}(S) = \{\xi \in M : S(\xi) = \xi\} \text{ with } \xi(S(\xi), S(q)) > 0.
\]

Theorem 1. Consider a complete bMS \((M, \xi, s)\) and a continuous self-map \( S \) on \( M \). Then, \( S \) has a fixed point in \( M \) if \( S \) is an extended interpolative CRR-type \( F \)-contraction.

Proof. Consider \( \xi_0 \in M \). Suppose \( \xi_n \) is a sequence. Consider \( \xi_n = S^n(\xi_0) \ \forall n \in \mathbb{N} \). Then, \( \xi_{n_0} \) becomes a fixed point of \( S \) if \( \exists n_0 \in \mathbb{N} \) such that \( \xi_{n_0} = \xi_{n_0+1} \). Hence, we assume that \( \xi_n \neq \xi_{n+1} \ \forall n \in \mathbb{N} \) with \( \xi = \xi_n \) and \( q = \xi_{n-1} \ \forall n \in \mathbb{N} \). Then, we have the following from the Equation (1):

\[
\tau + F(\xi(\xi_{n+1}, \xi_n)) = \tau + F(\xi(S(\xi_n), S(\xi_{n-1})))
\]

\[
\leq c_1 F(\xi(\xi_n, \xi_{n-1})) + c_2 F(\xi(\xi_n, S(\xi_n))) + (1 - c_1 - c_2) F\left(\frac{1}{s} \xi(\xi_{n-1}, S(\xi_{n-1}))\right)
\]

\[
= c_1 F(\xi(\xi_n, \xi_{n-1})) + c_2 F(\xi(S(\xi_n), \xi_{n-1})) + (1 - c_1 - c_2) F\left(\frac{1}{s} \xi(\xi_{n-1}, \xi_n)\right)
\]

\[
\leq c_1 F(\xi(\xi_n, \xi_{n-1})) + c_2 F(\xi(S(\xi_n), \xi_{n-1})) + (1 - c_1 - c_2) F(\xi(\xi_{n-1}, \xi_n)), \quad \text{for } s \geq 1.
\]

Suppose, \( \xi(\xi_{n-1}, \xi_n) < \xi(\xi_n, \xi_{n+1}) \); then, from (2), we have
\[
\tau + F(\xi(\xi_{n-1}, \xi_n)) < c_1 F(\xi(\xi_n, \xi_{n+1})) + c_2 F(\xi(\xi_{n+1}, \xi_1)) + (1 - c_1 - c_2) F(\xi(\xi_n, \xi_{n+1}))
= F(\xi(\xi_n, \xi_{n+1}))
\implies \tau + F(\xi(\xi_{n-1}, \xi_n)) < F(\xi(\xi_n, \xi_{n+1}))
\]
which is a contradiction.

Therefore,
\[
\xi(\xi_n, \xi_{n+1}) < \xi(\xi_{n-1}, \xi_n), \quad \forall \ n \geq 1.
\]
Hence,
\[
\tau + F(\xi(\xi_{n+1}, \xi_n)) \leq F(\xi(\xi_n, \xi_{n-1})).
\]
Consequently,
\[
F(\xi(\xi_n, \xi_{n+1})) \leq F(\xi(\xi_{n-1}, \xi_n)) - \tau \leq \cdots \leq F(\xi(\xi_0, \xi_1)) - n\tau, \forall \ n \geq 1.
\]
Then, by taking the limit as \( n \to \infty \), we have
\[
\lim_{n \to \infty} F(\xi(\xi_n, \xi_{n+1})) = -\infty.
\]
Thus, from condition \( F2 \) we obtain
\[
\lim_{n \to \infty} \xi(\xi_n, \xi_{n+1}) = 0.
\]
Let \( \theta_n = \xi(\xi_n, \xi_{n+1}) \).
Hence,
\[
\lim_{n \to \infty} \theta_n = 0.
\]
Next, from condition \( F3 \), \( \exists \ k \in (0, 1) \ \forall \ n \in \mathbb{N} \), and we have
\[
\theta_n^k (F(\theta_n) - F(\theta_0)) \leq -\theta_n^k n\tau < 0. \quad \text{(3)}
\]
Now, using condition \( F3 \) and taking the limit as \( n \to \infty \), we have
\[
\lim_{n \to \infty} n\theta_n^k = 0.
\]
Hence, \( \exists \ n_0 \in \mathbb{N} \) such that \( n\theta_n^k \leq 1, \forall n \geq n_0. \)
\[
\theta_n \leq \frac{1}{n\tau}, \quad \forall n \geq n_0.
\]
Next, to verify that \( \{\xi_n\} \) is a Cauchy sequence, we take \( m, n \in \mathbb{N} \) such that \( m > n \geq n_0. \)
From the triangular inequality, we have
\[
\xi(\xi_n, \xi_m) \leq \frac{1}{s^{m-1}} [\xi(\xi_n, \xi_{n+1}) + \xi(\xi_{n+1}, \xi_{n+2}) + \cdots + \xi(\xi_{m-1}, \xi_m)]
= \frac{1}{s^{m-1}} [\theta_n + \theta_{n+1} + \cdots + \theta_{m-1}]
= \sum_{i=1}^{m-1} \frac{1}{s^i} \sum_{i=n}^{m-1} \theta_i
\leq \sum_{i=1}^{\infty} \frac{1}{s^i} \sum_{i=n}^{\infty} \theta_i
\leq \sum_{i=1}^{\infty} \frac{1}{s^i} \sum_{i=n}^{\infty} \frac{1}{n^2}.
\]
If we take the limit as \( n \to \infty \) for the above inequality, we obtain
\[
\lim_{m, n \to \infty} \xi(\xi_n, \xi_m) = 0.
\]
Therefore, \( \{ \zeta_n \} \) is a Cauchy sequence in \( M \).

Because \( (M, \xi, s) \) is a complete \( b \)-MS, \( \exists \ z^* \in M \) such that \( \zeta_n \to z^* \in M \), as \( n \to \infty \).

Next, we have to show that \( z^* \) is a fixed point of \( S \).

Because \( \zeta_n \to z^* \in M \), as \( n \to \infty \),

\[
\lim_{n \to \infty} \zeta_n = z^*.
\]

Therefore, for a subsequence \( \zeta_{n_i} \) of \( \zeta_n \),

\[
\zeta^* = \lim_{n \to \infty} \zeta_{n_i+1} = \lim_{n \to \infty} S(\zeta_{n_i}) = S(\lim_{n \to \infty} \zeta_{n_i}) = S(z^*).
\]

Hence, \( z^* \) is a fixed point in \( S \).  

Next, we introduce the extended interpolative Kannan-type \( F \)-contraction and establish a fixed point result.

**Definition 5.** Consider the \( b \)-MS \( (M, \xi, s) \) and a self-map \( S : M \to M \). Then, the map \( S \) is called an extended interpolative modified Kannan-type \( F \)-contraction if for \( \tau > 0 \) \( \exists c_1, c_2 \in (0, 1) \) with \( c_1 + c_2 < 1 \) and \( F \in \mathcal{F} \) such that

\[
\tau + F(\xi(S(z), S(q))) \leq c_1 F(\xi(z, S(z))) + c_2 F\left(\frac{1}{s} \xi(q, S(q))\right)
\]

\[\forall \ z, q \in M \setminus \text{Fix}(S) \text{ with } S(z) \neq S(q), \text{ where Fix}(S) \text{ denotes the set of all fixed points of } S.\]

**Theorem 2.** Consider a complete \( b \)-MS \( (M, \xi, s) \) and a continuous self-map \( S \) on \( M \). Then, \( S \) has a unique fixed point in \( M \) if \( S \) is an extended interpolative Kannan-type \( F \)-contraction.

**Proof.** Consider \( \zeta_0 \in M \). Construct the sequence \( \zeta_n \) such that \( \zeta_n = S^n(\zeta_0) \) \( \forall n \geq 0 \). If \( \exists n_0 \in \mathbb{N} \) such that \( \zeta_{n_0} = \zeta_{n_0+1} \), then \( \zeta_{n_0} \) is clearly a fixed point of \( S \). Hence, assume that \( \zeta_n \neq \zeta_{n+1} \) \( \forall n \geq 0 \) with \( z = \zeta_n \) and \( q = \zeta_{n-1} \) \( \forall n \in \mathbb{N} \). Then, we have the following from Equation (4):

\[
\tau + F(\xi(\zeta_{n+1}, \zeta_n)) \leq c_1 F(\xi(\zeta_n, S(\zeta_n))) + c_2 F\left(\frac{1}{s} \xi(\zeta_{n-1}, S(\zeta_{n-1}))\right)
\]

\[
< c_1 F(\xi(\zeta_n, \zeta_{n+1})) + (1 - c_1) F\left(\frac{1}{s} \xi(\zeta_{n-1}, \zeta_n)\right)
\]

\[
< c_1 F(\xi(\zeta_n, \zeta_{n+1})) + (1 - c_1) F(\xi(\zeta_{n-1}, \zeta_n)), \text{ (for } s \geq 1). \quad (5)
\]

Suppose \( \xi(\zeta_n, \zeta_{n-1}) < \xi(\zeta_{n+1}, \zeta_n) \); then, from (5), we have

\[
\tau + F(\xi(\zeta_n, S(\zeta_{n-1}))) < c_1 F(\xi(\zeta_n, \zeta_{n+1})) + (1 - c_1) F(\xi(\zeta_{n+1}, \zeta_n))
\]

\[\implies \tau + F(\xi(\zeta_{n+1}, \zeta_n)) < F(\xi(\zeta_n, \zeta_{n+1})).\]

which is a contradiction.

Therefore,

\[\xi(\zeta_{n+1}, \zeta_n) < \xi(\zeta_n, \zeta_{n-1}), \quad \forall n \geq 1.\]
Hence,
\[ \tau + F(\xi(\zeta_{n+1}, \zeta_n)) < F(\xi(\zeta_n, \zeta_{n-1})). \]

Consequently,
\[ F(\xi(\zeta_n, \zeta_{n+1})) \leq F(\xi(\zeta_{n-1}, \zeta_n)) - \tau \leq \ldots \leq F(\xi(\zeta_0, \xi_1)) - n\tau, \quad \forall n \geq 1. \]

Then, taking the limit as \( n \to \infty \),
\[ \lim_{n \to \infty} F(\xi(\zeta_n, \zeta_{n+1})) = -\infty. \]

Thus, from condition \( F2 \) we obtain
\[ \lim_{n \to \infty} \xi(\zeta_n, \zeta_{n+1}) = 0. \]

Let \( \theta_n = \xi(\zeta_n, \zeta_{n+1}) \).

Hence,
\[ \lim_{n \to \infty} \theta_n = 0. \]

Next, from the condition \( F3 \), \( \exists k \in (0, 1) \forall n \in \mathbb{N} \), we have
\[ \theta_n^k (F(\theta_n) - F(\theta_0)) \leq -\theta_n^k n\tau < 0. \] (6)

Now, using condition \( F3 \) and taking the limit as \( n \to \infty \),
\[ \lim_{n \to \infty} n\theta_n^k = 0. \]

Hence, \( n_0 \in \mathbb{N} \) such that \( n\theta_n^k \leq 1, \forall n \geq n_0 \).
\[ \theta_n \leq \frac{1}{n^k}, \forall n \geq n_0. \]

Next, to verify that \( \{\zeta_n\} \) is a Cauchy sequence, we take \( m, n \in \mathbb{N} \) such that \( m > n \geq n_0 \).

From the triangular inequality, we have
\[
\xi(\zeta_n, \zeta_m) \leq \frac{1}{s^{m-1}}[\xi(\zeta_n, \zeta_{n+1}) + \xi(\zeta_{n+1}, \zeta_{n+2}) + \ldots + \xi(\zeta_{m-1}, \zeta_m)]
\]
\[ = \frac{1}{s^{m-1}}[\theta_n + \theta_{n+1} + \ldots + \theta_{m-1}] \]
\[ = \sum_{i=1}^{m-1} \frac{1}{s^i} \sum_{i=n}^{m-1} \theta_i \]
\[ \leq \sum_{i=1}^{\infty} \frac{1}{s^i} \sum_{i=n}^{\infty} \theta_i \]
\[ \leq \sum_{i=1}^{\infty} \frac{1}{s^i} \sum_{i=n}^{\infty} \frac{1}{n^k}. \]

If we take limit as \( n \to \infty \) for the above inequality, we obtain
\[ \lim_{n,m \to \infty} \xi(\zeta_n, \zeta_m) = 0. \]

Therefore, \( \{\zeta_n\} \) is a Cauchy sequence in \( M \).

Because \( (M, \xi, s) \) is a complete bMS,
there exists \( \zeta^* \in M \) such that \( \zeta_n \to \zeta^* \in M \), as \( n \to \infty \).

Next, we have to show that \( \zeta^* \) is a fixed point of \( S \).

Because \( \zeta_n \to \zeta^* \in M \) and \( n \to \infty \),
\[ \lim_{n \to \infty} \zeta_n = \zeta^*. \]

Therefore, for a subsequence \( \zeta_{n_k} \) of \( \zeta_n \),
\[
\zeta^* = \lim_{n \to \infty} \zeta_{n_k+1} = \lim_{n \to \infty} S(\zeta_{n_k}) = S(\lim_{n \to \infty} \zeta_{n_k}) = S(\zeta^*). \]
Hence, \( c^* \) is a fixed point in \( S \).

Finally, we have to show the uniqueness.

Consider two fixed points, \( c^* \) and \( q^* \), such that \( c^* \neq q^* \). Then, \( S(c^*) = c^* \neq q^* = S(q^*) \).

Thus, we have

\[
\tau + F(\xi(S(c^*), S(q^*))) \leq c_1 F(\xi(c^*, S(c^*))) + c_2 F\left(\frac{1}{S}(\xi(q^*, S(q^*)))\right)
\]

\[
= 0,
\]

which is a contradiction.

Therefore, \( c^* = q^* \).

Hence, \( S \) has a unique fixed point in \( bMS \).

\( \square \)

4. Examples

In this section, we provide examples to validate our results.

**Example 1.** Consider \( M = \{x, y, z\} \). Let \( \xi : M \times M \to [0, \infty) \) be defined as \( \xi(x, y) = 0 \) if \( x = y \), \( \xi(x, y) = \xi(y, x) \) \( \forall x, y \in M \) and \( \xi(x, z) = 1 \), \( \xi(x, z) = 2.3 \), \( \xi(y, z) = 1.1 \). Then, \( (M, \xi, \frac{21}{23}) \) is a complete \( bMS \) but not an \( MS \).

Construct a self-map \( S : M \to M \) such that

\[
S(x) = \begin{cases} 
\{x\}, & \text{if } x = x \text{ or } x = y \\
\{y\}, & \text{if } x = z.
\end{cases}
\]

Next, construct

\[
\xi(S(x), S(y)) = \begin{cases} 
\{\xi(x, x) = 0\}, & \text{if } x \neq z, y \neq z \\
\{\xi(y, x) = 1\}, & \text{if } x = z, y \neq z \\
\{\xi(x, y) = 1\}, & \text{if } x \neq z, y = z \\
\{\xi(y, y) = 0\}, & \text{if } x = z, y = z.
\end{cases}
\]

Hence, \( \xi(S(x), S(y)) \) attains its maximum value of 1.

Now, we consider \( x, y \in M \setminus Fix(S) \) with \( S(x) \neq S(y) \).

Consider \( c_1 = \frac{1}{4}, c_2 = \frac{1}{2}, \tau = \frac{1}{4}\ln\left(\frac{21}{23}\right) \) and \( F(x) = \ln(x) \) for \( x > 0 \).

For the Cirić–Reich–Rus type inequality, we have

\[
\tau + F(\xi(S(x), S(y))) = \frac{1}{4}\ln\left(\frac{21}{23}\right) + F(1)
\]

\[
= \frac{1}{4}\ln\left(\frac{21}{23}\right) + \ln(1)
\]

\[
= \frac{1}{4}\ln\left(\frac{21}{23}\right).
\]

Further,

\[
\frac{1}{2}[F(\xi(x, y))] + c_2[F(\xi(x, S(x)))] + (1 - c_1 - c_2)[F\left(\frac{1}{S}(\xi(y, S(y)))\right)]
\]

\[
= c_1 F(1) + c_2 F(1) + (1 - c_1 - c_2) F\left(\frac{21}{23}\right)
\]

\[
= (1 - c_1 - c_2) F\left(\frac{21}{23}\right)
\]

\[
= \frac{3}{10}\ln\left(\frac{21}{23}\right).
\]

Therefore,

\[
\tau + F(\xi(S(x), S(y))) < \frac{1}{2}[F(\xi(x, y))] + c_2[F(\xi(x, S(x)))] + (1 - c_1 - c_2)[F\left(\frac{1}{S}(\xi(y, S(y)))\right)].
\]

Hence, the conditions for a CRR-type F-contraction are satisfied.
Next, for a Kannan-type inequality, we have

\[ \tau + F(\xi(S(x), S(y))) = \frac{1}{4} \ln(\frac{21}{23}) + F(1) \]

\[ = \frac{1}{4} \ln(\frac{21}{23}) + \ln(1) \]

\[ = \frac{1}{4} \ln(\frac{21}{23}), \]

and

\[ c_1[F(\xi(x, S(x)))] + c_2[F(\frac{1}{2} \xi(y, S(y)))] = \frac{1}{5} F(1) + \frac{1}{2} F(\frac{21}{23}) \]

\[ = \frac{1}{2} \ln(\frac{21}{23}) \]

Therefore,

\[ \tau + F(\xi(S(x), S(y))) < c_1[F(\xi(x, S(x)))] + c_2[F(\frac{1}{2} \xi(y, S(y)))] \]

Hence, the conditions for a Kannan-type F-contraction are satisfied.

**Example 2.** Consider the space \( l_p(0 < p < 1) \), where

\[ l_p = \left\{ (x_n) \subset \mathbb{R} : \sum_{n=1}^{\infty} |x_n|^p < \infty \right\}. \]

Define a metric on this space such as

\[ \xi : l_p \times l_p \to \mathbb{R} \text{ defined by,} \]

\[ \xi(x, y) = \left( \sum_{n=1}^{\infty} |x_n - y_n|^p \right)^{\frac{1}{p}}, \text{ where } x = x_n, y = y_n \in l_p, \]

having the triangular inequality

\[ \xi(x, z) \leq 2^{\frac{1}{p}} [\xi(x, y) + \xi(y, z)]. \]

Then, \( (l_p, \xi, 2^{\frac{1}{p}}) \) is a complete bMS.

Construct a self-map \( S : l_p \to l_p \) such that

\[ S(x) = \begin{cases} 
\{0\}, & \text{if } x = (0) \text{ or } x = (x) \text{ where } x \in l_p \\
\{1\}, & \text{otherwise}. 
\end{cases} \]

Now, for the Cirić–Reich–Rus type inequality, we have

\[ \tau + F(\xi(S(x), S(y))) \leq \tau + F(2^{\frac{1}{p}}(\xi(S(x), x) + \xi(x, S(y)))) \]

\[ \leq \tau + F[2^{\frac{1}{p}}(\xi(S(x), x) + 2^{\frac{1}{p}}(\xi(x, y) + \xi(y, S(y))))] \]

\[ < \tau + c_1 F(\xi(x, y)) + c_2 F(\xi(x, S(x))) + (1 - c_1 - c_2)F(\frac{1}{2p}(y, S(y))) \]

where \( c_1, c_2 \in \mathbb{N} \).

Therefore the conditions for a CRR-type F-contraction are satisfied.
Next, for a Kannan-type inequality, we have
\[
\tau + F(\xi(S(x), S(y))) \leq \tau + F(2^\frac{1}{2} (\xi(S(x), x) + \xi(x, S(y))))
\]
\[
\leq \tau + F(2^\frac{1}{2} (\xi(x, y) + \xi(y, S(y))))
\]
\[
< \tau + c_1 F(\xi(x, S(x))) + c_2 F(\frac{1}{2p}\xi(y, S(y)))
\]
where \(c_1, c_2 \in \mathbb{N}\).
Therefore, the conditions for a Kannan-type \(F\)-contraction are satisfied.

5. Application to Integral Equations

Applications of fixed point theorems can be found in various areas of science, such as physics, engineering, etc., in terms of solutions of differential and integral equations. In this section, following the similar lines as in [25,26], we provide an application of Theorem 2 towards the solution of an integral equation.

Let \(I = [0, r], r > 0\) and \(M = C(I, \mathbb{R})\) be the set of all real-valued continuous functions defined on \(I\). Suppose that
\[
\xi(g, h) = \sup_{t \in I} |g(t) - h(t)| = \|g - h\|,
\]
where \(\xi\) is a \(b\)-metric on \(M\).

Consider the integral equation
\[
g(x) = q(x) + \int_0^r G(x, w)f(w, g(w))\,dw, x \in [0, r],
\]
where
(i) \(q : I \to \mathbb{R}\) and \(f : I \times \mathbb{R} \to \mathbb{R}\) are continuous;
(ii) for all \(x \in I\), \(G : I \times I \to \mathbb{R}\) is continuous and measurable at \(w \in I\);
(iii) for all \(x, w \in I\), \(G(x, w) > 0\) and for all \(x \in I\), \(\int_0^r G(x, w)f(w, g(w))\,dw \leq 1\).

**Theorem 3.** Assume that conditions (i) – (iii) hold. Let \(\tau > 0\) and \(c_1, c_2 \in (0, 1)\) with \(c_1 + c_2 < 1\) exist such that
\[
\left| f(x, g(x)) - f(x, h(x)) \right|
\leq \frac{1}{s^{c_2}} e^{-\tau} \left( \left\| g - \int_0^r G(x, w)f(w, g(w))\,dw \right\| \right)^{c_1}.
\]
\[
\left( \left\| h - \int_0^r G(x, w)f(w, h(w))\,dw \right\| \right)^{c_2},
\]
for each \(x \in I\) and for all \(g, h \in X\) such that
\[
g(x) \neq \int_0^r G(x, w)f(w, g(w))\,dw,
\]
\[
h(x) \neq \int_0^r G(x, w)f(w, h(w))\,dw,
\]
while \(g(x) \neq h(x)\) for all \(x \in I\). Then, the integral equation (7) has a solution in \(M\).

**Proof.** Let us define a self-map \(\Gamma : M \to M\) by
\[
\Gamma(g(x)) = q(x) + \int_0^r G(x, w)f(w, g(w))\,dw, x \in [0, r].
\]
For each \(x \in [0, r]\), we have
\[ |\Gamma(g(x)) - \Gamma(h(x))| = \left| \int_0^T G(x, w) \left\{ f(w, g(w)) - f(w, h(w)) \right\} dw \right| \]
\[ \leq \int_0^T G(x, w) \left| f(w, g(w)) - f(w, h(w)) \right| dw \]
\[ \leq \int_0^T G(x, w) \cdot \frac{1}{s^2} e^{-\tau} \left( \|g - \Gamma g\| \right)^{c_1} \cdot \left( \|h - \Gamma h\| \right)^{c_2} \int_0^T G(x, w) dw \]
\[ \leq \frac{1}{s^2} e^{-\tau} \cdot \left( \|g - \Gamma g\| \right)^{c_1} \cdot \left( \|h - \Gamma h\| \right)^{c_2}, \quad (9) \]

(because \( \int_0^T G(x, w) dw \leq 1 \) for all \( x \in I \).

Taking the supremum over \( x \in I \) on both sides of (9), we have

\[ \xi'(\Gamma g, \Gamma h) = \|\Gamma g - \Gamma h\| \leq \frac{1}{s^2} e^{-\tau} \cdot \left( \|g - \Gamma g\| \right)^{c_1} \cdot \left( \|h - \Gamma h\| \right)^{c_2} \]
\[ = \frac{1}{s^2} e^{-\tau} \cdot \left( \xi'(g, \Gamma g) \right)^{c_1} \cdot \left( \xi(h, \Gamma h) \right)^{c_2} \]
\[ \Rightarrow \quad \frac{\xi'(\Gamma g, \Gamma h)}{\xi'(g, \Gamma g)^{c_1} \cdot \xi(h, \Gamma h)^{c_2}} \leq e^{-\tau} \]
\[ \Rightarrow \ln \left[ \frac{\xi'(\Gamma g, \Gamma h)}{\xi'(g, \Gamma g)^{c_1} \cdot \xi(h, \Gamma h)^{c_2}} \right] \leq -\tau \]
\[ \Rightarrow \ln \left[ \frac{\xi'(\Gamma g, \Gamma h)}{\xi'(g, \Gamma g)^{c_1} \cdot \xi(h, \Gamma h)^{c_2}} \right] - \ln \left[ \frac{1}{s^2} e^{-\tau} \cdot \left( \xi'(g, \Gamma g) \right)^{c_1} \cdot \left( \xi(h, \Gamma h) \right)^{c_2} \right] \leq -\tau \]
\[ \Rightarrow \quad \tau + \ln \left[ \xi'(\Gamma g, \Gamma h) \right] - \ln \left[ \xi'(g, \Gamma g) \right] \leq \ln \left[ \left( \xi'(g, \Gamma g) \right)^{c_1} \cdot \left( \frac{1}{s} \xi(h, \Gamma h) \right)^{c_2} \right] \]
\[ \Rightarrow \quad \tau + \ln \left[ \xi'(\Gamma g, \Gamma h) \right] \leq c_1 \ln \left[ \xi'(g, \Gamma g) \right] + c_2 \ln \left[ \frac{1}{s} \xi(h, \Gamma h) \right]. \quad (10) \]

Taking \( F(\delta) = \ln(\delta) \), \( \delta > 0 \), we have form (10):

\[ \tau + F(\xi'(\Gamma g, \Gamma h)) \leq c_1 F(\xi'(g, \Gamma g)) + c_2 F(\frac{1}{s} \xi(h, \Gamma h)) \]

for all \( g, h \in M \setminus \text{Fix}(\Gamma) \) with \( \Gamma g \neq \Gamma h \).

Hence, per Theorem 2, \( \Gamma \) has a unique fixed point; therefore, the integral Equation (7) has a unique solution. \( \square \)

6. Conclusions

In this paper, we have introduced extended interpolative CRR-type \( F \)-contraction and interpolative Kannan-type \( F \)-contraction mappings in a \( b \)M. The existence of fixed point results has been established for the new contraction maps. We have provided examples involving both finite and infinite \( b \)MSs. This work can be extended in the future to investi-
gate discontinuity results at the fixed points. Applications of these extended results may point out new iterative schemes for the solution of differential and integral equations.

**Author Contributions:** N.K. contributed in Conceptualization, Investigation, Methodology and Writing the original draft; P.D. contributed in Investigation, Methodology, Validation, Supervision, Writing, and Editing. All authors have read and agreed to the published version of the manuscript.

**Funding:** This research received no external funding.

**Institutional Review Board Statement:** Not applicable.

**Informed Consent Statement:** Not applicable.

**Data Availability Statement:** Not applicable.

**Acknowledgments:** The authors are extremely grateful to the anonymous reviewers for their constructive remarks towards improvement of the manuscript.

**Conflicts of Interest:** The authors declare no conflict of interest.

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