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Unfolding a Hidden Lagrangian Structure of a Class of Evolution Equations

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Abstract: It is shown that a simple modification of the standard Lagrangian underlying the dynamics of Newtonian lattices enables one to infer the hidden Lagrangian structure of certain classes of first order in time evolution equations which lack the conventional Lagrangian structure. Implication to other setups is outlined and exemplified.

Keywords: inverse variational problem; Lagrangian; non-linear dispersion PDE

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1. The Problem

Unlike the Roman Deity Janus enshrined with two equal faces, in the calculus of variations most of the attention is given to the direct venue which emerges naturally when a scientific problem is ab initio introduced via the variational principle. However, in many scientific endeavors, in particular when the underlying problem is partially phenomenological, models are derived on the level of differential equations, thus raising a natural quest for an underlying Lagrangian structure which apart from "scientific beauty" would carry many advantages, with conservation laws being one of them. The 'upstream' path from Equations to the underlying variational structure is known as an inverse problem of calculus of variations. Unfortunately, our ability to tackle the inverse problem and thus deduce the underlying Lagrangian structure from a given dynamical system is far more modest in spite of the long-standing attempts, c.f., [1–5], which date back to Helmholtz, as those efforts, rather than provide us with a definite algorithm to construct the Lagrangian, provide us the conditions for its existence or statements of equivalence rather than with a specific 'how' (notably, there is a far richer literature on Hamiltonian structures than on Lagrangians, though a more balanced approach may be found in texts on dispersive waves. c.f., [6–8]).

In the present communication, we unfold the algorithm to determine the underlying Lagrangian of a particular dynamical system as it relates to the $K(n,n)$ equations

$$K(n,m) \quad u_t = (u^m)_x + (u^n)_{xxx}, \quad 1 < n, m, \tag{1}$$

and a variety of their extensions. Equation (1) were introduced by us some time ago, refs. [9,10] and for $1 < n$ beget the compaction, solitary wave with a finite span. In the particular case $m = n$, its compactons take a simple trigonometric form. For instance, when $n = 2$

$$u = \frac{4\lambda}{3}\cos^2\left(\frac{x+\lambda t}{4}\right)H(2\pi - |x+\lambda t|), \tag{2}$$

with H being the Heaviside function. However, besides compactons, the $K(m,n)$ equations had many other fascinating features; at the time the model was introduced, Lagrangian structure did not seem to be one of them. The apparent lack of a Lagrangian structure was the main critique directed at the $K(m,n)$ models by Cooper et al. [11], who introduced an

alternative, compacton-supporting model derived from an ab initio assumed Lagrangian. In what follows, for the $m = n$ and $m = n + 1$ cases we shall construct the seemingly 'missing' Lagrangian and thus, at least in part, refute their critique, but first let us note that to account for other physically relevant cases [12], the $K(m, n)$ model was extended to the $C(m; a, b)$ setup [10]

$$C(m; a, b) \quad u_t = (u^m)_x + \left[u^a (u^b)_{xx}\right]_x, \quad \textit{and} \quad n \doteq a + b, \tag{3}$$

which conserves

$$I_1 = \int u dx \quad \textit{and} \quad I_\omega = \int u^\omega dx, \tag{4}$$

where $\omega \doteq b + 1 - a$. For $a = 0$, Equation (3) reduces to the $K(m, n)$ setup and with model in [11] being a particular $b = a + 1$ case.

As shown in [10], for $\omega = 2$, i.e., $b = a + 1$, the $C(m; a, a + 1)$ equations follow from a conventional Lagrangian which in terms of $u = \psi_x$ reads

$$\mathcal{L} = \int dt \int \left\{ \frac{1}{2} \psi_t \psi_x + \frac{{\psi_x}^{m+1}}{m+1} - \frac{1}{2} \psi_x^{n-1} \psi_{xx}^2 \right\} dx, \tag{5}$$

adding to the conservation roster the conservation of energy. However, since in the $K(m, n)$ equations $a = 0$, in this setup only the semi-linear, Korteveg–deVries-like sub-cases are endowed with the Lagrangian structure (5).

Yet, insofar as the $m = n$ and $m = n + 1$ cases are concerned, i.e., for the $K(n, n)$ and $K(n + 1, n)$ cases, the claim in [11] was too sweeping as it *refers only to the conventional Lagrangian structure (5). We have recently demonstrated* [13], that both setups *admit a Lagrangian structure* which, however, is 'hidden from sight' and, as we shall see shortly, quite different from the conventional one (5). Consequently, the two conservation laws in (4) were appended with an additional, non-local, conservation law.

As we shall see shortly, in both $K(n, n)$ and $K(n + 1, n)$ setups, the derivation of the non-standard Lagrangian was based on non-obvious steps/tricks. The present communication aims to re-derive the underlying Lagrangian of the $K(n, n)$ setup via a more algorithmic approach, which exploits a certain formal affinity between the $K(n, n)$ equations and a dynamical system describing the motion of a Newtonian mass-spring chain which is not only easily deduced from a Lagrangian setup, but is often ab initio formulated via its Lagrangian. Although tying two very different setups may seem at first to be yet another ad hoc trick, as we shall see the idea may be easily extended to other dynamical systems, and thus has potentially a much wider scope. To avoid misunderstanding, we note that we assume the considered problems to be stated on the whole line with solutions vanishing at infinity and thus satisfying the natural boundary conditions implied by the variational derivation.

Let us first briefly summarize the derivation in [13]; see also [10]. To this end, the $K(n, n)$ equations are rewritten as a Hamiltonian system

$$u_t = \partial_x L^2 \frac{\delta}{\delta u} I_\omega \quad \textit{where} \quad L^2 = 1 + \frac{\partial^2}{\partial x^2}. \tag{6}$$

This step is deductive. The maneuver which leads to the breakthrough is to introduce a new variable v via

$$u = L[v], \tag{7}$$

with L understood in a pseudo-differential sense. In terms of L, we have

$$v_t = \partial_x \frac{\delta}{\delta v} \int \frac{1}{1+n} \left(L[v]\right)^{1+n} dx = \partial_x L \left(L[v]\right)^n. \tag{8}$$

The next, final step consists of introducing a 'potential' variable ψ [7–9], where $v = \psi_x$ that casts Equation (8) into a form which, as per ref. [4], is akin to the Lagrangian

$$\mathcal{L} = \int dt \int \Big\{ \frac{1}{2} \psi_t \psi_x - \frac{1}{n+1} \big(L[\psi_x] \big)^{n+1} \Big\} dx \tag{9}$$

of what, prior to their map, were $K(n, n,)$ equations.

Two remarks are in order:

(1) As noted, in the preceding discussion we have tacitly assumed that our system is given on a line with all data vanishing at infinity, thus the natural boundary conditions assumed tacitly in the variational derivation hold trivially. If non-trivial boundary conditions are at play, the issue may become more evolved. However, recalling the very recent work of Olver [14] who further developed the old observation [1–5] that one may modify the variational problem without altering the corresponding Euler–Lagrange equations by adding a null Lagrangian to the integrand, we note that since this modification changes the associated natural boundary conditions, it enables one to enlarge the range of boundary value problems akin to variational techniques. Thus, the core difficulty remains to find the underlying Lagrangian, whereas its extension to handle non-natural boundary conditions may be delegated to the unfolding of a proper null Lagrangian.

(2) Note that the last step of introducing a potential in (5) is well known and apart from its more recent use in dispersive systems [6–8], it could be found in the classical textbooks in electrodynamics., c.f., [15], or continuum [16,17].

With a Lagrangian and its invariant properties at hand, in addition to the conservation of mass and energy, we now also have conservation of the momentum $\int v^2 dx$, which in terms of the original variables, turns into the conservation of

$$I_2 = \int u L^{-2}[u] dx.$$

It is easily seen that though the presented approach was applied to a very particular problem it may be quite naturally extended. Thus, for instance, one may consider

$$u_t = \partial_x L_*^2 (F(u)), \tag{10}$$

where L_*^2 is any 'reasonable' linear operator and $F(u)$ any nice function [4]. For instance, in Ref. [18] addressing synchronisation of oscillators $F(u) = \cos u$ was assumed. Using the previous notation, in terms of v we have

$$v_t = \partial_x L_* \big(F(L_* v) \big). \tag{11}$$

In terms of ψ, $v = \psi_x$, the underlying Lagrangian reads,

$$\mathcal{L} = \int dt \int \{ \frac{1}{2} \psi_t \psi_x - G\big(L_*[\psi_x] \big) \} dx, \quad \textit{and} \quad F = G'. \tag{12}$$

As another extension, consider the *second order in time system* (c.f., Equation (27))

$$u_{tt} = \partial_x^2 L_*^2 (F(u)). \tag{13}$$

In terms of ψ, $v = \psi_x$, its Lagrangian reads,

$$\mathcal{L} = \int dt \int \{ \frac{1}{2} \psi_t^2 - G\big(L_*[\psi_x] \big) \} dx, \tag{14}$$

and $F = G'$.

In the next section, we take a detour needed to tie our problem with the obvious Lagrangian structure of a Newtonian mass-spring lattice which will render a more natural path toward the deduction of a Lagrangian structure of the $K(n, n)$ equations. Yet another take on the lattice–Lagrangian relations is briefly discussed in the Appendix A.

2. The Newtonian Chain

We start with the Hamiltonian of an unrelated mass-particle chain [19,20],

$$\mathcal{H} = \sum_{N}^{+N} \Big\{ \frac{m}{2} (\dot{y}_n)^2 + P\big(\frac{y_{n+1} - y_n}{h}\big) \Big\}. \tag{15}$$

Let $y_n = y(t, nh)$, $\ell = h/2$, $m = \rho h$, $\rho \downarrow 1$, and $D_x \doteq \ell \partial_x$, then expansion of the potential P yields

$$P\big(\frac{y_{n+1} - y_n}{h}\big) = P(y_x) - \frac{h^2}{24}(y_{xx})^2 P''(y_x) + \mathcal{O}(h^4).$$

To this order of expansion and the standard discrete-continuum association, the ab initio available Lagrangian density becomes

$$\mathcal{L} = \int\int \Big\{ \frac{1}{2}(y_t)^2 - P(y_x) + \frac{h^2}{24}(y_{xx})^2 P''(y_x) \Big\} dtdx, \tag{16}$$

with its equations of motion

$$y_{tt} = \frac{\partial}{\partial x}\Big[P'(y_x) + \frac{h^2}{12}\sqrt{P''(y_x)}\frac{\partial}{\partial x}\big(\sqrt{P''(y_x)} y_{xx}\big)\Big]. \tag{17}$$

Thus, if $P(s) = s^4/4$ and $u = y_x$, then after one differentiation the resulting PDE reads

$$u_{tt} = \big[u^3 + \frac{h^2}{8} u (u^2)_{xx}\big]_{xx}. \tag{18}$$

However, we shall also need an alternative description. Let

$$M(\ell) \doteq \frac{2\ell\partial_x}{e^{\ell\partial_x} - e^{-\ell\partial_x}}, \quad \mathcal{L}_{\mathcal{D}} \doteq M^{-1}, \tag{19}$$

and v a new variable defined *via*

$$y = M(\ell)[v] \quad or \;\; v = \mathcal{L}_{\mathcal{D}}[y], \tag{20}$$

then in terms of v, the Hamiltonian (13) reads

$$\mathcal{H} = \int\int \Big\{ \frac{1}{2}(M v_t)^2 + P(v_x) \Big\} dtdx, \tag{21}$$

which yields

$$M^2 v_{tt} = \big[P'(v_x)\big]_x. \tag{22}$$

Acting on (20) with ${\mathcal{L}_{\mathcal{D}}}^2$ and ∂_x we obtain

$$u_{tt} = \mathcal{L}_D^2 \big[P'(u)\big]_{xx} \quad where \;\; u = v_x. \tag{23}$$

Actually, without invoking v, we may use y ab initio and rewrite the Hamiltonian (13) in terms of y

$$\mathcal{H} = \int\int \Big\{ \frac{1}{2}(y_t)^2 + P(\mathcal{L}_{\mathcal{D}} y_x) \Big\} dtdx, \tag{24}$$

with its Lagrangian density

$$\mathcal{L} = \int\int \Big\{ \frac{1}{2}(y_t)^2 - P(\mathcal{L}_{\mathcal{D}} y_x) \Big\} dtdx, \tag{25}$$

which yields

$$y_{tt} = \Big[\mathcal{L}_{\mathcal{D}} P'(\mathcal{L}_{\mathcal{D}} y_x)\Big]_x. \tag{26}$$

Of course, since $u = v_x = \mathcal{L}_{\mathcal{D}} y_x$, acting with $\mathcal{L}_{\mathcal{D}}\partial x$, turns Equation (24) into (21).

Note that if, say, $P'(u) = u^3$ and $\mathcal{L}_{\mathcal{D}} = 1 + \frac{h^2}{12}\partial_x^2$, obtained after carrying the expansion of $\mathcal{L}_{\mathcal{D}}$ to fourth order, Equation (21) turns into a second order in time variant of the $K(n,n)$ equation

$$u_{tt} = \left[u^n + \frac{h^2}{12}(u^n)_{xx}\right]_{xx}. \tag{27}$$

With its Lagrangian structure given via (14). Both Equations (18) and (27) and their underlying Lagrangians provide a quasi-continuum rendition of the same lattice, with expansions centering around different nodal locations.

Bridging between the Two Systems

The Newtonian system is second order in time. To turn it into a structurally analogous, but of first order in time system, we modify the kinetic part in (23)

$$y_t^2 \to y_t y_x \tag{28}$$

with the resulting Lagrangian becoming

$$\mathcal{L} = \int\int \left\{\frac{1}{2} y_t y_x - P(\mathcal{L}_{\mathcal{D}} y_x)\right\} dt dx. \tag{29}$$

Identifying P with G and ψ with y, begets a Lagrangian form identical to (12), with a related dynamical system which is first order in time. Setting $u = \mathcal{L}_{\mathcal{D}} y_x$ yields Equation (11). In particular, $P(s) = \frac{1}{n+1}s^n$ yields the $K(n,n)$ setup with the resulting sought-after Lagrangian structure.

Finally, consider again the direct expansion of the lattice and its underlying standard Lagrangian (14). Assuming again that $P(s) = \frac{1}{n+1}s^{n+1}$, $m = n$ and repeating the $y_t^2 \to y_t y_x$ association, we recover module normalizable coefficients, the Lagrangian (5) and the underlying Equation (3); $C(m; a, a+1)$, with $2a = n - 1$.

Thus, as with Equations (21) and (25), with each of the expansions being based on a different nodal location on the lattice, we obtain a different equation of motion and a different corresponding Lagrangian: one which is conventional and readily available, and the other hidden and subject of the present exploration.

3. Closing Comments

The present note is a modest take on the inverse problem of the calculus of variations to seek from a particular partial differential system its variational antecedent. At the present state of affairs, the statements regarding the underlying Lagrangian structure of a given dynamical system are affirmative rather than constructive. Every success is ad hoc and relies on certain intuitive steps which have to be inferred independently for each problem.

Exploiting a formal structural affinity between two different physico-mathematical entities wherein one, being second order in time, is endowed with a natural Lagrangian structure, the Lagrangian of the other is constructed via a natural modification. What saves the presented approach from being yet another ad hoc spiel is the fact that the approach carries to a wider set of problems wherein the first order in time dynamical system lacks an obvious underlying Lagrangian, but a better chance may await us looking at structurally similar second-order equations in time endowed with an obvious Lagrangian structure.

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Appendix A

Another take on the discussed problem is provided by the following first-order system [8]

$$\frac{d}{dt}u_n = \sum_{-N}^{+N} \frac{f(u_{n+1}) - f(u_{n-1})}{2\ell}. \tag{A1}$$

Let $D_x = \ell\partial_x$ and $\mathcal{L}_+^2 \doteq \mathcal{L}_D$, then

$$u_t = \partial_x \mathcal{L}_+^2 f(u). \tag{A2}$$

If $u = \mathcal{L}_+ v$, then

$$v_t = \partial_x \mathcal{L}_+ f(\mathcal{L}_+ v). \tag{A3}$$

As before, let $v = \psi_x$, then the desired Lagrangian is

$$\mathcal{L} = \int\int \left\{ \frac{1}{2}\psi_t\psi_x - Q(\mathcal{L}_+\psi_x) \right\} dtdx, \tag{A4}$$

where $f(u) = Q'(u)$.

We may approximate $\mathcal{L}_+$ in two ways. In the standard approach we utilize the first two terms $\mathcal{L}_+^2 \simeq \mathcal{L}_a^2 \doteq 1 + \frac{D_x^2}{6}$ of the exact expression

$$\mathcal{L}_+^2 = 1 + \frac{D_x^2}{3!} + \frac{D_x^4}{5!} + \dots \tag{A5}$$

which renders $\mathcal{L}_a$ a pseudo differential operator. But if instead we adopt

$$\mathcal{L}_+^2 \simeq \mathcal{L}_b^2 \doteq (1 + \frac{D_x^2}{12})^2, \tag{A6}$$

with $\mathcal{L}_b = 1 + \frac{D_x^2}{12}$ being a nice differential operator, and within small error covering the first three terms in (A5).

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