Abstract: In this paper, we introduce and investigate new subclasses of bi-univalent functions with respect to the symmetric points in $U = \{z \in \mathbb{C} : |z| < 1\}$ defined by Bernoulli polynomials. We obtain upper bounds for Taylor–Maclaurin coefficients $|a_2|$, $|a_3|$ and Fekete–Szegö inequalities $|a_3 - \mu a_2^2|$ for these new subclasses.

Keywords: Fekete–Szegö inequality; Bernoulli polynomial; analytic and bi-univalent functions; subordination; symmetric points

MSC: 30C45; 30C50

1. Introduction

Let the class of analytic functions in $U = \{z \in \mathbb{C} : |z| < 1\}$, denoted by $A$, contain all the functions of the type

$$l(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad (z \in U), \quad (1)$$

which satisfy the usual normalization condition $l(0) = l'(0) - 1 = 0$.

Let $S$ be the subclass of $A$ consisting of all functions $l \in A$, which are also univalent in $U$. The Koebe one quarter theorem [1] ensures that the image of $U$ under every univalent function $l \in A$ contains a disk of radius $\frac{1}{4}$. Thus, every univalent function $l$ has an inverse $l^{-1}$ satisfying

$$l^{-1}(l(z)) = z, (z \in U) \quad \text{and} \quad l^{-1}(l^{-1}(\omega)) = \omega, \quad (|\omega| < r_0(l), r_0(l) \geq \frac{1}{4}).$$

If $l$ and $l^{-1}$ are univalent in $U$, then $l \in A$ is said to be bi-univalent in $U$, and the class of bi-univalent functions defined in the unit disk $U$ is denoted by $\Sigma$. Since $l \in \Sigma$ has the Maclaurin series given by (1), a computation shows that $m = l^{-1}$ has the expansion

$$m(\omega) = l^{-1}(\omega) = \omega - a_2 \omega^2 + \left(2a_2^2 - a_3\right)\omega^3 + \cdots. \quad (2)$$

The expression $\Sigma$ is a non-empty class of functions, as it contains at least the functions

$$l_1(z) = -\frac{z}{1 - z}, \quad l_2(z) = \frac{1}{2} \log \frac{1 + z}{1 - z},$$

with their corresponding inverses

$$l_1^{-1}(\omega) = \frac{\omega}{1 + \omega}, \quad l_2^{-1}(\omega) = \frac{e^{2\omega} - 1}{e^{2\omega} + 1}.$$
In addition, the Koebe function \( l(z) = \frac{z}{1-z^2} \not\in \Sigma \).

The study of analytical and bi-univalent functions is reintroduced in the publication of [2] and is then followed by work such as [3–8]. The initial coefficient constraints have been determined by several authors who have also presented new subclasses of bi-univalent functions (see [2–4,6,9–11]).

Consider \( \alpha \) and \( \beta \) to be analytic functions in \( U \). We say that \( \alpha \) is subordinate to \( \beta \), if a Schwarz function \( w \) exists that is analytic in \( U \) with \( w(0) = 0 \) and \( |w(z)| < 1 \), \( (z \in U) \) such that

\[
\alpha(z) = \beta(w(z)), \ (z \in U).
\]

This subordination is denoted by \( \alpha \prec \beta \) or \( \alpha(z) \prec \beta(z), \ (z \in U) \). Given that \( \beta \) is a univalent function in \( U \), then

\[
\alpha(z) \prec \beta(z) \iff \alpha(0) = \beta(0) \quad \text{and} \quad \alpha(U) \subset \beta(U).
\]

Using Loewner’s technique, the Fekete–Szegö problem for the coefficients of \( l \in S \) in [6] is

\[
|a_3 - \mu a_2^2| \leq 1 + 2 \exp \left( \frac{-2 \mu}{1 - \mu} \right) \quad \text{for} \quad 0 \leq \mu < 1.
\]

The elementary inequality \( |a_3 - a_2^2| \leq 1 \) is obtained as \( \mu \to 1 \). The coefficient functional

\[
F_\mu(l) = a_3 - \mu a_2^2
\]

on the normalized analytic functions \( l \) in the open unit disk \( U \) also has a significant impact on geometric function theory. The Fekete–Szegö problem is known as the maximization problem for functional \( |F_\mu(l)| \).

Researchers were concerned about several classes of univalent functions (see [12–15]) due to the Fekete–Szegö problem, proposed in 1933 ([16]); therefore, it stands to reason that similar inequalities were also discovered for bi-univalent functions, and fairly recent publications can be cited to back up the claim that the subject still yields intriguing findings [17–19].

Because of their importance in probability theory, mathematical statistics, mathematical physics, and engineering, orthogonal polynomials have been the subject of substantial research in recent years from a variety of angles. The classical orthogonal polynomials are the orthogonal polynomials that are most commonly used in applications (Hermite polynomials, Laguerre polynomials, Jacobi polynomials, and Bernoulli). We point out [17,18,20–24] as more recent examples of the relationship between geometric function theory and classical orthogonal polynomials.

Fractional calculus, a classical branch of mathematical analysis whose foundations were laid by Liouville in an 1832 paper and is currently a very active research field [25], is one of many special functions that are studied. This branch of mathematics is known as the Bernoulli polynomials, named after Jacob Bernoulli (1654–1705). A novel approximation method based on orthonormal Bernoulli’s polynomials has been developed to solve fractional order differential equations of the Lane–Emden type [26], whereas in [27–29], Bernoulli polynomials are utilized to numerically resolve Fredholm fractional integro-differential equations with right-sided Caputo derivatives.

The Bernoulli polynomials \( B_n(x) \) are often defined (see, e.g., [30]) using the generating function:

\[
F(x, t) = \frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} t^n, \ |t| < 2\pi,
\]

where \( B_n(x) \) are polynomials in \( x \), for each nonnegative integer \( n \).
The Bernoulli polynomials are easily computed by recursion since
\[ \sum_{j=0}^{n-1} \binom{n}{j} B_j(x) = nx^{n-1}, \; n = 2, 3, \cdots. \] (4)

The initial few polynomials of Bernoulli are
\[ B_0(x) = 1, \; B_1(x) = x - \frac{1}{2}, \; B_2(x) = x^2 - x + \frac{1}{6}, \; B_3(x) = x^3 - \frac{3}{2} x^2 + \frac{1}{2} x, \cdots. \] (5)

Sakaguchi [31] introduced the class \( S^*_s \) of functions starlike with respect to symmetric points, which consists of functions \( l \in S \) satisfying the condition
\[ \text{Re} \left\{ \frac{zl'(z)}{l(z) - l(-z)} \right\} > 0, \quad (z \in U). \]

In addition, Wang et al. [32] introduced the class \( C_s \) of functions convex with respect to symmetric points, which consists of functions \( l \in S \) satisfying the condition
\[ \text{Re} \left\{ \frac{[l'(z)]'}{[l(z) - l(-z)]'} \right\} > 0, \quad (z \in U). \]

In this paper, we consider two subclasses of \( \Sigma \): the class \( S\Sigma_s(x) \) of functions bi-starlike with respect to the symmetric points and the relative class \( C\Sigma_s(x) \) of functions bi-convex with respect to the symmetric points associated with Bernoulli polynomials. The definitions are as follows:

**Definition 1.** \( l \in S\Sigma_s(x) \), if the next subordinations hold:
\[ \frac{2zl'(z)}{l(-z) - l(z)} \prec F(x, z), \] (6)

and
\[ \frac{2\omega m'(\omega)}{m(\omega) - m(-\omega)} \prec F(x, \omega), \] (7)

where \( z, \omega \in U, F(x, z) \) is given by (3), and \( m = l^{-1} \) is given by (2).

**Definition 2.** \( l \in C\Sigma_s(x) \), if the following subordinations hold:
\[ \frac{2[zl'(z)]'}{[l(z) - l(-z)]'} \prec F(x, z), \] (8)

and
\[ \frac{2[\omega m'(\omega)']}{[m(\omega) - m(-\omega)]'} \prec F(x, \omega), \] (9)

where \( z, \omega \in U, F(x, z) \) is given by (3), and \( m = l^{-1} \) is given by (2).

**Lemma 1** ([33], p. 172). Suppose that \( c(z) = \sum_{n=1}^{\infty} c_n z^n, \; |c(z)| < 1, \; z \in U, \) is an analytic function in \( U. \) Then,
\[ |c_1| \leq 1, \; |c_n| \leq 1 - |c_1|^2, \; n = 2, 3, \cdots. \]

2. **Coefficients Estimates for the Class \( S\Sigma_s(x) \)**

We obtain upper bounds of \( |a_2| \) and \( |a_3| \) for the functions belonging to the class \( S\Sigma_s(x). \)
Theorem 1. If \( l \in S_\Sigma^x(x) \), then

\[
|a_2| \leq |B_1(x)| \sqrt{6 |B_1(x)|},
\]

and

\[
|a_3| \leq \frac{B_1(x)}{2} + \frac{|B_1(x)|^2}{4}.
\]

**Proof.** Let \( l \in S_\Sigma^x(x) \) and \( m = l^{-1} \). From definition in (6) and (7), we have

\[
\frac{2l'(z)z}{l(z) - l(-z)} = F(x, \varphi(z)),
\]

and

\[
\frac{2\omega m'(\omega)}{m(\omega) - m(-\omega)} = F(x, \chi(\omega)),
\]

where \( \varphi \) and \( \chi \) are analytic functions in \( U \) given by

\[
\varphi(z) = r_1 z + r_2 z^2 + \cdots,
\]

\[
\chi(\omega) = s_1 \omega + s_2 \omega^2 + \cdots,
\]

and \( \varphi(0) = \chi(0) = 0 \), and \( |\varphi(z)| < 1 \), \( |\chi(\omega)| < 1 \), \( z, \omega \in U \).

As a result of Lemma 1,

\[
|r_k| \leq 1 \quad \text{and} \quad |s_k| \leq 1, \quad k \in \mathbb{N}.
\]

If we replace (14) and (15) in (12) and (13), respectively, we obtain

\[
\frac{2zl'(z)}{l(z) - l(-z)} = B_0(x) + B_1(x)\varphi(z) + \frac{B_2(x)}{2!} \varphi^2(z) + \cdots,
\]

and

\[
\frac{2\omega m'(\omega)}{m(\omega) - m(-\omega)} = B_0(x) + B_1(x)\chi(\omega) + \frac{B_2(x)}{2!} \chi^2(\omega) + \cdots.
\]

In view of (1) and (2), from (17) and (18), we obtain

\[
1 + 2a_2 z + 2a_3 z^2 + \cdots = 1 + B_1(x) r_1 z + \left[ B_1(x) r_2 + \frac{B_2(x)}{2!} r_1^2 \right] z^2 + \cdots
\]

and

\[
1 - 2a_2 \omega + (4a_2^2 - 2a_3)\omega^2 + \cdots = 1 + B_1(x) s_1 \omega + \left[ B_1(x) s_2 + \frac{B_2(x)}{2!} s_1^2 \right] \omega^2 + \cdots,
\]

which yields the following relations:

\[
2a_2 = B_1(x) r_1,
\]

\[
2a_3 = B_1(x) r_2 + \frac{B_2(x)}{2!} r_1^2,
\]

and

\[
-2a_2 = B_1(x) s_1,
\]

\[
4a_2^2 - 2a_3 = B_1(x) s_2 + \frac{B_2(x)}{2!} s_1^2.
\]
From (19) and (21), it follows that
\[ r_1 = -s_1, \]  
(23)
and
\[ 8a_3^2 = [B_1(x)]^2 \left( r_1^2 + s_1^2 \right) \]
\[ a_2^2 = \frac{[B_1(x)]^2 (r_2^2 + s_2^2)}{8}. \]  
(24)
Adding (20) and (22), using (24), we obtain
\[ a_2^2 = \frac{[B_1(x)]^3 (r_2 + s_2)}{4([B_1(x)]^2 - B_2(x))}. \]  
(25)
Using relation (5), from (16) for \( r_2 \) and \( s_2 \), we get (10). Using (23) and (24), by subtracting (22) from relation (20), we get
\[ a_3 = \frac{B_1(x)(r_2 - s_2) + \frac{B_2(x)}{2}(r_1^2 - s_1^2) + a_2^2}{4} \]
\[ = \frac{B_1(x)(r_2 - s_2) + \frac{B_2(x)}{2}(r_1^2 - s_1^2) + [B_1(x)]^2 (r_2^2 + s_2^2)}{8}. \]  
(26)
Once again applying (23) and using (5), for the coefficients \( r_1, s_1, r_2, s_2 \), we deduce (11). □

3. The Fekete–Szegö Problem for the Function Class \( S^\mu_1(x) \)

We obtain the Fekete–Szegö inequality for the class \( S^\mu_1(x) \) due to the result of Zaprawa; see [19].

**Theorem 2.** If \( l \) given by (1) is in the class \( S^\mu_1(x) \) where \( \mu \in \mathbb{R} \), then we have
\[ \left| a_3 - \mu a_2^2 \right| \leq \begin{cases} 
\frac{B_1(x)}{2}, & \text{if } |h(\mu)| \leq \frac{1}{4}, \\
n 2B_1(x)|h(\mu)|, & \text{if } |h(\mu)| \geq \frac{1}{4}, 
\end{cases} \]
where
\[ h(\mu) = 3(1 - \mu)[B_1(x)]^2. \]

**Proof.** If \( l \in S^\mu_1(x) \) is given by (1), from (25) and (26), we have
\[ a_3 - \mu a_2^2 = \frac{B_1(x)(r_2 - s_2)}{4} + (1 - \mu)a_2^2 \]
\[ = \frac{B_1(x)(r_2 - s_2)}{4} + \frac{(1 - \mu)[B_1(x)]^3(r_2 + s_2)}{4([B_1(x)]^2 - B_2(x))} \]
\[ = B_1(x) \left[ \frac{r_2}{4} - \frac{s_2}{4} + \frac{(1 - \mu)[B_1(x)]^2 r_2}{4([B_1(x)]^2 - B_2(x))} + \frac{(1 - \mu)[B_1(x)]^2 s_2}{4([B_1(x)]^2 - B_2(x))} \right] \]
\[ = B_1(x) \left[ \left( h(\mu) + \frac{1}{4} \right)r_2 + \left( h(\mu) - \frac{1}{4} \right)s_2 \right], \]
where
\[ h(\mu) = \frac{(1 - \mu)[B_1(x)]^2}{4([B_1(x)]^2 - B_2(x))}. \]
Now, by using (5)
\[ a_3 - \mu a_2^2 = \left( x - \frac{1}{2} \right) \left[ \left( h(\mu) + \frac{1}{4} \right) r_2 + \left( h(\mu) - \frac{1}{4} \right) s_2 \right], \]
where
\[ h(\mu) = 3(1 - \mu) \left( x - \frac{1}{2} \right)^2. \]

Therefore, given (5) and (16), we conclude that the necessary inequality holds. \(\square\)

4. Coefficients Estimates for the Class \( C^E_\Sigma(x) \)

We will obtain upper bounds of \(|a_2|\) and \(|a_3|\) for the functions belonging to a class \( C^E_\Sigma(x) \).

**Theorem 3.** If \( l \in C^E_\Sigma(x) \), then
\[
|a_2| \leq \frac{|B_1(x)| \sqrt{|B_1(x)|}}{\sqrt{6|B_1(x)|^2 - 8B_2(x)}}, \tag{27}
\]
and
\[
|a_3| \leq \frac{B_1(x)}{6} + \frac{|B_1(x)|^2}{16}. \tag{28}
\]

**Proof.** Let \( l \in C^E_\Sigma(x) \) and \( m = l^{-1} \). From (8) and (9), we get
\[
\frac{2[zl'(z)]'}{[l(z) - l(-z)]'} = F(x, \varphi(z)), \tag{29}
\]
and
\[
\frac{2[\omega m'(\omega)]'}{[m(\omega) - m(-\omega)]'} = F(x, \chi(\omega)), \tag{30}
\]
where \( \varphi \) and \( \chi \) are analytic functions in \( U \) given by
\[
\varphi(z) = r_1 z + r_2 z^2 + \cdots, \tag{31}
\]
\[
\chi(\omega) = s_1 \omega + s_2 \omega^2 + \cdots, \tag{32}
\]
where \( \varphi(0) = \chi(0) = 0 \), and \( |\varphi(z)| < 1 \), \( |\chi(\omega)| < 1 \), \( z, \omega \in U \).

As a result of Lemma 1,
\[
|r_k| \leq 1 \text{ and } |s_k| \leq 1, \quad k \in \mathbb{N}. \tag{33}
\]

If we replace (31) and (32) in (29) and (30), respectively, we obtain
\[
\frac{2[zl'(z)]'}{[l(z) - l(-z)]'} = B_0(x) + B_1(x) \varphi(z) + \frac{B_2(x)}{2!} \varphi^2(z) + \cdots, \tag{34}
\]
and
\[
\frac{2[\omega m'(\omega)]'}{[m(\omega) - m(-\omega)]'} = B_0(x) + B_1(x) \chi(\omega) + \frac{B_2(x)}{2!} \chi^2(\omega) + \cdots. \tag{35}
\]

In view of (1) and (2), from (34) and (35), we obtain
\[
1 + 4a_2 z + 6a_3 z^2 + \cdots = 1 + B_1(x) r_1 z + \left[ B_1(x) r_2 + \frac{B_2(x)}{2!} r_1^2 \right] z^2 + \cdots
\]
and
\[1 - 4\omega + \left(12\omega^2 - 6\right)\omega^2 + \cdots = 1 + B_1(x)s_1\omega + \left[B_1(x)s_2 + \frac{B_2(x)}{2} s_1 \right] \omega^2 + \cdots,\]
which yields the following relations:
\[4\omega = B_1(x)r_1, \quad (36)\]
\[6\omega = B_1(x)r_2 + \frac{B_2(x)}{2} r_1^2, \quad (37)\]
and
\[-4\omega = B_1(x)s_1, \quad (38)\]
\[12\omega^2 - 6\omega = B_1(x)s_2 + \frac{B_2(x)}{2} s_1^2. \quad (39)\]
From (36) and (38), it follows that
\[r_1 = -s_1, \quad (40)\]
and
\[32\omega^2 = \left[B_1(x)^2 \left(r_1^2 + s_1^2\right)\right] \quad (41)\]
Adding (37) and (39), using (41), we obtain
\[a_2^2 = \frac{\left[B_1(x)^2 \left(r_2 + s_2\right)\right]}{4(3B_1(x)^2 - 4B_2(x))}. \quad (42)\]
Using relation (5), from (33) for \(r_2\) and \(s_2\), we get (27). Using (40) and (41), by subtracting (39) from relation (37), we get
\[a_3 = \frac{B_1(x)(r_2 - s_2) + \frac{B_2(x)}{2}(r_1^2 - s_1^2) + a_2^2}{12} \quad (43)\]
\[= \frac{B_1(x)(r_2 - s_2) + \frac{B_2(x)}{2}(r_1^2 - s_1^2) + \left[B_1(x)^2 \left(r_2 + s_2\right)\right]}{32}. \quad (44)\]
Once again applying (40) and using (5), for the coefficients \(r_1, s_1, r_2, s_2\), we deduce (28).

5. The Fekete–Szegö Problem for the Function Class \(\mathcal{C}_5^\Sigma(x)\)

We obtain the Fekete–Szegö inequality for the class \(\mathcal{C}_5^\Sigma(x)\) due to the result of Zaprawa; see [19].

Theorem 4. If \(l\) given by (1) is in the class \(\mathcal{C}_5^\Sigma(x)\) where \(\mu \in \mathbb{R}\), then, we have
\[
\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{B_1(x)}{6}, & \text{if } |h(\mu)| \leq \frac{1}{2}, \\ 2B_1(x)|h(\mu)|, & \text{if } |h(\mu)| \geq \frac{1}{2}, \end{cases}
\]
where
\[h(\mu) = \frac{(1 - \mu)|B_1(x)|^2}{4(3B_1(x)^2 - 4B_2(x))}.\]
Proof. If \( l \in C^2_s(x) \) is given by (1), from (42) and (43), we have

\[
a_3 - \mu a_2^2 = \frac{B_1(x)(r_2 - s_2)}{12} + (1 - \mu)a_2^2
\]

\[
= \frac{B_1(x)(r_2 - s_2)}{12} + \frac{(1 - \mu)|B_1(x)|^2(r_2 + s_2)}{4(3B_1(x)^2 - 4B_2(x))}
\]

\[
= B_1(x) \left[ \frac{r_2 - s_2}{12} + \frac{(1 - \mu)|B_1(x)|^2r_2}{4(3B_1(x)^2 - 4B_2(x))} + \frac{(1 - \mu)|B_1(x)|^2s_2}{4(3B_1(x)^2 - 4B_2(x))} \right]
\]

\[
= B_1(x) \left[ \left( h(\mu) + \frac{1}{12} \right) r_2 + \left( h(\mu) - \frac{1}{12} \right) s_2 \right],
\]

where

\[
h(\mu) = \frac{(1 - \mu)|B_1(x)|^2}{4(3B_1(x)^2 - 4B_2(x))}.
\]

Now, by using (5)

\[
a_3 - \mu a_2^2 = \left( x - \frac{1}{2} \right) \left[ \left( h(\mu) + \frac{1}{12} \right) r_2 + \left( h(\mu) - \frac{1}{12} \right) s_2 \right],
\]

where

\[
h(\mu) = \frac{(1 - \mu)\left( x - \frac{1}{2} \right)^2}{4\left( 3\left( x - \frac{1}{2} \right)^2 - 4\left( x^2 - x + \frac{1}{6} \right) \right)}.
\]

Therefore, given (5) and (33), we conclude that the required inequality holds. \( \square \)

6. Conclusions

We introduce and investigate new subclasses of bi-univalent functions in \( \mathcal{U} \) associated with Bernoulli polynomials and satisfying subordination conditions. Moreover, we obtain upper bounds for the initial Taylor–Maclaurin coefficients \(|a_2|, |a_3|\) and Fekete–Szegö problem \(|a_3 - \mu a_2^2|\) for functions in these subclasses.

The approach employed here has also been extended to generate new bi-univalent function subfamilies using the other special functions. The researchers may carry out the linked outcomes in practice.


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References


Amourah, A.; Frasin, B.A.; Ahmad, M.; Yousef, F. Exploiting the Pascal distribution series and Gegenbauer polynomials to construct and study a new subclass of analytic bi-univalent functions. *Symmetry* 2022, 14, 147.


