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Best Proximity Point for $\Gamma_\tau\mathcal{F}$-Fuzzy Proximal Contraction

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Abstract: In this writing, first, we disclose the first and second category of a $\Gamma_\tau\mathcal{F}$-fuzzy proximal contraction for a mapping $\mathcal{O} : \mathfrak{U} \to \mathfrak{V}$ which is nonself and also declare a fuzzy q-property to confirm the existence of the best proximity point for nonself function $\mathcal{O}$. Then, we discover a few results using the $\Gamma_\tau\mathcal{F}$-fuzzy proximal contraction of the first category for a continuous and discontinuous nonself function $\mathcal{O}$ in a non-Archimedean fuzzy metric space. Later, we discuss another result for the $\Gamma_\tau\mathcal{F}$-fuzzy proximal contraction of the second category as well. In between the fuzzy proximal theorems, many examples are presented in support of the definitions and theorems proved in this writing.

Keywords: $\Gamma_\tau\mathcal{F}$-fuzzy proximal contraction; fuzzy p- and q-properties; non-Archimedean fuzzy metric space

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1. Introduction

In 1969, Fan [1] established a crisp approximation theorem. Fan [1] asserted: "suppose that $\mathcal{O}$ is a continuous map $\mathcal{O} : \mathfrak{U} \to \mathfrak{X}$, where $\mathfrak{X}$ is Hausdorff locally convex topological vector space and $\mathfrak{U}$ is a nonempty compact convex subset with a semi-group norm p then there is a member $\mu \in \mathfrak{U}$ holding the assertion that $\rho_p(\mu, \mathcal{O}\mu) = \rho_p(\mathcal{O}\mu, \mathfrak{U})$". This theorem enhanced an approximate answer of the fixed-point equation $\mathcal{O}\mu = \mu$, where the function $\mathcal{O} : \mathfrak{U} \to \mathfrak{V}$ has no fixed point, and $\mathfrak{U}$ and $\mathfrak{V}$ are nonempty sets. The distance $\rho(\mu, \mathcal{O}\mu)$ should be at least $\rho(\mu, \nu) = \inf\{\rho(\mu, \nu) : \mu \in \mathfrak{U} \text{ and } \nu \in \mathfrak{V}\}$, and the best proximity point theorem guarantees the existence of a member of μ^* such that $\rho(\mu^*, \mathcal{O}\mu^*) = \rho(\mathfrak{U}, \mathfrak{V})$; this member is called the best proximity point for map $\mathcal{O}$.

The crisp set theory was enhanced by the mathematician Zadeh [2] in 1965 in his seminal paper by introducing a membership function. A membership function is defined from a nonempty set $\mathfrak{X}$ to a closed interval $[0, 1]$, which means that a membership value of any member from the set $\mathfrak{X}$ belongs to the closed interval $[0, 1]$; basically, a fuzzy set is a generalization of a characteristic function.

With the inspiration of this fuzzy theory, Kramosil and Michalak [3] introduced the notion of a new distance space called fuzzy metric space, and the concept of this distance space was improved by George and Veeramani [4] by defining a Hausdorff topology. Further, the fixed point theory associated with the fuzzy concept was first experimented with by Grabic [5] by demonstrating fuzzy Banach [6] and Edelstein [7] contraction theorems.

In this work, we are investigating the uniqueness and existence of the best proximity point in a non-Archimedean fuzzy (distance) metric space, and also extending, generalizing and fuzzifying the proved results in various distance spaces. We define a few proximal fuzzy contractions to prove the propositions for a nonself function. Moreover, many supportive examples are given to present the fruitfulness of the given theorems.

2. Preliminaries

First, we recollect some elementary concepts to support the main outcome.

Definition 1 ([8])**.** *A continuous triangular norm is a binary operation defined as* $\circledast : [0,1] \times [0,1] \to [0,1]$ *(t-norm in short) if* $\circledast$ *holds the following assertions:*

1. $\circledast$ *is commutative if* $\iota \circledast \kappa = \kappa \circledast \iota$ *and* $\circledast$ *is associative , if* $\iota \circledast (\kappa \circledast \gamma) = (\iota \circledast \kappa) \circledast \gamma$ *for all* $\iota, \kappa, \gamma \in [0,1]$;
2. *The binary operation* $\circledast$ *is continuous;*
3. $1 \circledast \iota = \iota$ *for all* $\iota \in [0,1]$;
4. $\iota \circledast \kappa \leq \gamma \circledast \zeta$ *when* $\iota \leq \gamma$ *and* $\kappa \leq \zeta$ *with* $\iota, \kappa, \gamma, \zeta \in [0,1]$.

Definition 2 ([4])**.** *The ordered triple is called a fuzzy metric space, if* $\mathfrak{X}$ *is a nonempty set,* $\circledast$ *is a continuous t-norm and* $\mathcal{S}$ *is defined as a fuzzy set* $\mathfrak{X} \times \mathfrak{X} \times (0, +\infty)$ *holding the following assertions;*

1. $\mathcal{S}(\mu, \nu, r) > 0$;
2. $\mu = \nu$ *if and only if* $\mathcal{S}(\mu, \nu, r) = 1$;
3. $\mathcal{S}(\mu, \nu, r) = \mathcal{S}(\nu, \mu, r)$;
4. $\mathcal{S}(\mu, \eta, r + s) \geq \mathcal{S}(\mu, \nu, r) \circledast \mathcal{S}(\nu, \eta, s)$;
5. $\mathcal{S}(\mu, \nu, \cdot) : (0, +\infty) \to (0, 1]$ *is continuous.*
   If we change 4. by
6. $\mathcal{S}(\mu, \eta, max\{r, s\}) \geq \mathcal{S}(\mu, \nu, r) \circledast \mathcal{S}(\nu, \eta, s)$,

for all $\mu, \nu, \eta \in \mathfrak{X}$ *and* $s, r > 0$*, then the ordered triplet* $(\mathfrak{X}, \mathcal{S}, \circledast)$ *is called a non-Archimedean fuzzy metric space. Every non-Archimedean fuzzy metric space is a fuzzy metric space, since assertion 6 implies assertion 4.*

Definition 3 ([4,8])**.** *Suppose that* $(\mathfrak{X}, \mathcal{S}, \circledast)$ *is a fuzzy metric space. Then, for every* $m, n \in \mathbb{N}$ *and for all* $r > 0$:

1. *A sequence* $\{\mu_n\}$ *is called convergent to* $\mu \in \mathfrak{X}$ *if* $\lim_{n \to +\infty} \mathcal{S}(\mu_n, \mu, r) \to 1$.
2. $\{\mu_n\}$ *is called a Cauchy sequence if* $\lim_{n \to +\infty} \mathcal{S}(\mu_n, \mu_{n+m}, r) = 1$.
3. *If every Cauchy sequence converges to* $\mu \in \mathfrak{X}$ *then the space is called complete.*

Recently, Patel and Radenović [9] introduced a new class of mappings, the Γ_τ-family, and they also gave $\Gamma_\tau\mathcal{F}$-fuzzy contractive mappings, which were weaker than the group of mappings by Huang [10].

Definition 4 ([9])**.** *Let* $\triangle_{\Gamma_\tau}$ *indicate the group of all continuous map* $\Gamma_\tau : (\mathbb{R}^+)^4 \to \mathbb{R}$ *satisfying:*

1. *For all* $t_1, t_2, t_3, t_4 \in \mathbb{R}^+$ *with* $max(t_1, t_2, t_3, t_4) = 1$*, there exists* $\tau \in (0,1)$ *such that* $\Gamma_\tau(t_1, t_2, t_3, t_4) = \tau$.

Examples:

1. $\Gamma_\tau(t_1, t_2, t_3, t_4) = \tau + \mathcal{L} \cdot log_e max(t_1, t_2, t_3, t_4)$, where $\mathcal{L} \in \mathbb{R}^+$.
2. $\Gamma_\tau(t_1, t_2, t_3, t_4) = \dfrac{\tau}{max(t_1, t_2, t_3, t_4)}$.
3. $\Gamma_\tau(t_1, t_2, t_3, t_4) = \dfrac{e \cdot \tau}{e^{max(t_1, t_2, t_3, t_4)}}$.
   Here $\tau \in (0,1)$ and then $\Gamma_\tau \in \triangle_{\Gamma_\tau}$.

The following lemma is essential to prove our key theorems.

Lemma 1 ([10])**.** *Suppose that* $\{\mu_n\}$ *is a sequence in a fuzzy metric space* $(\mathfrak{X}, \mathcal{S}, \circledast)$ *such that for every* $n \in \mathbb{N}$,

$$\lim_{r \to 0^+} \mathcal{S}(\mu_n, \mu_{n+1}, r) > 0,$$

and for any $r > 0$,

$$\lim_{n\to+\infty} \mathcal{S}(\mu_n, \mu_{n+1}, r) = 1.$$

If $\{\mu_n\}$ is not a Cauchy sequence in $\mathfrak{X}$, then there exists $\epsilon \in (0,1)$, $r_0 > 0$ and two subsequences of non-negative integers $\{n_k\}$, $\{m_k\}$, $n_k > m_k > k$ where $k \in \mathbb{N}$, such that

$$\{\mathcal{S}(\mu_{m_k}, \mu_{n_k}, r_0)\}, \{\mathcal{S}(\mu_{m_k}, \mu_{n_k+1}, r_0)\}, \{\mathcal{S}(\mu_{m_k-1}, \mu_{n_k}, r_0)\},$$
$$\{\mathcal{S}(\mu_{m_k-1}, \mu_{n_k+1}, r_0)\}, \{\mathcal{S}(\mu_{m_k+1}, \mu_{n_k+1}, r_0)\}$$

tend to $1-\epsilon$ as $k \to +\infty$.

3. Main Results

Consider two nonempty subsets $\mathfrak{U}$ and $\mathfrak{V}$ of a fuzzy metric space $(\mathfrak{X}, \mathcal{S}, \circledast)$. We use the following notations:

$$\begin{aligned} \mathfrak{U}_0(r) &= \{\mu \in \mathfrak{U} : \mathcal{S}(\mu, \nu, r) = \mathcal{S}(\mathfrak{U}, \mathfrak{V}, r) \text{ for some } \nu \in \mathfrak{V}, r > 0\}, \\ \mathfrak{V}_0(r) &= \{\nu \in \mathfrak{V} : \mathcal{S}(\mu, \nu, r) = \mathcal{S}(\mathfrak{U}, \mathfrak{V}, r) \text{ for some } \mu \in \mathfrak{U}, r > 0\} \end{aligned}$$

where

$$\mathcal{S}(\mathfrak{U}, \mathfrak{V}, r) = \sup\{\mathcal{S}(\mu, \nu, r) : \mu \in \mathfrak{U},\ \nu \in \mathfrak{V}, r > 0\}.$$

We recall that $\mu \in \mathfrak{U}$ is a best proximity point of the map $\mathcal{O} : \mathfrak{U} \to \mathfrak{V}$ if $\mathcal{S}(\mu, \mathcal{O}\mu, r) = \mathcal{S}(\mathfrak{U}, \mathfrak{V}, r)$. We may observe a best proximity point turn to a fixed point if $\mathfrak{U} = \mathfrak{V}$.

Definition 5 ([11])**.** *Suppose a pair of nonempty subsets $(\mathfrak{U}, \mathfrak{V})$ of a non-Archimedean fuzzy metric space $\mathfrak{X}$ with $\mathfrak{U}_0(r) \neq \phi$. Then, the pair $(\mathfrak{U}, \mathfrak{V})$ possesses the fuzzy p-property if*

$$\left.\begin{aligned} \mathcal{S}(\mu_1, \nu_1, r) = \mathcal{S}(\mathfrak{U}, \mathfrak{V}, r) \\ \mathcal{S}(\mu_2, \nu_2, r) = \mathcal{S}(\mathfrak{U}, \mathfrak{V}, r) \end{aligned}\right\} \text{implies } \mathcal{S}(\mu_1, \mu_2, , r) = \mathcal{S}(\nu_1, \nu_2, r).$$

where $\mu_1, \mu_2 \in \mathfrak{U}_0(r)$ and $\nu_1, \nu_2 \in \mathfrak{V}_0(r)$.

The pair $(\mathfrak{U}, \mathfrak{U})$ has the fuzzy p-property.

Definition 6 ([12])**.** *A set $\mathfrak{U}$ is said to be approximately compact with respect to $\mathfrak{V}$ if every sequence $\{\mu_n\}$ of $\mathfrak{U}$ satisfying the assertion $\mathcal{S}(\nu, \mu_n, r) \to \mathcal{S}(\nu, \mathfrak{U}, r)$ for some $\nu \in \mathfrak{V}$ has a convergent subsequence.*

Every set is approximately compact with respect to itself. Now, we define $\Gamma_\tau\mathcal{F}$-fuzzy proximal contractions of different categories.

Definition 7 ($\Gamma_\tau\mathcal{F}$-fuzzy proximal contraction of the first category)**.** *A mapping $\mathcal{O} : \mathfrak{U} \to \mathfrak{V}$ is said to be a $\Gamma_\tau\mathcal{F}$-fuzzy proximal contraction of the first category if*

$$\left.\begin{aligned} \mathcal{S}(\mu_1, \mathcal{O}\nu_1, r) = \mathcal{S}(\mathfrak{U}, \mathfrak{V}, r) \\ \mathcal{S}(\mu_2, \mathcal{O}\nu_2, r) = \mathcal{S}(\mathfrak{U}, \mathfrak{V}, r) \\ \mathcal{S}(\mu_1, \mu_2, r), \mathcal{S}(\nu_1, \nu_2, r) > 0 \end{aligned}\right\} \text{implies } \Gamma_\tau(\mathcal{S}(\mu_1, \nu_2, r), \mathcal{S}(\mu_2, \nu_1, r), \mathcal{S}(\mu_1, \nu_1, r), \mathcal{S}(\mu_2, \nu_2, r)) \cdot \mathcal{F}(\mathcal{S}(\mu_1, \mu_2, r)) \geq \mathcal{F}(\mathcal{S}(\nu_1, \nu_2, r)) \quad (1)$$

for all $\mu_1, \mu_2, \nu_1, \nu_2 \in \mathfrak{U}$, $\Gamma_\tau \in \triangle_{\Gamma_\tau}$ and $\mathcal{F} \in \triangle_{\mathcal{F}}$.

Definition 8 ($\Gamma_\tau\mathcal{F}$-fuzzy proximal contraction of the second category)**.** *A mapping $\mathcal{O}:\mathfrak{U}\to\mathfrak{V}$ is said to be a $\Gamma_\tau\mathcal{F}$-fuzzy proximal contraction of the second category if*

$$\left.\begin{array}{r}\mathcal{S}(\mu_1,\mathcal{O}\nu_1,r)=\mathcal{S}(\mathfrak{U},\mathfrak{V},r)\\ \mathcal{S}(\mu_2,\mathcal{O}\nu_2,r)=\mathcal{S}(\mathfrak{U},\mathfrak{V},r)\\ \mathcal{S}(\mu_1,\mu_2,r),\mathcal{S}(\nu_1,\nu_2,r)>0\end{array}\right\}\text{ implies }\Gamma_\tau(\mathcal{S}(\mathcal{O}\mu_1,\mathcal{O}\nu_2,r),\mathcal{S}(\mathcal{O}\mu_2,\mathcal{O}\nu_1,r),$$
$$\mathcal{S}(\mathcal{O}\mu_1,\mathcal{O}\nu_1,r),\mathcal{S}(\mathcal{O}\mu_2,\mathcal{O}\nu_2,r))\cdot\mathcal{F}(\mathcal{S}(\mathcal{O}\mu_1,\mathcal{O}\mu_2,r))\geq\mathcal{F}(\mathcal{S}(\mathcal{O}\nu_1,\mathcal{O}\nu_2,r)) \tag{2}$$

for all $\mu_1,\mu_2,\nu_1,\nu_2\in\mathfrak{U}$, $\Gamma_\tau\in\triangle_{\Gamma_\tau}$ and $\mathcal{F}\in\triangle_{\mathcal{F}}$.

Next we need to define the fuzzy q-property.

Definition 9. *Let $\mu_0\in\mathfrak{U}$ be any arbitrary point. Then, the mapping $\mathcal{O}:\mathfrak{U}\to\mathfrak{V}$ has a fuzzy q-property if for a sequence $\{\mu_n\}$ defined as*

$$\mathcal{S}(\mu_{n+1},\mathcal{O}\mu_n,r)=\mathcal{S}(\mathfrak{U},\mathfrak{V},r),$$

there exist two subsequences $\{\mu_{p(n)}\}_{n\in\mathbb{N}}$ and $\{\mu_{q(n)}\}_{n\in\mathbb{N}}$ of $\{\mu_n\}$ such that

$$\lim_{n\to+\infty}\mathcal{S}(\mu_{p(n)},\mu_{q(n)},r)=1$$

where $p(n)>q(n)>n$, $n\in\mathbb{N}$. Then,

$$\mathcal{S}(\mu_{p(n)},\mu_{p(n)-1},r)=\mathcal{S}(\mathfrak{U},\mathfrak{V},r)\quad\text{and}\quad\mathcal{S}(\mu_{q(n)},\mu_{q(n)-1},r)=\mathcal{S}(\mathfrak{U},\mathfrak{V},r).$$

Theorem 1. *Suppose that two nonempty closed subsets $\mathfrak{U}$ and $\mathfrak{V}$ of a complete non-Archimedean fuzzy metric space $(\mathfrak{X},\mathcal{S},\circledast)$ with $\mathfrak{U}_0(r)\neq\phi$ and $\mathcal{O}:\mathfrak{U}\to\mathfrak{V}$ satisfy the assertions:*

1. *$\mathcal{O}$ is continuous $\Gamma_\tau\mathcal{F}$-proximal contraction of the first category;*
2. *$\mathcal{O}(\mathfrak{U}_0(r))\subseteq\mathfrak{V}_0(r)$;*
3. *The pair $(\mathfrak{U},\mathfrak{V})$ has a fuzzy p-property;*
4. *The mapping $\mathcal{O}$ has a fuzzy q-property.*

Then there exists a unique $\mu^*\in\mathfrak{U}$ such that $\mathcal{S}(\mu^*,\mathcal{O}\mu^*,r)=\mathcal{S}(\mathfrak{U},\mathfrak{V},r)$.

Proof. Take $\mu_0\in\mathfrak{U}_0(r)$. Since $\mathcal{O}\mu_0\in\mathcal{O}(\mathfrak{U}_0(r))\subseteq\mathfrak{V}_0(r)$, there exists $\mu_1\in\mathfrak{U}_0(r)$ such that $\mathcal{S}(\mu_1,\mathcal{O}\mu_0,r)=\mathcal{S}(\mathfrak{U},\mathfrak{V},r)$. Moreover, since $\mathcal{O}\mu_1\in\mathcal{O}(\mathfrak{U}_0(r))\subseteq\mathfrak{V}_0(r)$, there exists $\mu_2\in\mathfrak{U}_0(r)$ such that $\mathcal{S}(\mu_2,\mathcal{O}\mu_1,r)=\mathcal{S}(\mathfrak{U},\mathfrak{V},r)$. Inductively, we can find a sequence $\{\mu_n\}$ in $\mathfrak{U}_0(r)$ such that

$$\mathcal{S}(\mu_{n+1},\mathcal{O}\mu_n,r)=\mathcal{S}(\mathfrak{U},\mathfrak{V},r)\text{ for all }n\in\mathbb{N}. \tag{3}$$

By assertion 3 and (3), we get

$$\mathcal{S}(\mu_n,\mu_{n+1},r)=\mathcal{S}(\mathcal{O}\mu_{n-1},\mathcal{O}\mu_n,r)\text{ for all }n\in\mathbb{N}. \tag{4}$$

Now we prove that $\{\mu_n\}$ is convergent in $\mathfrak{U}_0(r)$. If there exists $n_0\in\mathbb{N}$ such that

$$\mathcal{S}(\mathcal{O}\mu_{n_0-1},\mathcal{O}\mu_{n_0},r)=1,$$

by (4), we obtain $\mathcal{S}(\mu_{n_0},\mu_{n_0+1},r)=1$ implies $\mu_{n_0}=\mu_{n_0+1}$. Therefore,

$$\mathcal{O}\mu_{n_0}=\mathcal{O}\mu_{n_0+1}\text{ implies }\mathcal{S}(\mathcal{O}\mu_{n_0},\mathcal{O}\mu_{n_0+1},r)=1 \tag{5}$$

from (4) and (5),

$$\mathcal{S}(\mu_{n_0+2}, \mu_{n_0+1}, r) = \mathcal{S}(\mathcal{O}\mu_{n_0+1}, \mathcal{O}\mu_{n_0}, r) = 1 \text{ implies } \mu_{n_0+2} = \mu_{n_0+1}. \tag{6}$$

Therefore $\mu_n = \mu_{n_0}$ for all $n \geq n_0$ and $\{\mu_n\}$ is convergent in $\mathfrak{U}_0(r)$. In addition,

$$\mathcal{S}(\mu_{n_0}, \mathcal{O}\mu_{n_0}, r) = \mathcal{S}(\mu_{n_0+1}, \mathcal{O}\mu_{n_0}, r) = \mathcal{S}(\mathfrak{U}, \mathfrak{V}, r).$$

This means that μ_{n_0} is a best proximity point of a map $\mathcal{O}$, that is, the conclusion is immediate. Due to that, consider $\mathcal{S}(\mu_n, \mathcal{O}\mu_{n+1}, r) \neq 1$ for all $n \in \mathbb{N}$. By assumption *1*, $\mathcal{O}$ is a $\Gamma_\tau\mathcal{F}$-fuzzy proximal contraction of the first category,

$$\Gamma_\tau(\mathcal{S}(\mu_n, \mu_n, r), \mathcal{S}(\mu_{n+1}, \mu_{n-1}, r), \mathcal{S}(\mu_n, \mu_{n+1}, r), \mathcal{S}(\mu_{n+1}, \mu_n, r)) \cdot \mathcal{F}(\mathcal{S}(\mu_n, \mu_{n+1}, r)) \geq \mathcal{F}(\mathcal{S}(\mu_{n-1}, \mu_n, r)).$$

Since $\max(\mathcal{S}(\mu_{n-1}, \mu_n, r), \mathcal{S}(\mu_n, \mu_{n+1}, r), \mathcal{S}(\mu_{n-1}, \mu_n, r), \mathcal{S}(\mu_n, \mu_n, r)) = 1$, by definition of a Γ_τ-function, there exists $\tau \in (0,1)$ such that

$$\Gamma_\tau(\mathcal{S}(\mu_n, \mu_n, r), \mathcal{S}(\mu_n, \mu_{n+1}, r), \mathcal{S}(\mu_{n-1}, \mu_{n+1}, r), \mathcal{S}(\mu_{n-1}, \mu_n, r)) = \tau.$$

Therefore

$$\tau \cdot \mathcal{F}(\mathcal{S}(\mu_n, \mu_{n+1}, r)) \geq \mathcal{F}(\mathcal{S}(\mu_{n-1}, \mu_n, r)).$$

This implies that

$$\mathcal{F}(\mathcal{S}(\mu_n, \mu_{n+1}, r)) > \tau \cdot \mathcal{F}(\mathcal{S}(\mu_n, \mu_{n+1}, r)) \geq \mathcal{F}(\mathcal{S}(\mu_{n-1}, \mu_n, r)). \tag{7}$$

Since mapping $\mathcal{F}$ is strictly nondecreasing

$$\mathcal{S}(\mu_n, \mu_{n+1}, r) > \mathcal{S}(\mu_{n-1}, \mu_n, r). \tag{8}$$

Hence $\{\mathcal{S}(\mu_n, \mu_{n+1}, r)\}$ $(r > 0)$ is strictly nondecreasing, bounded from the above sequence, so $\{\mathcal{S}(\mu_n, \mu_{n+1}, r)\}$ $(r > 0)$ is convergent. Otherwise, there exists $a(r) \in [0,1]$ such that

$$\lim_{n\to+\infty} \mathcal{S}(\mu_n, \mu_{n+1}, r) = a(r) \tag{9}$$

for any $r > 0$ and $n \in \mathbb{N}$.

$$\mathcal{S}(\mu_n, \mu_{n+1}, r) < a(r), \tag{10}$$

and by (9) and (10),

$$\lim_{n\to+\infty} \mathcal{F}(\mathcal{S}(\mu_n, \mu_{n+1}, r)) = \mathcal{F}(a(r) - 0). \tag{11}$$

We must show that $a(r) = 1$. Assume $a(r) < 1$ for any $r > 0$ and by letting limit n tend to $+\infty$ in (7) and using (11),

$$\mathcal{F}(a(r) - 0) \geq \tau \cdot \mathcal{F}(a(r) - 0) \geq \mathcal{F}(a(r) - 0)$$

is a contradiction. Therefore,

$$\lim_{n\to+\infty} \mathcal{S}(\mu_n, \mu_{n+1}, r) = 1. \tag{12}$$

Next we must show that $\{\mu_n\}$ is a Cauchy sequence. Suppose $\{\mu_n\}$ is not a Cauchy sequence, by Lemma 1, there exists $\epsilon \in (0,1)$ and two subsequences $\{\mu_{m_k}\}$ and $\{\mu_{n_k}\}$ of $\{\mu_n\}$ such that

$$\lim_{k\to+\infty} \mathcal{S}(\mu_{m_k}, \mu_{n_k}, r) = 1 - \epsilon. \tag{13}$$

Again by using the fuzzy q-property,

$$\left.\begin{array}{r}\mathcal{S}(\mu_{m_k},\mathcal{O}\mu_{m_k-1},r)=\mathcal{S}(\mathfrak{U},\mathfrak{V},r)\\ \mathcal{S}(\mu_{n_k},\mathcal{O}\mu_{n_k-1},r)=\mathcal{S}(\mathfrak{U},\mathfrak{V},r)\end{array}\right\}\text{ implies that}$$

$$\Gamma_\tau(\mathcal{S}(\mu_{m_k},\mu_{n_k-1},r),\mathcal{S}(\mu_{n_k},\mu_{m_k-1},r),\mathcal{S}(\mu_{m_k},\mu_{m_k-1},r),\mathcal{S}(\mu_{n_k},\mu_{n_k-1},r))\cdot \mathcal{F}(\mathcal{S}(\mu_{m_k},\mu_{n_k},r))\geq\mathcal{F}(\mathcal{S}(\mu_{m_k-1},\mu_{n_k-1},r)).$$

Letting k tends to $+\infty$ and using (12),

$$\Gamma_\tau(\lim_{k\to+\infty}\mathcal{S}(\mu_{m_k},\mu_{n_k-1},r),\lim_{k\to+\infty}\mathcal{S}(\mu_{n_k},\mu_{m_k-1},r),1,1)\cdot \lim_{k\to+\infty}\mathcal{F}(\mathcal{S}(\mu_{m_k},\mu_{n_k},r))\geq\lim_{k\to+\infty}\mathcal{F}(\mathcal{S}(\mu_{m_k},\mu_{n_k},r)). \tag{14}$$

Since $\max(\lim_{k\to+\infty}\mathcal{S}(\mu_{m_k},\mu_{n_k-1},r),\lim_{k\to+\infty}\mathcal{S}(\mu_{n_k},\mu_{m_k-1},r),1,1)=1$, there exists $\tau\in(0,1)$ such that

$$\Gamma_\tau(\lim_{k\to+\infty}\mathcal{S}(\mu_{m_k},\mu_{n_k-1},r),\lim_{k\to+\infty}\mathcal{S}(\mu_{n_k},\mu_{m_k-1},r),1,1)=\tau.$$

Using (13) and (14) implies

$$\mathcal{F}((1-\epsilon)-0)>\tau\cdot\mathcal{F}((1-\epsilon)-0)\geq\mathcal{F}((1-\epsilon)-0).$$

This is a contradiction. Thus, the sequence $\{\mu_n\}$ is a Cauchy sequence in $\mathfrak{U}$. Since the space $(\mathfrak{X},\mathcal{S},\circledast)$ is complete, given $\mathfrak{U}$ is a closed subset of $\mathfrak{X}$, there exists $\mu^*\in\mathfrak{U}$ such that $\lim_{n\to+\infty}\mu_n=\mu^*$.

Since $\mathcal{O}$ is continuous, $\mathcal{O}\mu_n\to\mathcal{O}\mu^*$ and the continuity of $\mathcal{S}$ implies $\mathcal{S}(\mu_{n+1},\mathcal{O}\mu_n,r)\to\mathcal{S}(\mu^*,\mathcal{O}\mu^*,r)$. From (3),

$$\mathcal{S}(\mu^*,\mathcal{O}\mu^*,r)=\mathcal{S}(\mathfrak{U},\mathfrak{V},r).$$

Thus μ^* is a best proximity point of $\mathcal{O}$.

Suppose that $\mu^*,\nu^*\in\mathfrak{U}$ such that $\mu^*\neq\nu^*$, that is, $\mathcal{S}(\mu^*,\nu^*,r)\neq1$ and

$$\mathcal{S}(\mu^*,\mathcal{O}\mu^*,r)=\mathcal{S}(\nu^*,\mathcal{O}\nu^*,r)=\mathcal{S}(\mathfrak{U},\mathfrak{V},r).$$

Then by the p-property of the pair $(\mathfrak{U},\mathfrak{V})$, we write $\mathcal{S}(\mu^*,\nu^*,r)=\mathcal{S}(\mathcal{O}\mu^*,\mathcal{O}\nu^*,r)$.

$$\Gamma_\tau(\mathcal{S}(\mu^*,\nu^*,r),\mathcal{S}(\nu^*,\mu^*,r),\mathcal{S}(\mu^*,\mu^*,r),\mathcal{S}(\nu^*,\nu^*,r))\cdot\mathcal{F}(\mathcal{S}(\mu^*,\nu^*,r))\geq\mathcal{F}(\mathcal{S}(\mu^*,\nu^*,r))$$

implies that $\mathcal{F}(\mathcal{S}(\mu^*,\nu^*,r))\geq\tau\cdot\mathcal{F}(\mathcal{S}(\mu^*,\nu^*,r))\geq\mathcal{F}(\mathcal{S}(\mu^*,\nu^*,r))$ is a contradiction. Hence, the best proximity point is unique for the map $\mathcal{O}$. □

Example 1. *Let $\mathfrak{X}=\mathbb{R}^2$ and define the usual metric*

$$\rho(c,d)=\rho((\eta_1,\eta_2),(\zeta_1,\zeta_2))=|\eta_1-\zeta_1|+|\eta_2-\zeta_2|\quad\text{for all }\ (\eta_1,\eta_2),(\zeta_1,\zeta_2)\in\mathbb{R}^2.$$

Define a membership function

$$\mathcal{S}(c,d,r)=\frac{r}{r+\rho(c,d)}$$

where $c,d\in\mathfrak{X}$ and $r>0$. Clearly, $(\mathfrak{X},\mathcal{S},\circledast)$ is a complete non-Archimedean fuzzy metric space where $\circledast$ is a product t-norm.

Let $\mathfrak{U} = \{(\mu, -2) : \mu \in \mathbb{R}^+\}$ *and* $\mathfrak{V} = \{(\nu, 2) : \nu \in \mathbb{R}^+\}$. *Here, we have* $\mathfrak{U} = \mathfrak{U}_0(r)$ *and* $\mathfrak{V} = \mathfrak{V}_0(r)$. *Let* $\mathcal{O} : \mathfrak{U} \to \mathfrak{V}$ *be defined by*

$$\mathcal{O}(\mu, -2) = \left(\frac{\mu}{2}, 2\right) \text{ for all } (\mu, -2) \in \mathfrak{U}.$$

Consider $c_1 = (\mu_1, -2), c_2 = (\mu_2, -2)$ *and* $d_1 = (2\mu_1, -2), d_2 = (2\mu_2, -2)$.

Now

$$\begin{aligned} \mathcal{S}(c_1, \mathcal{O}d_1, r) &= \mathcal{S}((\mu_1, -2), \mathcal{O}(2\mu_1, -2), r) \\ &= \mathcal{S}((\mu_1, -2), (\mu_1, 2), r) \\ &= \frac{r}{r+4} = \mathcal{S}(\mathfrak{U}, \mathfrak{V}, r) \end{aligned}$$

$$\begin{aligned} \mathcal{S}(c_2, \mathcal{O}d_2, r) &= \mathcal{S}((\mu_2, -2), \mathcal{O}(2\mu_2, -2), r) \\ &= \mathcal{S}((\mu_2, -2), (\mu_2, 2), r) \\ &= \frac{r}{r+4} = \mathcal{S}(\mathfrak{U}, \mathfrak{V}, r) \end{aligned}$$

implies

$$\mathcal{S}(c_1, c_2, r) = \mathcal{S}(\mathcal{O}d_1, \mathcal{O}d_2, r).$$

Hence $(\mathfrak{U}, \mathfrak{V})$ *holds fuzzy p-property.*

Consider

$$c_n = \left(\frac{1}{2^n}, -2\right), n \in \mathbb{N} \text{ such that}$$

$$\begin{aligned} \mathcal{S}(c_{n+1}, \mathcal{O}c_n, r) &= \mathcal{S}\left(\left(\frac{1}{2^{n+1}}, -2\right), \mathcal{O}\left(\frac{1}{2^n}, -2\right), r\right) \\ &= \mathcal{S}\left(\left(\frac{1}{2^{n+1}}, -2\right), \left(\frac{1}{2^{n+1}}, 2\right), r\right) \\ &= \frac{r}{r+4} = \mathcal{S}(\mathfrak{U}, \mathfrak{V}, r). \end{aligned}$$

Then there exist two subsequences $c_{p(n)} = c_{3k} = \left(\frac{1}{2^{3k}}, -2\right)$ *and* $c_{q(n)} = c_{2k} = \left(\frac{1}{2^{2k}}, -2\right)$ *of* $\{c_n\}$ *where* $3k > 2k > k$ *such that*

$$\begin{aligned} \lim_{n\to+\infty} \mathcal{S}\left(c_{p(n)}, c_{q(n)}, r\right) &= \lim_{k\to+\infty} \mathcal{S}(c_{3k}, c_{2k}, r) \\ &= \lim_{k\to+\infty} \mathcal{S}\left(\left(\frac{1}{2^{3k}}, -2\right), \left(\frac{1}{2^{2k}}, -2\right), r\right) \\ &= \lim_{k\to+\infty} \frac{r}{r + |\frac{1}{2^{3k}} - \frac{1}{2^{2k}}|} = 1. \end{aligned}$$

Now

$$\begin{aligned}
\mathcal{S}(c_{3k},\mathcal{O}(c_{3k-1}),r) &= \mathcal{S}\left(\left(\frac{1}{2^{3k}},-2\right),\mathcal{O}\left(\frac{1}{2^{3k-1}},-2\right),r\right)\\
&= \mathcal{S}\left(\left(\frac{1}{2^{3k}},-2\right),\left(\frac{1}{2^{3k}},2\right),r\right)\\
&= \lim_{k\to+\infty}\frac{r}{r+|\frac{1}{2^{3k}}-\frac{1}{2^{3k}}|+|-2-2|}\\
&= \frac{r}{r+4}=\mathcal{S}(\mathfrak{U},\mathfrak{V},r).
\end{aligned}$$

Similarly,

$$\mathcal{S}(c_{2k},\mathcal{O}(c_{2k-1}),r) = \mathcal{S}\left(\left(\frac{1}{2^{2k}},-2\right),\mathcal{O}\left(\frac{1}{2^{2k-1}},-2\right),r\right)=\frac{r}{r+4}=\mathcal{S}(\mathfrak{U},\mathfrak{V},r).$$

Hence $\mathcal{O}$ satisfies the q-property, $\mathcal{O}$ is continuous and $\mathcal{O}(\mathfrak{U}_0)\subseteq\mathfrak{V}_0$. Define a strictly nondecreasing function $\mathcal{F}(s)=\dfrac{s}{1-s}$ for any $s\in(0,1)$ such that $\mathcal{F}\in\triangle_{\mathcal{F}}$ and suppose $\Gamma_\tau(l_1,l_2,l_3,l_4)=\tau$ where $\tau\in(0,1)$ such that $\Gamma_\tau\in\triangle_{\Gamma_\tau}$.

Consider $c_1=(\mu_1,-2),c_2=(\mu_2,-2)$ and $d_1=(2\mu_1,-2),d_2=(2\mu_2,-2)$,

$$\begin{aligned}
\tau\cdot\mathcal{F}(\mathcal{S}(c_1,c_2,r)) &= \tau\cdot\mathcal{F}(\mathcal{S}((\mu_1,-2),(\mu_2,-2),r))\\
&= \tau\cdot\mathcal{F}\left(\frac{r}{r+|\mu_1-\mu_2|}\right)\\
&= \tau\cdot\left(\frac{\frac{r}{r+|\mu_1-\mu_2|}}{1-\frac{r}{r+|\mu_1-\mu_2|}}\right)\\
&= \tau\cdot\frac{r}{|\mu_1-\mu_2|}\\
&\geq \frac{1}{2}\cdot\frac{r}{|\mu_1-\mu_2|}\text{ where }\tau\in[\frac{1}{2},1)\\
&= \frac{r}{|2\mu_1-2\mu_2|}\\
&= \frac{\frac{r}{r+|2\mu_1-2\mu_2|}}{1-\frac{r}{r+|2\mu_1-2\mu_2|}}\\
&= \mathcal{F}\left(\frac{r}{r+|2\mu_1-2\mu_2|}\right)\\
&= \mathcal{F}(\mathcal{S}((2\mu_1,-2),(2\mu_2,-2),r))=\mathcal{F}(\mathcal{S}(d_1,d_2,r)).
\end{aligned}$$

Thus $\mathcal{O}$ is a proximal $\Gamma_\tau\mathcal{F}$-fuzzy contraction of the first category. Thus, assumed assertions of Theorem 1 hold. Hence $\mathcal{O}$ has a unique best proximity point $(0,-2)$.

Now we insert the next theorem by avoiding the continuity of the $\mathcal{O}$ nonself function .

Theorem 2. *Suppose that two nonempty closed subsets $\mathfrak{U}$ and $\mathfrak{V}$ of a complete non-Archimedean fuzzy metric space $(\mathfrak{X},\mathcal{S},\circledast)$ with $\mathfrak{U}_0(r)\neq\phi$ and $\mathcal{O}:\mathfrak{U}\to\mathfrak{V}$ satisfy the assertions:*

1. *$\mathcal{O}(\mathfrak{U}_0(r))\subseteq\mathfrak{V}_0(r)$ and $(\mathfrak{U},\mathfrak{V})$ holds the fuzzy p-property;*
2. *$\mathcal{O}$ is a $\Gamma_\tau\mathcal{F}$- fuzzy proximal contraction of the first category and $\mathcal{F}$ is continuous;*
3. *The mapping $\mathcal{O}$ has a fuzzy q-property.*

4. *For any sequence* $\{\nu_n\}$ *in* $\mathfrak{V}_0(r)$ *and* $\mu \in \mathfrak{U}$ *satisfying* $\mathcal{S}(\mu, \nu_n, r) \to \mathcal{S}(\mathfrak{U}, \mathfrak{V}, r)$ *as n tends to* $+\infty$, *then* $\mu \in \mathfrak{U}_0(r)$.

Then there exists a unique $\mu^* \in \mathfrak{U}$ *such that* $\mathcal{S}(\mu^*, \mathcal{O}\mu^*, r) = \mathcal{S}(\mathfrak{U}, \mathfrak{V}, r)$ *for all* $r > 0$.

Proof. The construction of the sequence $\{\mu_n\}$ is similar to that in Theorem 1. Then, we must show that $\{\mu_n\}$ is a Cauchy sequence, whose proof is also similar to that of Theorem 1. The completeness property of $(\mathfrak{X}, \mathcal{S}, \circledast)$ and $\mathfrak{U}$ being a closed subset of $\mathfrak{X}$ ensure $\{\mu_n\}$ converges to $\mu^* \in \mathfrak{U}$,

$$\lim_{n \to +\infty} \mathcal{S}(\mu_n, \mu^*, r) = 1.$$

Moreover,

$$\begin{aligned} \mathcal{S}(\mathfrak{U}, \mathfrak{V}, r) &= \mathcal{S}(\mu_{n+1}, \mathcal{O}\mu_n, r) \\ &\geq \mathcal{S}(\mu_{n+1}, \mu^*, r) \circledast \mathcal{S}(\mu^*, \mathcal{O}\mu_n, r) \\ &\geq \mathcal{S}(\mu_{n+1}, \mu^*, r) \circledast \mathcal{S}(\mu^*, \mathcal{O}\mu_n, r) \\ &\geq \mathcal{S}(\mu_{n+1}, \mu^*, r) \circledast \mathcal{S}(\mu^*, \mu_{n+1}, r) \circledast \mathcal{S}(\mu_{n+1}, \mathcal{O}\mu_n, r), \end{aligned}$$

implies

$$\begin{aligned} \mathcal{S}(\mathfrak{U}, \mathfrak{V}, r) &\geq \mathcal{S}(\mu_{n+1}, \mu^*, r) \circledast \mathcal{S}(\mu^*, \mathcal{O}\mu_n, r) \\ &\geq \mathcal{S}(\mu_{n+1}, \mu^*, r) \circledast \mathcal{S}(\mu^*, \mu_{n+1}, r) \circledast \mathcal{S}(\mathfrak{U}, \mathfrak{V}, r). \end{aligned}$$

Letting n tend to $+\infty$,

$$\mathcal{S}(\mathfrak{U}, \mathfrak{V}, r) \geq 1 \circledast \lim_{n \to +\infty} \mathcal{S}(\mu^*, \mathcal{O}\mu_n, r) \geq 1 \circledast 1 \circledast \mathcal{S}(\mathfrak{U}, \mathfrak{V}, r)$$

implies

$$\lim_{n \to +\infty} \mathcal{S}(\mu^*, \mathcal{O}\mu_n, r) = \mathcal{S}(\mathfrak{U}, \mathfrak{V}, r), \tag{15}$$

and using 4, $\mu^* \in \mathfrak{U}_0(r)$. Since $\mathcal{O}(\mathfrak{U}_0(r)) \subseteq \mathfrak{V}_0(r)$, there exist $\eta \in \mathfrak{U}_0(r)$ such that $\mathcal{S}(\eta, \mathcal{O}\mu^*, r) = \mathcal{S}(\mathfrak{U}, \mathfrak{V}, r)$. Combining (15) with (3),

$$\left.\begin{aligned} \mathcal{S}(\eta, \mathcal{O}\mu^*, r) = \mathcal{S}(\mathfrak{U}, \mathfrak{V}, r) \\ \mathcal{S}(\mu_{n+1}, \mathcal{O}\mu_n, r) = \mathcal{S}(\mathfrak{U}, \mathfrak{V}, r) \end{aligned}\right\} \text{implies}$$

$$\begin{aligned} \Gamma_\tau(\mathcal{S}(\eta, \mu_n, r), \mathcal{S}(\mu_{n+1}, \mu^*, r), \mathcal{S}(\eta, \mu^*, r), \mathcal{S}(\mu_n, \mu_{n+1}, r)) \cdot \mathcal{F}(\mathcal{S}(\eta, \mu_{n+1}, r)) \\ \geq \mathcal{F}(\mathcal{S}(\mu^*, \mu_n, r)). \end{aligned}$$

Letting $n \to +\infty$,

$$\begin{aligned} \lim_{n \to +\infty} [\Gamma_\tau(\mathcal{S}(\eta, \mu_n, r), \mathcal{S}(\mu_{n+1}, \mu^*, r), \mathcal{S}(\eta, \mu^*, r), \mathcal{S}(\mu_n, \mu_{n+1}, r)) \cdot \\ \mathcal{F}(\mathcal{S}(\eta, \mu_{n+1}, r))] \geq \lim_{n \to +\infty} \mathcal{F}(\mathcal{S}(\mu^*, \mu_n, r)) \end{aligned}$$

implies

$$\begin{aligned} \Gamma_\tau(\lim_{n \to +\infty} \mathcal{S}(\eta, \mu_n, r), 1, \lim_{n \to +\infty} \mathcal{S}(\eta, \mu^*, r), 1) \cdot \lim_{n \to +\infty} \mathcal{F}(\mathcal{S}(\eta, \mu_{n+1}, r)) \\ \geq \lim_{n \to +\infty} \mathcal{F}(\mathcal{S}(\mu^*, \mu_n, r)). \end{aligned}$$

Since $\max(\lim_{n\to+\infty} \mathcal{S}(\eta,\mu_n,r),1,\lim_{k\to+\infty} \mathcal{S}(\eta,\mu^*,r),1) = 1$, there exists $\tau \in (0,1)$ such that

$$\Gamma_\tau(\lim_{n\to+\infty} \mathcal{S}(\eta,\mu_n,r),1,\lim_{n\to+\infty} \mathcal{S}(\eta,\mu^*,r),1) = \tau$$

implies

$$\tau \cdot \lim_{n\to+\infty} \mathcal{F}(\mathcal{S}(\eta,\mu_{n+1},r)) \geq \lim_{n\to+\infty} \mathcal{F}(\mathcal{S}(\mu^*,\mu_n,r))$$

$$\lim_{n\to+\infty} \mathcal{F}(\mathcal{S}(\eta,\mu_{n+1},r)) \geq \tau \cdot \lim_{n\to+\infty} \mathcal{F}(\mathcal{S}(\eta,\mu_{n+1},r)) \geq \lim_{n\to+\infty} \mathcal{F}(\mathcal{S}(\mu^*,\mu_n,r)).$$

Since $\mathcal{F}$ is continuous,

$$\lim_{n\to+\infty} \mathcal{S}(\eta,\mu_{n+1},r) \geq \lim_{n\to+\infty} \mathcal{S}(\mu^*,\mu_n,r) = 1$$

implies

$$\lim_{n\to+\infty} \mathcal{S}(\eta,\mu_{n+1},r) = 1,$$

but the limit of the sequence is unique, so we conclude that $\eta = \mu^*$, that is, $\mathcal{S}(\mu,\mathcal{O}\mu^*,r) = \mathcal{S}(\eta,\mathcal{O}\mu^*,r) = \mathcal{S}(\mathfrak{U},\mathfrak{V},r)$. The best proximity point of $\mathcal{O}$ is unique similarly to the proof of the uniqueness part of Theorem 1. □

Example 2. *Let $\mathfrak{X} = \mathbb{R}\times\mathbb{R}$. A membership function $\mathcal{S} : \mathfrak{X}\times\mathfrak{X}\times(0,+\infty) \to (0,1]$ is a complete non-Archimedean fuzzy metric space with a product t-norm defined by*

$$\mathcal{S}(c,d,r) = \left(\frac{r}{r+1}\right)^{\rho(c,d)}$$

for all $c,d \in \mathfrak{X}$ and $r > 0$ where ρ is a standard metric. Define the two sets

$$\mathfrak{U} = \left\{\left(0,\frac{1}{n}\right) : n \in \mathbb{N}\right\} \cup \{(0,0)\} \text{ and } \mathfrak{V} = \left\{\left(1,\frac{2}{n}\right) : n \in \mathbb{N}\right\} \cup \{(1,0)\},$$

so that $\rho(\mathfrak{U},\mathfrak{V}) = 1$ and $\mathcal{S}(\mathfrak{U},\mathfrak{V},r) = \frac{r}{r+1}$ for all $r > 0$. We can see that both nonempty subsets are $\mathfrak{U}$ and $\mathfrak{V}$ are closed in $\mathfrak{X}$.

Let us define $\mathcal{O} : \mathfrak{U} \to \mathfrak{V}$ by

$$\mathcal{O}(\alpha_1,\alpha_2) = \begin{cases} \left(1,\frac{2}{n}\right), & \text{if } (\alpha_1,\alpha_2) = (0,\frac{1}{n}) \text{ for all } n \in \mathbb{N}, \\ (1,0), & \text{if } (\alpha_1,\alpha_2) = (0,0). \end{cases} \tag{16}$$

Clearly, $\mathfrak{U}_0(r) = \mathfrak{U}$, $\mathfrak{V}_0(r) = \mathfrak{V}$, $\mathcal{O}(\mathfrak{U}_0(r)) \subseteq \mathfrak{V}_0(r)$ and the hypotheses of Theorem 2 hold. Let $\mathcal{F}(s) = log_e(s)$ where $s \in (0,1)$ such that $\mathcal{F} \in \triangle_{\mathcal{F}}$ and consider $\Gamma_\tau(t_1,t_2,t_3,t_4) = \tau$ where $\tau \in (0,1)$ such that $\Gamma_\tau \in \triangle_{\Gamma_\tau}$.

Consider $\mathcal{S}(u,\mathcal{O}v,r) = \mathcal{S}(\mathfrak{U},\mathfrak{V},r)$ for some $u,v \in \mathfrak{U}$. Then,

$$(u,v) = \left(((0,0),(0,0)),\left(\left(0,\frac{2}{n}\right),\left(0,\frac{1}{n}\right)\right) : n \in \mathbb{N}\right).$$

We investigate the following cases:

(1) *If* $(u_1, v_1) = \left(\left(0, \frac{2}{n}\right), \left(0, \frac{1}{n}\right)\right)$ *and* $(u_2, v_2) = \left(\left(0, \frac{2}{m}\right), \left(0, \frac{1}{m}\right)\right)$ *for all* $n, m \in \mathbb{N}$. *We have*

$$\begin{aligned}
\tau \cdot \mathcal{F}(\mathcal{S}(u_1, u_2, r)) &= \tau \cdot \mathcal{F}\left(\mathcal{S}\left(\left(0, \frac{2}{n}\right), \left(0, \frac{2}{m}\right), r\right)\right) \\
&= \tau \cdot \mathcal{F}\left(\left(\frac{r}{r+1}\right)^{|\frac{2}{n} - \frac{2}{m}|}\right) \\
&= \tau \cdot log_e\left(\frac{r}{r+1}\right)^{|\frac{2}{n} - \frac{2}{m}|} \\
&= \tau \cdot |\frac{2}{n} - \frac{2}{m}| \cdot log_e\left(\frac{r}{r+1}\right) \\
&\geq |\frac{1}{n} - \frac{1}{m}| \cdot log_e\left(\frac{r}{t+1}\right) \text{ where } \tau \in [1/2, 1) \\
&= log_e\left(\frac{r}{r+1}\right)^{|\frac{1}{n} - \frac{1}{m}|} \\
&= \mathcal{F}\left\{\left(\frac{r}{r+1}\right)^{|\frac{1}{n} - \frac{1}{m}|}\right\} \\
&= \mathcal{F}\left(\mathcal{S}\left(0, \frac{1}{n}\right), \left(0, \frac{1}{m}\right), r\right) \\
&= \mathcal{F}(\mathcal{S}(v_1, v_2, r)).
\end{aligned}$$

(2) *If* $(u_1, v_1) = ((0,0), (0,0))$ *and* $(u_2, v_2) = \left(\left((0, \frac{2}{m}\right), \left(0, \frac{1}{m}\right)\right)$ *for all* $m \in \mathbb{N}$,

$$\begin{aligned}
\tau \cdot \mathcal{F}(\mathcal{S}(u, v, r) &= \tau \cdot \mathcal{F}\left(\mathcal{S}\left(0,0\right), \left(0, \frac{2}{m}\right), r\right)\right) \\
&= \tau \cdot \mathcal{F}\left(\left(\frac{r}{r+1}\right)^{|\frac{2}{m}|}\right) \\
&= \tau \cdot |\frac{2}{m}| \cdot log_e\left(\frac{r}{r+1}\right) \\
&\geq |\frac{1}{m}| \cdot log_e\left(\frac{r}{r+1}\right) \text{ where } \tau \in [1/2, 1) \\
&= \mathcal{F}\left(\left(\frac{r}{r+1}\right)^{|\frac{1}{m}|}\right) \\
&= \mathcal{F}\left(\mathcal{S}\left((0,0), \left(0, \frac{1}{m}\right), r\right)\right) \\
&= \mathcal{F}(\mathcal{S}(v_1, v_2, r)).
\end{aligned}$$

(3) *If* $(u_1, v_1) = ((0,0), (0,0))$ *and* $(u_2, v_2) = ((0,0), (0,0))$, *the fuzzy proximal contraction condition holds.*
The property symmetry of membership function $\mathcal{S}$ *covers all the possible cases, so we leave the details of these parts. Now, we conclude that the hypotheses of Theorem 2 are satisfied, and there exist* $\alpha^* = (0,0) \in \mathfrak{U}$ *such that* $\mathcal{S}(\alpha^*, \mathcal{O}\alpha^*, r) = \mathcal{S}(\mathfrak{U}, \mathfrak{V}, r)$ *for all* $r > 0$.

Theorem 3. *Consider two nonempty closed subsets $\mathfrak{U}$ and $\mathfrak{V}$ of a complete non-Archimedean fuzzy metric space $(\mathfrak{X}, \mathcal{S}, \circledast)$ with $\mathfrak{U}_0(r) \neq \phi$. Assume that $\mathfrak{U}$ is approximately compact with respect to $\mathfrak{V}$ and $\mathcal{O} : \mathfrak{U} \to \mathfrak{V}$ satisfies the following assertions:*

1. *$\mathcal{O}(\mathfrak{U}_0(r)) \subseteq \mathfrak{V}_0(r)$ and $(\mathfrak{U}, \mathfrak{V})$ satisfies the fuzzy p-property;*
2. *$\mathcal{O}$ is a continuous $\Gamma_\tau\mathcal{F}$- fuzzy proximal contraction of the second category;*
3. *$\mathcal{O}$ has a fuzzy q-property.*

Then there exists a unique $\mu^* \in \mathfrak{U}$ such that $\mathcal{S}(\mu^*, \mathcal{O}\mu^*, r) = \mathcal{S}(\mathfrak{U}, \mathfrak{V}, r)$. Moreover, for any fixed element $\mu_0 \in \mathfrak{U}_0(r)$, the sequence $\{\mu_n\}$ defined by

$$\mathcal{S}(\mu_{n+1}, \mathcal{O}\mu_n, r) = \mathcal{S}(\mathfrak{U}, \mathfrak{V}, r),$$

converges to the best proximity point u. Further, if μ^* is another best proximity point of $\mathcal{O}$, then $\mathcal{O}u = \mathcal{O}\mu^*$.

Proof. Similar to Theorem 1, we formulate a sequence $\{\mu_n\}$ in $\mathfrak{U}_0(r)$ such that

$$\mathcal{S}(\mu_{n+1}, \mathcal{O}\mu_n, r) = \mathcal{S}(\mathfrak{U}, \mathfrak{V}, r) \tag{17}$$

for all positive integers n. From the fuzzy p-property of the pair of maps $\mathfrak{U}$ and $\mathfrak{V}$,

$$\mathcal{S}(\mu_n, \mu_{n+1}, r) = \mathcal{S}(\mathcal{O}\mu_{n-1}, \mathcal{O}\mu_n, r) \text{ for all } n \in \mathbb{N}.$$

If for some n_0, $\mathcal{S}(\mu_{n_0}, \mu_{n_0+1}, r) = 1$, then

$$\mathcal{S}(\mathcal{O}\mu_{n_0-1}, \mathcal{O}\mu_{n_0}, r) = 1 \text{ implies } \mathcal{O}\mu_{n_0-1} = \mathcal{O}\mu_{n_0} \text{ implies } \mathcal{S}(\mu_{n_0}, \mathcal{O}\mu_{n_0}, r) = \mathcal{S}(\mathfrak{U}, \mathfrak{V}, r),$$

thus the inclusion is immediate. Therefore, consider for any n in $\mathbb{N}$, $\mathcal{S}(\mathcal{O}\mu_n, \mathcal{O}\mu_{n+1}, r) > 0$. By 2, the mapping $\mathcal{O}$ is a $\Gamma_\tau\mathcal{F}$- fuzzy proximal contraction of the second category,

$$\Gamma_\tau(\mathcal{S}(\mathcal{O}\mu_n, \mathcal{O}\mu_n, r), \mathcal{S}(\mathcal{O}\mu_{n+1}, \mathcal{O}\mu_{n-1}, r), \mathcal{S}(\mathcal{O}\mu_n, \mathcal{O}\mu_{n-1}, r), \mathcal{S}(\mathcal{O}\mu_{n+1}, \mathcal{O}\mu_n, r)) \cdot \mathcal{F}(\mathcal{S}(\mathcal{O}\mu_n, \mathcal{O}\mu_{n+1}, r)) \geq \mathcal{F}(\mathcal{S}(\mathcal{O}\mu_{n-1}, \mathcal{O}\mu_n, r))$$

implies

$$\mathcal{F}(\mathcal{S}(\mathcal{O}\mu_n, \mathcal{O}\mu_{n+1}, r)) > \tau \cdot \mathcal{F}(\mathcal{S}(\mathcal{O}\mu_n, \mathcal{O}\mu_{n+1}, r)) \geq \mathcal{F}(\mathcal{S}(\mathcal{O}\mu_{n-1}, \mathcal{O}\mu_n, r)). \tag{18}$$

Since $\mathcal{F}$ is strictly nondecreasing,

$$\mathcal{S}(\mathcal{O}\mu_n, \mathcal{O}\mu_{n+1}, r) > \mathcal{S}(\mathcal{O}\mu_{n-1}, \mathcal{O}\mu_n, r).$$

Thus the sequence $\{\mathcal{S}(\mathcal{O}\mu_n, \mathcal{O}\mu_{n+1}, r)\}$, $(r > 0)$ is strictly nondecreasing and bounded from above, so the sequence $\{\mathcal{S}(\mathcal{O}\mu_n, \mathcal{O}\mu_{n+1}, r)\}$ $(r > 0)$ is convergent. In other words, there exists $a(r) \in [0, 1]$ such that

$$\lim_{n \to +\infty} \mathcal{S}(\mathcal{O}\mu_n, \mathcal{O}\mu_{n+1}, r) = a(r), \tag{19}$$

for $r > 0$ and $n \in \mathbb{N}$.

$$\mathcal{S}(\mathcal{O}\mu_n, \mathcal{O}\mu_{n+1}, r) < a(r), \tag{20}$$

by (19) and (20), for any $r > 0$, we have

$$\lim_{n \to +\infty} \mathcal{F}(\mathcal{S}(\mathcal{O}\mu_n, \mathcal{O}\mu_{n+1}, r)) = \mathcal{F}(a(r) - 0). \tag{21}$$

Next we must show that $a(r) = 1$. Assume $a(r) < 1$ for any $r > 0$ and take the limit as n tends to $+\infty$ in (18) and by (21), we obtain

$$\mathcal{F}(a(r) - 0) \geq \tau \cdot \mathcal{F}(a(r) - 0) \geq \mathcal{F}(a(r) - 0),$$

a contradiction. Therefore,

$$\lim_{n \to +\infty} \mathcal{S}(\mathcal{O}\mu_n, \mathcal{O}\mu_{n+1}, r) = 1. \tag{22}$$

Further we must prove that $\{\mathcal{O}\mu_n\}$ is a Cauchy sequence. Suppose $\{\mathcal{O}\mu_n\}$ is not a Cauchy sequence. By Lemma 1, there exists $\epsilon \in (0,1)$, $r > 0$ and subsequences $\{\mathcal{O}\mu_{m_k}\}$ and $\{\mathcal{O}\mu_{n_k}\}$ such that

$$\lim_{n \to +\infty} \mathcal{S}(\mathcal{O}\mu_{m_k}, \mathcal{O}\mu_{n_k}, r) = 1 - \epsilon. \tag{23}$$

By using the fuzzy q-property,

$$\left.\begin{aligned} \mathcal{S}(\mu_{m_k}, \mathcal{O}\mu_{m_k-1}, r) = \mathcal{S}(\mathfrak{U}, \mathfrak{B}, r) \\ \mathcal{S}(\mu_{n_k}, \mathcal{O}\mu_{n_k-1}, r) = \mathcal{S}(\mathfrak{U}, \mathfrak{B}, r) \end{aligned}\right\} \text{implies}$$

$$\begin{gathered} \Gamma_\tau(\mathcal{S}(\mathcal{O}\mu_{m_k}, \mathcal{O}\mu_{n_k-1}, r), \mathcal{S}(\mathcal{O}\mu_{n_k}, \mathcal{O}\mu_{m_k-1}, r), \mathcal{S}(\mathcal{O}\mu_{m_k}, \mathcal{O}\mu_{m_k-1}, r), \\ \mathcal{S}(\mathcal{O}\mu_{n_k}, \mathcal{O}\mu_{n_k-1}, r)) \cdot \mathcal{F}(\mathcal{S}(\mathcal{O}\mu_{m_k}, \mathcal{O}\mu_{n_k}, r)) \geq \mathcal{F}(\mathcal{S}(\mathcal{O}\mu_{m_k-1}, \mathcal{O}\mu_{n_k-1}, r)). \end{gathered}$$

Letting k tend to $+\infty$, using (23) and with the definition of the Γ_τ function,

$$\mathcal{F}((1-\epsilon) - 0)) \geq \tau \cdot \mathcal{F}((1-\epsilon) - 0)) \geq \mathcal{F}((1-\epsilon) - 0)),$$

a contradiction. Thus, $\{\mathcal{O}\mu_n\}$ is a Cauchy sequence in $\mathfrak{V}$. Since the space $(\mathfrak{X}, \mathcal{S}, \circledast)$ is complete, and $\mathfrak{V}$ is a closed subset of $\mathfrak{X}$, there exists $v \in \mathfrak{V}$ such that $\lim_{n \to +\infty} \mathcal{O}\mu_n = v$.

Furthermore,

$$\begin{aligned} \mathcal{S}(v, \mathfrak{U}, r) &\geq \mathcal{S}(v, \mu_{n+1}, r) \\ &\geq \mathcal{S}(v, \mathcal{O}\mu_n, r) \circledast \mathcal{S}(\mathcal{O}\mu_n, \mu_{n+1}, r) \\ &= \mathcal{S}(v, \mathcal{O}\mu_n, r) \circledast \mathcal{S}(\mathfrak{U}, \mathfrak{V}, r) \\ &\geq \mathcal{S}(v, \mathcal{O}\mu_n, r) \circledast \mathcal{S}(v, \mathfrak{U}, r), \end{aligned}$$

and taking the limit as $n \to +\infty$,

$$\lim_{n \to +\infty} \mathcal{S}(v, \mathcal{O}\mu_n, r) = \mathcal{S}(\mathfrak{U}, \mathfrak{V}, r).$$

Since $\mathfrak{U}$ is approximately compact with respect to $\mathfrak{V}$, there exists a subsequence $\{\mu_{n_k}\}$ of $\{\mu_n\}$ converging to element u in $\mathfrak{U}$. Thus,

$$\mathcal{S}(u, v, r) = \lim_{n \to +\infty} \mathcal{S}(\mu_{n_k}, \mathcal{O}\mu_{n_k-1}, r) = \mathcal{S}(v, \mathfrak{U}, r).$$

Hence it implies $u \in \mathfrak{U}_0(r)$, since $\lim_{k \to +\infty} \mu_{n_k} = u$. Since $\mathcal{O}$ is continuous and $\{\mathcal{O}\mu_n\}$ is convergent to v,

$$\lim_{k \to +\infty} \mathcal{O}\mu_{n_k} = \mathcal{O}u = v.$$

Hence
$$\mathcal{S}(u,\mathcal{O}u,r)=\lim_{n\to+\infty}\mathcal{S}(\mu_{n_k},\mathcal{O}\mu_{n_k},r)=\mathcal{S}(\mathfrak{U},\mathfrak{V},r).$$

Let μ^* be another best proximity point of the mapping $\mathcal{O}$ such that $\mathcal{S}(\mu^*,\mathcal{O}\mu^*,r)=\mathcal{S}(\mathfrak{U},\mathfrak{V},r)$. Since $\mathcal{O}$ is a $\Gamma_\tau\mathcal{F}$-fuzzy proximal contraction of the second category,

$$\left.\begin{aligned}\mathcal{S}(\mu^*,\mathcal{O}\mu^*,r)=\mathcal{S}(\mathfrak{U},\mathfrak{V},r)\\ \mathcal{S}(u,\mathcal{O}u,r)=\mathcal{S}(\mathfrak{U},\mathfrak{V},r)\end{aligned}\right\}\text{ implies that}$$

$$\Gamma_\tau(\mathcal{S}(\mathcal{O}\mu^*,\mathcal{O}u,r),\mathcal{S}(\mathcal{O}u,\mathcal{O}\mu^*,r),\mathcal{S}(\mathcal{O}\mu^*,\mathcal{O}\mu^*,r),\mathcal{S}(\mathcal{O}u,\mathcal{O}u,r))\cdot \mathcal{F}(\mathcal{S}(\mathcal{O}\mu^*,\mathcal{O}u,r))\geq\mathcal{F}(\mathcal{S}(\mathcal{O}\mu^*,\mathcal{O}u,r)),$$

by the definition of a Γ_τ-function,

$$\tau\cdot\mathcal{F}(\mathcal{S}(\mathcal{O}\mu^*,\mathcal{O}u,r))\geq\mathcal{F}(\mathcal{S}(\mathcal{O}\mu^*,\mathcal{O}u,r)).$$

Moreover,

$$\mathcal{F}(\mathcal{S}(\mathcal{O}\mu^*,\mathcal{O}u,r))>\tau\cdot\mathcal{F}(\mathcal{S}(\mathcal{O}\mu^*,\mathcal{O}u,r))\geq\mathcal{F}(\mathcal{S}(\mathcal{O}\mu^*,Tu,r))$$

implies that
$$\mathcal{S}(\mathcal{O}\mu^*,\mathcal{O}u,r)>\mathcal{S}(\mathcal{O}\mu^*,\mathcal{O}u,r),$$

which is a contradiction, that is, u and μ^* must be identical. Thus, $\mathcal{O}$ has a unique best proximity point. □

Our other result is for a nonself generalized $\Gamma_\tau\mathcal{F}$-fuzzy proximal contraction of the first category and second category.

Theorem 4. *Suppose that two nonempty closed subsets $\mathfrak{U}$ and $\mathfrak{V}$ of a complete non-Archimedean fuzzy metric space $(\mathfrak{X},\mathcal{S},\circledast)$ with $\mathfrak{U}_0(r)\neq\phi$ and $\mathcal{O}:\mathfrak{U}\to\mathfrak{V}$ satisfy the assertions:*

1. *$\mathcal{O}(\mathfrak{U}_0(r))\subseteq\mathfrak{V}_0(r)$;*
2. *$(\mathfrak{U},\mathfrak{V})$ satisfies fuzzy p-property;*
3. *$\mathcal{O}$ is a $\Gamma_\tau\mathcal{F}$-fuzzy proximal contraction of the first and second category both;*
4. *$\mathcal{O}$ has a fuzzy q-property.*

Then there exists a unique element $\mu\in\mathfrak{U}$ such that $\mathcal{S}(\mu,\mathcal{O}\mu,r)=\mathcal{S}(\mathfrak{U},\mathfrak{V},r)$. Moreover, for any fixed element $\mu_0\in\mathfrak{U}_0(r)$, a sequence $\{\mu_n\}$ defined by

$$\mathcal{S}(\mu_{n+1},\mathcal{O}\mu_n,r)=\mathcal{S}(\mathfrak{U},\mathfrak{V},r)$$

converges to the best proximity point μ. Further, if μ^* is another best proximity point of $\mathcal{O}$ then $\mathcal{O}\mu=\mathcal{O}\mu^*$.

Proof. Similar to Theorem 1, formulate a sequence $\{\mu_n\}$ in $\mathfrak{U}_0(r)$ such that

$$\mathcal{S}(\mu_{n+1},\mathcal{O}\mu_n,r)=\mathcal{S}(\mathfrak{U},\mathfrak{V},r)$$

for all non-negative integer n with $\mathcal{O}(\mathfrak{U}_0(r))\subseteq\mathfrak{V}_0(r)$. As in Theorem 1, we may show that the sequence $\{\mu_n\}$ is a Cauchy sequence. Thus, the sequence converges to any element μ in $\mathfrak{U}$. As in Theorem 2, the sequence $\{\mathcal{O}\mu_n\}$ can be shown to be a Cauchy sequence and to converge to some element v in $\mathfrak{V}$. Thus,

$$\mathcal{S}(\mu,v,r)=\lim_{n\to+\infty}\mathcal{S}(\mu_{n+1},\mathcal{O}\mu_n,r)=\mathcal{S}(\mathfrak{U},\mathfrak{V},r).\tag{24}$$

Thus μ becomes an element of $\mathfrak{U}_0(r)$. Since $\mathcal{O}(\mathfrak{U}_0(t)) \subseteq \mathfrak{V}_0(t)$,

$$\mathcal{S}(t_1, \mathcal{O}\mu, r) = \mathcal{S}(\mathfrak{U}, \mathfrak{V}, r) \tag{25}$$

for some element t_1 in $\mathfrak{U}$. By (24) and (25),

$$\mathcal{S}(\mu_{n+1}, t_1, r) = \mathcal{S}(\mathcal{O}\mu_n, \mathcal{O}\mu, r).$$

If for some n_0, $\mathcal{S}(t_1, \mu_{n_0+1}, r) = 1$, consequently $\mathcal{S}(\mathcal{O}\mu_{n_0}, \mathcal{O}\mu, r) = 1$ implies $\mathcal{O}\mu_{n_0} = \mathcal{O}\mu$, hence $\mathcal{S}(\mathfrak{U}, \mathfrak{V}, r) = \mathcal{S}(\mu, \mathcal{O}\mu, r)$. Thus, the inclusion is immediate. Therefore, let, for any $n \geq 0$, $\mathcal{S}(t_1, \mu_{n+1}, r) \neq 1$. Since $\mathcal{O}$ is a $\Gamma_\tau\mathcal{F}$-fuzzy proximal contraction of the first category,

$$\left.\begin{aligned} \mathcal{S}(t_1, \mathcal{O}\mu, r) &= \mathcal{S}(\mathfrak{U}, \mathfrak{V}, r) \\ \mathcal{S}(\mu_{n+1}, \mathcal{O}\mu_n, r) &= \mathcal{S}(\mathfrak{U}, \mathfrak{V}, r) \end{aligned}\right\} \text{implies that}$$

$$\begin{aligned} \Gamma_\tau(\mathcal{S}(t_1, \mu_n, r), \mathcal{S}(\mu_{n+1}, \mu, r), \mathcal{S}(t_1, \mu, r), \mathcal{S}(\mu_n, \mu_{n+1}, r)) \cdot \mathcal{F}(\mathcal{S}(t_1, \mu_{n+1}, r)) \\ \geq \mathcal{F}(\mathcal{S}(\mu, \mu_n, r)), \end{aligned}$$

implies

$$\mathcal{F}(\mathcal{S}(t_1, \mu_{n+1}, r)) \geq \tau \cdot \mathcal{F}(\mathcal{S}(t_1, \mu_{n+1}, r)) \geq \mathcal{F}(\mathcal{S}(\mu, \mu_n, r)).$$

Letting n tend to $+\infty$, we have $\mathcal{S}(t_1, \mu, r) = 1$, which implies that μ and t_1 must be identical. It follows that

$$\mathcal{S}(\mu, \mathcal{O}\mu, r) = \mathcal{S}(t_1, \mathcal{O}t_1, r) = \mathcal{S}(\mathfrak{U}, \mathfrak{V}, r).$$

Moreover, the uniqueness part will be similar to the steps followed for Theorem 1. □

Example 3. *Let $\mathfrak{X} = \mathbb{R}^2$. Define a fuzzy set as follows:*

$$\mathcal{S}(c, d, r) = \frac{r}{r + \rho(c, d)},$$

where $c, d \in \mathfrak{X}$ and $r > 0$, where ρ is a usual metric and $(\mathfrak{X}, \mathcal{S}, \circledast)$ is a complete non-Archimedean fuzzy metric with a product t-norm.

Let $\mathfrak{U} = \{(w, 0) : w \geq 0\}$ and $\mathfrak{V} = \{(g, 2) : g \geq 0\}$. Here, $\mathfrak{U} = \mathfrak{U}_0$ and $\mathfrak{V} = \mathfrak{V}_0$. Assume $\mathcal{O} : \mathfrak{U} \to \mathfrak{V}$ is

$$\mathcal{O}(w, 0) = \left(\frac{w}{w+1}, 2\right) \text{ for all } (w, 0) \in \mathfrak{U},$$

Consider

$$u_n = \left(\frac{1}{n}, 0\right) n \in \mathbb{N} \text{ such that}$$

$$\begin{aligned} \mathcal{S}(u_{n+1}, \mathcal{O}u_n, r) &= \mathcal{S}\left(\left(\frac{1}{n+1}, 0\right), \mathcal{O}\left(\frac{1}{n}, 0\right), r\right) \\ &= \mathcal{S}\left(\left(\frac{1}{n+1}, 0\right), \left(\frac{1}{n+1}, 2\right), r\right) \\ &= \frac{r}{r+2} = \mathcal{S}(\mathfrak{U}, \mathfrak{V}, r). \end{aligned}$$

There exist two subsequences $u_{n_k} = \left(\frac{1}{2k}, 0\right)$ *and* $u_{m_k} = \left(\frac{1}{3k}, 0\right)$ *of* $\{u_n\}$ *where* $3k > 2k > k$ *such that*

$$\begin{aligned} \lim_{k\to+\infty} \mathcal{S}(u_{n_k}, u_{m_k}, r) &= \lim_{k\to+\infty} \mathcal{S}\left(\left(\frac{1}{2k}, 0\right), \left(\frac{1}{3k}, 0\right), r\right) \\ &= \lim_{k\to+\infty} \frac{r}{r + |\frac{1}{2k} - \frac{1}{3k}|} = 1. \end{aligned}$$

Now

$$\begin{aligned} \mathcal{S}(u_{3k}, \mathcal{O}(u_{3k-1}), r) &= \mathcal{S}\left(\left(\frac{1}{3k}, 0\right), \mathcal{O}\left(\frac{1}{3k-1}, 0\right), r\right) \\ &= \mathcal{S}\left(\left(\frac{1}{3k}, 0\right), \left(\frac{1}{3k}, 2\right), r\right) \\ &= \lim_{k\to+\infty} \frac{r}{r + |\frac{1}{3k} - \frac{1}{3k}| + |0-2|} \\ &= \frac{r}{r+2} = \mathcal{S}(\mathfrak{U}, \mathfrak{V}, r). \end{aligned}$$

Similarly

$$\mathcal{S}(u_{2k}, \mathcal{O}(u_{2k-1}), r) = \mathcal{S}\left(\left(\frac{1}{2k}, 0\right), \mathcal{O}\left(\frac{1}{2k-1}, 0\right), r\right) = \mathcal{S}(\mathfrak{U}, \mathfrak{V}, r).$$

Hence $\mathcal{O}$ *satisfies the q-property. Now, for each* $c, d \geq 0$,

$$\begin{aligned} |\mathcal{O}c - \mathcal{O}d| &= |\mathcal{O}(w_1, 0) - \mathcal{O}(w_2, 0)| \\ &= \left|\frac{w_1}{1+w_1} - \frac{w_2}{1+w_2}\right| \\ &= \left|\frac{w_1 - w_2}{(1+w_1)(1+w_2)}\right| \\ &\leq |w_1 - w_2| + |0-0| \\ &= |(w_1, 0) - (w_2, 0)| \\ &= |c-d|. \end{aligned}$$

Hence $\mathfrak{U}$ *is approximately compact with regards to* $\mathfrak{V}$, $(\mathfrak{U}, \mathfrak{V})$ *satisfies the p-property,* $\mathcal{O}$ *is continuous and* $\mathcal{O}(\mathfrak{U}_0) \subseteq \mathfrak{V}_0$. *Define a function* $\mathcal{F}(s) = \dfrac{s}{1-s}$ *for any* $s \in (0,1)$ *such that* $\mathcal{F} \in \triangle_{\mathcal{F}}$ *and* $\Gamma_\tau(t_1, t_2, t_3, t_4) = \tau$ *where* $\tau \in (0,1)$ *such that* $\Gamma_\tau \in \triangle_{\Gamma_\tau}$.

Consider $u_1 = (w_1, 0), u_2 = (w_2, 0)$ *and* $x_1 = \left(\frac{w_1}{2}, 0\right), x_2 = \left(\frac{w_2}{2}, 0\right)$

$$\begin{aligned} \tau \cdot \mathcal{F}(\mathcal{S}(\mathcal{O}u_1, \mathcal{O}u_2, r)) &= \tau \cdot \mathcal{F}(\mathcal{S}(\mathcal{O}(w_1, 0), \mathcal{O}(w_2.0), r)) \\ &= \tau \cdot \mathcal{F}\left(\mathcal{S}\left(\left(\frac{w_1}{1+w_1}, 2\right), \left(\frac{w_2}{1+w_2}, 2\right), r\right)\right) \\ &= \tau \cdot \mathcal{F}\left(\frac{r}{r + |\frac{w_1}{1+w_1} - \frac{w_2}{1+w_2}|}\right) \\ &= \tau \cdot \left(\frac{\frac{r}{r + |\frac{w_1}{1+w_1} - \frac{w_2}{1+w_2}|}}{1 - \frac{r}{r + |\frac{w_1}{1+w_1} - \frac{w_2}{1+w_2}|}}\right) \\ &= \tau \cdot \frac{r}{|\frac{w_1}{1+w_1} - \frac{w_2}{1+w_2}|} \end{aligned}$$

Let us choose $\tau \in \left[\frac{1}{2}\frac{(2+w_1)(2+w_2)}{(1+w_1)(1+w_2)}, 1\right)$

$$
\begin{aligned}
&\geq \frac{1}{2}\frac{(2+w_1)(2+w_2)}{(1+w_1)(1+w_2)} \cdot \frac{r}{|\frac{w_1}{1+w_1} - \frac{w_2}{1+w_2}|} \\
&= \frac{r}{|\frac{w_1}{2+w_1} - \frac{w_2}{2+w_2}|} \\
&= \frac{r}{|\frac{\frac{w_1}{2}}{1+\frac{w_1}{2}} - \frac{\frac{w_2}{2}}{1+\frac{w_2}{2}}|} \\
&= \left(\frac{\dfrac{r}{r+|\frac{w_1/2}{1+w_1/2} - \frac{w_2/2}{1+w_2/2}|}}{1-\dfrac{r}{r+|\frac{w_1/2}{1+w_1/2} - \frac{w_2/2}{1+w_2/2}|}}\right) \\
&= \mathcal{F}\left(\frac{r}{r+|\frac{w_1/2}{1+w_1/2} - \frac{w_2/2}{1+w_2/2}|}\right) \\
&= \tau \cdot \mathcal{F}\left(\mathcal{S}\left(\left(\frac{w_1/2}{1+w_1/2}, 2\right), \left(\frac{w_2/2}{1+w_2/2}, 2\right), r\right)\right) \\
&= \mathcal{F}(\mathcal{S}(\mathcal{O}(w_1/2, 0), \mathcal{O}(w_2/2, 0), r)) = \mathcal{F}(\mathcal{S}(\mathcal{O}x_1, \mathcal{O}x_2, r))
\end{aligned}
$$

Hence $\mathcal{O}$ *is a proximal* $\Gamma_\tau\mathcal{F}$*-fuzzy contraction of the second category. Thus, all the assertions of Theorem 3 hold. Hence,* $\mathcal{O}$ *has a unique best proximity point* $(0,0)$.

Now, we can procure a few corollaries.

Corollary 1. *Suppose that two nonempty closed subsets* $\mathfrak{U}$ *and* $\mathfrak{V}$ *of a complete non-Archimedean fuzzy metric space* $(\mathfrak{X}, \mathcal{S}, \circledast)$ *with* $\mathfrak{U}_0(r) \neq \phi$, $\mathcal{O}(\mathfrak{U}_0(r)) \subseteq \mathfrak{V}_0(r)$ *and* $\mathcal{O} : \mathfrak{U} \to \mathfrak{V}$ *satisfy the assertions:*

$$\Gamma_\tau(\mathcal{S}(\mu, \mathcal{O}\mu, r), \mathcal{S}(\nu, \mathcal{O}\nu, r), \mathcal{S}(\mu, \mathcal{O}\nu, r), \mathcal{S}(\nu, \mathcal{O}\mu, r)) \cdot \mathcal{F}(\mathcal{S}(\mathcal{O}\mu, \mathcal{O}\nu, r)) \geq \mathcal{F}(\mathcal{S}(\mu, \nu, r)),$$

where $\mathcal{F} \in \triangle_{\mathcal{F}}$, $\Gamma_\tau \in \triangle_{\Gamma_\tau}$ *and* $(\mathfrak{U}, \mathfrak{V})$ *has the fuzzy p-property. Then, there exists a unique* $\mu^* \in \mathfrak{U}$ *such that* $\mathcal{S}(\mu^*, \mathcal{O}\mu^*, r) = \mathcal{S}(\mathfrak{U}, \mathfrak{V}, r)$.

Corollary 2. *Let* $\mathfrak{U}$ *and* $\mathfrak{V}$ *be nonempty closed subsets of a complete non-Archimedean fuzzy metric space* $(\mathfrak{X}, \mathcal{S}, \circledast)$ *such that* $\mathfrak{U}_0(r)$ *is nonempty. Let* $\mathcal{O} : \mathfrak{U} \to \mathfrak{V}$ *be a fuzzy* $\mathcal{F}$*- contraction for a nonself mapping such that* $\mathcal{O}(\mathfrak{U}_0(r)) \subseteq \mathfrak{V}_0(r)$. *Assume that the pair* $(\mathfrak{U}, \mathfrak{V})$ *has the p-property. Then, there exists a unique* $\mu^* \in \mathfrak{U}$ *such that* $\mathcal{S}(\mu^*, \mathcal{O}\mu^*, r) = \mathcal{S}(\mathfrak{U}, \mathfrak{V}, r)$.

Example 4. *Suppose* $\mathfrak{X} = \mathbb{R}^+ \times \mathbb{R}^+$ *and*

$$
\begin{aligned}
\mathfrak{U} &= \left\{\left(0, \frac{1}{n}\right) : n \in \mathbb{N}\right\} \cup \{(0,0)\} \\
\text{and } \mathfrak{V} &= \left\{\left(1, \frac{1}{n}\right) : n \in \mathbb{N}\right\} \cup \{(1,0)\}
\end{aligned}
$$

such that $\mathfrak{V}_0(r) = \mathfrak{V}$.

Consider a fuzzy set $\mathcal{S} : \mathfrak{X}^2 \times (0, +\infty) \to [0,1]$ *defined by* $\mathcal{S}(c,d,r) = e^{\frac{-\rho(c,d)}{r}}$ *for all* $c, d \in \mathfrak{X}$ *and* $r > 0$, *where* ρ *is the usual metric such that* $(\mathfrak{X}, \mathcal{S}, \circledast)$ *is a complete non-Archimedean fuzzy metric space with a product t-norm.*

Let $\mathcal{O} : \mathfrak{U} \to \mathfrak{V}$ *be defined as* $\mathcal{O}(0,x) = \left(1, \frac{x}{2}\right)$ *for all* $(0,x) \in \mathfrak{U}$ *such that* $\mathcal{O}(\mathfrak{U}_0(r)) \subseteq \mathfrak{V}_0(r)$.

Consider $u_1 = \left(0, \frac{1}{2n_1}\right)$, $x_1 = \left(0, \frac{1}{n_1}\right)$ *and* $u_2 = \left(0, \frac{1}{2n_2}\right)$, $x_2 = \left(0, \frac{1}{n_2}\right)$, *where* $n_1, n_2 \in \mathbb{N}$.

$$\begin{aligned}
\mathcal{S}(u_1, \mathcal{O}x_1, r) &= \mathcal{S}\left(\left(0, \frac{1}{2n_1}\right), \mathcal{O}\left(0, \frac{1}{n_1}\right), r\right) \\
&= \mathcal{S}\left(\left(0, \frac{1}{2n_1}\right), \left(1, \frac{1}{2n_1}\right), r\right) \\
&= e^{-\left(\frac{|0-1|+|\frac{1}{2n_1}-\frac{1}{2n_1}|}{r}\right)} \\
&= e^{-\frac{1}{r}} = \mathcal{S}(\mathfrak{U}, \mathfrak{V}, r).
\end{aligned}$$

Similarly

$$\mathcal{S}(u_2, \mathcal{O}x_2, r) = \mathcal{S}(\mathfrak{U}, \mathfrak{V}, r)$$

implies that

$$\begin{aligned}
\mathcal{S}(u_1, u_2, r) &= \mathcal{S}((0, \frac{1}{2n_1}), (0, \frac{1}{2n_2}), r) \\
&= e^{-}\left(\frac{|\frac{1}{2n_1} - \frac{1}{2n_2}|}{r}\right) \\
&= \mathcal{S}(\mathcal{O}x_1, \mathcal{O}x_2, r).
\end{aligned}$$

Thus the p-property $\mathcal{S}(u_1, u_2, r) = \mathcal{S}(\mathcal{O}x_1, \mathcal{O}x_2, r)$ *holds.*

Consider $\Gamma_\tau(l_1, l_2, l_3, l_4) = \tau$ *where* $\tau \in (0,1)$ *such that* $\Gamma_\tau \in \triangle_{\Gamma_\tau}$, *and define a strictly increasing function such that* $\mathcal{F}(s) = log_e(s)$ *where* $0 < s < 1$. *Now, we have to satisfy the* $\Gamma_\tau\mathcal{F}$*- fuzzy proximal contraction condition.*

$$\begin{aligned}
\tau \cdot \mathcal{F}(\mathcal{S}(u_1, u_2, r)) &= \tau \cdot \mathcal{F}\left(\mathcal{S}\left(\left(0, \frac{1}{2n_1}\right), \left(0, \frac{1}{2n_2}\right), r\right)\right) \\
&= \tau \cdot \mathcal{F}\left(e^{-\frac{1}{r}|\frac{1}{2n_1} - \frac{1}{2n_2}|}\right) \\
&= \tau \cdot log_e e^{-\frac{1}{r}|\frac{1}{2n_1} - \frac{1}{2n_2})|} \\
&= \tau \cdot (-\frac{1}{r}|\frac{1}{2n_1} - \frac{1}{2n_2})|) \\
&\geq -\frac{1}{r}|\frac{1}{n_1} - \frac{1}{n_2}| \\
&= \mathcal{F}(\mathcal{S}((0, \frac{1}{n_1}), (0, \frac{1}{n_2}), r)) = \mathcal{F}(\mathcal{S}(x_1, x_2, r)).
\end{aligned}$$

Hence $\mathcal{O}$ *is a proximal* $\Gamma_\tau\mathcal{F}$*-fuzzy contraction. Thus, all the assertions of Corollary 1 hold. Hence,* $\mathcal{O}$ *has a unique best proximity point* $(0,0)$.

4. Conclusions

The major contribution of this paper was to discuss the $\Gamma_\tau\mathcal{F}$-proximal contraction for the nonself map in fuzzy distance spaces. A few proximity theorems were proved for different proximal contraction in the fuzzy setting. In between the theorems, some sample examples were given to highlight the validity of the established results. As future work, we will consider and prove the unique best proximity point in various distance spaces with applications. The terms used here, such as $\mathfrak{U}_0(r)$ and $\mathfrak{V}_0(r)$, depended on the real parameters r, and a proximal contraction with some hypothesis could guarantee the existence of a unique best proximity point. Readers can investigate this hypothesis to obtain new fuzzy proximal theorems. Readers can extend these results in terms of cyclic proximal contractions in the fuzzy setting with applications, refer [13–24].

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