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Second-Ordered Parametric Duality for the Multi-Objective Programming Problem in Complex Space

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Abstract: We introduced and discussed a second-ordered parametric dual model of a complex multi-objective programming problem (P). Moreover, the authors constructed and proved the weak, strong and strictly converse duality theorems by the second-ordered generalized Θ-bonvexity.

Keywords: multi-objective programming; efficient solutions; generalized convexity; duality problem

MSC: 49K35; 90C29; 26A51; 90C46

1. Introduction

The complex optimization problem has been applied in many fields in electrical engineering, such as minimal entropy or maximum kurtosis. Levinson published their study on complex linear programming in 1966 [1]. Since then, case studies on complex nonlinear, fractional, and duality programming problems have been discussed [2–4]. Datta and Bhatia started their study on a complex minimax problem in 1984 [9]. Lai and Huang constructed various cases of complex minimax optimal problems. Following that, Huang et. al. constructed several types of second-order duality models for complex fractional and nonfractional minimax programming problems, and also derived the duality theorems under second-order generalized Θ-bonvexity [10–12].

All of the above, the complex optimal problems were focused on the real parts of complex objective functions. Youness and Elbrolosy considered the general case with both real and imaginary parts [13,14]. The complex extended programming problem is formulated as follows.

\[
(P_0) \quad \begin{array}{l}
\text{min} \quad f(z, \bar{z}) \\
\text{such that} \quad X = \{ (z, \bar{z}) \in Q \mid -g(z, \bar{z}) \in S \},
\end{array}
\]

where \( S \) is a polyhedral cone in \( \mathbb{C}^n \), \( f: \mathbb{C}^{2n} \rightarrow \mathbb{C} \) and \( g: \mathbb{C}^{2n} \rightarrow \mathbb{C}^m \) are analytic in \( z = (z, \bar{z}) \in Q \), and the set \( Q = \{ (z, \bar{z}) \mid z \in \mathbb{C}^n \} \subset \mathbb{C}^{2n} \) is a linear manifold over real field. Elbrolosy extended the complex multi-objective vector optimization problem (P), and also defined the concept of optimal efficient solutions and established the optimality conditions of the problem (P) by using the scalarization techniques as follows [15].

\[
(P) \quad \begin{array}{l}
\text{min} \quad f(z) = (f_1(z), \ldots, f_p(z)) \\
\text{such that} \quad z = (z, \bar{z}) \in X = \{ z \in Q \mid -g(z) \in S \},
\end{array}
\]

where \( S \subset \mathbb{C}^q \) is a polyhedral cone, and \( f: \mathbb{C}^{2n} \rightarrow \mathbb{C}^p \), \( g: \mathbb{C}^{2n} \rightarrow \mathbb{C}^q \) are analytic in \( z = (z, \bar{z}) \in Q = \{ (z, \bar{z}) \mid z \in \mathbb{C}^n \} \subset \mathbb{C}^{2n} \).

Recently, Huang and Tanaka established the sufficient optimality conditions of problem (P), formulated the parametric dual problem and proved their duality theorems under...
the generalized convexities [16]. Usually, the objective function in the complex programming problem was focused on the real part only. The novelty of this paper is extended the case of objective function from the real part to the case of both real and imaginary parts. Moreover, we would formulate the second-ordered parametric dual problem (D) with respect to the problem (P) and prove their duality theorems under the second-ordered generalized Θ-bonvexity.

2. Notations and Preliminary

Given $z \in \mathbb{C}^p$, the notations $\bar{z}, z^T$ and $z^H$ are the conjugate, transpose and conjugate transpose of $z$. Let $T = \{ z \in \mathbb{C}^p \mid \text{Re}(Kz) \geq 0 \} \subset \mathbb{C}^p$ be a polyhedral cone with matrix $K \in \mathbb{C}^{k \times p}$ where $k$ is a positive integer. The dual cone $T^*$ of the convex cone $T$ is defined by

$$T^* = \{ \eta \in \mathbb{C}^p \mid \text{Re}(z, \eta) \geq 0 \text{ for all } z \in T \},$$

where $\langle z, \eta \rangle = \eta^H z$ is defined to be the inner product of $z$ and $\eta$ in complex spaces. For $z_0 \in T$, the set $T(z_0)$ is the intersection of those closed half spaces that includes $z_0$ in their boundaries. Thus, if $z_0 \in \text{int}(T)$, $T(z_0)$ is the whole space $\mathbb{C}^p$.

Let $T \subset \mathbb{C}^p$ be a pointed, closed convex cone. For any $y, y_0 \in \mathbb{C}^p$, the ordered relation notation “$\leq_T$” with respect to cone $T$ is defined as:

$$y_0 \leq_T y \iff y - y_0 \in T.$$

Note that for a nonzero vector $\mu \in T^*$,

$$y_0 \leq_T y \Rightarrow \text{Re}[\mu^H(y - y_0)] \geq 0.$$

Definition 1. Duca [8], [Definition 3.3.1] (Optimal efficient solution)

Let $X$ be a nonempty subset of $Q = \{ z = (z, \bar{z}) \in \mathbb{C}^{2n} \mid z \in \mathbb{C}^n \} \subset \mathbb{C}^{2n}$, $T \subset \mathbb{C}^p$ be a pointed and closed convex cone, and $f : X \to \mathbb{C}^p$ be a map from $X$ to $\mathbb{C}^p$.

(1) The point $z_0 = (z_0, \bar{z_0}) \in X$ is a minimal efficient (or Pareto-minimal) solution of $f$ with respect to $T$ if there exists no other feasible point $z = (z, \bar{z}) \in X$ such that $f(z_0) - f(z) \in T \setminus \{0\}$.

(2) The point $z_0 = (z_0, \bar{z_0}) \in X$ is a maximal efficient (or Pareto-maximal) solution of $f$ with respect to $T$ if there exists no other feasible point $z = (z, \bar{z}) \in X$ such that $f(z) - f(z_0) \in T \setminus \{0\}$.

Note that $z_0 \in X$ is a minimal efficient solution of $f$ with respect to $T$ if $(f(X) - f(z_0)) \cap (-T) = \{0\}$; analogously, $z_0 \in X$ is a maximal efficient solution of $f$ with respect to $T$ if $(f(z_0) - f(X)) \cap (-T) = \{0\}$. The minimal efficient solution or maximal efficient solution of $f$ with respect to $T$ in a multi-objective programming problem is called the optimal efficient solution of $f$ with respect to $T$.

In order to establish the optimality conditions and duality properties, we re-called the gradient expression and second-order gradient expression of the complex functions. Given $z = (z, \bar{z}) \in \mathbb{C}^{2n}$ and a twice differentiable analytic function $f : \mathbb{C}^{2n} \to \mathbb{C}$, the gradient expression $\nabla f(z)$ is denoted by

$$\nabla f(z) = \left( \nabla_z f(z), \nabla_{\bar{z}} f(z) \right) \in \mathbb{C}^{p \times 2n}.$$
with \( \nabla_z f(z) = \begin{pmatrix} \frac{\partial}{\partial z_1} f_1(z) & \cdots & \frac{\partial}{\partial z_1} f_1(z) \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial z_p} f_p(z) & \cdots & \frac{\partial}{\partial z_p} f_p(z) \end{pmatrix} \), \( \nabla_z f(z) = \begin{pmatrix} \frac{\partial}{\partial z_1} f_1(z) & \cdots & \frac{\partial}{\partial z_1} f_1(z) \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial z_p} f_p(z) & \cdots & \frac{\partial}{\partial z_p} f_p(z) \end{pmatrix} \in \mathbb{C}^p \times \mathbb{H}.

The second-order gradient expression \( \nabla^2 f_k(z) \), \( k = 1, \ldots, p \) is denoted by

\[
\nabla^2 f_k(z) = \begin{pmatrix} \nabla_{zz} f_k(z) & \nabla_{z\overline{z}} f_k(z) \\ \nabla_{\overline{z}z} f_k(z) & \nabla_{\overline{z}\overline{z}} f_k(z) \end{pmatrix} \in \mathbb{C}^{2n \times 2n}
\]

with

\[
\nabla_{zz} f_k(z) = \left( \frac{\partial^2}{\partial z_i^2} f_k(z) \right)_{n \times n}, \quad i, j = 1, \ldots, n,
\]

\[
\nabla_{z\overline{z}} f_k(z) = \left( \frac{\partial^2}{\partial z_i \partial \overline{z}_j} f_k(z) \right)_{n \times n}, \quad i, j = 1, \ldots, n,
\]

\[
\nabla_{\overline{z}z} f_k(z) = \left( \frac{\partial^2}{\partial \overline{z}_i \partial z_j} f_k(z) \right)_{n \times n}, \quad i, j = 1, \ldots, n.
\]

We express the differential form of a complex function by using the gradient representations as the following lemma.

**Lemma 1.** Given \( z = (z, \overline{z}) \), \( z_0 = (z_0, \overline{z}_0) \subset \mathbb{C}^{2n} \) and \( \langle \overline{v}, \overline{w} \rangle = \langle z - z_0, \overline{z} - \overline{z}_0 \rangle \). Suppose that \( f(\cdot) : \mathbb{C}^{2n} \to \mathbb{C}^p \), \( \tau = (\tau_1, \ldots, \tau_p) \in \mathbb{C}^p \) and \( \Phi(z) = \langle f(z), \tau \rangle = \tau^H f(z) \). Then

\( \langle f(z), \tau \rangle - \langle f(z), \tau \rangle \)

\[
\text{Re}[\Phi'(z_0)(z - z_0)] = \text{Re}\left( \langle z - z_0, \tau^T \nabla_z f(z_0) + \tau^H \nabla_{\overline{z}} f(z_0) \rangle \right).
\]

\( \langle z - z_0, \tau^T \nabla_z f(z_0) + \tau^H \nabla_{\overline{z}} f(z_0) \rangle \) is the real part of Equation (b) is equal to

\[
\text{Re}\left( \langle v, \tau^H [\tau^T \nabla_z f(z_0) + \tau^H \nabla_{\overline{z}} f(z_0)] + v^T \left[ \tau^T \nabla_{\overline{z}} f(z_0) + \tau^H \nabla_{\overline{z}} f(z_0) \right] \right).
\]

**Proof.**

\( \langle x, y \rangle = y^H x \) is the inner product in complex space,

\[
\Phi'(z_0)(z - z_0) = \langle f'(z_0)(z - z_0), \tau \rangle = \langle \nabla_z f(z_0), \nabla_{\overline{z}} f(z_0) \rangle \left( \begin{array}{c} v \\ \overline{v} \end{array} \right), \tau
\]

\[
= \langle \nabla_z f(z_0) v + \nabla_{\overline{z}} f(z_0) \overline{v}, \tau \rangle
\]

\[
= \tau^H \nabla_z f(z_0) v + \tau^H \nabla_{\overline{z}} f(z_0) \overline{v}
\]

\[
= \sum_{j=1}^p \sum_{i=1}^n \overline{\tau}_j \frac{\partial}{\partial z_i} f_1(z_0) v_i + \sum_{j=1}^p \sum_{i=1}^n \tau_j \frac{\partial}{\partial \overline{z}_i} f_1(z_0) \overline{v}_i
\]

\[
= \sum_{j=1}^p \sum_{i=1}^n \overline{\tau}_j \frac{\partial}{\partial z_i} f_1(z_0) v_i + \sum_{j=1}^p \sum_{i=1}^n \tau_j \frac{\partial}{\partial \overline{z}_i} f_1(z_0) \overline{v}_i
\]

\[
= \langle v, \tau^T \nabla_z f(z_0) \rangle + \langle \tau^H \nabla_{\overline{z}} f(z_0), v \rangle.
\]

We obtain

\[
\Phi'(z_0)(z - z_0) = \langle z - z_0, \tau^T \nabla_z f(z_0) \rangle + \langle \tau^H \nabla_{\overline{z}} f(z_0), z - z_0 \rangle.
\]
Moreover, since \( \text{Re} \left[ \langle x, y \rangle \right] = \text{Re} \left[ \langle y, x \rangle \right] = \text{Re} \left[ \langle x, x \rangle \right] \), we have

\[
\text{Re} \left[ \Phi'(z_0)(z - z_0) \right] = \text{Re} \left\{ \left\langle z - z_0, \tau^T \nabla_z f(z_0) \right\rangle + \left\langle \tau^H \nabla_{\bar{z}} f(z_0), z - z_0 \right\rangle \right\} 
\]

\[
= \text{Re} \left\{ z - z_0, \tau^T \nabla_z f(z_0) \right\} + \text{Re} \left\{ z - z_0, \tau^H \nabla_{\bar{z}} f(z_0) \right\} 
\]

\[
= \text{Re} \left\{ z - z_0, \tau^T \nabla_z f(z_0) \right\} + \text{Re} \left\{ z - z_0, \tau^H \nabla_{\bar{z}} f(z_0) \right\} 
\]

\[
= \text{Re} \left\{ z - z_0, \tau^T \nabla_z f(z_0) + \tau^H \nabla_{\bar{z}} f(z_0) \right\}. 
\]

(b) Let \( \Phi(z) = \langle f(z), \tau \rangle = \sum_{k=1}^{p} \tau_k f_k(z_0) = \tau_1 f_1(z_0) + \cdots + \tau_p f_p(z_0) \), where \( f_k(z_0) \) is the mapping from \( \mathbb{C}^n \) to \( \mathbb{C} \) for \( k = 1, \cdots , p \). Then

\[
(z - z_0)^T \nabla^2 f(z_0), \tau \) \( (z - z_0) = (z - z_0)^T \nabla^2 f(z_0), \tau \) \( (z - z_0) + \cdots + (z - z_0)^T \nabla^2 f_p(z_0)(z - z_0). \) (1)

For \( j = 1, \ldots , p \), and \( z - z_0 = (z - z_0, \overline{z} - \overline{z_0}) = (v, \overline{v}), \)

\[
(z - z_0)^T \overline{\nabla}^2 f_j(z_0)(z - z_0) = \langle v, \overline{v} \rangle \left( \overline{\nabla}_{\overline{v}}^2 f_j(z_0), \overline{\nabla}_{\overline{v}}^2 f_j(z_0) \right) \left( \begin{array}{c} v \\ \overline{v} \end{array} \right) 
\]

\[
= \sum_{k=1}^{n} \sum_{l=1}^{n} \overline{v}_k \overline{v}_l \frac{\partial^2 f_j(z_0)}{\partial \overline{v}_k \partial \overline{v}_l} |v| + \sum_{k=1}^{n} \sum_{l=1}^{n} \overline{v}_k \overline{v}_l \frac{\partial^2 f_j(z_0)}{\partial \overline{v}_k \partial \overline{v}_l} |\overline{v}| + \sum_{k=1}^{n} \sum_{l=1}^{n} \overline{v}_k \overline{v}_l \frac{\partial^2 f_j(z_0)}{\partial \overline{v}_k \partial \overline{v}_l} |v| + \sum_{k=1}^{n} \sum_{l=1}^{n} \overline{v}_k \overline{v}_l \frac{\partial^2 f_j(z_0)}{\partial \overline{v}_k \partial \overline{v}_l} |\overline{v}| 
\]

By formula above, Equation (1) implies that

\[
(z - z_0)^T \nabla^2 f(z_0), \tau \) \( (z - z_0) = \langle v, v^H \rangle [\nabla_{\overline{z}}^2 f(z_0)], vs. \rangle + \langle v, v^H \rangle [\nabla_{\overline{z}}^2 f(z_0)], vs. \rangle + \langle v^H [\nabla_{\overline{z}}^2 f(z_0)], vs. \rangle \]

and the real part of the above identity is equal to

\[
\text{Re} \left\{ \langle v, v^H \left[ \tau^T \nabla_{\overline{z}}^2 f(z_0) + \tau^H \nabla_{\overline{z}}^2 f(z_0) \right] \right\} + \langle v, v^H \left[ \tau^T \nabla_{\overline{z}}^2 f(z_0) + \tau^H \nabla_{\overline{z}}^2 f(z_0) \right] \right\}. 
\]

\[\square\]

3. Optimality Conditions

We would like to find the minimum efficient solutions to the complex multi-objective programming problem (P). The scalarization technique is going to be applied to the multi-objective programming problem. We would obtain the existence of minimum efficient solutions of problem (P) above by scalarized programming problem (P\(_T\)) below, and the lemmas followed will be stated [15,16].

Given a nonzero vector \( \tau \in \mathbb{C}^p \), we consider the scalarized programming problem with respect to problem (P) as follows.

(P\(_T\)) \[
\begin{array}{ll}
\min & \text{Re}[\tau^H f(z)] \\
\text{such that} & \{z, \overline{z} \in Q | -g(z) \in S\}.
\end{array}
\]

Lemma 2. ([Elbrolosy [15], Theorem 4.4])

Let \( T \subset \mathbb{C}^p \) be a pointed, closed and convex cone and \( f(X) \) be a convex set. If point \( z_0 \) is a minimal efficient solution of (P) with respect to \( T \), then there exists a nonzero vector \( \tau \in T^* \) such that \( z_0 \) is an optimal solution of (P\(_T\)).
Lemma 3. (Elbrolosy [15], [Theorem 4.6])

Let $T \subset \mathbb{C}^p$ be a pointed, closed and convex cone, and $\tau \in T^*$ with $\tau \neq 0$. Assume that $z_0$ is an optimal solution of $(P_\tau)$, and anyone of the following conditions holds,

(i) nonzero vector $\tau \in \text{int}(T^*)$,

(ii) point $z_0$ is the unique optimal solution of $(P_\tau)$.

Then $z_0$ is the minimal efficient solution of $(P)$ with respect to $T$.

Elbrolosy [15] established the Kuhn-Tucker necessary optimality conditions of problem (P) by using the scalarization techniques, we described as follows.

Definition 2. (Lai and Huang [12], [Definition 3])

The problem $(P)$ is said to satisfy the constraint qualification at a point $z_0 = (z_0, \bar{z}_0)$, if for any nonzero $\mu \in S^* \subset \mathbb{C}^q$,

$$(g^*_{\bar{z}}(z_0) - z_0, \mu) \neq 0, \text{ for } z \neq z_0.$$ 

Under the gradient expression as in Lemma 1, the constraint qualification can be expressed by

$$\mu^T \nabla_{\bar{z}} g(z_0) + \mu^H \nabla_{\bar{z}} g(z_0) \neq 0, \text{ for } \mu \neq 0 \text{ in } S^*,$$

where $\mu^H = \mu^T$.

Theorem 1. (Elbrolosy [15], [Theorem 4.9]) (Necessary optimality conditions)

Let $T \subset \mathbb{C}^p$ be a pointed, closed and convex cone, $S$ be a polyhedral cone in $\mathbb{C}^q$ and $f(X)$ be a convex set. Suppose that the mappings $f(\cdot) : \mathbb{C}^{2n} \to \mathbb{C}^p$ and $g(\cdot) : \mathbb{C}^{2n} \to \mathbb{C}^q$ are analytic on $X \subset Q$, and $z_0$ is a minimal efficient solution of $(P)$ with respect to $T$. If problem $(P)$ possesses the constraint qualification at $z_0$, there are nonzero vectors $\tau \in T^* \subset \mathbb{C}^p$ and $\mu \in S^* \subset \mathbb{C}^q$ satisfying the following conditions:

$$\tau^T \nabla_{\bar{z}} f(z_0) + \tau^H \nabla_{\bar{z}} f(z_0) + \mu^T \nabla_{\bar{z}} g(z_0) + \mu^H \nabla_{\bar{z}} g(z_0) = 0,$$

$$\Re \mu^H g(z_0) = 0.$$

In order to formulate the sufficient optimality conditions and duality theorems, we introduce the generalized convexity in complex spaces as follows.

Definition 3. (Lai and Huang [12], [Definition 1])

The real part of an analytic function $f(\cdot)$ is said to be:

(i) convex (strictly) at $z_0 \in Q \subset \mathbb{C}^{2n}$ if for all $z \in Q$,

$$\Re [f(z) - f(z_0)] \geq (>) \Re [f'(z_0)(z - z_0)]$$

(ii) pseudoconvex (strictly) at $z_0 \in Q$ if for all $z \in Q$,

$$\Re [f'(z_0)(z - z_0)] \geq 0 \Rightarrow \Re [f(z) - f(z_0)] \geq 0 (> 0),$$

(iii) quasiconvex at $z_0 \in Q$ if for all $z \in Q$,

$$\Re [f(z) - f(z_0)] \leq 0 \Rightarrow \Re [f'(z_0)(z - z_0)] \leq 0.$$

Huang and Tanaka [16] established the sufficient optimality conditions below.

Theorem 2. [16], [Theorem 3.6] (Sufficient optimality conditions)

Let $T \subset \mathbb{C}^p$ be a pointed, closed and convex cone, $S$ be a polyhedral cone in $\mathbb{C}^q$, and $f(\cdot) : \mathbb{C}^{2n} \to \mathbb{C}^p$ and $g(\cdot) : \mathbb{C}^{2n} \to \mathbb{C}^q$ be two analytic mappings on $X \subset Q$, where $Q \subset \mathbb{C}^{2n}$. Suppose that $z_0$ is a feasible solution of $(P)$, and there are nonzero vectors $\tau \in T^* \subset \mathbb{C}^p$ and $\mu \in S^* \subset \mathbb{C}^q$ satisfying conditions (2) and (3) in Theorem 1. If any one of the following conditions (i)–(iii) holds:

(i) Either of $\Re [\tau^H f(\cdot)]$ or $\Re [\mu^H g(\cdot)]$ is strictly convex and the other is convex at $z_0 \in Q$, or both are strictly convex at $z_0 \in Q$.

(ii) $\Re [\tau^H f(\cdot)]$ is quasiconvex at $z_0 \in Q$ and $\Re [\mu^H g(\cdot)]$ is strictly pseudoconvex at $z_0 \in Q$. 

Axioms 2022, 11, 717
(iii) \( \text{Re}[\tau^H f(\cdot) + \mu^H g(\cdot)] \) is strictly pseudoconvex at \( z_0 \in Q \),
then \( z_0 \) is the minimal efficient solution of \( P \) with respect to \( T \).

4. The Second-Order Parametric Duality Model

We would like to use the following differential notations to simplify the expression. Let \( u = (u, \overline{u}) \in C^{2n}, \tau \in C^{p}, \mu \in C^{q} \), and \( f : C^{2n} \rightarrow C^{p}, g : C^{2n} \rightarrow C^{q} \) are analytic mappings:

\[
F^{(1)}(u, \tau) = \tau^T \nabla g(u) + \tau^H \nabla f(u); \\
F^{(2)}(u, \tau) = \tau^T \nabla g(u) + \tau^H \nabla f(u); \\
G^{(1)}(u, \mu) = \mu^T \nabla g(u) + \mu^H \nabla f(u); \\
G^{(2)}(u, \mu) = \mu^T \nabla g(u) + \mu^H \nabla f(u).
\]

The second-order parametric dual problem of problem \( (P) \) is considered as the following form.

\[
(D) \quad \max_{\mathcal{F}_D} \gamma = (\gamma_1, \ldots, \gamma_p),
\]

where \( \mathcal{F}_D \) is the set of all feasible solutions \((\tau, u, \mu, v, \gamma)\) satisfied the following conditions:

For \( u = (u, \overline{u}) \in Q, \tau \in C^{p}, v \in C^{2n} \) and \( \mu \in S^s \),

\[
[F^{(1)}(u, \tau) + G^{(1)}(u, \mu)] + v^H [F^{(2)}(u, \tau) + G^{(2)}(u, \mu)] + v^T [F^{(2)}(u, \tau) + G^{(2)}(u, \mu)] = 0,
\]

\[
\text{Re} \left( f(u) - \tau, \gamma \right) \geq \frac{1}{2} \text{Re} \left( v^H F^{(2)}(u, \tau) + v^T F^{(2)}(u, \tau) \right), \quad (5)
\]

\[
\text{Re} (g(u), \mu) \geq \frac{1}{2} \text{Re} \left( v^H G^{(2)}(u, \mu) + v^T G^{(2)}(u, \mu) \right). \quad (6)
\]

We introduce the second-ordered generalized \( \Theta \)-bonvexity as follows.

**Definition 4.** (Huang [10], [Definition 4.1])

The real part of an analytic function \( f(\cdot) \) is called,

(i) \( \Theta \)-bonvex (strictly) at \( z_0 \in Q \subseteq C^{2n} \) if there exists a suitable mapping \( \Theta : C^{2n} \times C^{2n} \rightarrow C^{2n} \) such that for any \( z \in Q \),

\[
\text{Re} \left\{ f(z) - f(z_0) + \frac{1}{2} (z - z_0)^T \nabla^2 f(z_0)(z - z_0) \right\} \geq (>) \text{Re} \left\{ (\nabla f(z_0) + (z - z_0)^T \nabla^2 f(z_0)) \Theta(z, z_0) \right\},
\]

(ii) \( \Theta \)-psuedobonvex (strictly) at \( z_0 \in Q \subseteq C^{2n} \) if there exists a suitable mapping \( \Theta : C^{2n} \times C^{2n} \rightarrow C^{2n} \) such that for any \( z \in Q \),

\[
\text{Re} \left\{ (\nabla f(z_0) + (z - z_0)^T \nabla^2 f(z_0)) \Theta(z, z_0) \right\} \geq 0 \quad (>) \text{Re} \left\{ f(z) - f(z_0) + \frac{1}{2} (z - z_0)^T \nabla^2 f(z_0)(z - z_0) \right\} \geq 0 \quad (>) \text{Re} \left\{ (\nabla f(z_0) + (z - z_0)^T \nabla^2 f(z_0)) \Theta(z, z_0) \right\},
\]

(iii) \( \Theta \)-quasibonvex at \( z_0 \in Q \) if there exists a suitable mapping \( \Theta : C^{2n} \times C^{2n} \rightarrow C^{2n} \) such that for any \( z \in Q \),

\[
\text{Re} \left\{ f(z) - f(z_0) + \frac{1}{2} (z - z_0)^T \nabla^2 f(z_0)(z - z_0) \right\} \leq 0 \quad (>) \text{Re} \left\{ (\nabla f(z_0) + (z - z_0)^T \nabla^2 f(z_0)) \Theta(z, z_0) \right\} \leq 0.
\]

Using the generalized \( \Theta \)-bonvexities, we could obtain the weak, strong and strictly converse duality theorem of dual problem \( (D) \) with respect to primary problem \( (P) \).
Theorem 3. (Weak Duality)
Let \( z = (z, z) \) be \((P)\)-feasible solution, and \((\tau, u, \nu, \gamma)\) be \((D)\)-feasible solution. Suppose that any one of the conditions holds:

(i) Either one of \( \text{Re}[\tau^H f(\cdot)] \) or \( \text{Re}[\mu^H g(\cdot)] \) is strictly \( \Theta\)-bonvex and the other is \( \Theta\)-bonvex at \( u \in Q \), or both are strictly \( \Theta\)-bonvex at \( u \in Q \).

(ii) \( \text{Re}[\tau^H f(\cdot)] \) is \( \Theta\)-quasibonvex at \( u \in Q \) and \( \text{Re}[\mu^H g(\cdot)] \) is strictly \( \Theta\)-pseudoconvex at \( u \in Q \).

(iii) \( \text{Re}[\tau^H f(\cdot) + \mu^H g(\cdot)] \) is strictly \( \Theta\)-pseudoconvex at \( u \in Q \).

Then
\[
 f(z) \leq_T \gamma.
\]

Proof. Suppose on the contrary that
\[
\gamma - f(z) \in T \setminus \{0\}. 
\]

We could pick a nonzero vector \( \tau \in T^* \), such that \( \text{Re}\langle \gamma - f(z), \tau \rangle \geq 0 \), or
\[
\text{Re}\langle f(z) - \gamma, \tau \rangle \leq 0.
\]

By inequality (5), then
\[
\text{Re}\langle f(z) - \gamma, \tau \rangle \leq 0 \leq \text{Re}\langle f(u) - \gamma, \tau \rangle - \frac{1}{2} \text{Re}\left\langle u, v^H \eta_1^{(2)}(u, \tau) + v^T F_2^{(2)}(u, \tau) \right\rangle.
\]

That is
\[
\text{Re}\langle f(z) - f(u), \tau \rangle + \frac{1}{2} \text{Re}\left\langle u, v^H \eta_1^{(2)}(u, \tau) + v^T F_2^{(2)}(u, \tau) \right\rangle \leq 0.
\]

Since the feasibility of \( z \) for problem \((P)\) and the inequality (6),
\[
\text{Re}\langle g(z), \mu \rangle \leq 0 \leq \text{Re}\langle g(u), \mu \rangle - \frac{1}{2} \text{Re}\left\langle u, v^H \eta_1^{(2)}(u, \mu) + v^T G_2^{(2)}(u, \mu) \right\rangle.
\]

We get the following inequality
\[
\text{Re}\langle g(z) - g(u), \mu \rangle + \frac{1}{2} \text{Re}\left\langle u, v^H \eta_1^{(2)}(u, \mu) + v^T G_2^{(2)}(u, \mu) \right\rangle \leq 0.
\]

(a) If hypothesis (i) holds, without loss of generality, assume that \( \text{Re}[\tau^H f(\cdot)] \) is strictly \( \Theta\)-bonvex and \( \text{Re}[\mu^H g(\cdot)] \) is \( \Theta\)-bonvex at \( u \in Q \), and let \( (v, \overline{\tau}) = z - u \).

From inequality (8) and \( \text{Re}[\tau^H f(\cdot)] \) is strictly \( \Theta\)-bonvex at \( u \in Q \), then there is a mapping \( \Theta : \mathbb{C}^{2n} \times \mathbb{C}^{2n} \rightarrow \mathbb{C}^{2n} \) such that
\[
\text{Re}\{[\nabla \tau^H f(u) + (v, \overline{\tau})^T \nabla^2 \tau^H f(u)]\Theta(z, u)\} < 0.
\]

From inequality (9) and \( \text{Re}[\mu^H g(\cdot)] \) is \( \Theta\)-bonvex at \( u \in Q \), then there is a mapping \( \Theta : \mathbb{C}^{2n} \times \mathbb{C}^{2n} \rightarrow \mathbb{C}^{2n} \) such that
\[
\text{Re}\{[\nabla \mu^H g(u) + (v, \overline{\tau})^T \nabla^2 \mu^H g(u)]\Theta(z, u)\} \leq 0.
\]

Combine inequalities (10) and (11), then
\[
\text{Re}\{[\nabla \tau^H f(u) \mu^H g(u)] + (v, \overline{\tau})^T \nabla^2 [\tau^H f(u) + \mu^H g(u)]\Theta(z, u)\} < 0.
\]

This implies that
\[
[F^{(1)}(u, \nu) + G^{(1)}(u, \mu)] + v^H \left[F_1^{(2)}(u, \nu) + G_1^{(2)}(u, \mu)\right] + v^T \left[F_2^{(2)}(u, \nu) + G_2^{(2)}(u, \mu)\right] \neq 0,
\]

where
this contradicts the equality (4).

(b) If hypothesis (ii) holds, \( \Re[\tau^H f(\cdot)] \) is \( \Theta \)-quasibonvex at \( u \) and according to inequality (8), then there is a mapping \( \Theta : \mathbb{C}^{2n} \times \mathbb{C}^{2n} \to \mathbb{C}^{2n} \) such that

\[
\Re \{ [\nabla \tau^H f(u) + (v, \bar{v})^T \nabla^2 \tau^H f(u)]\Theta(z, u) \} \leq 0.
\]

By inequality (9) and \( \Re[\mu^H g(\cdot)] \) is strictly \( \Theta \)-pseudobonvex at \( u \in Q \), then there is a mapping \( \Theta : \mathbb{C}^{2n} \times \mathbb{C}^{2n} \to \mathbb{C}^{2n} \) such that

\[
\Re \{ [\nabla \mu^H g(u) + (v, \bar{v})^T \nabla^2 \mu^H g(u)]\Theta(z, u) \} < 0.
\]

We obtain inequality (12) by summing up the two inequalities above, and then this contradicts the equality of (4).

(c) Combine inequalities (8) and (9), and since \( \Re[\tau^H f(\cdot) + \mu^H g(\cdot)] \) is strictly \( \Theta \)-pseudoconvex at \( u \in Q \), then we get the same inequality (12), which contradicts the equality (4).

Therefore, the result of theorem is proved.

\( \square \)

**Theorem 4. (Strong Duality)**

Let \( T \subset \mathbb{C}^p \) is a pointed, closed and convex cone. Suppose that \( z_0 \) is a minimal efficient solution of (P) with respect to \( T \), and the problem (P) satisfies the constraint qualification at \( z_0 \). Then there exists \( (\tau, z_0, \mu, \nu, \gamma) \) a feasible solution of the dual problem (D). Moreover, if the hypotheses of Theorem 3 are fulfilled, then \( (\tau, z_0, \mu, \nu, \gamma) \) is also an optimal solution of (D) with respect to \( T \), and the two problems (P) and (D) have the same optimal values.

**Proof.** Let \( z_0 = (z_0, \nu) \in Q \) is a minimal efficient solution of problem (P) with optimal value \( \gamma \), and take \( v = z_0 - z_0 = 0 \). By using Theorem 1 (Necessary optimality conditions), there exist \( \tau \in T^* \subset \mathbb{C}^p \) and \( \mu \in S^* \subset \mathbb{C}^q \) such that

\[
\tau^T \nabla z f(z_0) + \tau^H \nabla z f(z_0) + \mu^T \nabla z g(z_0) + \mu^H \nabla z g(z_0) = 0,
\]

then conditions (4) and (6) of dual problem (D) are hold. Because \( \gamma \) is the optimal value of problem (P), that is \( \gamma = \min f(z) = f(z_0) \). It implies that \( \Re(f(z_0) - \gamma, \tau) = 0 \), the condition (5) of problem (D) holds. Hence, \( (\tau, z_0, \mu, \nu, \gamma) \) is a feasible solution of the dual problem (D). From Theorem 3, the optimality of the feasible solution \( (\tau, z_0, \mu, \nu, \gamma) \) for (D) reduces to be the optimal value of (D). Indeed, if there exists a feasible solution \( (\tau', \mu, \nu, \gamma') \) of (D) such that \( \gamma' - \gamma \in T \setminus \{0\} \). Since \( \gamma = f(z_0) \) is the optimal value of problem (P), we obtain

\[
\gamma' - f(z_0) \in T \setminus \{0\},
\]

which contradicts to Theorem 3. \( \square \)

**Theorem 5. (Strictly Converse Duality)**

Let \( T \subset \mathbb{C}^p \) is a pointed, closed and convex cone. Suppose that \( \tilde{z} \) and \( (\tau, \tilde{u}, \mu, \nu, \gamma) \) are optimal efficient solutions of (P) and (D) with respect to \( T \), respectively, and assume that the assumptions of Theorem 4 are fulfilled. Meanwhile, if \( \Re[\tau^H f(\cdot)] \) is strictly \( \Theta \)-pseudobonvex at \( \tilde{u} \in Q \) and \( \Re[\mu^H g(\cdot)] \) is \( \Theta \)-quasibonvex at \( \tilde{u} \in Q \), then \( \tilde{z} = \hat{u} \), and the problems of (P) and (D) with the same optimal values.

**Proof.** We assume that \( \tilde{z} \neq \hat{u} \). Since \( \tilde{z} \) is an optimal efficient solution of (P) with optimal value \( \gamma \), and from Theorem 4, then

\[
\gamma = \min f(\tilde{z}) = (f_1(\tilde{z}), \ldots, f_p(\tilde{z})).
\]
So, we get \( \text{Re}(f(\hat{z}) - \gamma, \tau) = 0 \) for nonzero \( \tau \in T^* \). By condition (5) and the above inequality,

\[
\text{Re}(f(\hat{z}) - \gamma, \tau) = 0 \leq \text{Re}(f(\hat{u}) - \gamma, \tau) - \frac{1}{2} \text{Re} \left< v, v^H F_1^{(2)}(\hat{u}, \tau) + v^T F_2^{(2)}(\hat{u}, \tau) \right>.
\]

That is,

\[
\text{Re}(f(\hat{z}) - f(\hat{u}), \tau) + \frac{1}{2} \text{Re} \left< v, v^H F_1^{(2)}(\hat{u}, \tau) + v^T F_2^{(2)}(\hat{u}, \tau) \right> \leq 0. \tag{13}
\]

Using the feasibility of \( \hat{z} \) of (P) with \( \mu \in S^* \), and inequality (6),

\[
\text{Re}[\mu^H g(\hat{z})] \leq 0 \leq \text{Re}[\mu^H g(\hat{u})] - \frac{1}{2} \text{Re} \left< v, v^H G_1^{(2)}(u, \mu) + v^T G_2^{(2)}(u, \mu) \right>.
\]

Then

\[
\text{Re}[\mu^H g(\hat{z})] - \mu^H g(\hat{u})] + \frac{1}{2} \text{Re} \left< v, v^H G_1^{(2)}(u, \mu) + v^T G_2^{(2)}(u, \mu) \right] \leq 0. \tag{14}
\]

If \( \text{Re}[\tau^H f(\cdot)] \) is strictly \( \Theta \)-pseudobonvex at \( \hat{u} \in Q \) and by inequality (13), there is a mapping \( \Theta: \mathbb{C}^{2n} \times \mathbb{C}^{2n} \to \mathbb{C} \) such that

\[
\text{Re} \left\{ \langle \nabla \tau^H f(u) + (v, r)^T \nabla^2 \tau^H f(u) \rangle \Theta(z, u) \right\} < 0. \tag{15}
\]

If \( \text{Re}[\mu^H g(\cdot)] \) is \( \Theta \)-quasibonvex at \( \hat{u} \in Q \) and by inequality (14), there is a mapping \( \Theta: \mathbb{C}^{2n} \times \mathbb{C}^{2n} \to \mathbb{C} \) such that

\[
\text{Re} \left\{ \langle \nabla \mu^H g(u) + (v, r)^T \nabla^2 \mu^H g(u) \rangle \Theta(z, u) \right\} \leq 0. \tag{16}
\]

By considering inequalities (15) and (16), we could obtain the following inequality:

\[
[F^{(1)}(\hat{u}, \tau) + G^{(1)}(\hat{u}, \mu)] + \nu^H [F_1^{(2)}(\hat{u}, \tau) + G_1^{(2)}(\hat{u}, \mu)] + \nu^T F_2^{(2)}(\hat{u}, \tau) + G_2^{(2)}(\hat{u}, \mu) \neq 0,
\]

which contradicts the equality (4). This completed the proof. \( \Box \)

5. Conclusions

In this paper, we state the necessary and sufficient optimality conditions of (P), establish the second-ordered parameter dual model (D) with respect to problem (P), and discuss their duality theorems.

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References