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Interval-Valued General Residuated Lattice-Ordered Groupoids and Expanded Triangle Algebras

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Abstract: As an extension of interval-valued pseudo t-norms, interval-valued pseudo-overlap functions (IPOFs) play a vital role in solving interval-valued multi-attribute decision making problems. However, their corresponding interval-valued algebraic structure has not been studied yet. On the other hand, with the development of non-commutative (non-associative) fuzzy logic, the study of residuated lattice theory is gradually deepening. Due to the conditions of operators being weakened, the algebraic structures are gradually expanding. Therefore, on the basis of interval-valued residuated lattice theory, we generalize and research the related contents of interval-valued general, residuated, lattice-ordered groupoids. In this paper, the concept of interval-valued, general, residuated, lattice-ordered groupoids is given, and some examples are presented to illustrate the relevance of IPOFs to them. Then, in order to further study them, we propose the notions of expanded, interval-valued, general, residuated lattice-ordered groupoids and expanded triangle algebras, and explain that there is one-to-one correspondence between them through a specific proposition. Some of their properties are also analyzed. Lastly, we show the definitions of the filters on the expanded triangle algebras, and investigate the congruence and quotient structure through them.

Keywords: fuzzy logic; interval-valued pseudo-overlap function; interval-valued general residuated lattice-ordered groupoid; expanded triangle algebra; filter

MSC: 03B52; 03G10; 06B05

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1. Introduction

Zadeh put forward the idea of interval-valued fuzzy sets (see [1]), which extends the fuzzy sets from interval [0, 1] to the set of all intervals on [0, 1]. This makes an interval value take the place of an exact value, thereby reducing the restrictions in applicability. With the deepening of research, it is also proved that interval-valued fuzzy sets do play a significant role in decision making [2,3], fuzzy rule-based classification systems [4,5], etc. Later, Deschrijver et al. discussed the related theory of interval-valued residuated lattices (IVRLs) in [6]. Residuated lattices were first proposed by scholars in [7], which is about the study of ideal lattices of rings. After that, other scholars considered them in fuzzy logic (see [8]) and extended them to the non-commutative or non-associative cases, thereby studying more general residuated lattice structures of fuzzy logic, such as pseudo-BL-algebras [9] and non-associative residuated lattices [10]. In addition, Yong Chan Kim [11] performed research on generalized triangle algebras based on non-commutative generalized residuated lattices and their properties. However, there is no relevant conclusion on the extension of other algebraic structures on interval values.

With the development of fuzzy logic, some aggregation operators, such as t-norms [12], pseudo-t-norms [9], overlap functions [13] and pseudo-overlap functions [14], have also been extended to interval-valued functions for research, and some results are obtained (see [15–19]). For example, as an extension of interval-valued t-norms, IPOFs no longer require commutativity, associativity and unit element, and have good application effect in interval-valued multi-attribute decision making problems. In particular, some scholars introduced the logical structures corresponding to interval-valued t-norms, and gave their equivalent characterization (see [6,20,21]). However, the algebraic structure formed by other aggregation operators on interval-valued is still not involved.

Based on the above reasons, we expand IVRLs to a more general form, study interval-valued general, residuated, lattice-ordered groupoids, show that they can be generated by interval-valued pseudo-overlap functions that meet certain conditions, and give their equivalent descriptions. This enrichs the content of interval-valued fuzzy logic.

In the article, we discuss bounded residuated ordered groupoids (see [22]) with lattice order on interval-valued functions; that is, we study interval-valued general, residuated, lattice-ordered groupoids. First, we explain that they can be generated by some representable IPOFs. Then, on the basis of further expansion, we discuss their properties and equivalent characterization, and give related propositions. Finally, we analyze the properties of the expanded triangle algebras and study their filters and quotient structures.

The contents of article are arranged as follows. Section 2 summarizes of basic knowledge involved in the article, mainly including the definitions of some functions and algebraic structures. We discuss the contents of interval-valued general, residuated, lattice-ordered groupoids in Section 3, including their definitions, constructions and extensions. At the same time, we also describe the concept of expanded triangle algebras. Through a specific proposition, we show that there is one-to-one correspondence between them and interval-valued, general, residuated lattice-ordered groupoids, and then illustrate that every expanded interval-valued general, residuated, lattice-ordered groupoid is an expanded triangle algebra. Finally, some examples are given, and their properties are analyzed. In Section 4, we focus on the filters for the expanded triangle algebras. After giving the definitions and properties, we investigate the congruence relationship and quotient structure on the expanded triangle algebras through them, and show an example. At the end of the article, present conclusions and references.

2. Preliminaries

Several concepts involved in the following text are reviewed in this part. We first present the definitions of (interval-valued) pseudo-overlap functions and their (interval-valued) residuated implications, which are related to the algebraic structure we will discuss later.

Definition 1 ([14])**.** *Given an operation PO:* $[0,1]^2 \to [0,1]$*, PO is known as a pseudo-overlap function (briefly POF) if it meets the requirements as follows:*
$(PO1)$ *The first boundary condition, i.e., the value of PO, is 0 when and only when the product of the values of two variables is 0;*
$(PO2)$ *The second boundary condition, i.e., the value of PO, is 1 when and only when the product of the values of two variables is 1;*
$(PO3)$ *Monotonicity, i.e., PO, is monotonically increasing;*
$(PO4)$ *Continuity, i.e., PO, is continuous with respect to two variables.*

Definition 2 ([14])**.** *Given a POF O, two operators* $I_O^{(1)}$ *and* $I_O^{(2)}$*, defined as follows, are called residuated implications (RI for short) induced by O:*

$$I_O^{(1)}(p,q) = sup\{r|O(p,r) \le q\}$$
$$I_O^{(2)}(p,q) = sup\{r|O(r,p) \le q\}$$

for arbitrary $p,q,r \in [0,1]$.

Definition 3 ([18]). *Given an interval-valued function IPO on* $IV([0,1]) = \{[x,y]|0 \leq x \leq y \leq 1\}$, *it is called an interval-valued pseudo-overlap function (IPOF for short) if IPO meets the following requirements:*
$(IPO1)$ *The value of IPO is* $[0,0]$ *when and only when the product of the values of two variables is* $[0,0]$;
$(IPO2)$ *The value of IPO is* $[1,1]$ *when and only when the product of the values of two variables is* $[1,1]$;
$(IPO3)$ *IPO is increasing with respect to two variables;*
$(IPO4)$ *IPO is Moore continuous, for any* $([x_1,x_2],[x_3,x_4])$, $([y_1,y_2],[y_3,y_4]) \in IV([0,1])^2$, $\epsilon > 0$, $\exists\delta > 0$ *such that* $\sqrt{(max\{|x_1-y_1|,|x_2-y_2|\})^2+(max\{|x_3-y_3|,|x_4-y_4|\})^2} < \delta \Rightarrow max\{|z_1-z_3|,|z_2-z_4|\}$, *where* $IPO([x_1,x_2],[x_3,x_4]) = [z_1,z_2]$, $IPO([y_1,y_2],[y_3,y_4]) = [z_3,z_4]$.

Definition 4 ([18]). *Given an IPOF O, the interval-valued operators* $IR_O^{(1)}$ *and* $IR_O^{(2)}$, *defined as below, are said to be interval-valued residuated implications (IVRIs for short) induced by O:*

$$IR_O^{(1)}([a_1,b_1],[a_2,b_2]) = sup\{[z_1,z_2] \in IV([0,1])|O([a_1,b_1],[z_1,z_2]) \leq [a_2,b_2]\}$$
$$IR_O^{(2)}([a_1,b_1],[a_2,b_2]) = sup\{[z_1,z_2] \in IV([0,1])|O([z_1,z_2],[a_1,b_1]) \leq [a_2,b_2]\}$$

for any $[a_1,b_1]$, $[a_2,b_2]$, $[z_1,z_2] \in IV([0,1])$.

In [18], a special class of IPOFs, called representable, is also described and defined as follows.

Definition 5 ([18]). *Given an IPOF O, it is said to be representable when there exist POFs* O_1 *and* O_2 *such that* $O = \widetilde{O_1O_2}$; *that is,* $O([x,y],[x',y']) = [O_1(x,x'),O_2(y,y')]$.

At the same time, the IVRIs induced by representable IPOFs can also be expressed in another form.

Theorem 1. *Given a representable IPOF O,* $O([x,y],[x',y']) = [O_1(x,x'),O_2(y,y')]$ *and* $I_{O_1}^{(1)}$, $I_{O_1}^{(2)}$ *are RIs induced from* O_1, $I_{O_2}^{(1)}$, $I_{O_2}^{(2)}$ *are RIs induced from* O_2; *then, IVRIs* $IR_O^{(1)}$, $IR_O^{(2)}$ *induced from O are defined as:*

$$IR_O^{(1)}([x,y],[x',y']) = [I_{O_1}^{(1)}(x,x') \wedge I_{O_2}^{(1)}(y,y'), I_{O_2}^{(1)}(y,y')],$$
$$IR_O^{(2)}([x,y],[x',y']) = [I_{O_1}^{(2)}(x,x') \wedge I_{O_2}^{(2)}(y,y'), I_{O_2}^{(2)}(y,y')].$$

Van Gasse et al. [20,21] defined a new type of algebraic structure based on interval-valued fuzzy logic, namely, triangle algebras, to characterize interval-valued residuated lattices. The related notions are as follows.

Definition 6 ([7,8]). *Given a structure* $(L,\wedge,\vee,*,\rightarrow,0,1)$, *it is known as a residuated lattice (RL for short) when it meets requirements as below:*
$(RL1)$ $(L,\wedge,\vee,0,1)$ *is a bounded lattice and has lower bound 0 and upper bound 1;*
$(RL2)$ $(L,*)$ *is a monoid satisfying commutativity;*
$(RL3)$ $(*,\rightarrow)$ *meets the residuation principle, i.e.,* $x*y \leq z \Leftrightarrow y \leq x \rightarrow z$ *when taking arbitrary* $x,y,z \in L$.

Definition 7 ([21]). *Given the lattice* $L = (L,\wedge,\vee)$, *a structure* $T(L)$ *defined by* $T(L) = (In(L),\sqcap,\sqcup)$ *is called triangularization of L, where* $In(L) = \{[m,n]|(m,n) \in L^2, m \leq n\}$ *and* $[m_1,n_1] \sqcap [m_2,n_2] = [m_1 \wedge m_2, n_1 \wedge n_2]$, $[m_1,n_1] \sqcup [m_2,n_2] = [m_1 \vee m_2, n_1 \vee n_2]$.

The order relation on In(L) is defined as: $[x_1, y_1] \leq [x_2, y_2] \Leftrightarrow x_1 \leq x_2$ and $y_1 \leq y_2$. Additionally, the diagonal set of $T(L)$ denoted as $D_L = \{[c,d] | c, d \in L \text{ and } c = d\}$.

Definition 8 ([21]). *Given a lattice L with boundary and its triangularization $T(L)$, the RL $(In(L), \bigcap, \bigcup, \odot, \rightarrow_{\odot}, [0,0], [1,1])$ on $T(L)$ is called an interval-valued RL when diagonal set is closed with respect to operations $\odot$ and $\rightarrow_{\odot}$.*

Definition 9 ([20,21]). *Given an algebra $(A, \sqcap, \sqcup, *, \rightarrow, \alpha, \beta, 0, u, 1)$, where $(A, \sqcap, \sqcup, *, \rightarrow, 0, 1)$ is a RL, α and β are univariate operations, and u is a constant; it is known as a triangle algebra if it meets the requirements as below:*
$(T1)$ $\alpha c \leq c$, $c \leq \beta c$;
$(T2)$ $\alpha c \leq \alpha\alpha c$, $\beta\beta c \leq \beta c$;
$(T3)$ $\alpha(c \sqcap d) = \alpha c \sqcap \alpha d$, $\beta(c \sqcap d) = \beta c \sqcap \beta d$;
$(T4)$ $\alpha(c \sqcup d) = \alpha c \sqcup \alpha d$, $\beta(c \sqcup d) = \beta c \sqcup \beta d$;
$(T5)$ $\alpha u = 0$, $\beta u = 1$;
$(T6)$ $\alpha\beta c = \beta c$, $\beta\alpha c = \alpha c$;
$(T7)$ $\alpha(c \rightarrow d) \leq \alpha c \rightarrow \alpha d$;
$(T8)$ $(\alpha c \Leftrightarrow \alpha d) * (\beta c \Leftrightarrow \beta d) \leq c \Leftrightarrow d$;
$(T9)$ $\alpha c \rightarrow \alpha d \leq \alpha(\alpha c \rightarrow \alpha d)$
where $c \Leftrightarrow d = (c \rightarrow d) \sqcap (d \rightarrow c)$, $c, d \in A$.

Finally, we show the concept of commutative, residuated lattice-ordered groupoids, where conjunctive operators no longer require associativity.

Definition 10 ([10]). *Given a structure $(L, \wedge, \vee, *, \rightarrow, 0, 1)$, it is known as a non-associative RL when it meets conditions as follows:*
$(naRL1)$ *It meets* $(RL1)$;
$(naRL2)$ $(L, *)$ *is a commutative groupoid with identity element 1;*
$(naRL3)$ *It meets the residuation principle.*

3. Interval-Valued, General, Residuated Lattice-Ordered Groupoids

We first introduce the concept of general, residuated, lattice-ordered groupoids. In fact, it is also a residuated ordered groupoid that satisfies lattice order and is bounded, as mentioned in [22].

Definition 11 ([22]). *An algebra $(G, \wedge, \vee, *, \rightarrow, \rightsquigarrow, 0, 1)$ is called a general, residuated, lattice-ordered groupoid (GRLG for short) when meeting representations as below:*
$(GRLG1)$ *It meets* $(RL1)$;
$(GRLG2)$ $(G, *)$ *is a groupoid with the binary operation $*$, where G is a nonempty set;*
$(GRLG3)$ $(*, \rightarrow, \rightsquigarrow)$ *meets the 2-residuation principle for all elements of G:* $x * y \leq z \Leftrightarrow y \leq x \rightarrow z \Leftrightarrow x \leq y \rightsquigarrow z$.

The binary operator $*$ on a GRLG is non-commutative, non-associative and has no unit element in general. In addition, we know that operation $*$ is monotonically increasing, and operations $\rightarrow$ and $\rightsquigarrow$ are monotonically non-increasing concerning the first element and monotonically increasing concerning the second element according to (GRLG3), and the proof is omitted.

Example 1. *Given an algebra $([0,1], min, max, *, \rightarrow, \rightsquigarrow, 0, 1)$, where $x * y = xy^2$, and*

$$x \rightarrow y = \begin{cases} 1, & x \leq y \\ \sqrt{\frac{y}{x}}, & otherwise \end{cases}, x \rightsquigarrow y = \begin{cases} 1, & x^2 \leq y \\ \frac{y}{x^2}, & x^2 > y \end{cases}.$$

Then, it meets $(GRLG1) \sim (GRLG3)$.

In fact, given a bounded lattice $([0,1], min, max, 0, 1)$, if we take any POF as the operator $*$ and its residuated implications as operations $\rightarrow$ and $\rightsquigarrow$, then the algebra $([0,1], min, max, *, \rightarrow, \rightsquigarrow, 0, 1)$ must be a GRLG.

About the triangularization of lattices, we have the following proposition.

Proposition 1. *If $L = (L, \wedge, \vee, *, \rightarrow, \rightsquigarrow, 0, 1)$ is a GRLG, then the GRLG can be generated on $T(L)$.*

Proof. Assume that $L = (L, \wedge, \vee, *, \rightarrow, \rightsquigarrow, 0, 1)$ is a GRLG; then $(*, \rightarrow, \rightsquigarrow)$ meets $x * y \leq z \Leftrightarrow y \leq x \rightarrow z \Leftrightarrow x \leq y \rightsquigarrow z$ for any $x, y, z \in L$. Given a structure $S = (\mathrm{In}(L), \bigcap, \bigcup, \circledast, \rightarrow_{\circledast}, \rightsquigarrow_{\circledast}, [0,0], [1,1])$ on $T(L)$, where $[x,y] \circledast [x',y']$ are denoted by $[x * x', y * y']$, $[x,y] \rightarrow_{\circledast} [x',y']$ are denoted by $[(x \rightarrow x') \wedge (y \rightarrow y'), y \rightarrow y']$ and $[x,y] \rightsquigarrow_{\circledast} [x',y']$ are denoted by $[(x \rightsquigarrow x') \wedge (y \rightsquigarrow y'), y \rightsquigarrow y']$. Then, we verify that S is a GRLG. It is obvious that S is a lattice and $[0,0] \leq [x,y] \leq [1,1]$ for arbitrary $(x,y) \in L^2$. Additionally, $\mathrm{In}(L)$ is nonempty; $\circledast$ is a binary operation on $\mathrm{In}(L)$. As $[x,y] \circledast [x',y'] \leq [w_1, w_2] \Leftrightarrow [x * x', y * y'] \leq [w_1, w_2] \Leftrightarrow x * x' \leq w_1$ and $y * y' \leq w_2 \Leftrightarrow x' \leq x \rightarrow w_1$ and $y' \leq y \rightarrow w_2$ (considering that $x' \leq y'$) $\Leftrightarrow [x',y'] \leq [(x \rightarrow w_1) \wedge (y \rightarrow w_2), y \rightarrow w_2] \Leftrightarrow [x',y'] \leq [x,y] \rightarrow_{\circledast} [w_1, w_2]$, and similarly, $[x,y] \circledast [x',y'] \leq [w_1, w_2] \Leftrightarrow [x,y] \leq [x',y'] \rightsquigarrow_{\circledast} [w_1, w_2]$, $(\circledast, \rightarrow_{\circledast}, \rightsquigarrow_{\circledast})$ meets the 2-residuation principle. Thus, S is a GRLG. $\square$

Then we show the notion of interval-valued general, residuated, lattice-ordered groupoids.

Definition 12. *Given a bounded lattice $L = (L, \wedge, \vee, 0, 1)$ and its triangularization $T(L)$, a GRLG $(In(L), \bigcap, \bigcup, \circledast, \rightarrow_{\circledast}, \rightsquigarrow_{\circledast}, [0,0], [1,1])$ on $T(L)$ is called an interval-valued, general, residuated, lattice-ordered groupoid ($IGRLG$ for short), where a diagonal set has closed under binary operations $\circledast, \rightarrow_{\circledast}$ and $\rightsquigarrow_{\circledast}$.*

Example 2. *Given a lattice $C = ([0,1], min, max)$ and its triangularization $T(C) = (In([0,1]), \bigcap, \bigcup)$. For arbitrary $[x_1, y_1], [x_2, y_2] \in In([0,1])$, we define $[x_1, y_1] \circledast [x_2, y_2] = [O(x_1, x_2), O(y_1, y_2)]$, $[x_1, y_1] \rightarrow_{\circledast} [x_2, y_2] = [I_O^{(1)}(x_1, x_2) \wedge I_O^{(1)}(y_1, y_2), I_O^{(1)}(y_1, y_2)]$ and $[x_1, y_1] \rightsquigarrow_{\circledast} [x_2, y_2] = [I_O^{(2)}(x_1, x_2) \wedge I_O^{(2)}(y_1, y_2), I_O^{(2)}(y_1, y_2)]$, where O is an arbitrary POF; $I_O^{(1)}$ and $I_O^{(2)}$ are two RIs induced from O. Then, $(In(C), \bigcap, \bigcup, \circledast, \rightarrow_{\circledast}, \rightsquigarrow_{\circledast}, [0,0], [1,1])$ is an $IGRLG$.*

Obviously, the binary operations in the above example can be replaced by IPOFs and their interval-valued residuated implications, as shown in the following proposition.

Proposition 2. *Given a bounded lattice $L = (L, \wedge, \vee, 0, 1)$, $T(L) = (In(L), \bigcap, \bigcup)$ is a triangularization of L. Then, the algebra $(In(L), \bigcap, \bigcup, \circledast, \rightarrow_{\circledast}, \rightsquigarrow_{\circledast}, [0,0], [1,1])$ is an $IGRLG$, in which $S \circledast E = IO(S, E)$, $S \rightarrow_{\circledast} E = IR_O^{(1)}(S, E)$, $S \rightsquigarrow_{\circledast} E = IR_O^{(2)}(S, E)$ for arbitrary $S, E \in In(L)$, and IO is a representable $IPOF$ obtained by a POF; $IR_O^{(1)}$ and $IR_O^{(2)}$ are two interval-valued residuated implications induced by IO.*

Proof. We denote algebra $(\mathrm{In}(L), \bigcap, \bigcup, \circledast, \rightarrow_{\circledast}, \rightsquigarrow_{\circledast}, [0,0], [1,1])$ as G, and prove that it is a GRLG. As $[0,0] \leq [x,y] \leq [1,1]$ for every $[x,y] \in \mathrm{In}(L)$, G meets (GRLG1) by definition. It is obvious that G also meets (GRLG2) due to $[x,y] \circledast [x',y'] = IO([x,y],[x',y'])$ for arbitrary $[x,y], [x',y'] \in \mathrm{In}(L)$. Since IO is representable, $IO = \widetilde{OO}$, where O is a POF. We suppose $I_O^{(1)}$, $I_O^{(2)}$ are RIs induced from O. Evidently, $[x,y] \circledast [x',y'] \leq [t_1, t_2] \Leftrightarrow IO([x,y],[x',y']) \leq [t_1, t_2] \Leftrightarrow O(x,x') \leq t_1$ and $O(y,y') \leq t_2 \Leftrightarrow I_O^{(1)}(x, t_1) \geq x'$ and $I_O^{(1)}(y, t_2) \geq y' \geq x' \Leftrightarrow [x',y'] \leq [I_O^{(1)}(x, t_1) \wedge I_O^{(1)}(y, t_2), I_O^{(1)}(y, t_2)] = IR_O^{(1)}([x,y],[t_1,t_2]) = [x,y] \rightarrow_{\circledast} [t_1, t_2]$, and on the other hand, $[x,y] \circledast [x',y'] \leq [t_1, t_2] \Leftrightarrow IO([x,y],[x',y']) \leq [t_1, t_2] \Leftrightarrow O(x,x') \leq t_1$ and $O(y,y') \leq t_2 \Leftrightarrow x \leq I_O^{(2)}(x', t_1)$ and $x \leq y \leq I_O^{(2)}(y', t_2) \Leftrightarrow [x,y] \leq [I_O^{(2)}(x', t_1) \wedge I_O^{(2)}(y', t_2), I_O^{(2)}(y', t_2)] = IR_O^{(2)}([x',y'],[t_1,t_2]) = [x',y'] \rightsquigarrow_{\circledast} [t_1, t_2]$, i.e., $[x,y] \circledast [x',y'] \leq [t_1, t_2] \Leftrightarrow [x',y'] \leq [x,y] \rightarrow_{\circledast} [t_1, t_2] \Leftrightarrow [x,y] \leq [x',y'] \rightsquigarrow_{\circledast} [t_1, t_2]$ for any $[x,y],[x',y'],[t_1,t_2] \in \mathrm{In}(L)$, so G meets (GRLG3). When take arbitrary $[x,x],[x',x'] \in D_L = \{[x,y] | x,y \in L, x=y\}$, $[x,x] \circledast [x',x'] = IO([x,x],[x',x']) = [O(x,x'),O(x,x')] \in D_L$, $[x,x] \rightarrow_{\circledast} [x',x'] = [I_O^{(1)}(x,x') \wedge I_O^{(1)}(x,x'), I_O^{(1)}(x,x')] = [I_O^{(1)}(x,x'), I_O^{(1)}(x,x')] \in D_L$ and $[x,x] \rightsquigarrow_{\circledast} [x',x'] = IR_O^{(2)}([x,x],[x',x']) = [I_O^{(2)}(x,x') \wedge I_O^{(2)}(x,x'), I_O^{(2)}(x,x')] = [I_O^{(2)}(x,x'), I_O^{(2)}(x,x')] \in D_L$; that is, D_L is closed under operators $\circledast$, $\rightarrow_{\circledast}$ and $\rightsquigarrow_{\circledast}$. Thereby, G is an IGRLG by definition. □

Remark 1. *Given a bounded lattice B and its triangularization $T(B) = (In(B), \bigcap, \bigcup)$. When operations $\circledast$, $\rightarrow_{\circledast}$ and $\rightsquigarrow_{\circledast}$ are replaced by the representable IPOF and its IVRIs, respectively, the algebra $(In(B), \bigcap, \bigcup, \circledast, \rightarrow_{\circledast}, \rightsquigarrow_{\circledast}, [0,0], [1,1])$ may not be an IGRLG. A counterexample is as follows.*

Example 3. *Let $L = ([0,1], min, max, 0, 1)$ be a bounded lattice, $T(L) = (In([0,1]), \bigcap, \bigcup)$. We make operator $\circledast$ as the representable IPOF IO, where $IO([a_1,b_1],[a_2,b_2]) = [a_1^2 a_2, b_1 \wedge b_2]$ for arbitrary $[a_1,b_1]$, $[a_2,b_2] \in In([0,1])$ and $\rightarrow_{\circledast}$ and $\rightsquigarrow_{\circledast}$ are defined as interval-valued residuated implications of IO. When we take $[0.1,0.1],[0.2,0.2] \in \{[x,x] | x \in [0,1]\}$, $[0.1,0.1] \circledast [0.2,0.2] = IO([0.1,0.1],[0.2,0.2]) = [0.002,0.1] \notin \{[x,x] | x \in [0,1]\}$; i.e., diagonal set is not closed under operation $\circledast$. Thus, the algebra $(In([0,1]), \bigcap, \bigcup, \circledast, \rightarrow_{\circledast}, \rightsquigarrow_{\circledast}, [0,0], [1,1])$ is not an IGRLG.*

In order to further study the relevant properties of IGRLGs, we introduce the concept of expanded triangle algebras. Next, we show the definition of the expanded IGRLG first of all.

Definition 13. *Given a bounded lattice $G = (G, \wedge, \vee)$ and its triangularization $T(G) = (In(G), \bigcap, \bigcup)$, the algebra $(In(G), \bigcap, \bigcup, \circledast, \rightarrow_{\circledast}, \rightsquigarrow_{\circledast}, [0,0], [1,1])$ is an IGRLG on $T(G)$. A structure $(In(G), \bigcap, \bigcup, \circledast, \rightarrow_{\circledast}, \rightsquigarrow_{\circledast}, l, r, [0,0], [0,1], [1,1])$ is called an expanded IGRLG, where operator l is denoted by $l([x,y]) = [x,x]$, and r is denoted by $r([x,y]) = [y,y]$ for arbitrary $[x,y] \in In(G)$, and $[0,1]$ is a constant.*

A few properties of expanded IGRLGs are uncovered.

Proposition 3. *Given an expanded IGRLG $G = (In(G), \bigcap, \bigcup, \circledast, \rightarrow_{\circledast}, \rightsquigarrow_{\circledast}, l, r, [0,0], [0,1], [1,1])$, and then some requirements are satisfied as follows:*
(E1) $l(l([g,i])) = l([g,i]) \leq [g,i]$, $[g,i] \leq r([g,i]) = r(r([g,i]))$ for arbitrary $[g,i] \in In(G)$;
(E2) $l([g_1,i_1] \bigcap [g_2,i_2]) = l([g_1,i_1]) \bigcap l([g_2,i_2])$, $r([g_1,i_1] \bigcap [g_2,i_2]) = r([g_1,i_1]) \bigcap r([g_2,i_2])$ for arbitrary $[g_1,i_1],[g_2,i_2] \in In(G)$;
(E3) $l([g_1,i_1] \bigcup [g_2,i_2]) = l([g_1,i_1]) \bigcup l([g_2,i_2])$, $r([g_1,i_1] \bigcup [g_2,i_2]) = r([g_1,i_1]) \bigcup r([g_2,i_2])$ for arbitrary $[g_1,i_1],[g_2,i_2] \in In(G)$;
(E4) $l([0,1]) = r([0,0]) = [0,0]$, $l([1,1]) = r([0,1]) = [1,1]$;
(E5) $l(r([g,i])) = r([g,i])$, $r(l([g,i])) = l([g,i])$ for arbitrary $[g,i] \in In(G)$;
(E6) $[g_2,g_2] \leq [g_1,i_1] \rightarrow_{\circledast} [z_1,z_2]$ and $[g_1,g_1] \leq [g_2,i_2] \rightsquigarrow_{\circledast} [z_1,z_2]$ if $[g_1,i_1] \circledast [g_2,i_2] \leq [z_1,z_2]$ for arbitrary $[g_1,i_1],[g_2,i_2],[z_1,z_2] \in In(G)$;
(E7). If $[g_1,i_1] \leq [g_2,i_2] \rightsquigarrow_{\circledast} [z_1,z_2]$ or $[g_2,i_2] \leq [g_1,i_1] \rightarrow_{\circledast} [z_1,z_2]$, then $l([g_1,i_1] \circledast [g_2,i_2]) \leq [z_2,z_2]$ for arbitrary $[g_1,i_1],[g_2,i_2],[z_1,z_2] \in In(G)$;
(E8) $l([g_1,i_1]) \leq l([g_2,i_2])$ and $r([g_1,i_1]) \leq r([g_2,i_2])$ if $[g_1,i_1] \leq [g_2,i_2]$ for arbitrary $[g_1,i_1]$, $[g_2,i_2] \in In(G)$;
(E9) $[g,g] \leq [g,i] \rightarrow_{\circledast} [1,1]$ and $[g,g] \leq [g,i] \rightsquigarrow_{\circledast} [1,1]$ for arbitrary $[g,i] \in In(G)$;
(E10) $[g_2,g_2] \leq [g_1,i_1] \rightarrow_{\circledast} ([g_1,i_1] \circledast [g_2,i_2])$ and $[g_1,g_1] \leq [g_2,i_2] \rightsquigarrow_{\circledast} ([g_1,i_1] \circledast [g_2,i_2])$ for arbitrary $[g_1,i_1],[g_2,i_2] \in In(G)$;
(E11) $[g_1,i_1] \circledast ([g_1,i_1] \rightarrow_{\circledast} [g_2,i_2]) \leq [i_2,i_2]$ and $([g_1,i_1] \rightsquigarrow_{\circledast} [g_2,i_2]) \circledast [g_1,i_1] \leq [i_2,i_2]$ for arbitrary $[g_1,i_1],[g_2,i_2] \in In(G)$;
(E12) $[g_1,g_1] \leq ([g_1,i_1] \rightarrow_{\circledast} [g_2,i_2]) \rightsquigarrow_{\circledast} [g_2,i_2]$ and $[g_1,g_1] \leq ([g_1,i_1] \rightsquigarrow_{\circledast} [g_2,i_2]) \rightarrow_{\circledast} [g_2,i_2]$ for arbitrary $[g_1,i_1],[g_2,i_2] \in In(G)$;
(E13) $l([g_1,i_1]) \circledast l([g_2,i_2]) \leq l([g_1,i_1] \circledast [g_2,i_2])$ if and only if $l([g_1,i_1] \rightarrow_{\circledast} [g_2,i_2]) \leq$

$l([g_1,i_1]) \to_{\circledast} l([g_2,i_2])$ if and only if $l([g_1,i_1] \rightsquigarrow_{\circledast} [g_2,i_2]) \leq l([g_1,i_1]) \rightsquigarrow_{\circledast} l([g_2,i_2])$ for arbitrary $[g_1,i_1],[g_2,i_2] \in In(G)$;
$(E14)$ $[g,i] = l([g,i])$ if and only if $[g,i] = r([g,i])$ for arbitrary $[g,i] \in In(G)$;
$(E15)$ $[g,i] \to_{\circledast} [1,1] = [g,i] \rightsquigarrow_{\circledast} [1,1] = [1,1]$ for arbitrary $[g,i] \in In(G)$;
$(E16)$ $[g_1,g_1] \leq ([g_1,i_1] \to_{\circledast} [g_2,i_2]) \rightsquigarrow_{\circledast} [i_2,i_2]$ and $[g_1,g_1] \leq ([g_1,i_1] \rightsquigarrow_{\circledast} [g_2,i_2]) \to_{\circledast} [i_2,i_2]$ for arbitrary $[g_1,i_1],[g_2,i_2] \in In(G)$.

Proof. (1) As for arbitrary $[g,i] \in \mathrm{In}(G)$, $g \leq i$, then $l([g,i]) = [g,g] \leq [g,i]$ and $r([g,i]) = [i,i] \geq [g,i]$. Naturally, $l(l([g,i])) = l([g,g]) = [g,g] = l([g,i])$, $r(r([g,i])) = r([i,i]) = [i,i] = r([g,i])$.
(2) For arbitrary $[g_1,i_1],[g_2,i_2] \in \mathrm{In}(G)$, $l([g_1,i_1] \bigcap [g_2,i_2]) = l([g_1 \wedge g_2, i_1 \wedge i_2]) = [g_1 \wedge g_2, g_1 \wedge g_2]$, and $l([g_1,i_1]) \bigcap l([g_2,i_2]) = [g_1,g_1] \bigcap [g_2,g_2] = [g_1 \wedge g_2, g_1 \wedge g_2]$, so $l([g_1,i_1] \bigcap [g_2,i_2]) = l([g_1,i_1]) \bigcap l([g_2,i_2])$. Analogously, $r([g_1,i_1] \bigcap [g_2,i_2]) = [i_1 \wedge i_2, i_1 \wedge i_2] = r([g_1,i_1]) \bigcap r([g_2,i_2])$.
(3) Same as above.
(4) By definition, it is obvious.
(5) For arbitrary $[g,i] \in \mathrm{In}(G)$, $l(r([g,i])) = l([i,i]) = [i,i] = r([g,i])$, $r(l([g,i])) = r([g,g]) = [g,g] = l([g,i])$.
(6) For arbitrary $[g_1,i_1],[g_2,i_2],[z_1,z_2] \in \mathrm{In}(G)$, when $[g_1,i_1] \circledast [g_2,i_2] \leq [z_1,z_2]$, by the 2-residuation principle, $[g_1,i_1] \leq [g_2,i_2] \rightsquigarrow_{\circledast} [z_1,z_2]$ and $[g_2,i_2] \leq [g_1,i_1] \to_{\circledast} [z_1,z_2]$, then $[g_1,g_1] \leq [g_1,i_1] \leq [g_2,i_2] \rightsquigarrow_{\circledast} [z_1,z_2]$ and $[g_2,g_2] \leq [g_2,i_2] \leq [g_1,i_1] \to_{\circledast} [z_1,z_2]$.
(7) For arbitrary $[g_1,i_1],[g_2,i_2],[z_1,z_2] \in \mathrm{In}(G)$, if $[g_1,i_1] \leq [g_2,i_2] \rightsquigarrow_{\circledast} [z_1,z_2]$, by the 2-residuation principle, $[g_1,i_1] \circledast [g_2,i_2] \leq [z_1,z_2]$, then by (E1), $l([g_1,i_1] \circledast [g_2,i_2]) \leq [z_1,z_2]$. Similarly, when $[g_2,i_2] \leq [g_1,i_1] \to_{\circledast} [z_1,z_2]$, $l([g_1,i_1] \circledast [g_2,i_2]) \leq [z_1,z_2]$.
(8) For arbitrary $[g_1,i_1],[g_2,i_2] \in \mathrm{In}(G)$, if $[g_1,i_1] \leq [g_2,i_2]$, then $g_1 \leq g_2$ and $i_1 \leq i_2$. Further, $l([g_1,i_1]) = [g_1,g_1] \leq [g_2,g_2] = l([g_2,i_2])$, and $r([g_1,i_1]) = [i_1,i_1] \leq [i_2,i_2] = r([g_2,i_2])$. Thus, operators l,r are increasing.
(9) For arbitrary $[g,i] \in \mathrm{In}(G)$, since $[g,i] \circledast [g,i] \leq [1,1]$, by the 2-residuation principle, $[g,i] \leq [g,i] \to_{\circledast} [1,1]$ and $[g,i] \leq [g,i] \rightsquigarrow_{\circledast} [1,1]$. Furthermore, $[g,g] \leq [g,i] \to_{\circledast} [1,1]$ and $[g,g] \leq [g,i] \rightsquigarrow_{\circledast} [1,1]$.
(10) For arbitrary $[g_1,i_1],[g_2,i_2] \in \mathrm{In}(G)$, because $[g_1,i_1] \circledast [g_2,i_2] \leq [g_1,i_1] \circledast [g_2,i_2]$, by the 2-residuation principle, $[g_1,i_1] \leq [g_2,i_2] \rightsquigarrow_{\circledast} ([g_1,i_1] \circledast [g_2,i_2])$ and $[g_2,i_2] \leq [g_1,i_1] \to_{\circledast} ([g_1,i_1] \circledast [g_2,i_2])$. Further, $[g_1,g_1] \leq [g_2,i_2] \rightsquigarrow_{\circledast} ([g_1,i_1] \circledast [g_2,i_2])$ and $[g_2,g_2] \leq [g_1,i_1] \to_{\circledast} ([g_1,i_1] \circledast [g_2,i_2])$.
(11) For arbitrary $[g_1,i_1],[g_2,i_2] \in \mathrm{In}(G)$, since $[g_1,i_1] \to_{\circledast} [g_2,i_2] \leq [g_1,i_1] \to_{\circledast} [g_2,i_2]$ and $[g_1,i_1] \rightsquigarrow_{\circledast} [g_2,i_2] \leq [g_1,i_1] \rightsquigarrow_{\circledast} [g_2,i_2]$, by the 2-residuation principle, we have $[g_1,i_1] \circledast ([g_1,i_1] \to_{\circledast} [g_2,i_2]) \leq [g_2,i_2] \leq [i_2,i_2]$ and $([g_1,i_1] \rightsquigarrow_{\circledast} [g_2,i_2]) \circledast [g_1,i_1] \leq [g_2,i_2] \leq [i_2,i_2]$.
(12) For arbitrary $[g_1,i_1],[g_2,i_2] \in \mathrm{In}(G)$, because $[g_1,i_1] \to_{\circledast} [g_2,i_2] \leq [g_1,i_1] \to_{\circledast} [g_2,i_2]$, it holds that $[g_1,i_1] \circledast ([g_1,i_1] \to_{\circledast} [g_2,i_2]) \leq [g_2,i_2] \Leftrightarrow [g_1,i_1] \leq ([g_1,i_1] \to_{\circledast} [g_2,i_2]) \rightsquigarrow_{\circledast} [g_2,i_2]$ according to the 2-residuation principle. Then, $[g_1,g_1] \leq ([g_1,i_1] \to_{\circledast} [g_2,i_2]) \rightsquigarrow_{\circledast} [g_2,i_2]$. Similarly, $[g_1,i_1] \rightsquigarrow_{\circledast} [g_2,i_2] \leq [g_1,i_1] \rightsquigarrow_{\circledast} [g_2,i_2] \Leftrightarrow ([g_1,i_1] \rightsquigarrow_{\circledast} [g_2,i_2]) \circledast [g_1,i_1] \leq [g_2,i_2] \Leftrightarrow [g_1,i_1] \leq ([g_1,i_1] \rightsquigarrow_{\circledast} [g_2,i_2]) \to_{\circledast} [g_2,i_2]$, and further, $[g_1,g_1] \leq ([g_1,i_1] \rightsquigarrow_{\circledast} [g_2,i_2]) \to_{\circledast} [g_2,i_2]$.
(13) For arbitrary $[g_1,i_1],[g_2,i_2] \in \mathrm{In}(G)$, if $l([g_1,i_1]) \circledast l([g_2,i_2]) \leq l([g_1,i_1] \circledast [g_2,i_2])$, then $l([g_1,i_1]) \circledast l([g_1,i_1] \to_{\circledast} [g_2,i_2]) \leq l([g_1,i_1] \circledast ([g_1,i_1] \to_{\circledast} [g_2,i_2])) \leq l([g_2,i_2])$ according to 2-residuation principle; further, $l([g_1,i_1] \to_{\circledast} [g_2,i_2]) \leq l([g_1,i_1]) \to_{\circledast} l([g_2,i_2])$. Similarly, $l([g_1,i_1] \rightsquigarrow_{\circledast} [g_2,i_2]) \circledast l([g_1,i_1]) \leq l(([g_1,i_1] \rightsquigarrow_{\circledast} [g_2,i_2]) \circledast [g_1,i_1]) \leq l([g_2,i_2]) \Rightarrow l([g_1,i_1] \rightsquigarrow_{\circledast} [g_2,i_2]) \leq l([g_1,i_1]) \rightsquigarrow_{\circledast} l([g_2,i_2])$. On the other hand, when $l([g_1,i_1] \to_{\circledast} [g_2,i_2]) \leq l([g_1,i_1]) \to_{\circledast} l([g_2,i_2])$, we have $l([g_1,i_1] \to_{\circledast} ([g_1,i_1] \circledast [g_2,i_2])) \leq l([g_1,i_1]) \to_{\circledast} l([g_1,i_1] \circledast [g_2,i_2])$, since $l([g_2,i_2]) \leq l([g_1,i_1] \to_{\circledast} ([g_1,i_1] \circledast [g_2,i_2]))$, and according to (GRLG3), it holds that $l([g_2,i_2]) \leq l([g_1,i_1]) \to_{\circledast} l([g_1,i_1] \circledast [g_2,i_2])$, further, $l([g_1,i_1]) \circledast l([g_2,i_2]) \leq l([g_1,i_1] \circledast [g_2,i_2])$. Analogously, $l([g_1,i_1]) \leq l([g_2,i_2] \rightsquigarrow_{\circledast} ([g_1,i_1] \circledast [g_2,i_2])) \leq l([g_2,i_2]) \rightsquigarrow_{\circledast} l([g_1,i_1] \circledast [g_2,i_2]) \Rightarrow l([g_1,i_1]) \circledast l([g_2,i_2]) \leq l([g_1,i_1] \circledast [g_2,i_2])$.

(14) For arbitrary $[g,i] \in \mathrm{In}(G)$, $[g,i] = l([g,i])$ iff $[g,i] = [g,g]$ iff $g = i$ iff $[g,i] = [i,i]$ iff $[g,i] = r([g,i])$. In addition, if $[g,i] = l([g,i])$, then by (E5) $r([g,i]) = r(l([g,i])) = l([g,i]) = [g,i]$, for the same reason, $l([g,i]) = l(r([g,i])) = r([g,i]) = [g,i]$ when $[g,i] = r([g,i])$.
(15) For arbitrary $[g,i] \in \mathrm{In}(G)$, since $[g,i] \circledast [1,1] \leq [1,1]$ and $[1,1] \circledast [g,i] \leq [1,1]$, by the 2-residuation principle, $[1,1] \leq [g,i] \rightarrow_{\circledast} [1,1] \leq [1,1]$ and $[1,1] \leq [g,i] \rightsquigarrow_{\circledast} [1,1] \leq [1,1]$, $[g,i] \rightarrow_{\circledast} [1,1] = [1,1]$ and $[g,i] \rightsquigarrow_{\circledast} [1,1] = [1,1]$.
(16) For arbitrary $[g_1,i_1],[g_2,i_2] \in \mathrm{In}(G)$, by (E11) $[g_1,i_1] \circledast ([g_1,i_1] \rightarrow_{\circledast} [g_2,i_2]) \leq [i_2,i_2]$ and $([g_1,i_1] \rightsquigarrow_{\circledast} [g_2,i_2]) \circledast [g_1,i_1] \leq [i_2,i_2]$, we have $[g_1,i_1] \leq ([g_1,i_1] \rightarrow_{\circledast} [g_2,i_2]) \rightsquigarrow_{\circledast} [i_2,i_2]$ and $[g_1,i_1] \leq ([g_1,i_1] \rightsquigarrow_{\circledast} [g_2,i_2]) \rightarrow_{\circledast} [i_2,i_2]$. Further, $[g_1,g_1] \leq ([g_1,i_1] \rightarrow_{\circledast} [g_2,i_2]) \rightsquigarrow_{\circledast} [i_2,i_2]$ and $[g_1,g_1] \leq ([g_1,i_1] \rightsquigarrow_{\circledast} [g_2,i_2]) \rightarrow_{\circledast} [i_2,i_2]$. □

The definition of expanded triangle algebra is as follows.

Definition 14. *An algebra structure $(T,\wedge,\vee,*,\rightarrow,\rightsquigarrow,\omega,\theta,0,c,1)$ is called an expanded triangle algebra (briefly ET-algebra) when it meets requirements as below:*
$(ET1)$ $(T,\wedge,\vee,*,\rightarrow,\rightsquigarrow,0,1)$ *is a GRLG;*
$(ET2)$ $\omega(\omega(x)) = \omega(x) \leq x$, $\theta(\theta(x)) = \theta(x) \geq x$;
$(ET3)$ $\omega(x \wedge y) = \omega(x) \wedge \omega(y)$, $\theta(x \wedge y) = \theta(x) \wedge \theta(y)$;
$(ET4)$ $\omega(x \vee y) = \omega(x) \vee \omega(y)$, $\theta(x \vee y) = \theta(x) \vee \theta(y)$;
$(ET5)$ $\omega(c) = 0$, $\theta(c) = 1$;
$(ET6)$ $\omega(\theta(x)) = \theta(x)$, $\theta(\omega(x)) = \omega(x)$;
$(ET7)$ $\omega(x \rightarrow y) \leq \omega(x) \rightarrow \omega(y)$, $\omega(x \rightsquigarrow y) \leq \omega(x) \rightsquigarrow \omega(y)$;
$(ET8)$ $x = y$ *if* $\omega(x) = \omega(y)$ *and* $\theta(x) = \theta(y)$;
$(ET9)$ $\omega(x) \rightarrow \omega(y) \leq \omega(\omega(x) \rightarrow \omega(y))$, $\omega(x) \rightsquigarrow \omega(y) \leq \omega(\omega(x) \rightsquigarrow \omega(y))$
where ω,θ are unitary operations and c is a constant.

Evidently, operators ω,θ are increasing, as $\omega(x \wedge y) \leq \omega(x)$ and $\theta(x \wedge y) \leq \theta(x)$ by (ET3).

Definition 15. *The exact set $W(T)$ of ET-algebra $T = (T,\wedge,\vee,*,\rightarrow,\rightsquigarrow,\omega,\theta,0,c,1)$ defined as $W(T) = \{x \in T | \omega(x) = x\}$.*

Remark 2. *Given an ET-algebra $T = (T,\wedge,\vee,*,\rightarrow,\rightsquigarrow,\omega,\theta,0,c,1)$ and its exact set $W(T)$,*
(1) *$\theta(a) = a$ when taking arbitrary $a \in W(T)$;*
(2) *$a * b \in W(T)$ if $a,b \in W(T)$;*
(3) *$a \rightarrow b$, $a \rightsquigarrow b \in W(T)$ if $a,b \in W(T)$;*
(4) *$\theta(a \rightarrow b) = \theta(a) \rightarrow \theta(b)$, $\theta(a \rightsquigarrow b) = \theta(a) \rightsquigarrow \theta(b)$ when taking any $a,b \in W(T)$.*

Proof. (1) As $a \in W(T)$, $a = \omega(a)$. Then by (ET6), $\theta(a) = \theta(\omega(a)) = \omega(a) = a$.
(2) By (ET7), $\omega(a \rightarrow (a * b)) \leq \omega(a) \rightarrow \omega(a * b)$. Since $a * b \leq a * b \Leftrightarrow b \leq a \rightarrow (a * b)$ through 2-residuation principle, $\omega(b) \leq \omega(a \rightarrow (a * b))$. Then, $\omega(b) \leq \omega(a) \rightarrow \omega(a * b)$, through (GRLG3), $\omega(a) * \omega(b) \leq \omega(a * b)$. As $a,b \in W(T)$, $\omega(a) = a$ and $\omega(b) = b$, $a * b = \omega(a) * \omega(b) \leq \omega(a * b) \leq a * b$ through (ET2), and thereby $\omega(a * b) = \omega(a) * \omega(b)$; i.e., $a * b \in W(T)$.
(3) According to (ET9) and (ET2), we have $\omega(a) \rightarrow \omega(b) \leq \omega(\omega(a) \rightarrow \omega(b)) \leq \omega(a) \rightarrow \omega(b)$, $\omega(a) \rightsquigarrow \omega(b) \leq \omega(\omega(a) \rightsquigarrow \omega(b)) \leq \omega(a) \rightsquigarrow \omega(b)$—that is, $\omega(a) \rightarrow \omega(b) = \omega(\omega(a) \rightarrow \omega(b))$ and $\omega(a) \rightsquigarrow \omega(b) = \omega(\omega(a) \rightsquigarrow \omega(b))$. As $a,b \in W(T)$, $a \rightarrow b = \omega(a) \rightarrow \omega(b) = \omega(\omega(a) \rightarrow \omega(b)) = \omega(a \rightarrow b)$ and similar $a \rightsquigarrow b = \omega(a \rightsquigarrow b)$, i.e., $a \rightarrow b$, $a \rightsquigarrow b \in W(T)$.
(4) By (3), $a \rightarrow b$, $a \rightsquigarrow b \in W(T)$ when $a,b \in W(T)$, then by (1), $\theta(a \rightarrow b) = a \rightarrow b$ and $\theta(a \rightsquigarrow b) = a \rightsquigarrow b$. Additionally, because $a = \theta(a), b = \theta(b)$ according to (1), it is clear $\theta(a \rightarrow b) = \theta(a) \rightarrow \theta(b)$ and $\theta(a \rightsquigarrow b) = \theta(a) \rightsquigarrow \theta(b)$. □

After that, we prove that ET-algebras is correspond one-to-one to the IGRLGs.

Proposition 4. *Given an IGRLG, there is a corresponding ET-algebra, and vice versa.*

Proof. Assume that $G = (\mathrm{In}(G), \bigcap, \bigcup, \circledast, \rightarrow_{\circledast}, \rightsquigarrow_{\circledast}, [0,0], [1,1])$ is an IGRLG. We denote operations $\circledast, \rightarrow_{\circledast}, \rightsquigarrow_{\circledast}$ as $[x,x] \circledast [y,y] = [x*y, x*y]$, $[x,x] \rightarrow_{\circledast} [y,y] = [x \rightarrow y, x \rightarrow y]$, $[x,x] \rightsquigarrow_{\circledast} [y,y] = [x \rightsquigarrow y, x \rightsquigarrow y]$ for arbitrary $[x,x], [y,y] \in D_G$, where D_G is diagonal set of G, and define the mapping φ as $\varphi(\mathrm{In}(G), \bigcap, \bigcup, \circledast, \rightarrow_{\circledast}, \rightsquigarrow_{\circledast}, [0,0], [1,1]) = (\mathrm{In}(G), \bigcap, \bigcup, \circledast, \rightarrow_{\circledast}, \rightsquigarrow_{\circledast}, l, r, [0,0], [0,1], [1,1])$, where $l([a,b]) = [a,a]$ and $r([a,b]) = [b,b]$ when taking arbitrary $[a,b] \in \mathrm{In}(G)$. It is obvious that $(\mathrm{In}(G), \bigcap, \bigcup, \circledast, \rightarrow_{\circledast}, \rightsquigarrow_{\circledast}, l, r, [0,0], [0,1], [1,1])$ is an expanded IGRLG.

Additionally, assume $T = (T, \wedge, \vee, *, \rightarrow, \rightsquigarrow, \omega, \theta, 0, c, 1)$ is an ET-algebra. We denote the set $\{x \in T | \omega(x) = x\}$ as $W(T)$. Define the mapping ψ as $\psi(T, \wedge, \vee, *, \rightarrow, \rightsquigarrow, \omega, \theta, 0, c, 1) = (\mathrm{In}(W(T)), \bigcap, \bigcup, \circledast, \rightarrow_{\circledast}, \rightsquigarrow_{\circledast}, [0,0], [1,1])$, where

$\mathrm{In}(W(T)) = \{[x,y] | (x,y) \in T^2, x \leq y\}$, $[x,y] \bigcap [x',y'] = [x \wedge x', y \wedge y']$, $[x,y] \bigcup [x',y'] = [x \vee x', y \vee y']$,

$[x,y] \circledast [x',y'] = [\omega((x \vee (y \wedge c)) * (x' \vee (y' \wedge c))), \theta((x \vee (y \wedge c)) * (x' \vee (y' \wedge c)))]$,

$[x,y] \rightarrow_{\circledast} [x',y'] = [\omega((x \vee (y \wedge c)) \rightarrow (x' \vee (y' \wedge c))), \theta((x \vee (y \wedge c)) \rightarrow (x' \vee (y' \wedge c)))]$,

$[x,y] \rightsquigarrow_{\circledast} [x',y'] = [\omega((x \vee (y \wedge c)) \rightsquigarrow (x' \vee (y' \wedge c))), \theta((x \vee (y \wedge c)) \rightsquigarrow (x' \vee (y' \wedge c)))]$, for arbitrary $[x,y], [x',y'] \in \mathrm{In}(W(T))$. In addition, $[x,y] \leq [x',y'] \Leftrightarrow x \leq x'$ and $y \leq y'$.

Then, we define operation ρ: $T \rightarrow \mathrm{In}(W(T))$ as $\rho(x) = [\omega(x), \theta(x)]$; it is obvious that ρ is increasing, and thereby, $[x,y] = \rho(x \vee (y \wedge c))$ when taking arbitrary $[x,y] \in \mathrm{In}(W(T))$.
(1) We verify $\varphi(G) = (\mathrm{In}(G), \bigcap, \bigcup, \circledast, \rightarrow_{\circledast}, \rightsquigarrow_{\circledast}, l, r, [0,0], [0,1], [1,1])$ as an ET-algebra. Obviously, $\varphi(G)$ meets (ET1). Then, it also meets (ET2) $\sim$ (ET6) according to (E1) $\sim$ (E5) of Proposition 3.

As $l([x,y]) \circledast l([x',y']) = [x,x] \circledast [x',x'] = [x*x', x*x'] = l([x*x', x*x']) = l([x,x] \circledast [x',x']) \leq l([x,y] \circledast [x',y'])$, then by (E13) we have $l([x,y] \rightarrow_{\circledast} [x',y']) \leq l([x,y]) \rightarrow_{\circledast} l([x',y'])$ and $l([x,y] \rightsquigarrow_{\circledast} [x',y']) \leq l([x,y]) \rightsquigarrow_{\circledast} l([x',y'])$; i.e., (ET7) is met.

For random $[x,y], [x',y'] \in \mathrm{In}(G)$, when $l([x,y]) = l([x',y'])$ and $r([x,y]) = r([x',y'])$, $[x,x] = [x',x']$ and $[y,y] = [y',y'] \Rightarrow x = x'$ and $y = y' \Rightarrow [x,y] = [x',y']$; that is, $\varphi(G)$ meets (ET8).

Additionally, $l([x,y]) \rightarrow_{\circledast} l([x',y']) = [x,x] \rightarrow_{\circledast} [x',x'] = [x \rightarrow x', x \rightarrow x'] = l([x \rightarrow x', x \rightarrow x']) = l([x,x] \rightarrow_{\circledast} [x',x']) = l(l([x,y]) \rightarrow_{\circledast} l([x',y']))$, and similarly $l([x,y]) \rightsquigarrow_{\circledast} l([x',y']) = l(l([x,y]) \rightsquigarrow_{\circledast} l([x',y']))$, so it meets (ET9).

Thus, $(\mathrm{In}(G), \bigcap, \bigcup, \circledast, \rightarrow_{\circledast}, \rightsquigarrow_{\circledast}, l, r, [0,0], [0,1], [1,1])$ is an ET-algebra.
(2) We verify $\psi(T) = (\mathrm{In}(W(T)), \bigcap, \bigcup, \circledast, \rightarrow_{\circledast}, \rightsquigarrow_{\circledast}, [0,0], [1,1])$ is an IGRLG. Obviously, it meets (GRLG1) and (GRLG2). Since for arbitrary $[x,y], [x',y'], [t_1,t_2] \in \mathrm{In}(W(T))$, $[x,y] \circledast [x',y'] \leq [t_1,t_2]$
$\Leftrightarrow [\omega((x \vee (y \wedge c)) * (x' \vee (y' \wedge c))), \theta((x \vee (y \wedge c)) * (x' \vee (y' \wedge c)))] \leq [t_1,t_2]$
$\Leftrightarrow \rho((x \vee (y \wedge c)) * (x' \vee (y' \wedge c))) \leq \rho(t_1 \vee (t_2 \wedge c))$
$\Leftrightarrow (x \vee (y \wedge c)) * (x' \vee (y' \wedge c)) \leq t_1 \vee (t_2 \wedge c)$
$\Leftrightarrow x' \vee (y' \wedge c) \leq (x \vee (y \wedge c)) \rightarrow [t_1 \vee (t_2 \wedge c)]$ $(x \vee (y \wedge c) \leq (x' \vee (y' \wedge c)) \rightsquigarrow [t_1 \vee (t_2 \wedge c)])$
$\Leftrightarrow \rho(x' \vee (y' \wedge c)) \leq \rho((x \vee (y \wedge c)) \rightarrow [t_1 \vee (t_2 \wedge c)])$ $(\rho(x \vee (y \wedge c)) \leq \rho((x' \vee (y' \wedge c)) \rightsquigarrow [t_1 \vee (t_2 \wedge c)]))$
$\Leftrightarrow [x',y'] \leq [\omega((x \vee (y \wedge c)) \rightarrow [t_1 \vee (t_2 \wedge c)]), \theta((x \vee (y \wedge c)) \rightarrow [t_1 \vee (t_2 \wedge c)])]$ $([x,y] \leq [\omega((x' \vee (y' \wedge c)) \rightsquigarrow [t_1 \vee (t_2 \wedge c)]), \theta((x' \vee (y' \wedge c)) \rightsquigarrow [t_1 \vee (t_2 \wedge c)])])$
$\Leftrightarrow [x',y'] \leq [x,y] \rightarrow_{\circledast} [t_1,t_2]$ $([x,y] \leq [x',y'] \rightsquigarrow_{\circledast} [t_1,t_2])$, it also meets (GRLG3). Thus $(\mathrm{In}(W(T)), \bigcap, \bigcup, \circledast, \rightarrow_{\circledast}, \rightsquigarrow_{\circledast}, [0,0], [1,1])$ is a GRLG.

Additionally, because for arbitrary $[e,e], [f,f] \in \mathrm{In}(W(T))$, $[e,e] \circledast [f,f] = [\omega((e \vee (e \wedge c)) * (f \vee (f \wedge c))), \theta((e \vee (e \wedge c)) * (f \vee (f \wedge c)))] = [\omega(e*f), \theta(e*f)] = [e*f, e*f]$,
$[e,e] \rightarrow_{\circledast} [f,f] = [\omega((e \vee (e \wedge c)) \rightarrow (f \vee (f \wedge c))), \theta((e \vee (e \wedge c)) \rightarrow (f \vee (f \wedge c)))] = [\omega(e \rightarrow f), \theta(e \rightarrow f)] = [e \rightarrow f, e \rightarrow f]$ and
$[e,e] \rightsquigarrow_{\circledast} [f,f] = [\omega((e \vee (e \wedge c)) \rightsquigarrow (f \vee (f \wedge c))), \theta((e \vee (e \wedge c)) \rightsquigarrow (f \vee (f \wedge c)))] =$

$[\omega(e \rightsquigarrow f), \theta(e \rightsquigarrow f)] = [e \rightsquigarrow f, e \rightsquigarrow f]$ by Remark 2, the diagonal set is closed under operations $\circledast$, $\rightarrow_{\circledast}$ and $\rightsquigarrow_{\circledast}$.

Thus, (In$(W(T))$, $\bigcap, \bigcup, \circledast, \rightarrow_{\circledast}, \rightsquigarrow_{\circledast}, [0,0], [1,1]$) is an IGRLG.

In addition, using $\rho'(x) = [l(x), r(x)]$ we also have $\psi($ In$(G); \bigcap, \bigcup, \circledast, \rightarrow_{\circledast}, \rightsquigarrow_{\circledast}, l, r, [0,0], [0,1], [1,1])$ can be regarded as an IGRLG; and $\varphi($ In$(W(T)), \bigcap, \bigcup, \circledast, \rightarrow_{\circledast}, \rightsquigarrow_{\circledast}, [0,0], [1,1])$ is an ET-algebra due to $l(\rho(x)) = l([\omega(x), \theta(x)]) = [\omega(x), \omega(x)] = [\omega(\omega(x)), \theta(\omega(x))]$ (by (ET2) and (ET6)) $= \rho(\omega(x))$ and $r(\rho(x)) = r([\omega(x), \theta(x)]) = [\theta(x), \theta(x)] = [\omega(\theta(x)), \theta(\theta(x))]$ (by (ET6) and (ET2)) $= \rho(\theta(x))$. Therefore, there is one-to-one correspondence between IGRLGs and ET-algebras. □

Corollary 1. *Each expanded IGRLG is an ET-algebra.*

Proof. It is obviously based on (1) of Proposition 4 above. □

We state some examples of ET-algebras as below.

Example 4. *Given an IGRLG* $(In(L), \bigcap, \bigcup, \circledast, \rightarrow_{\circledast}, \rightsquigarrow_{\circledast}, [0,0], [1,1])$ *as stated in Example 2, we add two operators* l, r *as* $l([a,b]) = [a,a]$ *and* $r([a,b]) = [b,b]$ *for any* $[a,b] \in In(L)$*, and take constant* $[0,1]$ *as* c*. Since the structure* $(In(L), \bigcap, \bigcup, \circledast, \rightarrow_{\circledast}, \rightsquigarrow_{\circledast}, l, r, [0,0], [0,1], [1,1])$ *meets* $(ET1) \sim (ET9)$*; in fact,* $(ET1) \sim (ET6)$ *and* $(ET8)$ *are clear, for* $(ET7)$*,* $l([e,f] \rightarrow_{\circledast} [e',f']) = l([I_O^{(1)}(e,e') \wedge I_O^{(1)}(f,f'), I_O^{(1)}(f,f')]) = [I_O^{(1)}(e,e') \wedge I_O^{(1)}(f,f'), I_O^{(1)}(e,e') \wedge I_O^{(1)}(f,f')]$ *and* $l([e,f]) \rightarrow_{\circledast} l([e',f']) = [e,e] \rightarrow_{\circledast} [e',e'] = [I_O^{(1)}(e,e'), I_O^{(1)}(e,e')]$*; i.e.,* $l([e,f] \rightarrow_{\circledast} [e',f']) \leq l([e,f]) \rightarrow_{\circledast} l([e',f'])$ *for any* $[e,f], [e',f'] \in In(L)$*; analogously,* $l([e,f] \rightsquigarrow_{\circledast} [e',f']) \leq l([e,f]) \rightsquigarrow_{\circledast} l([e',f'])$*; for (ET9),* $l([e,f]) \rightarrow_{\circledast} l([e',f']) = [I_O^{(1)}(e,e'), I_O^{(1)}(e,e')] = l(l([e,f]) \rightarrow_{\circledast} l([e',f']))$ *and* $l([e,f]) \rightsquigarrow_{\circledast} l([e',f']) = [I_O^{(2)}(e,e'), I_O^{(2)}(e,e')] = l(l([e,f]) \rightsquigarrow_{\circledast} l([e',f']))$ *for any* $[e,f], [e',f'] \in In(L)$*. Thus, it is an ET-algebra.*

Example 5. *Given an algebra* $(T, \wedge, \vee, *, \rightarrow, \rightsquigarrow, \omega, \theta, [0,0], c, [1,1])$*, its structure is as Figure 1 below:*

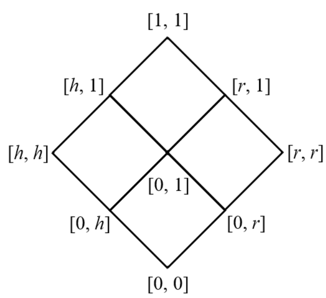

Figure 1. Structure of algebra in Example 5.

The operators $*$, $\rightarrow$ *and* $\rightsquigarrow$ *are shown in Tables 1–3 below:*

Table 1. The operator $*$ of algebra in Example 5.

$*$	$[0,0]$	$[0,h]$	$[0,r]$	$[h,h]$	$[0,1]$	$[r,r]$	$[h,1]$	$[r,1]$	$[1,1]$
$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$
$[0,h]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,h]$	$[0,h]$	$[0,0]$	$[0,h]$	$[0,h]$	$[0,h]$
$[0,r]$	$[0,0]$	$[0,0]$	$[0,r]$	$[0,0]$	$[0,r]$	$[0,r]$	$[0,r]$	$[0,r]$	$[0,r]$
$[h,h]$	$[0,0]$	$[0,h]$	$[0,0]$	$[h,h]$	$[0,h]$	$[0,0]$	$[h,h]$	$[0,h]$	$[h,h]$
$[0,1]$	$[0,0]$	$[0,0]$	$[0,r]$	$[0,h]$	$[0,h]$	$[0,r]$	$[0,1]$	$[0,1]$	$[0,1]$
$[r,r]$	$[0,0]$	$[0,0]$	$[0,r]$	$[0,0]$	$[0,r]$	$[r,r]$	$[0,r]$	$[r,r]$	$[r,r]$
$[h,1]$	$[0,0]$	$[0,h]$	$[0,r]$	$[h,h]$	$[0,1]$	$[0,r]$	$[h,1]$	$[0,1]$	$[h,1]$
$[r,1]$	$[0,0]$	$[0,h]$	$[0,r]$	$[0,h]$	$[0,1]$	$[r,r]$	$[0,1]$	$[r,1]$	$[r,1]$
$[1,1]$	$[0,0]$	$[0,h]$	$[0,r]$	$[h,h]$	$[0,1]$	$[r,r]$	$[h,1]$	$[r,1]$	$[1,1]$

Table 2. The operator $\rightarrow$ of algebra in Example 5.

$\rightarrow$	$[0,0]$	$[0,h]$	$[0,r]$	$[h,h]$	$[0,1]$	$[r,r]$	$[h,1]$	$[r,1]$	$[1,1]$
$[0,0]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$
$[0,h]$	$[r,1]$	$[1,1]$	$[r,1]$	$[1,1]$	$[1,1]$	$[r,1]$	$[1,1]$	$[1,1]$	$[1,1]$
$[0,r]$	$[h,h]$	$[h,h]$	$[1,1]$	$[h,h]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$
$[h,h]$	$[r,r]$	$[r,1]$	$[r,r]$	$[1,1]$	$[r,1]$	$[r,r]$	$[1,1]$	$[r,1]$	$[1,1]$
$[0,1]$	$[0,h]$	$[h,1]$	$[r,1]$	$[h,1]$	$[1,1]$	$[r,1]$	$[1,1]$	$[1,1]$	$[1,1]$
$[r,r]$	$[h,h]$	$[h,h]$	$[h,1]$	$[h,h]$	$[h,1]$	$[1,1]$	$[h,1]$	$[1,1]$	$[1,1]$
$[h,1]$	$[0,0]$	$[0,h]$	$[r,r]$	$[h,h]$	$[r,1]$	$[r,r]$	$[1,1]$	$[r,1]$	$[1,1]$
$[r,1]$	$[0,0]$	$[h,h]$	$[0,r]$	$[h,h]$	$[h,1]$	$[r,r]$	$[h,1]$	$[1,1]$	$[1,1]$
$[1,1]$	$[0,0]$	$[0,h]$	$[0,r]$	$[h,h]$	$[0,1]$	$[r,r]$	$[h,1]$	$[r,1]$	$[1,1]$

Table 3. The operator $\rightsquigarrow$ of algebra in Example 5.

$\rightsquigarrow$	$[0,0]$	$[0,h]$	$[0,r]$	$[h,h]$	$[0,1]$	$[r,r]$	$[h,1]$	$[r,1]$	$[1,1]$
$[0,0]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$
$[0,h]$	$[r,1]$	$[1,1]$	$[r,1]$	$[1,1]$	$[1,1]$	$[r,1]$	$[1,1]$	$[1,1]$	$[1,1]$
$[0,r]$	$[h,h]$	$[h,h]$	$[1,1]$	$[h,h]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$
$[h,h]$	$[r,r]$	$[r,1]$	$[r,r]$	$[1,1]$	$[r,1]$	$[r,r]$	$[1,1]$	$[r,1]$	$[1,1]$
$[0,1]$	$[0,0]$	$[h,1]$	$[r,r]$	$[h,1]$	$[1,1]$	$[r,r]$	$[1,1]$	$[1,1]$	$[1,1]$
$[r,r]$	$[h,h]$	$[h,h]$	$[h,1]$	$[h,h]$	$[h,1]$	$[1,1]$	$[h,1]$	$[1,1]$	$[1,1]$
$[h,1]$	$[0,0]$	$[0,h]$	$[r,r]$	$[h,h]$	$[r,1]$	$[r,r]$	$[1,1]$	$[r,1]$	$[1,1]$
$[r,1]$	$[0,0]$	$[h,h]$	$[0,r]$	$[h,h]$	$[h,1]$	$[r,r]$	$[h,1]$	$[1,1]$	$[1,1]$
$[1,1]$	$[0,0]$	$[0,h]$	$[0,r]$	$[h,h]$	$[0,1]$	$[r,r]$	$[h,1]$	$[r,1]$	$[1,1]$

Additionally, $\omega([x,y]) = [x,x]$, $\theta([x,y]) = [y,y]$ *for arbitrary* $[x,y] \in T$, *and* $c = [0,1]$. *After verification, it is clearly an ET-algebra.*

Next, we show a few properties that ET-algebras satisfy.

Proposition 5. *Given an ET-algebra* $(T, \wedge, \vee, *, \rightarrow, \rightsquigarrow, \omega, \theta, 0, c, 1)$,
(1) $\omega(x) \vee c = x \vee c$;
(2) $\theta(x) \wedge c = x \wedge c$;
(3) $\omega(x) * \omega(y) \leq \omega(x * y)$;
(4) $\theta(x * y) \leq \theta(x) * \theta(y)$ *when* $\omega(x * y) \leq \omega(x) * \omega(y)$;
(5) $\omega(1) = 1, \theta(0) = 0$;
(6) $\omega(x) \rightarrow \theta(c) = \omega(x) \rightsquigarrow \theta(c) = 1$;
(7) $\theta(x * y) \geq \theta(x) * \theta(y)$ *is equivalent to* $\theta(x \rightarrow y) \leq \theta(x) \rightarrow \theta(y)$ *is equivalent to* $\theta(x \rightsquigarrow y) \leq \theta(x) \rightsquigarrow \theta(y)$;
(8) $\omega(\omega(x) * \omega(y)) = \omega(x) * \omega(y)$;
(9) $\theta(\theta(x) * \theta(y)) = \theta(x) * \theta(y)$ *when* $\omega(x * y) \leq \omega(x) * \omega(y)$;
(10) $\theta(\theta(x) \rightarrow \theta(y)) = \theta(x) \rightarrow \theta(y)$ *and* $\theta(\theta(x) \rightsquigarrow \theta(y)) = \theta(x) \rightsquigarrow \theta(y)$ *if* $\omega(x * y) \leq \omega(x) * \omega(y)$ *when taking arbitrary* $x, y \in T$.

Proof. (1) By (ET4), (ET2) and (ET5), $\omega(\omega(x) \vee c) = \omega(x) \vee \omega(c) = \omega(x)$, along with $\omega(x \vee c) = \omega(x)$, $\omega(\omega(x) \vee c) = \omega(x \vee c)$. Additionally, $\theta(\omega(x) \vee c) = \theta(\omega(x)) \vee \theta(c) = 1$, $\theta(x \vee c) = \theta(x) \vee \theta(c) = 1$, i.e., $\theta(\omega(x) \vee c) = \theta(x \vee c)$. Then, by (ET8). we have $\omega(x) \vee c = x \vee c$.
(2) As above according to (ET2), (ET3) and (ET5).
(3) By (ET7), $\omega(t \rightsquigarrow (s * t)) \leq \omega(t) \rightsquigarrow \omega(s * t)$. As $s \leq t \rightsquigarrow (s * t)$ for arbitrary $s, t \in T$, $\omega(s) \leq \omega(t \rightsquigarrow (s * t)) \leq \omega(t) \rightsquigarrow \omega(s * t)$, by (GRLG3) $\omega(s) * \omega(t) \leq \omega(s * t)$.
(4) When $\omega(x * y) \leq \omega(x) * \omega(y)$, by (3), we have $\omega(x * y) = \omega(x) * \omega(y)$. Further, $\omega(\theta(x) * \theta(y)) = \omega(\theta(x)) * \omega(\theta(y)) = \theta(x) * \theta(y)$ according to (ET6). Then, by Remark 2, $\theta(x) * \theta(y) = \theta(\theta(x) * \theta(y))$. Since $x \leq \theta(x)$ and $y \leq \theta(y)$ by (ET2), $\theta(x * y) \leq \theta(\theta(x) * \theta(y))$. Thus, $\theta(x * y) \leq \theta(x) * \theta(y)$.

(5) It is clear that $\omega(1) = \omega(\theta(c)) = \theta(c) = 1$ by (ET5) and (ET6); similarly, $\theta(0) = \theta(\omega(c)) = \omega(c) = 0$.
(6) As for arbitrary $x \in T$, $x \to 1 = x \rightsquigarrow 1 = 1$, $\omega(x \to 1) = \omega(x \rightsquigarrow 1) = \omega(1) = 1$ by (5). By (ET7), $\omega(x \to 1) \leq \omega(x) \to \omega(1)$, $\omega(x \rightsquigarrow 1) \leq \omega(x) \rightsquigarrow \omega(1)$; i.e., $\omega(x) \to \omega(1) = 1$ and $\omega(x) \rightsquigarrow \omega(1) = 1$. Thus, $\omega(x) \to \theta(c) = \omega(x) \rightsquigarrow \theta(c) = 1$ according to (ET5).
(7) First of all, $\theta(x * y) \geq \theta(x) * \theta(y) \Rightarrow \theta(x * (x \to y)) \geq \theta(x) * \theta(x \to y)$, since $\theta(x * (x \to y)) \leq \theta(y)$, so it holds that $\theta(x) * \theta(x \to y) \leq \theta(y)$, by the 2-residuation principle $\theta(x \to y) \leq \theta(x) \to \theta(y)$, Moreover, $\theta(y) \geq \theta((x \rightsquigarrow y) * x) \geq \theta(x \rightsquigarrow y) * \theta(x) \Rightarrow \theta(x \rightsquigarrow y) \leq \theta(x) \rightsquigarrow \theta(y)$. Conversely, $\theta(x \to y) \leq \theta(x) \to \theta(y) \Rightarrow \theta(x \to (x * y)) \leq \theta(x) \to \theta(x * y)$, because $\theta(x \to (x * y)) \geq \theta(y)$, $\theta(y) \leq \theta(x) \to \theta(x * y)$, and thereby $\theta(x) * \theta(y) \leq \theta(x * y)$ according to the 2-residuation principle. Similarly, $\theta(x) \leq \theta(y \rightsquigarrow (x * y)) \leq \theta(y) \rightsquigarrow \theta(x * y) \Rightarrow \theta(x) * \theta(y) \leq \theta(x * y)$.
(8) By (3), $\omega(\omega(x) * \omega(y)) \geq \omega(\omega(x)) * \omega(\omega(y)) = \omega(x) * \omega(y)$ due to (ET2). Since $\omega(x) * \omega(y) \geq \omega(\omega(x) * \omega(y))$; that is, $\omega(x) * \omega(y) = \omega(\omega(x) * \omega(y))$.
(9) By (3), $\omega(x * y) = \omega(x) * \omega(y)$ when $\omega(x * y) \leq \omega(x) * \omega(y)$. Further, we have $\omega(\theta(x) * \theta(y)) = \omega(\theta(x)) * \omega(\theta(y)) = \theta(x) * \theta(y)$ by (ET6), so $\theta(\theta(x) * \theta(y)) = \theta(x) * \theta(y)$ by Remark 2.
(10) By (9), if $\omega(x * y) \leq \omega(x) * \omega(y)$, we have $\theta(\theta(x) * \theta(y)) = \theta(x) * \theta(y)$, and thereby $\theta(\theta(x) * \theta(y)) \geq \theta(x) * \theta(y)$. By (7), $\theta(\theta(x) \to \theta(y)) \leq \theta(x) \to \theta(y)$, and since $\theta(x) \to \theta(y) \leq \theta(\theta(x) \to \theta(y))$ according to (ET2), $\theta(\theta(x) \to \theta(y)) = \theta(x) \to \theta(y)$. Analogously, $\theta(\theta(x) \rightsquigarrow \theta(y)) = \theta(x) \rightsquigarrow \theta(y)$. □

Apparently, the class of ET-algebras includes triangle algebras, as described next.

Proposition 6. *Each triangle algebra is a commutative ET-algebra.*

Proof. Assume the structure $(S, \sqcap, \sqcup, *, \to, \eta, \xi, 0, u, 1)$ is a triangle algebra, by definition. We just need to demonstrate that it meets (ET8)—that is, $n = q$ when $\eta(n) = \eta(q)$ and $\xi(n) = \xi(q)$ for any $n, q \in S$. In fact, since operation $*$ with identity element 1, $n \leq q \Leftrightarrow n \to q = 1$. Thus, $\eta(n) = \eta(q) \Leftrightarrow \eta(n) \to \eta(q) = 1$, $\eta(q) \to \eta(n) = 1$ and $\xi(n) = \xi(q) \Leftrightarrow \xi(n) \to \xi(q) = 1$, $\xi(q) \to \xi(n) = 1$, then by (T8) $(n \to q) \sqcap (q \to n) \geq 1$, i.e., $(n \to q) \sqcap (q \to n) = 1$. Thereby, $n \to q = q \to n = 1$; that is, $n = q$. □

Now we give an example of ET-algebra but not triangle algebra.

Example 6. *We take an example of ET-algebra but not triangle algebra as follows: given a structure $(T, \wedge, \vee, *, \to, \rightsquigarrow, \omega, \theta, [0,0], c, [1,1])$, in which $T = \{[a,b] | (a,b) \in [0,1]^2, a \leq b\}$, when taking arbitrary $[a_1, b_1]$, $[a_2, b_2] \in T$, $[a_1, b_1] \wedge [a_2, b_2] = [min\{a_1, a_2\}, min\{b_1, b_2\}]$, $[a_1, b_1] \vee [a_2, b_2] = [max\{a_1, a_2\}, max\{b_1, b_2\}]$, $[a_1, b_1] * [a_2, b_2] = [a_1 a_2^2, b_1 b_2^2]$,*

$$[a_1, b_1] \to [a_2, b_2] = \begin{cases} [1,1], & a_1 \leq a_2 \text{ and } b_1 \leq b_2 \\ [\sqrt{\frac{b_2}{b_1}}, \sqrt{\frac{b_2}{b_1}}], & a_1 \leq a_2 \text{ and } b_1 > b_2 \\ [\sqrt{\frac{a_2}{a_1}}, 1], & a_1 > a_2 \text{ and } b_1 \leq b_2{}' \\ [\sqrt{\frac{b_2}{b_1}} \wedge \sqrt{\frac{a_2}{a_1}}, \sqrt{\frac{b_2}{b_1}}], & a_1 > a_2 \text{ and } b_1 > b_2 \end{cases}$$

$$[a_1, b_1] \rightsquigarrow [a_2, b_2] = \begin{cases} [1,1], & a_1^2 \leq a_2 \text{ and } b_1^2 \leq b_2 \\ [\frac{b_2}{b_1^2}, \frac{b_2}{b_1^2}], & a_1^2 \leq a_2 \text{ and } b_1^2 > b_2 \\ [\frac{a_2}{a_1^2}, 1], & a_1^2 > a_2 \text{ and } b_1^2 \leq b_2{}' \\ [\frac{b_2}{b_1^2} \wedge \frac{a_2}{a_1^2}, \frac{b_2}{b_1^2}], & a_1^2 > a_2 \text{ and } b_1^2 > b_2 \end{cases}$$

i.e., $(T, \wedge, \vee, *, \to, \rightsquigarrow, [0,0], [1,1])$ is a GRLG, and $\omega([a_1, b_1]) = [a_1, a_1]$, $\theta([a_1, b_1]) = [b_1, b_1]$, $c = [0,1]$, so $(T, \wedge, \vee, *, \to, \rightsquigarrow, \omega, \theta, [0,0], c, [1,1])$ is an ET-algebra due to $(ET1) \sim (ET9)$

being met. It is clear that $(T, \wedge, \vee, *, \rightarrow, \rightsquigarrow, \omega, \theta, [0,0], c, [1,1])$ *is not a triangle algebra, because* $(T, \wedge, \vee, *, \rightarrow, \rightsquigarrow, [0,0], [1,1])$ *is not a RL; that is, operation* $*$ *is non-commutative and non-associative and has no unit element, so that the structure does not meet* $(T8)$.

4. Filters of ET-Algebras

In [23], several different kinds of filters and their relations on triangle algebras are introduced. In this part, we put forward filters of ET-algebras to construct quotient structure and discuss some properties they satisfy. First, the definitions of filters are given below.

Definition 16. *Given an ET-algebra* $(T, \wedge, \vee, *, \rightarrow, \rightsquigarrow, \omega, \theta, 0, c, 1)$*, a nonempty set* $S \subseteq T$ *is called an expanded interval-valued filter (briefly EIF) of T if it meets the requirements that follow:*
$(EIF1)$ $y \in S$ *when* $x \in S, y \in T$ *and* $x \leq y$;
$(EIF2)$ $x * y \in S$ *when* $x \in S, y \in S$;
$(EIF3)$ $x \rightarrow x \in S$ *when taking arbitrary* $x \in T$;
$(EIF4)$ $x \wedge y \in S$ *for every* $x, y \in S$;
$(EIF5)$ $a * (x * y) \leq (a * x) * y$ *for arbitrary* $x, y \in S$ *and* $a \in T$;
$(EIF6)$ *there exists* $k_i \in S$ $(i = 1, 2, \ldots, 6)$ *satisfying* $(m * n) * k_1 \leq m * (n * x)$, $(m * n) * k_2 \leq (m * x) * n$, $m * (n * k_3) \leq (m * n) * x$, $m * (n * k_4) \leq (m * x) * n$, $(k_5 * m) * n \leq m * (x * n)$ *and* $(k_6 * m) * n \leq x * (m * n)$ *for arbitrary* $x \in S$, $m, n \in T$;
$(EIF7)$ $\omega(x) \in S$ *when* $x \in S$;
$(EIF8)$ $\theta(x * y) \geq \theta(x) * \theta(y)$ *when taking any* $x, y \in S$;
$(EIF9)$ $x \rightarrow y \in S$ *when* $\omega(x) \rightarrow \omega(y) \in S$ *and* $\theta(x) \rightarrow \theta(y) \in S$ *for arbitrary* $x, y \in T$.
In addition, S is called a normal expanded interval-valued filter (NEIF for short) when it meets the requirement as below:
$(EIF10)$ $x \rightarrow y \in S \Leftrightarrow x \rightsquigarrow y \in S$.

Example 7. *Given an algebra* $(T, \wedge, \vee, *, \rightarrow, \rightsquigarrow, \omega, \theta, [0,0], c, [1,1])$*, its structure as Figure 2 below:*

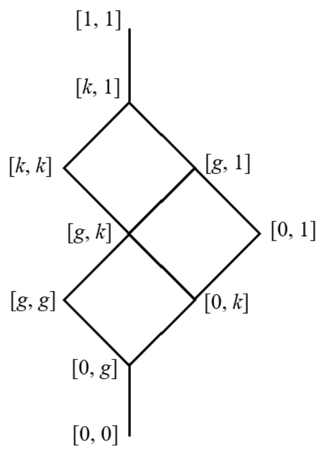

Figure 2. Structure of algebra in Example 7.

where operations $*$, $\rightarrow$ *and* $\rightsquigarrow$ *on T are as shown in Tables 4–6 below:*

Table 4. The operator $*$ of algebra in Example 7.

$*$	$[0,0]$	$[0,g]$	$[g,g]$	$[0,k]$	$[g,k]$	$[0,1]$	$[k,k]$	$[g,1]$	$[k,1]$	$[1,1]$
$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$
$[0,g]$	$[0,0]$	$[0,0]$	$[0,g]$	$[0,0]$	$[0,g]$	$[0,0]$	$[0,g]$	$[0,g]$	$[0,g]$	$[0,g]$
$[g,g]$	$[0,0]$	$[0,g]$	$[g,g]$	$[0,g]$	$[g,g]$	$[0,g]$	$[g,g]$	$[g,g]$	$[g,g]$	$[g,g]$
$[0,k]$	$[0,0]$	$[0,g]$	$[0,g]$	$[0,g]$	$[0,k]$	$[0,g]$	$[0,k]$	$[0,k]$	$[0,k]$	$[0,k]$
$[g,k]$	$[0,0]$	$[0,g]$	$[g,g]$	$[0,k]$	$[g,k]$	$[0,k]$	$[g,k]$	$[g,k]$	$[g,k]$	$[g,k]$
$[0,1]$	$[0,0]$	$[0,g]$	$[0,g]$	$[0,g]$	$[0,k]$	$[0,g]$	$[0,k]$	$[0,k]$	$[0,k]$	$[0,1]$
$[k,k]$	$[0,0]$	$[0,g]$	$[g,g]$	$[0,k]$	$[g,k]$	$[0,k]$	$[k,k]$	$[g,k]$	$[k,k]$	$[k,k]$
$[g,1]$	$[0,0]$	$[0,g]$	$[g,g]$	$[0,k]$	$[g,k]$	$[0,k]$	$[g,k]$	$[g,1]$	$[g,1]$	$[g,1]$
$[k,1]$	$[0,0]$	$[0,g]$	$[g,g]$	$[0,k]$	$[g,k]$	$[0,k]$	$[k,k]$	$[g,1]$	$[k,1]$	$[k,1]$
$[1,1]$	$[0,0]$	$[0,g]$	$[g,g]$	$[0,k]$	$[g,k]$	$[0,1]$	$[k,k]$	$[g,1]$	$[k,1]$	$[1,1]$

Table 5. The operator $\to$ of algebra in Example 7.

$\to$	$[0,0]$	$[0,g]$	$[g,g]$	$[0,k]$	$[g,k]$	$[0,1]$	$[k,k]$	$[g,1]$	$[k,1]$	$[1,1]$
$[0,0]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$
$[0,g]$	$[0,k]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$
$[g,g]$	$[0,0]$	$[0,1]$	$[1,1]$	$[0,1]$	$[1,1]$	$[0,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$
$[0,k]$	$[0,0]$	$[g,1]$	$[g,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$
$[g,k]$	$[0,0]$	$[0,g]$	$[g,g]$	$[0,1]$	$[1,1]$	$[0,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$
$[0,1]$	$[0,0]$	$[g,1]$	$[g,1]$	$[k,1]$	$[k,1]$	$[1,1]$	$[k,1]$	$[1,1]$	$[1,1]$	$[1,1]$
$[k,k]$	$[0,0]$	$[0,g]$	$[g,g]$	$[0,1]$	$[g,1]$	$[0,1]$	$[1,1]$	$[g,1]$	$[1,1]$	$[1,1]$
$[g,1]$	$[0,0]$	$[0,g]$	$[g,g]$	$[0,1]$	$[k,1]$	$[0,1]$	$[k,1]$	$[1,1]$	$[1,1]$	$[1,1]$
$[k,1]$	$[0,0]$	$[0,g]$	$[g,g]$	$[0,1]$	$[g,1]$	$[0,1]$	$[k,1]$	$[g,1]$	$[1,1]$	$[1,1]$
$[1,1]$	$[0,0]$	$[0,g]$	$[g,g]$	$[0,k]$	$[g,k]$	$[0,1]$	$[k,k]$	$[g,1]$	$[k,1]$	$[1,1]$

Table 6. The operator $\rightsquigarrow$ of algebra in Example 7.

$\rightsquigarrow$	$[0,0]$	$[0,g]$	$[g,g]$	$[0,k]$	$[g,k]$	$[0,1]$	$[k,k]$	$[g,1]$	$[k,1]$	$[1,1]$
$[0,0]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$
$[0,g]$	$[0,g]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$
$[g,g]$	$[0,0]$	$[0,1]$	$[1,1]$	$[0,1]$	$[1,1]$	$[0,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$
$[0,k]$	$[0,g]$	$[g,1]$	$[g,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$
$[g,k]$	$[0,0]$	$[0,g]$	$[g,g]$	$[0,1]$	$[1,1]$	$[0,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$
$[0,1]$	$[0,g]$	$[g,1]$	$[g,1]$	$[k,1]$	$[k,1]$	$[1,1]$	$[k,1]$	$[1,1]$	$[1,1]$	$[1,1]$
$[k,k]$	$[0,0]$	$[0,g]$	$[g,g]$	$[0,1]$	$[g,1]$	$[0,1]$	$[1,1]$	$[g,1]$	$[1,1]$	$[1,1]$
$[g,1]$	$[0,0]$	$[0,g]$	$[g,g]$	$[0,1]$	$[k,1]$	$[0,1]$	$[k,1]$	$[1,1]$	$[1,1]$	$[1,1]$
$[k,1]$	$[0,0]$	$[0,g]$	$[g,g]$	$[0,1]$	$[g,1]$	$[0,1]$	$[k,1]$	$[g,1]$	$[1,1]$	$[1,1]$
$[1,1]$	$[0,0]$	$[0,g]$	$[g,g]$	$[0,k]$	$[g,k]$	$[0,1]$	$[k,k]$	$[g,1]$	$[k,1]$	$[1,1]$

Additionally, $\omega([s,t]) = [s,s]$, $\theta([s,t]) = [t,t]$ *and* $c = [0,1]$ *for arbitrary* $[s,t] \in T$. *After verification, it is an ET-algebra. Additionally,* $S = \{[k,k],[k,1],[1,1]\}$ *is an NEIF of T.*

Let us discuss some properties satisfied by EIFs of ET-algebras.

Proposition 7. *Given an ET-algebra* $(T, \wedge, \vee, *, \to, \rightsquigarrow, \omega, \theta, 0, c, 1)$, *and an EIF E of T,*
(1) $1 \in E$;
(2) $x \to 1 \in E$, $x \rightsquigarrow 1 \in E$ *for arbitrary* $x \in T$;
(3) $0 \to x \in E$, $0 \rightsquigarrow x \in E$ *when taking arbitrary* $x \in T$;
(4) $\omega(x) \to \theta(c) \in E$, $\omega(x) \rightsquigarrow \theta(c) \in E$ *for arbitrary* $x \in T$;
(5) $\theta(x) \in E$ *when* $x \in E$;
(6) $\theta(x * y) \in E$, $\theta(x \wedge y) \in E$ *when taking arbitrary* $x, y \in E$;
(7) $\omega(x * y) \in E$, $\omega(x \wedge y) \in E$ *when taking random* $x, y \in E$;
(8) *For arbitrary* $r, y, n \in T$, $r \to (y * n) \in E$ *when* $r \to y \in E$, $n \in E$;
(9) *For arbitrary* $x, y \in T$, $x \to y \in E$ *when* $(x * y) \to y \in E$, $y \in E$;

(10) *If E is an NEIF of T, then $\omega(x) \to \omega(y) \in E$ and $\omega(x) \rightsquigarrow \omega(y) \in E$ when $x \to y \in E$ or $x \rightsquigarrow y \in E$.*

Proof. (1) Obviously, since E is a nonempty set and meets (EIF1), $x \leq 1$ for arbitrary $x \in T$, which means there must be $1 \in E$.
(2) Since $x * 1 \leq 1$ and $1 * x \leq 1$ for any $x \in T$, by the 2-residuation principle $1 \leq x \to 1$ and $1 \leq x \rightsquigarrow 1$; i.e., $x \to 1 = x \rightsquigarrow 1 = 1$. Then, by (1), $x \to 1 \in E$, $x \rightsquigarrow 1 \in E$.
(3) As $0 \leq x \to 0$ for arbitrary $x \in T$, by the 2-residuation principle, $x * 0 \leq 0$, i.e., $x * 0 = 0$. Then, we have $1 * 0 = 0$—that is, $1 * 0 \leq x$ for arbitrary $x \in T$, so $1 \leq 0 \rightsquigarrow x$ according to the 2-residuation principle; that is, $0 \rightsquigarrow x = 1 \in E$. Analogously, due to $0 \leq x \rightsquigarrow 0$ for arbitrary $x \in T$, we can get $0 \to x = 1 \in E$.
(4) By (6) of Proposition 5, $\omega(x) \to \theta(c) = \omega(x) \rightsquigarrow \theta(c) = 1$ for arbitrary $x \in T$, and then by (1) $\omega(x) \to \theta(c) \in E$, $\omega(x) \rightsquigarrow \theta(c) \in E$.
(5) Since $x \leq \theta(x)$, by (EIF1) $\theta(x) \in E$ when $x \in E$.
(6) For arbitrary $x, y \in E$, by (EIF2) and (EIF4) $x * y \in E$, $x \wedge y \in E$, so by (5), $\theta(x * y) \in E$, $\theta(x \wedge y) \in E$.
(7) For arbitrary $x, y \in E$, by (EIF2) and (EIF4) $x * y \in E$ and $x \wedge y \in E$, according to (EIF7), $\omega(x * y) \in E$, $\omega(x \wedge y) \in E$. In fact, since $\omega(x) \leq x$ by (ET2), $x \in E$ when $\omega(x) \in E$, and in combination with (EIF7), we have $x \in E \Leftrightarrow \omega(x) \in E$.
(8) Since $r \to y \in E$, $n \in E$ and $r \in T$, by (EIF5) $r * ((r \to y) * n) \leq [r * (r \to y)] * n \leq y * n$. Then, by the 2-residuation principle, $(r \to y) * n \leq r \to (y * n)$. According to (EIF2) $(r \to y) * n \in E$, further, $r \to (y * n) \in E$ by (EIF1).
(9) As $(x * y) * [(x * y) \to y] \leq y$, when $(x * y) \to y \in E$, by (EIF6) there exists $c \in E$ satisfying $x * (y * c) \leq (x * y) * [(x * y) \to y]$; that is, $x * (y * c) \leq y$. By the 2-residuation principle, $y * c \leq x \to y$. When $y \in E$, $y * c \in E$ according to (EIF2), then by (EIF1) $x \to y \in E$.
(10) If E is an NEIF, then it meets (EIF10). When $x \to y \in E$, by (EIF7) $\omega(x \to y) \in E$. Furthermore, by (ET7) and (EIF1), $\omega(x) \to \omega(y) \in E$; meanwhile, by (EIF10) $x \to y \in E \Rightarrow x \rightsquigarrow y \in E$—that is, $\omega(x \rightsquigarrow y) \leq \omega(x) \rightsquigarrow \omega(y) \in E$. On the other hand, when $x \rightsquigarrow y \in E$, we can obtain the same result. □

Then, we study the congruence relation on ET-algebras through EIFs. We first give the following proposition.

Proposition 8. *Given an ET-algebra $(T, \wedge, \vee, *, \to, \rightsquigarrow, \omega, \theta, 0, c, 1)$, then the below conditions are true for any $r, m, n \in T$:*
(1) $r \to (m \wedge n) = (r \to m) \wedge (r \to n)$;
(2) $(r \vee m) \to n = (r \to n) \wedge (m \to n)$.

Proof. We know $(T, \wedge, \vee, *, \to, \rightsquigarrow, 0, 1)$ is a GRLG according to the definition, so $(*, \to, \rightsquigarrow)$ meets the 2-residuation principle. According to (A9) and (A10) of Proposition 7 in [24], (1) and (2) are obvious. In fact, it is not difficult to find that some conditions are obviously valid as long as the 2-residuation principle is satisfied. □

Proposition 9. *Given an ET-algebra $(T, \wedge, \vee, *, \to, \rightsquigarrow, \omega, \theta, 0, c, 1)$, and N is an NEIF of T. The relation $\equiv$ on T defined as $x \equiv y$ when and only when $x \to y \in N$, $y \to x \in N$, so it is a congruence relation.*

Proof. We demonstrate it in two steps. Firstly, we prove that it is equivalent, then prove that it is congruent with respect to these operations: $\wedge, \vee, *, \to, \rightsquigarrow, \omega$ and θ.
(1) By (EIF3) $x \to x \in N$—i.e., $x \equiv x$—so it is reflexive. Evidently, $x \to y \in N$ and $y \to x \in N$ when $x \equiv y$; that is, $y \equiv x$ by definition. It is symmetrical. Suppose $x \equiv y$ and $y \equiv z$. We prove that $x \equiv z$. As $x \to y \in N$, $y \to z \in N$ and $x \in T$, by (EIF5), $x * [(x \to y) * (y \to z)] \leq [x * (x \to y)] * (y \to z)$. Since $[x * (x \to y)] * (y \to z) \leq y * (y \to z) \leq z$, and further, $x * [(x \to y) * (y \to z)] \leq z$, by the 2-residuation principle,

$(x \to y) * (y \to z) \le x \to z$, by (EIF2) and (EIF1), $x \to z \in N$. Analogously, $z \to y \in N$, $y \to x \in N \Rightarrow z \to x \in N$; thus, $x \equiv z$. Consequently, it is transitive.
(2) Assume $x \equiv y$ and $a \equiv b$; then, $x \to y \in N, y \to x \in N, a \to b \in N$ and$b \to a \in N$.
(i) We prove $(x \wedge a) \equiv (y \wedge b)$. Since $(x \wedge a) \to (y \wedge b) = [(x \wedge a) \to y] \wedge [(x \wedge a) \to b] \ge (x \to y) \wedge (a \to b)$ according to Proposition 8 above, then by (EIF4) and (EIF1), $(x \wedge a) \to (y \wedge b) \in N$. For the same reason, $(y \to x) \wedge (b \to a) \in N \Rightarrow (y \wedge b) \to (x \wedge a) \in N$.
(ii) We prove that $(x \vee a) \equiv (y \vee b)$. As $(x \vee a) \to (y \vee b) = [x \to (y \vee b)] \wedge [a \to (y \vee b)] \ge (x \to y) \wedge (a \to b)$ according to Proposition 8 above, then by (EIF4) and (EIF1) $(x \vee a) \to (y \vee b) \in N$. Analogously, $(y \vee b) \to (x \vee a) \in N$.
(iii) we prove that $(x * a) \equiv (y * b)$. As $y * (a * (a \to b)) \le y * b$, where $y, a \in T$ and $a \to b \in N$, by (EIF6), there exists $c_1 \in N$ such that $(y * a) * c_1 \le y * (a * (a \to b))$; i.e., $(y * a) * c_1 \le y * b$. Since $x * (x \to y) \le y$ by the 2-residuation principle, $[(x * (x \to y)) * a] * c_1 \le y * b$. Additionally, by (EIF6), there exists $c_2 \in N$ satisfying $(x * a) * c_2 \le (x * (x \to y)) * a$, and further, $[(x * a) * c_2] * c_1 \le y * b$. Finally, by (EIF5), $[(x * a) * c_2] * c_1 \ge (x * a) * (c_2 * c_1)$, so $(x * a) * (c_2 * c_1) \le y * b$, and thus $c_2 * c_1 \le (x * a) \to (y * b)$ by the 2-residuation principle, so $(x * a) \to (y * b) \in N$ by (EIF2) and (EIF1). Similarly, because $x * (b * (b \to a)) \le x * a$, there exists $c_1 \in N$ making $(x * b) * c_1 \le x * (b * (b \to a)) \le x * a$ by (EIF6). Then, by $[(y * (y \to x)) * b] * c_1 \le x * a$, there exists $c_2 \in N$ such that $(y * b) * c_2 \le (y * (y \to x)) * b$ by (EIF6); further, $[(y * b) * c_2] * c_1 \le x * a$, according to (EIF5), and $(y * b) * (c_2 * c_1) \le [(y * b) * c_2] * c_1 \le x * a$; thus, $c_2 * c_1 \le (y * b) \to (x * a) \in N$.
(iv) We verify $(x \to a) \equiv (y \to b)$. Firstly, because for arbitrary $x, y, z \in T$, $[y * (y \to x)] * (x \to z) \le z$, by (EIF6), there exists $c_1 \in N$ such that $y * ((x \to z) * c_1) \le [y * (y \to x)] * (x \to z)$, so then $y * ((x \to z) * c_1) \le z$. According to the 2-residuation principle, $(x \to z) * c_1 \le y \to z$; further, $c_1 \le (x \to z) \to (y \to z)$, so $(x \to z) \to (y \to z) \in N$ by (EIF1). In addition, because $[x * (x \to y)] * (y \to z) \le z$, by (EIF6) there exists $c_2 \in N$ satisfying $x * ((y \to z) * c_2) \le [x * (x \to y)] * (y \to z) \le z$, according to the 2-residuation principle, $c_2 \le (y \to z) \to (x \to z)$; i.e., $(y \to z) \to (x \to z) \in N$. Hence, $(x \to z) \equiv (y \to z)$. Secondly, for arbitrary $x, y, z \in T$, since $[z * (z \to x)] * (x \to y) \le y$, by (EIF6) there exists $c_3 \in N$, such that $z * ((z \to x) * c_3) \le [z * (z \to x)] * (x \to y) \le y$. Thereby, using the 2-residuation principle, $(z \to x) * c_3 \le z \to y$, and $c_3 \le (z \to x) \to (z \to y) \in N$ through (EIF1). On the other hand, there exists $c_4 \in N$ such that $z * [(z \to y) * c_4] \le (z * (z \to y)) * (y \to x)$ by (EIF6), and further, $z * [(z \to y) * c_4] \le y * (y \to x) \le x$. Then, by the 2-residuation principle and (EIF1), $c_4 \le (z \to y) \to (z \to x) \in N$. Hence, $(z \to y) \equiv (z \to x)$. As $(x \to a) \equiv (y \to a)$ and $(y \to a) \equiv (y \to b)$, using transitivity, we obtain $(x \to a) \equiv (y \to b)$.
(v) We verify $(x \rightsquigarrow a) \equiv (y \rightsquigarrow b)$. As N meets (EIF10), $x \to y \in N \Rightarrow x \rightsquigarrow y \in N$ and $y \to x \in N \Rightarrow y \rightsquigarrow x \in N$. Firstly, because $(x \rightsquigarrow z) * [(y \rightsquigarrow x) * y] \le (x \rightsquigarrow z) * x \le z$, by (EIF6) there exists $c_1 \in N$ such that $(c_1 * (x \rightsquigarrow z)) * y \le (x \rightsquigarrow z) * [(y \rightsquigarrow x) * y]$, so $(c_1 * (x \rightsquigarrow z)) * y \le z$; then by (GRLG3), $c_1 \le (x \rightsquigarrow z) \rightsquigarrow (y \rightsquigarrow z)$. Due to (EIF1), $(x \rightsquigarrow z) \rightsquigarrow (y \rightsquigarrow z) \in N$, using (EIF10), $(x \rightsquigarrow z) \to (y \rightsquigarrow z) \in N$. Moreover, $(y \rightsquigarrow z) * [(x \rightsquigarrow y) * x] \le (y \rightsquigarrow z) * y \le z$ according to the 2-residuation principle. Since by (EIF6), there exists $c_2 \in N$ such that $(c_2 * (y \rightsquigarrow z)) * x \le (y \rightsquigarrow z) * [(x \rightsquigarrow y) * x]$, $(c_2 * (y \rightsquigarrow z)) * x \le z$. Then, by the 2-residuation principle and (EIF1), $c_2 \le (y \rightsquigarrow z) \rightsquigarrow (x \rightsquigarrow z) \in N$. Thus, by (EIF10), $(y \rightsquigarrow z) \to (x \rightsquigarrow z) \in N$. Hence, $(x \rightsquigarrow z) \equiv (y \rightsquigarrow z)$. Secondly, because $(x \rightsquigarrow y) * [(z \rightsquigarrow x) * z] \le (x \rightsquigarrow y) * x \le y$, and by (EIF6) there exists $c_3 \in N$ such that $(c_3 * (z \rightsquigarrow x)) * z \le (x \rightsquigarrow y) * [(z \rightsquigarrow x) * thez]$, we get $(c_3 * (z \rightsquigarrow x)) * z \le y$. According to 2-residuation principle, $c_3 \le (z \rightsquigarrow x) \rightsquigarrow (z \rightsquigarrow y)$. Then, according to (EIF1) and (EIF10), $(z \rightsquigarrow x) \to (z \rightsquigarrow y) \in N$. In addition, because $(y \rightsquigarrow x) * [(z \rightsquigarrow y) * z] \le x$, and there exists $c_4 \in N$ such that $(c_4 * (z \rightsquigarrow y)) * z \le (y \rightsquigarrow x) * [(z \rightsquigarrow y) * z]$ considering (EIF6), it means $(c_4 * (z \rightsquigarrow y)) * z \le x$. By 2-residuation principle, we have $c_4 \le (z \rightsquigarrow y) \rightsquigarrow (z \rightsquigarrow x)$. Then, $(z \rightsquigarrow y) \to (z \rightsquigarrow x) \in N$. Hence, $(z \rightsquigarrow x) \equiv (z \rightsquigarrow y)$. Furthermore, by transitivity we can get $(x \rightsquigarrow a) \equiv (y \rightsquigarrow a)$, $(y \rightsquigarrow a) \equiv (y \rightsquigarrow b) \Rightarrow (x \rightsquigarrow a) \equiv (y \rightsquigarrow b)$.
(vi) wW prove $\omega(x) \equiv \omega(y)$. By (EIF7), $\omega(x \to y) \in N$. As $\omega(x \to y) \le \omega(x) \to \omega(y)$ by (ET7), $\omega(x) \to \omega(y) \in N$ by (EIF1). Similarly, $\omega(y \to x) \le \omega(y) \to \omega(x) \in N$. Therefore,

$\omega(x) \equiv \omega(y)$.
(vii) We prove that $\theta(x) \equiv \theta(y)$. As $\theta(x \to y) \geq x \to y \in N$ according to (ET2), then $\theta(x \to y) \in N$ by (EIF1). By (EIF8) and (7) of Proposition 5, we have $\theta(x \to y) \leq \theta(x) \to \theta(y)$, and as a result $\theta(x) \to \theta(y) \in N$ by (EIF1). For the same reason, $y \to x \leq \theta(y \to x) \leq \theta(y) \to \theta(x) \in N$. Thereby, $\theta(x) \equiv \theta(y)$. □

On the basis of the above proposition, we investigate the quotient set generated by the congruence relation of ET-algebras. The definition of quotient algebra is as below.

Definition 17. *Given an ET-algebra* $(T, \wedge, \vee, *, \to, \rightsquigarrow, \omega, \theta, 0, c, 1)$ *and an NEIF* J *of* T. *The relation* $\equiv$ *is a congruence relation on* T, *defined as:* $x \equiv y$ *iff* $x \to y \in J$, $y \to x \in J$. *A structure* $T_{\equiv} = \{[x]_J | x \in T\}$ *is called as a quotient algebra of* T, *where* $[x]_J$ *represents the equivalent class of* x *regarding* $\equiv$, *and order relation on* $T_{\equiv}$ *is denoted by:* $[x]_J \leq [y]_J \Leftrightarrow x \to y \in J$. *Some operations on* $T_{\equiv}$ *are defined as below:* $[x]_J \sqcap [y]_J$ *denoted by* $[x \wedge y]_J$, $[x]_J \sqcup [y]_J$ *denoted by* $[x \vee y]_J$, $[x]_J \circledast [y]_J$ *denoted by* $[x * y]_J$, $[x]_J \to_J [y]_J$ *denoted by* $[x \to y]_J$, $[x]_J \rightsquigarrow_J [y]_J$ *denoted by* $[x \rightsquigarrow y]_J$, $\omega_J([x]_J)$ *denoted by* $[\omega(x)]_J$ *and* $\theta_J([x]_J)$ *denoted by* $[\theta(x)]_J$.

Remark 3. *Take an ET-algebra* $(T, \wedge, \vee, *, \to, \rightsquigarrow, \omega, \theta, 0, c, 1)$ *and an NEIF* N *of* T. $\equiv$ *is a congruence relation on* T; $T_{\equiv} = \{[t]_N | t \in T\}$ *is a quotient algebra of* T. *Then, when taking arbitrary* $s, t \in T$, *if* $s \leq t$, *then* $[s]_N \leq [t]_N$. *Since when* $s \leq t$, $s \to s \leq s \to t$, *and then according to* $(EIF3)$ *and* $(EIF1)$, $s \to t \in N$. *Thus,* $[s]_N \leq [t]_N$. *The converse is not necessarily true. For instance, in Example 7 above,* $[0,1] \to [0,k] = [k,1] \in S$; *i.e.,* $[0,1]_S \leq [0,k]_S$, *but* $[0,1] > [0,k]$.

In the following, we certify that the quotient algebra of ET-algebra is also an ET-algebra.

Lemma 1. *Taken an ET-algebra* $(T, \wedge, \vee, *, \to, \rightsquigarrow, \omega, \theta, 0, c, 1)$, *and an NEIF* N *of* T. *Then, for arbitrary* $x, y, z \in T$, $(x * y) \to z \in N$ *is equivalent to* $y \to (x \to z) \in N$, *and* $(x * y) \rightsquigarrow z \in N$ *is equivalent to* $x \to (y \rightsquigarrow z) \in N$.

Proof. (1) Assume $(x * y) \to z \in N$, and because $(x * y) * [(x * y) \to z] \leq z$, by (EIF6), there exists $c_1 \in N$ such that $x * (y * c_1) \leq (x * y) * [(x * y) \to z]$; further, $x * (y * c_1) \leq z$. Then, by the 2-residuation principle, $c_1 \leq y \to (x \to z)$, so $y \to (x \to z) \in N$ according to (EIF1). Conversely, assume $y \to (x \to z) \in N$, and because $x * [y * (y \to (x \to z))] \leq z$, according to (EIF6) there exists $c_2 \in N$ satisfying $(x * y) * c_2 \leq x * [y * (y \to (x \to z))] \leq z$. Then, by the 2-residuation principle, $c_2 \leq (x * y) \to z$. By (EIF1), $(x * y) \to z \in N$.
(2) Assume $(x * y) \rightsquigarrow z \in N$. Since $[(x * y) \rightsquigarrow z] * (x * y) lez$, by (EIF6) there exists $c_1 \in N$ satisfying $(c_1 * x) * y \leq [(x * y) \rightsquigarrow z] * (x * y) \leq z$, and then $c_1 * x \leq y \rightsquigarrow z$ through (GRLG3), and further, $c_1 \leq x \rightsquigarrow (y \rightsquigarrow z)$. Thus, by (EIF1) $x \rightsquigarrow (y \rightsquigarrow z) \in N$. Since N meets (EIF10), $x \to (y \rightsquigarrow z) \in N$. Conversely, assume $x \to (y \rightsquigarrow z) \in N$. As $(x * [x \to (y \rightsquigarrow z)]) * y \leq z$, using (EIF6) there exists $c_2 \in N$ satisfying $(x * y) * c_2 \leq (x * [x \to (y \rightsquigarrow z)]) * y \leq z$, and then by the 2-residuation principle, $c_2 \leq (x * y) \to z$. Thus, $(x * y) \to z \in N$ by (EIF1); further, $(x * y) \rightsquigarrow z \in N$ by (EIF10). □

Proposition 10. *Given an ET-algebra,* $(T, \wedge, \vee, *, \to, \rightsquigarrow, \omega, \theta, 0, c, 1)$, *and an NEIF* N *of* T. *The congruence relation* $\equiv$ *on* T *denoted by* $e \equiv f$ *iff* $e \to f \in N$, $f \to e \in N$. *Then, the quotient algebra* $(T_{\equiv}, \sqcap, \sqcup, \circledast, \to_N, \rightsquigarrow_N, \omega_N, \theta_N, [0]_N, [c]_N, [1]_N)$ *is an ET-algebra.*

Proof. (1) We first prove that $(T_{\equiv}, \sqcap, \sqcup, \circledast, \to_N, \rightsquigarrow_N, [0]_N, [1]_N)$ is a GRLG. Obviously, because $0 \to x \in N$ and $x \to 1 \in N$ for arbitrary $x \in T$, it means $[0]_N \leq [x]_N \leq [1]_N$ for arbitrary $[x]_N \in T_{\equiv}$, and then it meets (GRLG1). (GRLG2) is also obviously satisfied. Then, on the one hand, for arbitrary $[l]_N, [m]_N \in T_{\equiv}$, $[l]_N \circledast [m]_N \leq [z]_N \Leftrightarrow [l * m]_N \leq [z]_N \Leftrightarrow (l * m) \to z \in N \Leftrightarrow m \to (l \to z) \in N \Leftrightarrow [m]_N \leq [l \to z]_N \Leftrightarrow [m]_N \leq [l]_N \to_N [z]_N$ by Lemma 1 above. On the other hand, according to the definition and (EIF10), $[l]_N \circledast [m]_N \leq [z]_N \Leftrightarrow [l * m]_N \leq [z]_N \Leftrightarrow (l * m) \rightsquigarrow z \in N$. Then, using Lemma 1,

$(l * m) \rightsquigarrow z \in N \Leftrightarrow l \to (m \rightsquigarrow z) \in N \Leftrightarrow [l]_N \leq [m \rightsquigarrow z]_N \Leftrightarrow [l]_N \leq [m]_N \rightsquigarrow_N [z]_N$, so it meets (GRLG3); that is, $T_\equiv = (T_\equiv, \sqcap, \sqcup, \circledast, \to_N, \rightsquigarrow_N, \omega_N, \theta_N, [0]_N, [c]_N, [1]_N)$ meets (ET1).
(2) By (EIF3), $x \to x \in N$ for arbitrary $x \in T$. Since $\omega(x) \leq x$, $x \to x \leq \omega(x) \to x$, and then by (EIF1), $\omega(x) \to x \in N$. Thus, $[\omega(x)]_N \leq [x]_N$; i.e., $\omega_N([x]_N) \leq [x]_N$. Similarly, because $x \leq \theta(x) \Rightarrow x \to x \leq x \to \theta(x)$, by (EIF1) $x \to \theta(x) \in N$; i.e., $[x]_N \leq [\theta(x)]_N = \theta_N([x]_N)$. Moreover, it is obvious that $\omega_N(\omega_N([x]_N)) \leq \omega_N([x]_N)$ and $\theta_N([x]_N) \leq \theta_N(\theta_N([x]_N))$. Meanwhile, since T meets (ET2), $\omega(\omega(x)) = \omega(x)$, $\theta(\theta(x)) = \theta(x)$, and we have $\omega(x) \to \omega(\omega(x)) \in N$ and $\theta(\theta(x)) \to \theta(x) \in N$ by (EIF3); i.e., $[\omega(x)]_N \leq [\omega(\omega(x))]_N$ and $[\theta(\theta(x))]_N \leq [\theta(x)]_N$; further, $\omega_N([x]_N) \leq \omega_N(\omega_N([x]_N))$ and $\theta_N(\theta_N([x]_N)) \leq \theta_N([x]_N)$. Thus, $\omega_N(\omega_N([x]_N)) = \omega_N([x]_N)$, $\theta_N(\theta_N([x]_N)) = \theta_N([x]_N)$, $T_\equiv$ meets (ET2). In fact, for arbitrary $[e]_N, [f]_N \in T_\equiv$, when $[e]_N \leq [f]_N$, i.e., $e \to f \in N$, by (EIF7), $\omega(e \to f) \in N$, since T meets (ET7), and by (EIF1), $\omega(e \to f) \leq \omega(e) \to \omega(f) \in N$, which means $[\omega(e)]_N \leq [\omega(f)]_N$—that is, $\omega_N([e]_N) \leq \omega_N([f]_N)$. Thus, ω_N is increasing on $T_\equiv$.
(3) Since T meets (ET3), by definition $\omega_N([e]_N \sqcap [f]_N) = \omega_N([e \wedge f]_N) = [\omega(e \wedge f)]_N = [\omega(e) \wedge \omega(f)]_N = [\omega(e)]_N \sqcap [\omega(f)]_N = \omega_N([e]_N) \sqcap \omega_N([f]_N)$ when taking arbitrary $[e]_N, [f]_N \in T_\equiv$. Similarly, we have $\theta_N([e]_N \sqcap [f]_N) = \theta_N([e]_N) \sqcap \theta_N([f]_N)$. Hence, $T_\equiv$ meets (ET3).
(4) As above, $T_\equiv$ meets (ET4).
(5) Evidently, $\omega_N([c]_N) = [\omega(c)]_N = [0]_N$ and $\theta_N([c]_N) = [\theta(c)]_N = [1]_N$.
(6) For arbitrary $[x]_N \in T_\equiv$, by definition, $\omega_N(\theta_N([x]_N)) = \omega_N([\theta(x)]_N) = [\omega(\theta(x))]_N$; then because T meets (ET6), $[\omega(\theta(x))]_N = [\theta(x)]_N = \theta_N([x]_N)$, and for the same reason, $\theta_N(\omega_N([x]_N)) = \omega_N([x]_N)$. Hence, $T_\equiv$ meets (ET6).
(7) Since T meets (ET7), by definition and Remark 3 $\omega_N([e]_N \to_N [f]_N) = \omega_N([e \to f]_N) = [\omega(e \to f)]_N \leq [\omega(e) \to \omega(f)]_N = [\omega(e)]_N \to_N [\omega(f)]_N = \omega_N([e]_N) \to_N \omega_N([f]_N)$ when taking arbitrary $[e]_N, [f]_N \in T_\equiv$. Similarly, $\omega_N([e]_N \rightsquigarrow_N [f]_N) \leq \omega_N([e]_N) \rightsquigarrow_N \omega_N([f]_N)$. Hence, $T_\equiv$ meets (ET7).
(8) When $\omega_N([e]_N) = \omega_N([f]_N)$ and $\theta_N([e]_N) = \theta_N([f]_N)$ for every $[e]_N, [f]_N \in T_\equiv$; on the one hand, $\omega_N([e]_N) \leq \omega_N([f]_N)$ and $\theta_N([e]_N) \leq \theta_N([f]_N)$ iff $[\omega(e)]_N \leq [\omega(f)]_N$ and $[\theta(e)]_N \leq [\theta(f)]_N$ iff $\omega(e) \to \omega(f) \in N$ and $\theta(e) \to \theta(f) \in N$ by definition; then by (EIF9), we have $e \to f \in N$; i.e., $[e]_N \leq [f]_N$. On the other hand, $\omega_N([e]_N) \geq \omega_N([f]_N)$ and $\theta_N([e]_N) \geq \theta_N([f]_N) \Rightarrow [f]_N \leq [e]_N$. Thus, $[e]_N = [f]_N$, $T_\equiv$ meets (ET8).
(9) Since T meets (ET9), by definition and Remark 3 $\omega_N([e]_N) \to_N \omega_N([f]_N) = [\omega(e)]_N \to_N [\omega(f)]_N = [\omega(e) \to \omega(f)]_N \leq [\omega(\omega(e) \to \omega(f))]_N = \omega_N([\omega(e) \to \omega(f)]_N) = \omega_N([\omega(e)]_N \to_N [\omega(f)]_N) = \omega_N(\omega_N([e]_N) \to_N \omega_N([f]_N))$ when taking random $[e]_N$, $[f]_N \in T_\equiv$. Similarly, $\omega_N([e]_N) \rightsquigarrow_N \omega_N([f]_N) \leq \omega_N(\omega_N([e]_N) \rightsquigarrow_N \omega_N([f]_N))$. Hence, $T_\equiv$ meets (ET9). □

Example 8. *Given an ET-algebra $(T, \wedge, \vee, *, \to, \rightsquigarrow, \omega, \theta, [0,0], c, [1,1])$ and its NEIF M, as shown in Example 7, we define the congruence relation $\equiv$ as $e \equiv f$ when and only when $e \to f \in M$; $f \to e \in M$; and operators $\circledast$, $\to_M$ and $\rightsquigarrow_M$ of the quotient algebra $T_\equiv = \{[0,0]_M, [0,g]_M, [g,g]_M, [0,1]_M, [g,1]_M, [1,1]_M\}$ are as shown in Tables 7–9 below:*

Table 7. The operation $\circledast$ of algebra in Example 8.

$\circledast$	$[0,0]_M$	$[0,g]_M$	$[g,g]_M$	$[0,1]_M$	$[g,1]_M$	$[1,1]_M$
$[0,0]_M$	$[0,0]_M$	$[0,0]_M$	$[0,0]_M$	$[0,0]_M$	$[0,0]_M$	$[0,0]_M$
$[0,g]_M$	$[0,0]_M$	$[0,0]_M$	$[0,g]_M$	$[0,0]_M$	$[0,g]_M$	$[0,g]_M$
$[g,g]_M$	$[0,0]_M$	$[0,g]_M$	$[g,g]_M$	$[0,g]_M$	$[g,g]_M$	$[g,g]_M$
$[0,1]_M$	$[0,0]_M$	$[0,g]_M$	$[0,g]_M$	$[0,g]_M$	$[0,1]_M$	$[0,1]_M$
$[g,1]_M$	$[0,0]_M$	$[0,g]_M$	$[g,g]_M$	$[0,1]_M$	$[g,1]_M$	$[g,1]_M$
$[1,1]_M$	$[0,0]_M$	$[0,g]_M$	$[g,g]_M$	$[0,1]_M$	$[g,1]_M$	$[1,1]_M$

Table 8. The operation $\rightarrow_M$ of algebra in Example 8.

$\rightarrow_M$	$[0,0]_M$	$[0,g]_M$	$[g,g]_M$	$[0,1]_M$	$[g,1]_M$	$[1,1]_M$
$[0,0]_M$	$[1,1]_M$	$[1,1]_M$	$[1,1]_M$	$[1,1]_M$	$[1,1]_M$	$[1,1]_M$
$[0,g]_M$	$[0,1]_M$	$[1,1]_M$	$[1,1]_M$	$[1,1]_M$	$[1,1]_M$	$[1,1]_M$
$[g,g]_M$	$[0,0]_M$	$[0,1]_M$	$[1,1]_M$	$[0,1]_M$	$[1,1]_M$	$[1,1]_M$
$[0,1]_M$	$[0,0]_M$	$[g,1]_M$	$[g,1]_M$	$[1,1]_M$	$[1,1]_M$	$[1,1]_M$
$[g,1]_M$	$[0,0]_M$	$[0,g]_M$	$[g,g]_M$	$[0,1]_M$	$[1,1]_M$	$[1,1]_M$
$[1,1]_M$	$[0,0]_M$	$[0,g]_M$	$[g,g]_M$	$[0,1]_M$	$[g,1]_M$	$[1,1]_M$

Table 9. The operation $\rightsquigarrow_M$ of algebra in Example 8.

$\rightsquigarrow_M$	$[0,0]_M$	$[0,g]_M$	$[g,g]_M$	$[0,1]_M$	$[g,1]_M$	$[1,1]_M$
$[0,0]_M$	$[1,1]_M$	$[1,1]_M$	$[1,1]_M$	$[1,1]_M$	$[1,1]_M$	$[1,1]_M$
$[0,g]_M$	$[0,g]_M$	$[1,1]_M$	$[1,1]_M$	$[1,1]_M$	$[1,1]_M$	$[1,1]_M$
$[g,g]_M$	$[0,0]_M$	$[0,1]_M$	$[1,1]_M$	$[0,1]_M$	$[1,1]_M$	$[1,1]_M$
$[0,1]_M$	$[0,g]_M$	$[g,1]_M$	$[g,1]_M$	$[1,1]_M$	$[1,1]_M$	$[1,1]_M$
$[g,1]_M$	$[0,0]_M$	$[0,g]_M$	$[g,g]_M$	$[0,1]_M$	$[1,1]_M$	$[1,1]_M$
$[1,1]_M$	$[0,0]_M$	$[0,g]_M$	$[g,g]_M$	$[0,1]_M$	$[g,1]_M$	$[1,1]_M$

Additionally, where $\omega_M([x]_M) = [\omega(x)]_M$, $\theta_M([x]_M) = [\theta(x)]_M$ *and* $[c]_M = [0,1]_M$ *for arbitrary* $[x]_M \in T_{\equiv}$*. Since* $(T_{\equiv}, \sqcap, \sqcup, \circledast, \rightarrow_M, \rightsquigarrow_M, \omega_M, \theta_M, [0,0]_M, [c]_M, [1,1]_M)$ *meets* $(ET1) \sim (ET9)$*, it is also an ET-algebra.*

5. Conclusions

Overall, we mainly presented IGRLGs and ET-algebras as the generalized algebraic structures of interval-valued, residuated lattices associated with interval-valued pseudo-overlap functions, and enriched the content of interval-valued fuzzy logic. Next, we declare the main conclusions: (1) The concept of IGRLGs was given, and the representable IPOFs composed of a POF and their IVRIs can generate the IGRLGs. (2) The notion of IGRLGs was generalized, and the expanded IGRLGs and their properties were stated. (3) The ET-algebras corresponding to the expanded IGRLGs were proposed. We analyzed their properties and gave the definition of filters. Finally, the congruence relation and induced quotient structures of them were discussed, and the quotient algebra of an ET-algebra was still an ET-algebra. At the same time, there were some shortcomings in the article—for example, can IGRLGs only be generated by representable IPOFs, and what are the operators corresponding to them one by one? Moreover, only one kind of filter of ET-algebra was given, and other special filters need to be studied.

Furthermore, we can also study the extension of other operators and algebraic structures for interval-valued fuzzy logic and explore the logic corresponding to algebraic structures (see [24,25]). In addition, other expansions of fuzzy sets and their applications are also a new research direction (see [26–30]).

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