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Article

TOPSIS Method Based on Hamacher Choquet-Integral Aggregation Operators for Atanassov-Intuitionistic Fuzzy Sets and Their Applications in Decision-Making

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Abstract: The collection of Hamacher t-norms was created by Hamacher in 1970, which played a critical and significant role in computing aggregation operators. All aggregation operators that are derived based on Hamacher norms are very powerful and are beneficial because of the parameter $0 \leq \zeta \leq +\infty$. Choquet first posited the theory of the Choquet integral (CI) in 1953, which is used for evaluating awkward and unreliable information to address real-life problems. In this manuscript, we analyze several aggregation operators based on CI, aggregation operators, the Hamacher t-norm and t-conorm, and Atanassov intuitionistic fuzzy (A-IF) information. These are called A-IF Hamacher CI averaging (A-IFHCIA), A-IF Hamacher CI ordered averaging (A-IFHCIOA), A-IF Hamacher CI geometric (A-IFHCIG), and A-IF Hamacher CI ordered geometric (A-IFHCIOG) operators; herein, we identify their most beneficial and valuable results according to their main properties. Working continuously, we developed a multi-attribute decision-making (MADM) procedure for evaluating awkward and unreliable information, with the help of the TOPSIS technique for order performance by similarity to the ideal solution, and derive operators to enhance the worth and value of the present information. Finally, by comparing the pioneering information with some of the existing operators, we illustrate some examples for evaluating the real-life problems related to enterprises, wherein the owner of a company appointed four senior board members of the enterprise to decide what was the best Asian company in which to invest money, to show the supremacy and superiority of the invented approaches.

Keywords: Hamacher t-norm and t-conorm; aggregation operators; Choquet integral; intuitionistic fuzzy sets; decision-making problem based on TOPSIS methods

MSC: 03B52; 68T27; 68T37; 94D05; 03E72; 28E10

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1. Introduction

A group decision-making procedure is a technique whereby a collection of experts mutually identifies the best decision in a scenario. Usually, the decision is based on valuable discussion and after soliciting advice from knowledgeable people. This type of information is preferable to and more valuable than pattern recognition and is utilized in medical diagnosis. There are a large variety of collective decision-making techniques that can be used in the environment of classical information. One of the most valuable and dominant parts of any decision-making process is to confirm that all the experts in the group are comfortable with the method chosen and that it is optimal for the decision that is to be made. Effective and commonly employed methods include the nominal group technique, the brainstorming method, and the Delphi technique. However, these methods have given rise to many complications for experts because when they find the solution to any awkward and complicated problems using classical information, this method fails or does not take into account much of the information. For managing such problems, the theory of fuzzy set (FS) analysis was first reported by Zadeh [1] in 1965. Fuzzy information is one of the most valuable and realistic modifications of classical information that are used for addressing awkward and vague information in many real-life problems. Furthermore, a multi-objective hierarchical genetic algorithm considering fuzzy interpretable rules in the presence of knowledge extraction was developed by Wang et al. [2], while Chalco-Cano and Roman-Flores [3] found the solution of complicated differential equations based on fuzzy logic and its valuable capabilities. Dehghan et al. [4] discovered a computation solution for some linear systems based on fuzzy information, and finally, the theory of the segmentation of protein surfaces via fuzzy logic was discovered by Heiden and Brickmann [5].

Classical information and fuzzy information are identical in shape but are very different in practice because the range of the fuzzy set is much better than the range of the classical set theory. Furthermore, it is also clear that the fuzzy information contains or deals only with one element, called the membership grade, and ignores the concept of negative or non-membership grade. The non-membership grade has the same or equal role in our problem as a membership grade; without a falsity grade, we cannot evaluate problems in our daily life effectively. Therefore, the theory of FS is not enough or sufficient for accommodating awkward and unreliable information in genuine problems; for this purpose, the main theory of the Atanassov-intuitionistic fuzzy (A-IF) set was invented by Atanassov [6] in 1986. The theory of IF information is one of the most valuable and realistic modifications of FS and classical information and is used for addressing awkward and vague information related to many real-life problems. Elsewhere, Liu et al. [7] offered three different perspectives for evaluating three-way opinions, based on linguistic A-IF information. Xie et al. [8] invented a modified version of information quality for an A-IF set and evaluated their main features, while Liu et al. [9] identified the internet public preference emergency within the context of cubic A-IF values and described it in terms of decision-making problems. Garg and Rani [10] presented an algorithm for evaluating the awkward and unreliable decision-making procedure, taking into consideration the correlation coefficient for A-IF information. Wang et al. [11] proposed a three-way decision approach with probabilistic dominance relationships, based on A-IF information. Ecer [12] evaluated a modified form of the MAIRCA "multi-attribute ideal-real comparative analysis" technique for an A-IF set and evaluated several problems in the context of COVID-19; finally, Panda and Nagwani [13] described the modeling or exploitation of different types of problems, based on A-IF information and its applications in decision-making.

The TOPSIS technique utilized a combination of valuable and effective information models, such as the ideal positive solution, the negative ideal solution, discrimination measures, and closeness measures; we evaluated the ranking values based on the closeness information from several valuable distance measures. The TOPSIS method was first presented by various scholars in 1981. The concept behind the TOPSIS technique was developed by Hwang and Yoon [14]; in 1987, the theory of the TOPSIS method was first presented by Yoon [15], then, in 1993, the method was evaluated by Hwang et al. [16]. To illustrate the above theory, we consider the example of buying a mobile phone; we assume that a person wants to buy a new mobile phone and, for this reason, he goes to the shop and investigates the options based on the following five features, comprising RAM, memory, display size, battery, and price. At first, the customer is confused, after seeing so many complicated features, about how to decide which mobile is most suitable. The TOPSIS technique is the most effective method to evaluate this type of problem because TOPSIS is the best way to rank results based on the weights and impacts of the considered features. Later, the fuzzy TOPSIS technique was discovered by Chu and Lin [17]; a TOPSIS method based on FS was also employed by Wang and Elhag [18] for evaluating bridge risk assessment problems. Chen and Tsao [19], Sun and Lin [20], Dymoya et al. [21], Ashtiani et al. [22], and Memari et al. [23] also utilized the TOPSIS method for FS theory. Moreover, Shen et al. [24] invented a TOPSIS method for A-IF set theory, while Joshi and

Kumar [25] utilized the TOPSIS technique and entropy measures for A-IF information. Liu [26] developed a TOPSIS method for evaluating the problems of physical education, while Zulqarnain and Dayan [27] presented a TOPSIS method for addressing the problems of automotive enterprises.

Many valuable and dominant norms have been identified by different scholars, but the collection of Hamacher t-norms that were presented by Hamacher in 1970 played a critical and much-appreciated role in the area of computing aggregation operators. Elsewhere, Bellman and Zadeh [28] developed the theory of Hamacher aggregation with the help of decision-making based on fuzzy information. Huang [29] discovered the Hamacher aggregation operators for A-IF sets and evaluated their application in decision-making, while Garg [30] presented the Hamacher aggregation operator and entropy measures for A-IF information. More recently, Cakir and Ulukan [31] developed the theory of A-IF Hamacher aggregation operators. All aggregation operators that are derived based on Hamacher norms are very powerful because of the parameter $0 \leq \zeta \leq +\infty$. Choquet [32] first posited the theory of CI in 1953, used for evaluating awkward and unreliable information in real-life problems. Later, Xu [33] invented a theory of CI based on weighted A-IF information, while Tan and Chen [34] presented CI operators in the context of A-IF sets, Wu et al. [35] examined CI in terms of A-IF set theory, while Liu et al. [36] evaluated the CI operators for A-IF information; finally, an exploration of CI operators based on an A-IF set was performed by Wang et al. [37]. Because of ambiguity and uncertainty, many structures have been developed for depicting awkward and unreliable problems in real-life scenarios. Combining any two or three different structures in fuzzy set theory is a very complicated and challenging task for scholars. The main influence of this theory is to combine the ideas of the Choquet integral, aggregation operator, and Hamacher t-norm and t-conorm, based on A-IF information. Many people have explored the theory of CI based on an A-IF set and Hamacher aggregation operators, also based on an A-IF set, but to date, no one has explored the theory of Hamacher Choquet integral aggregation operators based on A-IF sets. Retaining the advantages of the Choquet integral, aggregation operators, and Hamacher t-norm and t-conorm, we have found a gap in the literature for a new idea. For this manuscript, we addressed the following aims:

1. To compute the idea of the A-IFHCIA, A-IFHCIOA, A-IFHCIG, and A-IFHCIOG operators, and benefit from the results accordingly.
2. To examine the TOPSIS method, based on A-IF information.
3. To derive a MADM procedure for evaluating awkward and unreliable information with the help of the TOPSIS method and derived operators, to enhance the worth and value of the presented information.
4. To compare our novel technique with existing operators, we needed to illustrate some examples to show the supremacy and superiority of the invented approaches.

The structure of this paper is as follows. In Section 2, we revise the theory of the IF set and the Hamacher operational laws with fuzzy measures and the Choquet integral. In Section 3, we compute the results using the A-IFHCIA, A-IFHCIOA, A-IFHCIG, and A-IFHCIOG operators. Furthermore, we offer the properties and main results of the proposed technique. In Section 4, we explore the TOPSIS method based on A-IF information. Then, we demonstrate a MADM procedure for evaluating awkward and unreliable information with the help of the TOPSIS method and derived operators, to enhance the worth and value of the presented information. In Section 5, we compare the novel technique with some of the existing operators; for this purpose, we offer several examples to show the supremacy and superiority of the invented approaches. In Section 6, we explain and evaluate our findings and offer final conclusions and offer helpful remarks.

2. Preliminaries

Here, we mainly aim to revise the theory of the IF set and their Hamacher operational laws with fuzzy measures and Choquet integral.

Definition 1 ([6]). *Here, we explain the theory of A-IFS η_{if}, based on the universal set X and discovered by:*

$$\eta_{if} = \{(\Xi_s(\sigma), \Lambda_{sa}(\sigma)) : \sigma \in X\} \tag{1}$$

Furthermore, we explain the mathematical term in Equation (1) as $\Xi_s(\sigma), \Lambda_{sa}(\sigma) : X \to [0,1]$, represented the information for and against each choice, with $0 \leq \Xi_s(\sigma) + \Lambda_{sa}(\sigma) \leq 1$. The refusal grade is derived from: $^{\circ}\mathrm{F}_r(\sigma) = 1 - (\Xi_s(\sigma) + \Lambda_{sa}(\sigma))$ and the mathematical information $\eta^j_{if} = \left(\Xi_{s_j}, \Lambda_{sa_j}\right), j = 1, 2, \ldots, n$ represents the A-IF number (A-IFN).

Definition 2 ([29]). *We assume that $\eta^j_{if} = \left(\Xi_{s_j}, \Lambda_{sa_j}\right), j = 1, 2, \ldots, n$ be the family or collection of A-IFNs. Then the mathematical information can be given as:*

$$\eta^j_{S-if} = \Xi_{s_j}(\sigma) - \Lambda_{sa_j}(\sigma) \in [-1, 1] \tag{2}$$

$$\eta^j_{H-if} = \Xi_{s_j}(\sigma) + \Lambda_{sa_j}(\sigma) \in [0, 1] \tag{3}$$

The score function η^j_{S-if} and the η^j_{S-if} can be represented as an accuracy function with the following characteristics:

1. If $\eta^1_{S-if} > \eta^2_{S-if}$, then $\eta^1_{if} > \eta^2_{if}$.
2. If $\eta^1_{S-if} < \eta^2_{S-if}$, then $\eta^1_{if} < \eta^2_{if}$.
3. If $\eta^1_{S-if} = \eta^2_{S-if}$, then:
   (1) If $\eta^1_{H-if} > \eta^2_{H-if}$, then $\eta^1_{if} > \eta^2_{if}$.
   (2) If $\eta^1_{H-if} < \eta^2_{H-if}$, then $\eta^1_{if} < \eta^2_{if}$.

Definition 3 ([29]). *We assume that $\eta^j_{if} = \left(\Xi_{s_j}, \Lambda_{sa_j}\right), j = 1, 2, \ldots, n$ should be the family or collection of A-IFNs. Then, we can describe certain operational laws, such that:*

$$\eta^1_{if} \oplus \eta^2_{if} \oplus \ldots \oplus \eta^n_{if} = \begin{pmatrix} \frac{\prod_{j=1}^n \left(1+(\zeta-1)\Xi_{s_j}\right) - \prod_{j=1}^n \left(1-\Xi_{s_j}\right)}{\prod_{j=1}^n \left(1+(\zeta-1)\Xi_{s_j}\right) + (\zeta-1)\prod_{j=1}^n \left(1-\Xi_{s_j}\right)}, \\ \frac{\zeta \prod_{j=1}^n \left(1-\Xi_{s_j}\right) - \zeta \prod_{j=1}^n \left(1-\Xi_{s_j}-\Lambda_{sa_j}\right)}{\prod_{j=1}^n \left(1+(\zeta-1)\Xi_{s_j}\right) + (\zeta-1)\prod_{j=1}^n \left(1-\Xi_{s_j}\right)} \end{pmatrix} \tag{4}$$

$$\eta^1_{if} \otimes \eta^2_{if} \otimes \ldots \otimes \eta^n_{if} = \begin{pmatrix} \frac{\zeta \prod_{j=1}^n \left(1-\Lambda_{sa_j}\right) - \zeta \prod_{j=1}^n \left(1-\Xi_{s_j}-\Lambda_{sa_j}\right)}{\prod_{j=1}^n \left(1+(\zeta-1)\Lambda_{sa_j}\right) + (\zeta-1)\prod_{j=1}^n \left(1-\Lambda_{sa_j}\right)}, \\ \frac{\prod_{j=1}^n \left(1+(\zeta-1)\Lambda_{sa_j}\right) - \prod_{j=1}^n \left(1-\Lambda_{sa_j}\right)}{\prod_{j=1}^n \left(1+(\zeta-1)\Lambda_{sa_j}\right) + (\zeta-1)\prod_{j=1}^n \left(1-\Lambda_{sa_j}\right)} \end{pmatrix} \tag{5}$$

$$\delta\eta^1_{if} = \left(\frac{\left(1+(\zeta-1)\Xi_{s_1}\right)^\delta - \left(1-\Xi_{s_1}\right)^\delta}{\left(1+(\zeta-1)\Xi_{s_1}\right)^\delta - (\zeta-1)\left(1-\Xi_{s_1}\right)^\delta}, \frac{\zeta\left(1-\Xi_{s_1}\right)^\delta - \zeta\left(1-\Xi_{s_1}-\Lambda_{sa_1}\right)^\delta}{\left(1+(\zeta-1)\Xi_{s_1}\right)^\delta - (\zeta-1)\left(1-\Xi_{s_1}\right)^\delta}\right) \tag{6}$$

$$\eta^{1\,\delta}_{if} = \left(\frac{\zeta\left(1-\Lambda_{sa_1}\right)^\delta - \zeta\left(1-\Xi_{s_1}-\Lambda_{sa_1}\right)^\delta}{\left(1+(\zeta-1)\Lambda_{sa_1}\right)^\delta - (\zeta-1)\left(1-\Lambda_{sa_1}\right)^\delta}, \frac{\left(1+(\zeta-1)\Lambda_{sa_1}\right)^\delta - \left(1-\Lambda_{sa_1}\right)^\delta}{\left(1+(\zeta-1)\Lambda_{sa_1}\right)^\delta - (\zeta-1)\left(1-\Lambda_{sa_1}\right)^\delta}\right) \tag{7}$$

Definition 4 ([37]). *The mathematical version of the fuzzy measure is stated by:*

$$\int \psi d\mathfrak{V} = \sum_{j=1}^n \left(\mathfrak{V}\left(\bar{\bar{\eta}}_{0(j)}\right) - \mathfrak{V}\left(\bar{\bar{\eta}}_{0(j-1)}\right)\right)\psi_{0(j)} \tag{8}$$

$$\mathfrak{V}(\overline{\overline{\eta}}) = \mathfrak{V}\left(\coprod_{j=1}^{n} \sigma_j\right) = \begin{cases} \frac{1}{\delta}\left[\prod_{j=1}^{n}\left(1+\gamma\mathfrak{V}(\sigma_j)\right) - 1\right] & \delta \neq 0 \\ \sum_{\sigma_j \in A} \mathfrak{V}(\sigma_j) & \delta = 0 \end{cases} \tag{9}$$

where $0(j)$ is used for the permutation of $(1,2,\ldots,n)$ in the context of $\psi_{0(1)} \geq \psi_{0(2)} \geq \ldots \geq \psi_{0(n)}$ with $\overline{\overline{\eta}} = \theta$, $\overline{\overline{\eta}}_{0(j)} = \{\eta'_{0(1)}, \eta'_{0(2)}, \ldots, \eta'_{0(j)}\}$.

3. Hamacher–Choquet Integral Aggregation Operators for A-IFSs

In this analysis, we considered the theory of A-IFHCIA, A-IFHCIOA, A-IFHCIG, and A-IFHCIOG operators, and highlighted their most beneficial and valuable results and their main properties.

Definition 5. *We assume $\eta_{if}^{j} = \left(\Xi_{s_j}, \Lambda_{sa_j}\right), j = 1,2,\ldots,n$ to be the family or collection of A-IFNs. The mathematical shape of the A-IFHCIA operator is derived from:*

$$\begin{aligned} \int \eta_{if} d\mathfrak{V} = A - \quad & IFHCIA\left(\eta_{if}^{1}, \eta_{if}^{2}, \ldots, \eta_{if}^{n}\right) \\ & = \left(\mathfrak{V}(\overline{\overline{\eta}}_{0(1)}) - \mathfrak{V}(\overline{\overline{\eta}}_{0(0)})\right)\eta_{if}^{1} \oplus \left(\mathfrak{V}(\overline{\overline{\eta}}_{0(2)}) - \mathfrak{V}(\overline{\overline{\eta}}_{0(1)})\right)\eta_{if}^{2} \oplus \ldots \\ & \oplus\left(\mathfrak{V}(\overline{\overline{\eta}}_{0(n)}) - \mathfrak{V}(\overline{\overline{\eta}}_{0(n-1)})\right)\eta_{if}^{n} = \sum_{j=1}^{n}\left(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)}) - \mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)})\right)\eta_{if}^{j}. \\ & = \oplus_{j=1}^{n}\left(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)}) - \mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)})\right)\eta_{if}^{j} \end{aligned} \tag{10}$$

Theorem 1. *We assume $\eta_{if}^{j} = \left(\Xi_{s_j}, \Lambda_{sa_j}\right), j = 1,2,\ldots,n$to be the family or collection of A-IFNs. Then, we assume that the aggregate values of Equation (10) are again in the shape of IFN, such that:*

$$\begin{aligned} & A - IFHCIA\left(\eta_{if}^{1}, \eta_{if}^{2}, \ldots, \eta_{if}^{n}\right) \\ & = \left(\begin{array}{c} \frac{\prod_{j=1}^{n}\left(1+(\zeta-1)\Xi_{s_j}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))} - \prod_{j=1}^{n}\left(1-\Xi_{s_j}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}}{\prod_{j=1}^{n}\left(1+(\zeta-1)\Xi_{s_j}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))} + (\zeta-1)\prod_{j=1}^{n}\left(1-\Xi_{s_j}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}}, \\ \frac{\zeta\prod_{j=1}^{n}\left(1-\Xi_{s_j}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))} - \zeta\prod_{j=1}^{n}\left(1-\Xi_{s_j}-\Lambda_{sa_j}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}}{\prod_{j=1}^{n}\left(1+(\zeta-1)\Xi_{s_j}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))} + (\zeta-1)\prod_{j=1}^{n}\left(1-\Xi_{s_j}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}} \end{array}\right) \end{aligned} \tag{11}$$

Proof. By considering mathematical induction, we derive the theory in Equation (11). First, we use $n = 2$, then we have:

$$\left(\mathfrak{V}(\overline{\overline{\eta}}_{0(1)}) - \mathfrak{V}(\overline{\overline{\eta}}_{0(0)})\right)\eta_{if}^{1} = \left(\begin{array}{c} \frac{\left(1+(\zeta-1)\Xi_{s_1}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(1)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(1-1)}))} - \left(1-\Xi_{s_1}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(1)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(1-1)}))}}{\left(1+(\zeta-1)\Xi_{s_1}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(1)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(1-1)}))} + (\zeta-1)\left(1-\Xi_{s_1}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(1)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(1-1)}))}}, \\ \frac{\zeta\left(1-\Xi_{s_1}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(1)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(1-1)}))} - \zeta\left(1-\Xi_{s_1}-\Lambda_{sa_1}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(1)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(1-1)}))}}{\left(1+(\zeta-1)\Xi_{s_1}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(1)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(1-1)}))} + (\zeta-1)\left(1-\Xi_{s_1}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(1)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(1-1)}))}} \end{array}\right)$$

$$\left(\mathfrak{V}(\overline{\overline{\eta}}_{0(2)}) - \mathfrak{V}(\overline{\overline{\eta}}_{0(1)})\right)\eta_{if}^{2} = \left(\begin{array}{c} \frac{\left(1+(\zeta-1)\Xi_{s_2}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(2)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(2-1)}))} - \left(1-\Xi_{s_2}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(2)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(2-1)}))}}{\left(1+(\zeta-1)\Xi_{s_2}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(2)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(2-1)}))} + (\zeta-1)\left(1-\Xi_{s_2}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(2)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(2-1)}))}}, \\ \frac{\left(1+(\zeta-1)\Xi_{s_2}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(2)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(2-1)}))} - \left(1-\Xi_{s_2}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(2)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(2-1)}))}}{\left(1+(\zeta-1)\Xi_{s_2}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(2)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(2-1)}))} + (\zeta-1)\left(1-\Xi_{s_2}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(2)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(2-1)}))}}, \end{array}\right)$$

Then,

$$= \left(\mathfrak{V}(\overline{\overline{\eta}}_{0(1)}) - \mathfrak{V}(\overline{\overline{\eta}}_{0(0)})\right)\eta_{if}^{1} \oplus \left(\mathfrak{V}(\overline{\overline{\eta}}_{0(2)}) - \mathfrak{V}(\overline{\overline{\eta}}_{0(1)})\right)\eta_{if}^{2}$$

$$= \left(\begin{array}{c} \frac{\left(1+(\zeta-1)\Xi_{s_1}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(1)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(1-1)}))}-\left(1-\Xi_{s_1}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(1)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(1-1)}))}}{\left(1+(\zeta-1)\Xi_{s_1}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(1)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(1-1)}))}+(\zeta-1)\left(1-\Xi_{s_1}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(1)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(1-1)}))}}, \\ \frac{\zeta\left(1-\Xi_{s_1}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(1)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(1-1)}))}-\zeta\left(1-\Xi_{s_1}-\Lambda_{sa_1}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(1)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(1-1)}))}}{\left(1+(\zeta-1)\Xi_{s_1}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(1)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(1-1)}))}+(\zeta-1)\left(1-\Xi_{s_1}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(1)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(1-1)}))}} \end{array} \right)$$

$$\oplus \left(\begin{array}{c} \frac{\left(1+(\zeta-1)\Xi_{s_2}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(2)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(2-1)}))}-\left(1-\Xi_{s_2}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(2)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(2-1)}))}}{\left(1+(\zeta-1)\Xi_{s_2}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(2)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(2-1)}))}+(\zeta-1)\left(1-\Xi_{s_2}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(2)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(2-1)}))}}, \\ \frac{\zeta\left(1-\Xi_{s_2}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(2)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(2-1)}))}-\zeta\left(1-\Xi_{s_2}-\Lambda_{sa_2}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(2)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(2-1)}))}}{\left(1+(\zeta-1)\Xi_{s_2}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(2)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(2-1)}))}+(\zeta-1)\left(1-\Xi_{s_2}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(2)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(2-1)}))}} \end{array} \right)$$

$$= \left(\begin{array}{c} \frac{\prod_{j=1}^{2}\left(1+(\zeta-1)\Xi_{s_j}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}-\prod_{j=1}^{2}\left(1-\Xi_{s_j}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}}{\prod_{j=1}^{2}\left(1+(\zeta-1)\Xi_{s_j}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}+(\zeta-1)\prod_{j=1}^{2}\left(1-\Xi_{s_j}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}}, \\ \frac{\zeta\prod_{j=1}^{2}\left(1-\Xi_{s_j}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}-\zeta\prod_{j=1}^{2}\left(1-\Xi_{s_j}-\Lambda_{sa_j}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}}{\prod_{j=1}^{2}\left(1+(\zeta-1)\Xi_{s_j}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}+(\zeta-1)\prod_{j=1}^{2}\left(1-\Xi_{s_j}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}} \end{array} \right)$$

Moreover, we assume that the information in Equation (11) is also valid for $n = k$; then we have:

$$A - IFHCIA\left(\eta_{if}^{1}, \eta_{if}^{2}, \ldots, \eta_{if}^{k}\right)$$
$$= \left(\begin{array}{c} \frac{\prod_{j=1}^{k}\left(1+(\zeta-1)\Xi_{s_j}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}-\prod_{j=1}^{k}\left(1-\Xi_{s_j}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}}{\prod_{j=1}^{k}\left(1+(\zeta-1)\Xi_{s_j}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}+(\zeta-1)\prod_{j=1}^{k}\left(1-\Xi_{s_j}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}}, \\ \frac{\zeta\prod_{j=1}^{k}\left(1-\Xi_{s_j}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}-\zeta\prod_{j=1}^{k}\left(1-\Xi_{s_j}-\Lambda_{sa_j}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}}{\prod_{j=1}^{k}\left(1+(\zeta-1)\Xi_{s_j}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}+(\zeta-1)\prod_{j=1}^{k}\left(1-\Xi_{s_j}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}} \end{array} \right)$$

Then, we assume that the information in Equation (11) is also valid for $n = k+1$, such that:

$$\begin{aligned} A - IFHCIA\left(\eta_{if}^{1}, \eta_{if}^{2}, \ldots, \eta_{if}^{n}\right) \\ &= \left(\mathfrak{V}\left(\overline{\overline{\eta}}_{0(1)}\right) - \mathfrak{V}\left(\overline{\overline{\eta}}_{0(0)}\right)\right)\eta_{if}^{1} \oplus \left(\mathfrak{V}\left(\overline{\overline{\eta}}_{0(2)}\right) - \mathfrak{V}\left(\overline{\overline{\eta}}_{0(1)}\right)\right)\eta_{if}^{2} \oplus \ldots \\ &\oplus \left(\mathfrak{V}\left(\overline{\overline{\eta}}_{0(k)}\right) - \mathfrak{V}\left(\overline{\overline{\eta}}_{0(k-1)}\right)\right)\eta_{if}^{k} \oplus \left(\mathfrak{V}\left(\overline{\overline{\eta}}_{0(k+1)}\right) - \mathfrak{V}\left(\overline{\overline{\eta}}_{0(k+1-1)}\right)\right)\eta_{if}^{k+1} \\ &= \sum_{j=1}^{k}\left(\mathfrak{V}\left(\overline{\overline{\eta}}_{0(j)}\right) - \mathfrak{V}\left(\overline{\overline{\eta}}_{0(j-1)}\right)\right)\eta_{if}^{j} \oplus \left(\mathfrak{V}\left(\overline{\overline{\eta}}_{0(k+1)}\right) - \mathfrak{V}\left(\overline{\overline{\eta}}_{0(k+1-1)}\right)\right)\eta_{if}^{k+1} \\ &= \oplus_{j=1}^{k}\left(\mathfrak{V}\left(\overline{\overline{\eta}}_{0(j)}\right) - \mathfrak{V}\left(\overline{\overline{\eta}}_{0(j-1)}\right)\right)\eta_{if}^{j} \oplus \left(\mathfrak{V}\left(\overline{\overline{\eta}}_{0(k+1)}\right) - \mathfrak{V}\left(\overline{\overline{\eta}}_{0(k+1-1)}\right)\right)\eta_{if}^{k+1} \end{aligned}$$

$$= \left(\begin{array}{c} \frac{\prod_{j=1}^{k}\left(1+(\zeta-1)\Xi_{s_j}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}-\prod_{j=1}^{k}\left(1-\Xi_{s_j}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}}{\prod_{j=1}^{k}\left(1+(\zeta-1)\Xi_{s_j}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}+(\zeta-1)\prod_{j=1}^{k}\left(1-\Xi_{s_j}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}}, \\ \frac{\zeta\prod_{j=1}^{k}\left(1-\Xi_{s_j}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}-\zeta\prod_{j=1}^{k}\left(1-\Xi_{s_j}-\Lambda_{sa_j}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}}{\prod_{j=1}^{k}\left(1+(\zeta-1)\Xi_{s_j}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}+(\zeta-1)\prod_{j=1}^{k}\left(1-\Xi_{s_j}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}} \end{array} \right)$$
$$\oplus \left(\mathfrak{V}\left(\overline{\overline{\eta}}_{0(k+1)}\right) - \mathfrak{V}\left(\overline{\overline{\eta}}_{0(k+1-1)}\right)\right)\eta_{if}^{k+1}$$

$$= \begin{pmatrix} \frac{\prod_{j=1}^{k}\left(1+(\zeta-1)\Xi_{s_j}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}-\prod_{j=1}^{k}\left(1-\Xi_{s_j}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}}{\prod_{j=1}^{k}\left(1+(\zeta-1)\Xi_{s_j}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}+(\zeta-1)\prod_{j=1}^{k}\left(1-\Xi_{s_j}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}}, \\ \frac{\zeta\prod_{j=1}^{k}\left(1-\Xi_{s_j}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}-\zeta\prod_{j=1}^{k}\left(1-\Xi_{s_j}-\Lambda_{sa_j}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}}{\prod_{j=1}^{k}\left(1+(\zeta-1)\Xi_{s_j}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}+(\zeta-1)\prod_{j=1}^{k}\left(1-\Xi_{s_j}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}} \end{pmatrix}$$

$$\oplus \begin{pmatrix} \frac{\left(1+(\zeta-1)\Xi_{s_{k+1}}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(k+1)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(k+1-1)}))}-\left(1-\Xi_{s_{k+1}}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(k+1)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(k+1-1)}))}}{\left(1+(\zeta-1)\Xi_{s_{k+1}}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(k+1)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(k+1-1)}))}+(\zeta-1)\left(1-\Xi_{s_{k+1}}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(k+1)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(k+1-1)}))}}, \\ \frac{\zeta\left(1-\Xi_{s_{k+1}}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(k+1)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(k+1-1)}))}-\zeta\left(1-\Xi_{s_{k+1}}-\Lambda_{sa_{k+1}}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(k+1)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(k+1-1)}))}}{\left(1+(\zeta-1)\Xi_{s_{k+1}}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(k+1)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(k+1-1)}))}+(\zeta-1)\left(1-\Xi_{s_{k+1}}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(k+1)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(k+1-1)}))}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\prod_{j=1}^{k+1}\left(1+(\zeta-1)\Xi_{s_j}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}-\prod_{j=1}^{k+1}\left(1-\Xi_{s_j}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}}{\prod_{j=1}^{k+1}\left(1+(\zeta-1)\Xi_{s_j}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}+(\zeta-1)\prod_{j=1}^{k+1}\left(1-\Xi_{s_j}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}}, \\ \frac{\zeta\prod_{j=1}^{k+1}\left(1-\Xi_{s_j}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}-\zeta\prod_{j=1}^{k+1}\left(1-\Xi_{s_j}-\Lambda_{sa_j}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}}{\prod_{j=1}^{k+1}\left(1+(\zeta-1)\Xi_{s_j}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}+(\zeta-1)\prod_{j=1}^{k+1}\left(1-\Xi_{s_j}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}} \end{pmatrix}$$

Hence, we can assume that the information in Equation (11) is valid for all possible values of n. □

Moreover, we can derive the idempotency, monotonicity, and boundedness from consideration of the theory in Equation (11).

Property 1. We assume that $\eta_{if}^{j} = \left(\Xi_{s_j}, \Lambda_{sa_j}\right), j = 1, 2, \ldots, n$ should be the family or collection of A-IFNs. Then:

1. ***Idempotency:*** When $\eta_{if}^{j} = \eta_{if} = (\Xi_s, \Lambda_{sa}), j = 1, 2, \ldots, n$, then:

$$A - IFHCIA\left(\eta_{if}^{1}, \eta_{if}^{2}, \ldots, \eta_{if}^{n}\right) = \eta_{if}.$$

2. ***Monotonicity:*** When $\eta_{if}^{j} = \left(\Xi_{s_j}, \Lambda_{sa_j}\right) \leq \eta^{*j}_{if} = \left(\Xi^{*}_{s_j}, \Lambda^{*}_{sa_j}\right)$, then:

$$A - IFHCIA\left(\eta_{if}^{1}, \eta_{if}^{2}, \ldots, \eta_{if}^{n}\right) \leq A - IFHCIA\left(\eta^{*1}_{if}, \eta^{*2}_{if}, \ldots, \eta^{*n}_{if}\right).$$

3. ***Boundedness:*** When $\eta_{if}^{-} = \left(\min_{j}\Xi_{s_j}, \max_{j}\Lambda_{sa_j}\right)$ and $\eta_{if}^{+} = \left(\max_{j}\Xi_{s_j}, \min_{j}\Lambda_{sa_j}\right)$, then:

$$\eta_{if}^{-} \leq A - IFHCIA\left(\eta_{if}^{1}, \eta_{if}^{2}, \ldots, \eta_{if}^{n}\right) \leq \eta_{if}^{+}.$$

Proof. We assume that $\eta_{if}^{j} = \left(\Xi_{s_j}, \Lambda_{sa_j}\right), j = 1, 2, \ldots, n$ should be the family or collection of A-IFNs. Then:

1. When $\eta_{if}^{j} = \eta_{if} = (\Xi_s, \Lambda_{sa}), j = 1, 2, \ldots, n$, then:

$$A - IFHCIA\left(\eta_{if}^{1}, \eta_{if}^{2}, \ldots, \eta_{if}^{n}\right)$$

$$= \left(\begin{array}{c} \frac{\prod_{j=1}^{n}(1+(\zeta-1)\Xi_s)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}-\prod_{j=1}^{n}(1-\Xi_s)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}}{\prod_{j=1}^{n}(1+(\zeta-1)\Xi_s)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}+(\zeta-1)\prod_{j=1}^{n}(1-\Xi_s)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}}, \\ \frac{\zeta\prod_{j=1}^{n}(1-\Xi_s)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}-\zeta\prod_{j=1}^{n}(1-\Xi_s-\Lambda_{sa})^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}}{\prod_{j=1}^{n}(1+(\zeta-1)\Xi_s)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}+(\zeta-1)\prod_{j=1}^{n}(1-\Xi_s)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}} \end{array} \right)$$

$$= \left(\begin{array}{c} \frac{(1+(\zeta-1)\Xi_s)^{\sum_{j=1}^{n}(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}-(1-\Xi_s)^{\sum_{j=1}^{n}(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}}{(1+(\zeta-1)\Xi_s)^{\sum_{j=1}^{n}(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}+(\zeta-1)(1-\Xi_s)^{\sum_{j=1}^{n}(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}}, \\ \frac{\zeta(1-\Xi_s)^{\sum_{j=1}^{n}(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}-\zeta(1-\Xi_s-\Lambda_{sa})^{\sum_{j=1}^{n}(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}}{(1+(\zeta-1)\Xi_s)^{\sum_{j=1}^{n}(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}+(\zeta-1)(1-\Xi_s)^{\sum_{j=1}^{n}(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}} \end{array} \right)$$

$$= \left(\begin{array}{c} \frac{(1+(\zeta-1)\Xi_s)-(1-\Xi_s)}{(1+(\zeta-1)\Xi_s)+(\zeta-1)(1-\Xi_s)}, \\ \frac{\zeta(1-\Xi_s)-\zeta(1-\Xi_s-\Lambda_{sa})}{(1+(\zeta-1)\Xi_s)+(\zeta-1)(1-\Xi_s)} \end{array} \right), \sum_{j=1}^{n}\left(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)}) - \mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)})\right) = 1$$

$$= \left(\begin{array}{c} \frac{1+\zeta\Xi_s-\Xi_s-1+\Xi_s}{1+\zeta\Xi_s-\Xi_s+\zeta-\zeta\Xi_s-1+\Xi_s}, \\ \frac{\zeta-\zeta\Xi_s-\zeta1+\zeta\Xi_s+\zeta\Lambda_{sa}}{1+\zeta\Xi_s-\Xi_s+\zeta-\zeta\Xi_s-1+\Xi_s} \end{array} \right)$$

$$= (\Xi_s, \Lambda_{sa}) = \eta_{if}$$

2. When $\eta_{if}^{j} = \left(\Xi_{s_j}, \Lambda_{sa_j}\right) \leq \eta^{*j}_{if} = \left(\Xi^*_{s_j}, \Lambda^*_{sa_j}\right)$, then we can further explain the considered information, such that $\Xi_{s_j} \leq \Xi^*_{s_j}$ and $\Lambda_{sa_j} \geq \Lambda^*_{sa_j}$; then:

$$\Xi_{s_j} \leq \Xi^*_{s_j} \Rightarrow 1-\Xi_{s_j} \geq 1-\Xi^*_{s_j} \Rightarrow \prod_{j=1}^{n}\left(1-\Xi_{s_j}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))} \geq \prod_{j=1}^{n}\left(1-\Xi^*_{s_j}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}$$

$$\Rightarrow \prod_{j=1}^{n}\left(1+(\zeta-1)\Xi_{s_j}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))} - \prod_{j=1}^{n}\left(1-\Xi_{s_j}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}$$
$$\leq \prod_{j=1}^{n}\left(1+(\zeta-1)\Xi^*_{s_j}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))} - \prod_{j=1}^{n}\left(1-\Xi^*_{s_j}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}$$

$$\Rightarrow \frac{\prod_{j=1}^{n}\left(1+(\zeta-1)\Xi_{s_j}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}-\prod_{j=1}^{n}\left(1-\Xi_{s_j}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}}{\prod_{j=1}^{n}\left(1+(\zeta-1)\Xi_{s_j}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}+(\zeta-1)\prod_{j=1}^{n}\left(1-\Xi_{s_j}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}}$$
$$\leq \frac{\prod_{j=1}^{n}\left(1+(\zeta-1)\Xi^*_{s_j}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}-\prod_{j=1}^{n}\left(1-\Xi^*_{s_j}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}}{\prod_{j=1}^{n}\left(1+(\zeta-1)\Xi^*_{s_j}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}+(\zeta-1)\prod_{j=1}^{n}\left(1-\Xi^*_{s_j}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}}$$

Furthermore,

$$\Lambda_{sa_j} \geq \Lambda^*_{sa_j} \Rightarrow 1-\Lambda_{sa_j} \leq 1-\Lambda^*_{sa_j} \Rightarrow 1-\Xi_{s_j}-\Lambda_{sa_j} \geq 1-\Xi^*_{s_j}-\Lambda^*_{sa_j}$$

$$\Rightarrow \prod_{j=1}^{n}\left(1-\Xi_{s_j}-\Lambda_{sa_j}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))} \geq \prod_{j=1}^{n}\left(1-\Xi^*_{s_j}-\Lambda^*_{sa_j}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}$$

$$\Rightarrow -\zeta\prod_{j=1}^{n}\left(1-\Xi_{s_j}-\Lambda_{sa_j}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))} \leq -\zeta\prod_{j=1}^{n}\left(1-\Xi^*_{s_j}-\Lambda^*_{sa_j}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}$$

$$\Rightarrow \zeta \prod_{j=1}^{n}\left(1-\Xi_{s_j}\right)^{\left(\mathfrak{V}\left(\overline{\overline{\eta}}_{0(j)}\right)-\mathfrak{V}\left(\overline{\overline{\eta}}_{0(j-1)}\right)\right)}-\zeta \prod_{j=1}^{n}\left(1-\Xi_{s_j}-\Lambda_{sa_j}\right)^{\left(\mathfrak{V}\left(\overline{\overline{\eta}}_{0(j)}\right)-\mathfrak{V}\left(\overline{\overline{\eta}}_{0(j-1)}\right)\right)}$$
$$\geq \zeta \prod_{j=1}^{n}\left(1-\Xi^*_{s_j}\right)^{\left(\mathfrak{V}\left(\overline{\overline{\eta}}_{0(j)}\right)-\mathfrak{V}\left(\overline{\overline{\eta}}_{0(j-1)}\right)\right)}-\zeta \prod_{j=1}^{n}\left(1-\Xi^*_{s_j}-\Lambda^*_{sa_j}\right)^{\left(\mathfrak{V}\left(\overline{\overline{\eta}}_{0(j)}\right)-\mathfrak{V}\left(\overline{\overline{\eta}}_{0(j-1)}\right)\right)}$$

$$\Rightarrow \frac{\zeta \prod_{j=1}^{n}\left(1-\Xi_{s_j}\right)^{\left(\mathfrak{V}\left(\overline{\overline{\eta}}_{0(j)}\right)-\mathfrak{V}\left(\overline{\overline{\eta}}_{0(j-1)}\right)\right)}-\zeta \prod_{j=1}^{n}\left(1-\Xi_{s_j}-\Lambda_{sa_j}\right)^{\left(\mathfrak{V}\left(\overline{\overline{\eta}}_{0(j)}\right)-\mathfrak{V}\left(\overline{\overline{\eta}}_{0(j-1)}\right)\right)}}{\prod_{j=1}^{n}\left(1+(\zeta-1)\Xi_{s_j}\right)^{\left(\mathfrak{V}\left(\overline{\overline{\eta}}_{0(j)}\right)-\mathfrak{V}\left(\overline{\overline{\eta}}_{0(j-1)}\right)\right)}+(\zeta-1)\prod_{j=1}^{n}\left(1-\Xi_{s_j}\right)^{\left(\mathfrak{V}\left(\overline{\overline{\eta}}_{0(j)}\right)-\mathfrak{V}\left(\overline{\overline{\eta}}_{0(j-1)}\right)\right)}}$$
$$\geq \frac{\zeta \prod_{j=1}^{n}\left(1-\Xi^*_{s_j}\right)^{\left(\mathfrak{V}\left(\overline{\overline{\eta}}_{0(j)}\right)-\mathfrak{V}\left(\overline{\overline{\eta}}_{0(j-1)}\right)\right)}-\zeta \prod_{j=1}^{n}\left(1-\Xi^*_{s_j}-\Lambda^*_{sa_j}\right)^{\left(\mathfrak{V}\left(\overline{\overline{\eta}}_{0(j)}\right)-\mathfrak{V}\left(\overline{\overline{\eta}}_{0(j-1)}\right)\right)}}{\prod_{j=1}^{n}\left(1+(\zeta-1)\Xi^*_{s_j}\right)^{\left(\mathfrak{V}\left(\overline{\overline{\eta}}_{0(j)}\right)-\mathfrak{V}\left(\overline{\overline{\eta}}_{0(j-1)}\right)\right)}+(\zeta-1)\prod_{j=1}^{n}\left(1-\Xi^*_{s_j}\right)^{\left(\mathfrak{V}\left(\overline{\overline{\eta}}_{0(j)}\right)-\mathfrak{V}\left(\overline{\overline{\eta}}_{0(j-1)}\right)\right)}}$$

Then, by using the information in Equations (2) and (3), we can easily derive the following result, such that:

$$A-IFHCIA\left(\eta^1_{if},\eta^2_{if},\ldots,\eta^n_{if}\right)\leq A-IFHCIA\left({\eta^*}^1_{if},{\eta^*}^2_{if},\ldots,{\eta^*}^n_{if}\right).$$

3. When $\eta^-_{if}=\left(\min_j \Xi_{s_j},\max_j \Lambda_{sa_j}\right)$ and $\eta^+_{if}=\left(\max_j \Xi_{s_j},\min_j \Lambda_{sa_j}\right)$, then, using point 1 and point 2, we have:

$$A-IFHCIA\left(\eta^1_{if},\eta^2_{if},\ldots,\eta^n_{if}\right)\leq A-IFHCIA\left({\eta^+}^1_{if},{\eta^+}^2_{if},\ldots,{\eta^+}^n_{if}\right)=\eta^+_{if}$$

$$A-IFHCIA\left(\eta^1_{if},\eta^2_{if},\ldots,\eta^n_{if}\right)\geq A-IFHCIA\left({\eta^-}^1_{if},{\eta^-}^2_{if},\ldots,{\eta^-}^n_{if}\right)=\eta^-_{if}$$

Then, we can derive:

$$\eta^-_{if}\leq A-IFHCIA\left(\eta^1_{if},\eta^2_{if},\ldots,\eta^n_{if}\right)\leq \eta^+_{if}.\square$$

Definition 6. *We assume* $\eta^j_{if}=\left(\Xi_{s_j},\Lambda_{sa_j}\right), j=1,2,\ldots,n$ *to be the family or collection of A-IFNs. The mathematical shape of the A-IFHCIOA operator is derived from:*

$$\begin{aligned}\int \eta_{if}d\mathfrak{V} = A-IFHCIOA&\left(\eta^1_{if},\eta^2_{if},\ldots,\eta^n_{if}\right)\\ &=\left(\mathfrak{V}\left(\overline{\overline{\eta}}_{0(1)}\right)-\mathfrak{V}\left(\overline{\overline{\eta}}_{0(0)}\right)\right)\eta^{0(1)}_{if}\oplus\left(\mathfrak{V}\left(\overline{\overline{\eta}}_{0(2)}\right)-\mathfrak{V}\left(\overline{\overline{\eta}}_{0(1)}\right)\right)\eta^{0(2)}_{if}\oplus\ldots\\ &\oplus\left(\mathfrak{V}\left(\overline{\overline{\eta}}_{0(n)}\right)-\mathfrak{V}\left(\overline{\overline{\eta}}_{0(n-1)}\right)\right)\eta^{0(n)}_{if}=\sum_{j=1}^{n}\left(\mathfrak{V}\left(\overline{\overline{\eta}}_{0(j)}\right)-\mathfrak{V}\left(\overline{\overline{\eta}}_{0(j-1)}\right)\right)\eta^{0(j)}_{if}\\ &=\oplus_{j=1}^{n}\left(\mathfrak{V}\left(\overline{\overline{\eta}}_{0(j)}\right)-\mathfrak{V}\left(\overline{\overline{\eta}}_{0(j-1)}\right)\right)\eta^{0(j)}_{if}\end{aligned} \tag{12}$$

where $0(j)\leq 0(j-1)$ represents the permutations of $j=1,2,\ldots,n$.

Theorem 2. *We assume that* $\eta^j_{if}=\left(\Xi_{s_j},\Lambda_{sa_j}\right), j=1,2,\ldots,n$ *is the family or collection of A-IFNs. Then, we can see that the aggregate values of Equation (12) are again in the shape of an IFN, such that:*

$$A-IFHCIA\,(\eta_{if}^{1},\eta_{if}^{2},\ldots,\eta_{if}^{n}\,)\\=\left(\begin{array}{c}\frac{\prod_{j=1}^{n}\left(1+(\zeta-1)\Xi_{s_{0(j)}}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}-\prod_{j=1}^{n}\left(1-\Xi_{s_{0(j)}}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}}{\prod_{j=1}^{n}\left(1+(\zeta-1)\Xi_{s_{0(j)}}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}+(\zeta-1)\prod_{j=1}^{n}\left(1-\Xi_{s_{0(j)}}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}},\\ \frac{\zeta\prod_{j=1}^{n}\left(1-\Xi_{s_{0(j)}}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}-\zeta\prod_{j=1}^{n}\left(1-\Xi_{s_{0(j)}}-\Lambda_{sa_{0(j)}}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}}{\prod_{j=1}^{n}\left(1+(\zeta-1)\Xi_{s_{0(j)}}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}+(\zeta-1)\prod_{j=1}^{n}\left(1-\Xi_{s_{0(j)}}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}}\end{array}\right). \quad (13)$$

Moreover, we can derive the idempotency, monotonicity, and boundedness within the exploration of the theory in Equation (13).

Property 2. We assume that $\eta_{if}^{j}=\left(\Xi_{s_j},\Lambda_{sa_j}\right), j=1,2,\ldots,n$ represents the family or collection of A-IFNs. Then:

1. ***Idempotency:*** When $\eta_{if}^{j}=\eta_{if}=(\Xi_s,\Lambda_{sa}), j=1,2,\ldots,n$, then:

$$A-IFHCIOA\,(\eta_{if}^{1},\eta_{if}^{2},\ldots,\eta_{if}^{n}\,)=\eta_{if}.$$

2. ***Monotonicity:*** When $\eta_{if}^{j}=\left(\Xi_{s_j},\Lambda_{sa_j}\right)\leq{\eta^*}_{if}^{j}=\left(\Xi_{s_j}^{*},\Lambda_{sa_j}^{*}\right)$, then:

$$A-IFHCIOA\,(\eta_{if}^{1},\eta_{if}^{2},\ldots,\eta_{if}^{n}\,)\leq A-IFHCIOA\left({\eta^*}_{if}^{1},{\eta^*}_{if}^{2},\ldots,{\eta^*}_{if}^{n}\right).$$

3. ***Boundedness:*** When $\eta_{if}^{-}=(\min_j\Xi_{s_j},\max_j\Lambda_{sa_j})$ and $\eta_{if}^{+}=(\max_j\Xi_{s_j},\min_j\Lambda_{sa_j})$, then:

$$\eta_{if}^{-}\leq A-IFHCIOA\,(\eta_{if}^{1},\eta_{if}^{2},\ldots,\eta_{if}^{n}\,)\leq\eta_{if}^{+}.$$

Definition 7. *We assume that* $\eta_{if}^{j}=\left(\Xi_{s_j},\Lambda_{sa_j}\right), j=1,2,\ldots,n$ *will be the family or collection of A-IFNs. The mathematical shape of the A-IFHCIG operator is derived by:*

$$\int\eta_{if}d\mathfrak{V}=A-IFHCIG\left(\eta_{if}^{1},\eta_{if}^{2},\ldots,\eta_{if}^{n}\right)\\=\eta_{if}^{1\,(\mathfrak{V}(\overline{\overline{\eta}}_{0(1)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(0)}))}\otimes\eta_{if}^{2\,(\mathfrak{V}(\overline{\overline{\eta}}_{0(2)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(1)}))}\otimes\ldots\otimes\eta_{if}^{n\,(\mathfrak{V}(\overline{\overline{\eta}}_{0(n)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(n-1)}))}\\=\prod_{j=1}^{n}\eta_{if}^{j\,(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}=\otimes_{j=1}^{n}\eta_{if}^{j\,(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))} \quad (14)$$

Theorem 3. *We assume that* $\eta_{if}^{j}=\left(\Xi_{s_j},\Lambda_{sa_j}\right), j=1,2,\ldots,n$ *is the family or collection of A-IFNs. Then we can see that the aggregate values of Equation (14) are again in the shape of IFN, such that:*

$$A-IFHCIG\left(\eta_{if}^{1},\eta_{if}^{2},\ldots,\eta_{if}^{n}\right)\\=\left(\begin{array}{c}\frac{\zeta\prod_{j=1}^{n}\left(1-\Lambda_{sa_j}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}-\zeta\prod_{j=1}^{n}\left(1-\Xi_{s_j}-\Lambda_{sa_j}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}}{\prod_{j=1}^{n}\left(1+(\zeta-1)\Lambda_{sa_j}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}+(\zeta-1)\prod_{j=1}^{n}\left(1-\Lambda_{sa_j}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}},\\ \frac{\prod_{j=1}^{n}\left(1+(\zeta-1)\Lambda_{sa_j}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}-\prod_{j=1}^{n}\left(1-\Lambda_{sa_j}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}}{\prod_{j=1}^{n}\left(1+(\zeta-1)\Lambda_{sa_j}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}+(\zeta-1)\prod_{j=1}^{n}\left(1-\Lambda_{sa_j}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}}\end{array}\right) \quad (15)$$

Moreover, we derive the idempotency, monotonicity, and boundedness by considering the theory offered in Equation (15).

Property 3. We assume that $\eta_{if}^{j}=\left(\Xi_{s_j},\Lambda_{sa_j}\right), j=1,2,\ldots,n$ is the family or collection of A-IFNs. Then:

1. ***Idempotency:*** When $\eta_{if}^{j} = \eta_{if} = (\Xi_s, \Lambda_{sa}), j = 1, 2, \ldots, n$, then:

$$A - IFHCIA\,(\eta_{if}^{1}, \eta_{if}^{2}, \ldots, \eta_{if}^{n}) = \eta_{if}.$$

2. ***Monotonicity:*** When $\eta_{if}^{j} = \left(\Xi_{s_j}, \Lambda_{sa_j}\right) \leq \eta^{*j}_{if} = \left(\Xi^{*}_{s_j}, \Lambda^{*}_{sa_j}\right)$, then:

$$A - IFHCIA\,(\eta_{if}^{1}, \eta_{if}^{2}, \ldots, \eta_{if}^{n}) \leq A - IFHCIA\left(\eta^{*1}_{if}, \eta^{*2}_{if}, \ldots, \eta^{*n}_{if}\right).$$

3. ***Boundedness:*** When $\eta_{if}^{-} = (\min_j \Xi_{s_j}, \max_j \Lambda_{sa_j})$ and $\eta_{if}^{+} = (\max_j \Xi_{s_j}, \min_j \Lambda_{sa_j})$, then:

$$\eta_{if}^{-} \leq A - IFHCIA\,(\eta_{if}^{1}, \eta_{if}^{2}, \ldots, \eta_{if}^{n}) \leq \eta_{if}^{+}.$$

Definition 8. *We assume that $\eta_{if}^{j} = \left(\Xi_{s_j}, \Lambda_{sa_j}\right), j = 1, 2, \ldots, n$ is the family or collection of A-IFNs. The mathematical shape of the A-IFHCIOG operator is derived by:*

$$\begin{aligned}\int \eta_{if} d\mathfrak{V} = A - IFHCIOG\,(\eta_{if}^{1}, \eta_{if}^{2}, \ldots, \eta_{if}^{n}) \\ = \eta_{if}^{0(1)\,(\mathfrak{V}(\overline{\overline{\eta}}_{0(1)}) - \mathfrak{V}(\overline{\overline{\eta}}_{0(0)}))} \otimes \eta_{if}^{0(2)\,(\mathfrak{V}(\overline{\overline{\eta}}_{0(2)}) - \mathfrak{V}(\overline{\overline{\eta}}_{0(1)}))} \otimes \ldots \otimes \eta_{if}^{0(n)\,(\mathfrak{V}(\overline{\overline{\eta}}_{0(n)}) - \mathfrak{V}(\overline{\overline{\eta}}_{0(n-1)}))} \\ = \prod_{j=1}^{n} \eta_{if}^{0(j)\,(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)}) - \mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))} = \otimes_{j=1}^{n} \eta_{if}^{0(j)\,(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)}) - \mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}\end{aligned} \tag{16}$$

where $0(j) \leq 0(j-1)$ represents the permutations of $j = 1, 2, \ldots, n$.

Theorem 4. *We assume that $\eta_{if}^{j} = \left(\Xi_{s_j}, \Lambda_{sa_j}\right), j = 1, 2, \ldots, n$ should be the family or collection of A-IFNs. Then we derive that the aggregate values of Equation (16) are again in the shape of IFN, such as*

$$\begin{aligned} & A - IFHCIG\,(\eta_{if}^{1}, \eta_{if}^{2}, \ldots, \eta_{if}^{n}) \\ & = \left(\begin{array}{c} \frac{\zeta \prod_{j=1}^{n}\left(1-\Lambda_{sa_{0(j)}}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))} - \zeta \prod_{j=1}^{n}\left(1-\Xi_{s_{0(j)}}-\Lambda_{sa_{0(j)}}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}}{\prod_{j=1}^{n}\left(1+(\zeta-1)\Lambda_{sa_{0(j)}}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))} + (\zeta-1)\prod_{j=1}^{n}\left(1-\Lambda_{sa_{0(j)}}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}}, \\ \frac{\prod_{j=1}^{n}\left(1+(\zeta-1)\Lambda_{sa_{0(j)}}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))} - \prod_{j=1}^{n}\left(1-\Lambda_{sa_{0(j)}}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}}{\prod_{j=1}^{n}\left(1+(\zeta-1)\Lambda_{sa_{0(j)}}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))} + (\zeta-1)\prod_{j=1}^{n}\left(1-\Lambda_{sa_{0(j)}}\right)^{(\mathfrak{V}(\overline{\overline{\eta}}_{0(j)})-\mathfrak{V}(\overline{\overline{\eta}}_{0(j-1)}))}} \end{array}\right) \end{aligned} \tag{17}$$

Moreover, we derive the idempotency, monotonicity, and boundedness after consideration of the theory in Equation (17).

Property 4. We assume that $\eta_{if}^{j} = \left(\Xi_{s_j}, \Lambda_{sa_j}\right), j = 1, 2, \ldots, n$ represents the family or collection of A-IFNs. Then

1. ***Idempotency:*** When $\eta_{if}^{j} = \eta_{if} = (\Xi_s, \Lambda_{sa}), j = 1, 2, \ldots, n$, then:

$$A - IFHCIA\,(\eta_{if}^{1}, \eta_{if}^{2}, \ldots, \eta_{if}^{n}) = \eta_{if}.$$

2. ***Monotonicity:*** When $\eta_{if}^{j} = \left(\Xi_{s_j}, \Lambda_{sa_j}\right) \leq \eta^{*j}_{if} = \left(\Xi^{*}_{s_j}, \Lambda^{*}_{sa_j}\right)$, then:

$$A - IFHCIA\,(\eta_{if}^{1}, \eta_{if}^{2}, \ldots, \eta_{if}^{n}) \leq A - IFHCIA\left(\eta^{*1}_{if}, \eta^{*2}_{if}, \ldots, \eta^{*n}_{if}\right).$$

3. ***Boundedness:*** When $\eta_{if}^{-} = (\min_j \Xi_{s_j}, \max_j \Lambda_{sa_j})$ and $\eta_{if}^{+} = (\max_j \Xi_{s_j}, \min_j \Lambda_{sa_j})$, then:

$$\eta_{if}^{-} \leq A - IFHCIA\,(\eta_{if}^{1}, \eta_{if}^{2}, \ldots, \eta_{if}^{n}) \leq \eta_{if}^{+}.$$

4. MADM Technique Based on the TOPSIS Method

In this section, we aim to construct a theory of the TOPSIS procedure under the Hamacher CI aggregation operators for IFSs. For this, we consider the theory of derived operators and the most important steps of the TOPSIS method, from which we derive our novel TOPSIS techniques. Therefore, the main and most valuable points of the TOPSIS method are described below.

Point 1: First, we arrange the decision matrices using the information from IFNs. Therefore, here, we considered the collection of alternatives $\eta_{if}^{AL} = \{\eta_{if}^{a-1}, \eta_{if}^{a-2}, \ldots, \eta_{if}^{a-m}\}$ and their finite values of attributes $\eta_{if}^{AT} = \{\eta_{if}^{at-1}, \eta_{if}^{at-2}, \ldots, \eta_{if}^{at-n}\}$. Then by using this information, we compute a matrix by including intuitionistic fuzzy information, such as $\eta_{if}^{j} = \left(\Xi_{s_j}, \Lambda_{sa_j}\right), j = 1, 2, \ldots, n$, which represented the A-IF number (A-IFN), where $\Xi_s(\sigma), \Lambda_{sa}(\sigma) : X \to [0,1]$ represented the information for and against each choice, information where $0 \leq \Xi_s(\sigma) + \Lambda_{sa}(\sigma) \leq 1$. The refusal grade is derived from: $°\mathrm{F}_r(\sigma) = 1 - (\Xi_s(\sigma) + \Lambda_{sa}(\sigma))$. Each decision matrix contained two types of information, such as benefit or cost types; if the information in the decision matrix is of the cost type, then we normalize it thus:

$$F = \begin{cases} \left(\Xi_{s_j}, \Lambda_{sa_j}\right) & for\ benefit \\ \left(\Lambda_{sa_j}, \Xi_{s_j}\right) & for\ cost \end{cases}$$

If the information in the decision matrix is of the benefit type, then we do not normalize it.

Point 2: Furthermore, we aggregate the information in decision matrices to aggregate one matrix, using the theory of an $A - IFHCIA$ operator or $A - IFHCIG$ operator.

Point 3: Moreover, we derive the ideal positive and negative solutions by using the information in the decision matrix.

$$\eta_{if}^{+} = \left\{ \left(\max_j \Xi_{s_{j1}}, \min_j \Lambda_{sa_{j1}}\right), \left(\max_j \Xi_{s_{j2}}, \min_j \Lambda_{sa_{j2}}\right), \ldots, \left(\max_j \Xi_{s_{jn}}, \min_j \Lambda_{sa_{jn}}\right) \right\}$$

$$\eta_{if}^{-} = \left\{ \left(\min_j \Xi_{s_{j1}}, \max_j \Lambda_{sa_{j1}}\right), \left(\min_j \Xi_{s_{j2}}, \max_j \Lambda_{sa_{j2}}\right), \ldots, \left(\min_j \Xi_{s_{jn}}, \max_j \Lambda_{sa_{jn}}\right) \right\}$$

Point 4: Then, we aim to evaluate the discrimination measures, based on the ideal positive and negative solutions, using the information in the decision matrix.

$$D^{+}\left(\eta_{if}^{j}, \eta_{if}^{+}\right) = \frac{1}{n}\sum_{j=1}^{n}\left(\frac{\left(\Xi_{s_j}\Xi_{s_j}^{+} + \Xi_{s_j}\Xi_{s_j}^{+} + °\mathrm{F}_{r_j}°\mathrm{F}_{r_j}^{+}\right)}{\sqrt{\Xi_{s_j}^{2} + \Xi_{s_j}^{2} + °\mathrm{F}_{r_j}^{2}} \times \sqrt{\Xi_{s_j}^{+2} + \Xi_{s_j}^{+2} + °\mathrm{F}_{r_j}^{+2}}}\right)$$

$$D^{-}\left(\eta_{if}^{j}, \eta_{if}^{-}\right) = \frac{1}{n}\sum_{j=1}^{n}\left(\frac{\left(\Xi_{s_j}\Xi_{s_j}^{-} + \Xi_{s_j}\Xi_{s_j}^{-} + °\mathrm{F}_{r_j}°\mathrm{F}_{r_j}^{-}\right)}{\sqrt{\Xi_{s_j}^{2} + \Xi_{s_j}^{2} + °\mathrm{F}_{r_j}^{2}} \times \sqrt{\Xi_{s_j}^{-2} + \Xi_{s_j}^{-2} + °\mathrm{F}_{r_j}^{-2}}}\right)$$

Point 5: Additionally, we derive the closeness measure with the help of the obtained discrimination measures:

$$C^{j} = \frac{D^{-}\left(\eta_{if}^{j}, \eta_{if}^{-}\right)}{D^{+}\left(\eta_{if}^{j}, \eta_{if}^{+}\right) + D^{-}\left(\eta_{if}^{j}, \eta_{if}^{-}\right)}$$

where $0 \leq C^{j} \leq 1$.

Point 6: We determine or evaluate the ranking values, based on the closeness measures, and try to discover the best one.

Furthermore, we aim to verify the derivation procedure with the help of some practical examples, to discover the supremacy and proficiency of the evaluated operators and method.

A. Illustrative example

Here, we discuss a MADM technique in the presence of the Hamacher Choquet integral aggregation operators for IF information. For this, we select a decision-making dilemma in an investment enterprise where they are planning to create a strategy for the next few years. Therefore, they appointed four board members of the enterprise to choose where to invest their money in the following four companies located in Asian markets, representing alternatives such as:

η_{if}^{a-1}: Southern.

η_{if}^{a-2}: Eastern.

η_{if}^{a-3}: Northern.

η_{if}^{a-4}: Local market.

To evaluate the companies, they used four main predictors, which represent attributes such as:

η_{if}^{at-1}: Growth.

η_{if}^{at-2}: Social impact.

η_{if}^{at-3}: Political impact.

η_{if}^{at-4}: Environmental impact.

Furthermore, to address the above problem, we created artificial information and tried to evaluate it with the help of our derived procedure. Therefore, the process used in the TOPSIS method is described below.

Point 1: Here, we arranged the decision matrices using the information regarding IFNs. Therefore, we considered the collection of alternatives $\eta_{if}^{AL} = \{\eta_{if}^{a-1}, \eta_{if}^{a-2}, \ldots, \eta_{if}^{a-m}\}$ and their finite values for the attributes $\eta_{if}^{AT} = \{\eta_{if}^{at-1}, \eta_{if}^{at-2}, \ldots, \eta_{if}^{at-n}\}$. Then, using this information, we computed a matrix, including intuitionistic fuzzy information, such as $\eta_{if}^{j} = \left(\Xi_{s_j}, \Lambda_{sa_j}\right), j = 1, 2, \ldots, n$ represented the A-IF number (A-IFN), where $\Xi_s(\sigma), \Lambda_{sa}(\sigma) : X \to [0,1]$ represented the information for and against each choice, with $0 \leq \Xi_s(\sigma) + \Lambda_{sa}(\sigma) \leq 1$. The refusal grade is derived by: $°\text{F}_r(\sigma) = 1 - (\Xi_s(\sigma) + \Lambda_{sa}(\sigma))$. Each decision matrix contained two types of information, such as benefit or cost types; if the information in the decision matrix is of the cost type, then we normalize it, such that:

$$F = \begin{cases} \left(\Xi_{s_j}, \Lambda_{sa_j}\right) & for\ benefit \\ \left(\Lambda_{sa_j}, \Xi_{s_j}\right) & for\ cost \end{cases}$$

If the information in the decision matrix is of the benefit type, then we do not normalize it. However, the information in Tables 1–4 does not need to be normalized. Here, we use the information on CI given in Ref. [34].

Table 1. Decision matrix 1.

	η_{if}^{at-1}	η_{if}^{at-2}	η_{if}^{at-3}	η_{if}^{at-4}
η_{if}^{a-1}	$(0.3, 0.2)$	$(0.5, 0.4)$	$(0.8, 0.1)$	$(0.7, 0.1)$
η_{if}^{a-2}	$(0.4, 0.1)$	$(0.4, 0.4)$	$(0.7, 0.3)$	$(0.3, 0.3)$
η_{if}^{a-3}	$(0.5, 0.4)$	$(0.3, 0.3)$	$(0.6, 0.3)$	$(0.4, 0.3)$
η_{if}^{a-4}	$(0.7, 0.2)$	$(0.5, 0.2)$	$(0.5, 0.4)$	$(0.5, 0.4)$

Table 2. Decision matrix 2.

	η_{if}^{at-1}	η_{if}^{at-2}	η_{if}^{at-3}	η_{if}^{at-4}
η_{if}^{a-1}	$(0.3, 0.2)$	$(0.7, 0.1)$	$(0.8, 0.1)$	$(0.1, 0.1)$
η_{if}^{a-2}	$(0.4, 0.4)$	$(0.6, 0.2)$	$(0.7, 0.2)$	$(0.2, 0.1)$
η_{if}^{a-3}	$(0.5, 0.3)$	$(0.5, 0.3)$	$(0.6, 0.3)$	$(0.3, 0.2)$
η_{if}^{a-4}	$(0.6, 0.2)$	$(0.7, 0.1)$	$(0.5, 0.4)$	$(0.4, 0.3)$

Table 3. Decision matrix 3.

	η_{if}^{at-1}	η_{if}^{at-2}	η_{if}^{at-3}	η_{if}^{at-4}
η_{if}^{a-1}	$(0.6, 0.2)$	$(0.8, 0.1)$	$(0.2, 0.1)$	$(0.5, 0.3)$
η_{if}^{a-2}	$(0.5, 0.3)$	$(0.7, 0.2)$	$(0.6, 0.2)$	$(0.7, 0.1)$
η_{if}^{a-3}	$(0.7, 0.1)$	$(0.5, 0.4)$	$(0.4, 0.3)$	$(0.5, 0.3)$
η_{if}^{a-4}	$(0.3, 0.2)$	$(0.3, 0.2)$	$(0.2, 0.1)$	$(0.7, 0.1)$

Table 4. Decision matrix 4.

	η_{if}^{at-1}	η_{if}^{at-2}	η_{if}^{at-3}	η_{if}^{at-4}
η_{if}^{a-1}	$(0.6, 0.2)$	$(0.7, 0.2)$	$(0.2, 0.1)$	$(0.7, 0.2)$
η_{if}^{a-2}	$(0.5, 0.3)$	$(0.6, 0.3)$	$(0.3, 0.2)$	$(0.4, 0.1)$
η_{if}^{a-3}	$(0.1, 0.1)$	$(0.6, 0.2)$	$(0.7, 0.2)$	$(0.2, 0.1)$
η_{if}^{a-4}	$(0.3, 0.2)$	$(0.5, 0.3)$	$(0.6, 0.3)$	$(0.3, 0.2)$

Point 2: Furthermore, we aggregate the information in the decision matrices to aggregate a single matrix, using the theory of the $A - IFHCIA$ operator, as shown in Table 5.

Table 5. Aggregated matrix.

	η_{if}^{at-1}	η_{if}^{at-2}	η_{if}^{at-3}	η_{if}^{at-4}
η_{if}^{a-1}	$(0.1608, 0.1313)$	$(0.2504, 0.3183)$	$(0.3362, 0.1317)$	$(0.3135, 0.1389)$
η_{if}^{a-2}	$(0.1800, 0.1433)$	$(0.1985, 0.2967)$	$(0.2654, 0.7344)$	$(0.1072, 0.1271)$
η_{if}^{a-3}	$(0.1356, 0.3107)$	$(0.1793, 0.1549)$	$(0.3066, 0.2917)$	$(0.1237, 0.1327)$
η_{if}^{a-4}	$(0.2752, 0.2104)$	$(0.2426, 0.1784)$	$(0.2530, 0.3701)$	$(0.1601, 0.3134)$

Point 3: Moreover, we derive the ideal positive and negative ideal solutions by using the information in the decision matrix.

$$\eta_{if}^{+} = \{(0.2752, 0.1313), (0.2504, 0.1549), (0.3362, 0.1317), (0.3135, 0.1271)\}$$

$$\eta_{if}^{-} = \{(0.1356, 0.3107), (0.1793, 0.3183), (0.2530, 0.7344), (0.1072, 0.3134)\}$$

Point 4: Then, we aim to evaluate the discrimination measures, based on the ideal positive and negative solutions, with the information in the decision matrix.

$$D^{+}\left(\eta_{if}^{1}, \eta_{if}^{+}\right) = 0.9791, D^{+}\left(\eta_{if}^{2}, \eta_{if}^{+}\right) = 0.8622.D^{+}\left(\eta_{if}^{3}, \eta_{if}^{+}\right) = 0.9548, D^{+}\left(\eta_{if}^{4}, \eta_{if}^{+}\right) = 0.9508$$

$$D^-\left(\eta_{if}^1, \eta_{if}^-\right) = 0.8071, D^-\left(\eta_{if}^2, \eta_{if}^-\right) = 0.9696, D^-\left(\eta_{if}^3, \eta_{if}^-\right) = 0.8883, D^-\left(\eta_{if}^4, \eta_{if}^-\right) = 0.9176$$

Point 5: Additionally, we derive the closeness measurement with the help of the obtained discrimination measures:

$$C^1 = \frac{D^-\left(\eta_{if}^1, \eta_{if}^-\right)}{D^+\left(\eta_{if}^1, \eta_{if}^+\right) + D^-\left(\eta_{if}^1, \eta_{if}^-\right)} = \frac{0.8071}{0.8071 + 0.9791} = 0.4518, C^2 = 0.5293, C^3 = 0.4819, C^4 = 0.4911$$

where $0 \leq C^j \leq 1$.

Point 6: We determine or evaluate the ranking values, based on the closeness measures, and try to discover the best one.

$$C^2 \geq C^4 \geq C^3 \geq C^1$$

Hence, we establish that the best decision is C^2. Furthermore, we evaluate the information in Table 5 with the help of the A-IFHCIA and A-IFHCIG operators and try to compare them with the obtained results of the TOPSIS method. Therefore, the final aggregated results are given in Table 6.

Table 6. Aggregated matrix.

	$\boldsymbol{A-IFHCIA}$	$\boldsymbol{A-IFHCIG}$
η_{if}^{a-1}	$(0.0843, 0.0813)$	$(0.1037, 0.0913)$
η_{if}^{a-2}	$(0.0581, 0.0106)$	$(0.0397, 0.0363)$
η_{if}^{a-3}	$(0.0459, 0.1072)$	$(0.0540, 0.0915)$
η_{if}^{a-4}	$(0.0940, 0.1144)$	$(0.1177, 0.0671)$

Finally, by using the information in Equation (2), we can evaluate the score or net result (see the information in Table 7).

Table 7. Score matrix.

	$\boldsymbol{A-IFHCIA}$	$\boldsymbol{A-IFHCIG}$
η_{if}^{a-1}	0.0029	0.0124
η_{if}^{a-2}	0.0474	0.0034
η_{if}^{a-3}	−0.0612	−0.0374
η_{if}^{a-4}	−0.0204	0.0506

Moreover, we can determine or evaluate the ranking values, based on the information in Table 7, and try to discover the best one.

$$\eta_{if}^{a-2} \geq \eta_{if}^{a-1} \geq \eta_{if}^{a-4} \geq \eta_{if}^{a-3}$$

$$\eta_{if}^{a-4} \geq \eta_{if}^{a-1} \geq \eta_{if}^{a-2} \geq \eta_{if}^{a-3}$$

Hence, we established that the best decision is η_{if}^{a-2}, according to the A-IFHCIA operator, and the best decision is η_{if}^{a-4}, according to the A-IFHCIG operator, where the final result obtained from the TOPSIS method and A-IFHCIA operator are the same but the final result of the A-IFHCIG operator is different. Finally, we derive the comparative analysis of the presented information with some existing operators, to find the supremacy and real worth of the mentioned operator and TOPSIS methods.

5. Comparative Analysis

A-IF information plays a very critical and valuable role in the environment of fuzzy set theory because it can deal with awkward and vague information far better, compared with FS and classical information. In this section, we compare the derived operator with some existing operators that were computed by different scholars. To compare the derived work with some of the prevailing works in the literature, we used the following data, such as that of Shen et al. [24], who invented the TOPSIS method for A-IF set theory. Huang [29] discovered the Hamacher aggregation operators for A-IF sets and evaluated their applications in decision-making, Tan and Chen [34] presented the CI operators taking into consideration the A-IF sets, while Wu et al. [35] diagnosed the CI in the presence of A-IF set theory. Therefore, the comparative analysis for taking the information from Table 5 is illustrated in Table 8.

Table 8. Comparative analysis matrix.

Methods	Score Values/Similarity Values	Ranking Values
Shen et al. [24]	0.4518, 0.5293, 0.4819, 0.4911	$\eta_{if}^{a-2} \geq \eta_{if}^{a-4} \geq \eta_{if}^{a-3} \geq \eta_{if}^{a-1}$
Huang [29]	0.0417, −0.15, −0.068, 0.0017	$\eta_{if}^{a-1} \geq \eta_{if}^{a-4} \geq \eta_{if}^{a-3} \geq \eta_{if}^{a-2}$
	0.0386, −0.047, −0.054, 0.0226	$\eta_{if}^{a-1} \geq \eta_{if}^{a-4} \geq \eta_{if}^{a-2} \geq \eta_{if}^{a-3}$
Tan and Chen [34]	−0.378, −0.376, −0.488, −0.448	$\eta_{if}^{a-1} \geq \eta_{if}^{a-2} \geq \eta_{if}^{a-4} \geq \eta_{if}^{a-3}$
Wu et al. [35]	0.4569, 0.4283, 0.3316, 0.452	$\eta_{if}^{a-1} \geq \eta_{if}^{a-4} \geq \eta_{if}^{a-2} \geq \eta_{if}^{a-3}$
TOPSIS Method	0.4518, 0.5293, 0.4819, 0.4911	$\eta_{if}^{a-2} \geq \eta_{if}^{a-4} \geq \eta_{if}^{a-3} \geq \eta_{if}^{a-1}$
A-IFHCIA	0.0029, 0.0474, −0.0612, −0.0204	$\eta_{if}^{a-2} \geq \eta_{if}^{a-1} \geq \eta_{if}^{a-4} \geq \eta_{if}^{a-3}$
A-IFHCIG	0.0124, 0.0034, −0.0374, 0.0506	$\eta_{if}^{a-4} \geq \eta_{if}^{a-1} \geq \eta_{if}^{a-2} \geq \eta_{if}^{a-3}$

We obtained three different types of ranking results, comprising η_{if}^{a-1}, η_{if}^{a-2}, and η_{if}^{a-4}, but most operators were given the best preference as first η_{if}^{a-1}, then η_{if}^{a-2}, and only the proposed A-IFHCIG operator is given η_{if}^{a-4} as the best decision. To enhance the worth of the derived information, we took the information from the work of Tan and Chen [34] and tried to evaluate it using the derived work. Here, we consider the information in Table 1 from Ref. [34], and their final ranking results are stated below.

$$\pi_{S-IF}^{1} = 0.56 - 0.23 = 0.33, \pi_{S-IF}^{2} = 0.68 - 021 = 0.47, \pi_{S-IF}^{3} = 0.49 - 0.22 = 0.27, \pi_{S-IF}^{4} = 0.51 - 0.22 = 0.29$$

The final and ranking results of the existing information in Ref. [34] are stated below:

$$\pi_{S-I}^{2} \geq \pi_{S-I}^{1} \geq \pi_{S-I}^{4} \geq \pi_{S-I}^{3}$$

They identify the preferred choice as a π_{S-I}^{2}, further using the information in Table 1; the ranking results of the proposed work are stated in Table 9.

We have obtained three different types of ranking results, as η_{if}^{a-2}, η_{if}^{a-3}, and η_{if}^{a-4}, but most operators are given the best preference as η_{if}^{a-4}. Therefore, here, we consider that η_{if}^{a-4} represented the preferred choice.

Many scholars have derived different types of operators by combining two or three different structures, based on the IF set, while other scholars have tried to modify or utilize them based on the generalization of the IF set. The proposed model is massively modified according to the many aggregation operators, such as the averaging/geometric aggregation operators.

Table 9. Comparative analysis matrix.

Methods	Score Values/Similarity Values	Ranking Values
Shen et al. [24]	0.4274, 0.5146, 0.4889, 0.4532	$\eta_{if}^{a-2} \geq \eta_{if}^{a-3} \geq \eta_{if}^{a-4} \geq \eta_{if}^{a-1}$
Huang [29]	0.1977, 0.0114, 0.0104, 0.3014	$\eta_{if}^{a-4} \geq \eta_{if}^{a-1} \geq \eta_{if}^{a-2} \geq \eta_{if}^{a-3}$
	0.1795, 0.0387, 0.204, 0.2743	$\eta_{if}^{a-4} \geq \eta_{if}^{a-1} \geq \eta_{if}^{a-3} \geq \eta_{if}^{a-2}$
Tan and Chen [34]	−0.3471, −0.5541, −0.1226, −0.1702	$\eta_{if}^{a-3} \geq \eta_{if}^{a-4} \geq \eta_{if}^{a-1} \geq \eta_{if}^{a-2}$
Wu et al. [35]	0.6524, 0.431, 0.7543, 0.6539	$\eta_{if}^{a-3} \geq \eta_{if}^{a-4} \geq \eta_{if}^{a-1} \geq \eta_{if}^{a-2}$
TOPSIS Method	0.4274, 0.5146, 0.4889, 0.4532	$\eta_{if}^{a-2} \geq \eta_{if}^{a-3} \geq \eta_{if}^{a-4} \geq \eta_{if}^{a-1}$
A-IFHCIA	0.1977, 0.0114, 0.0104, 0.3014	$\eta_{if}^{a-4} \geq \eta_{if}^{a-1} \geq \eta_{if}^{a-2} \geq \eta_{if}^{a-3}$
A-IFHCIG	0.1795, 0.0387, 0.204, 0.2743	$\eta_{if}^{a-4} \geq \eta_{if}^{a-1} \geq \eta_{if}^{a-3} \geq \eta_{if}^{a-2}$

6. Conclusions

The key findings of this analysis can be summarized below:

1. The derived information uses the theory of A-IFSs to manage vague, unreliable, and awkward information by selecting the appropriate truth and falsity grades.
2. To aggregate, the collection of a finite number of preferences is a very challenging task for scholars when considering A-IFS. Therefore, in this analysis, a novel theory using Hamacher aggregation operators, based on the Choquet integral, for A-IFSs was evaluated. By combining these theories, we derived a theory of averaging and geometric aggregation operators by considering A-IFSs, Hamacher aggregation operators, and Choquet integrals, such as A-IFHCIA, A-IFHCIOA, A-IFHCIG, and A-IFHCIOG operators.
3. Several valuable properties, such as idempotency, monotonicity, and boundedness are also derived.
4. By constructing the decision matrix and finding the ideal positive and negative situations based on derived operators, we computed the theory of the TOPSIS method for A-IFS, which is very valuable for evaluating the closeness between any number of attributes.
5. To evaluate the decision-making problem, we computed or derived a new MADM procedure under the consideration of stated operators for A-IFSs. Furthermore, we also derived the technique of the TOPSIS method, based on the ideal positive and ideal negative solution and considering the A-IFSs.
6. Finally, we compared the presented operators with a few existing operators to discuss the superiority and effectiveness of the derived approaches.

In the future, we will utilize the discovered operators to evaluate various other complicated decision-making problems, such as clustering analysis, as well as artificial intelligence. The derived analysis can easily be used for evaluating the interrelationship among any finite number of attributes into a singleton set and can be improved in future work; we will extend the theory of the fuzzy superior Mandelbrot set [38], spherical fuzzy sets [39], and m-polar fuzzy Hamacher aggregation operators [40]. Furthermore, we will aim to utilize the derived information in the field of game theory, decision-making theory, neural networks, database and data mining, artificial intelligence, road signals, pattern recognition, medical diagnosis, and clustering analysis, to improve the quality of the presented work.,

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