axioms

MDPI

Article

On Intuitionistic Fuzzy Temporal Topological Structures

Krassimir Atanassov [1,2]

[1] Department of Bioinformatics and Mathematical Modelling, Institute of Biophysics and Biomedical Engineering, Bulgarian Academy of Sciences, Acad. Georgi Bonchev Str., 1113 Sofia, Bulgaria; krat@bas.bg or k.t.atanassov@gmail.com
[2] Intelligent Systems Laboratory, Prof. Dr. Assen Zlatarov University, 1 "Prof. Yakimov" Blvd., 8010 Burgas, Bulgaria

Abstract: In the present paper, four different intuitionistic fuzzy temporal topological structures are introduced, and some of their properties are discussed. These topological structures are based on the intuitionistic fuzzy topological operators and on the temporal intuitionistic fuzzy topological operators, which exist in intuitionistic fuzzy sets theory. The new structures are direct extensions of the IFTSs and will be the basis for introducing of a new type of topological structures.

Keywords: temporal intuitionistic fuzzy operation; temporal intuitionistic fuzzy operator; temporal intuitionistic fuzzy set; intuitionistic fuzzy topological structure topology

MSC: 03E72

Citation: Atanassov, K. On Intuitionistic Fuzzy Temporal Topological Structures. *Axioms* **2023**, *12*, 182. https://doi.org/10.3390/axioms12020182

Academic Editors: Darko Božanić and Adis Puška

Received: 21 December 2022
Revised: 6 February 2023
Accepted: 7 February 2023
Published: 9 February 2023

1. Introduction

The idea for development of the intuitionistic fuzzy topological structures (in the sense of [1–3]) started from a discussion between the Bulgarian Prof. Doychin Doychinov (1926–1996) and the author in 1989, when Prof. Doychinov was the Head of the Department of Geometry and Topology at the Institute of Mathematics and Informatics of the Bulgarian Academy of Sciences. Unfortunately, back then, none of the Master's or PhD students in the department were interested in this idea. That is why no progress was made until the end of the 20th century, when Dogan Čoker (1951–2003) started working on the topic (see [4,5]). The author discussed with him the possibility of joint research, but sadly he passed away too soon. The next important leg of the research in this area was made by F. Gallego Lupiañez in [6,7]. During the last 20 years, a lot of research in this area has been published, but it contains the intuitionistic fuzzy interpretations of existing objects in standard topology (see, e.g., [8–21]).

In [22], the author introduced the idea of intuitionistic fuzzy modal topological structures (IFMTS). On the basis on Kazimierz Kuratowski's (1896–1980) definition for a topological structure [2] that satisfies the conditions

C1 $cl(A\Delta B) = cl(A)\Delta cl(B)$,
C2 $A \subseteq cl(A)$,
C3 $cl(cl(A)) = cl(A)$,
C4 $cl(O) = O$,

where $A, B \in X$, X is some fixed set of sets with a minimal element O, cl is the topological operator "closure", and $\Delta : X \times X \to X$ is the operation that generates cl; additionally,

I1 $in(A\nabla B) = in(A)\nabla in(B)$,
I2 $in(A) \subseteq A$,
I3 $in(in(A)) = in(A)$,
I4 $in(I) = I$,

where $A, B \in X$, X is the same set and I is its maximal element, in is the topological operator "interior", and $\nabla : X \times X \to X$ is the operation that generates in.

On a side note, it seems that the first time when the term "temporal topology" was used was in Robin Le Poidevin's paper [23] from 1990, which, however discusses the concept in a philosophical context, bearing no relation to the usage meant in intuitionistic fuzzy sets. Additionally, we note that another term, which is unrelated, is "temporal intuitionistic fuzzy topology" in Šostak's sense, as studied by Fatih Kutlu, Tunay Bilgin and A.A. Ramadan in [24–26]).

Later, the operators that satisfy the conditions C1–C4 will be termed as "operators of a closure type", and those that satisfy the conditions I1–I4 as "operators of an interior type".

The idea for IFMTS has been discussed in a series of papers [22,27–29], in which some types of these structures are constructed.

The present paper is a continuation of the author's research. It is based on the definitions and the notations of the intuitionistic fuzzy sets (IFSs, see, e.g., [30,31]), introduced by the author 40 years ago, in 1983. In it, the tempotal intuitionistic fuzzy operators (see [30,31]) play the role of the modal operators in the IFMTSs.

In the present paper, we combine the ideas and definitions from the areas of (general) topology (see, e.g., [1–3]) and of temporal intuitionistic fuzzy sets (see, e.g., [30–32]) and introduce, in a novel way, the concept of an intuitionistic fuzzy temporal topological structure (IFTTS). We must immediately mention that the object temporal topology is fuzzified in the sense of Brouwer's (1881–1966) intuitionism, while the name "temporal intuitionistic fuzzy topology" points to the idea for the temporalization of an intuitionistic fuzzy topology that has another meaning.

The present paper shows the possibility for a new direction in the development of topological structures, and more precisely of the IFTSs. In future, it will be the basis of a series of modifications and extensions.

Initially, short remarks over intuitionistic fuzzy sets (IFSs) and temporal IFSs (TIFSs) are given (in Section 2). After this, in Section 3, the new objects are introduced, and some of their basic properties will be discussed. In Section 4 an example of the application of IFTTSs in a Multi-Criteria Decision Making procedure is presented, and finally Section 5 outlines a conclusion and discusses new directions for the development of the present ideas.

2. Short Remarks about Temporal Intuitionistic Fuzzy Sets

The IFSs are one of the early extensions of Lotfi Zadeh's (1921–2017) fuzzy sets [33] . When a set E, called a "universum", is fixed and A is its subset, each IFS in E has the form

$$A^* = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\},$$

where functions $\mu_A : E \to [0,1]$ and $\nu_A : E \to [0,1]$ define the degree of membership and of non-membership of element $x \in E$, respectively, to the set $A \subseteq E$, and for each $x \in E$:

$$0 \le \mu_A(x) + \nu_A(x) \le 1.$$

As usual, instead of A^* for brevity, the notation A is used.

Let the above (fixed) set E be given and let T be a non-empty set (finite or infinite), called the temporal scale with upper and lower boundaries. The elements of T are called "time-moments".

Following [30–32], we define the Temporal IFS (TIFS) as follows:

$$A^*(T) = \{\langle x, \mu_A(x,t), \nu_A(x,t)\rangle | \langle x, t\rangle \in E \times T\},$$

where

(a) $A \subseteq E$ is a fixed set,
(b) $\mu_A(x,t) + \nu_A(x,t) \le 1$ for every $\langle x, t\rangle \in E \times T$,
(c) $\mu_A(x,t)$ and $\nu_A(x,t)$ are the degrees of membership and non-membership, respectively, of the element $x \in E$ at the time-moment $t \in T$.

As usual, instead of $A^*(T)$ for brevity, the notation $A(T)$ is used.

As is mentioned in [31], each ordinary IFS can be regarded as a TIFS for which T is a singleton set. Additionally, it is mentioned there that all operations and operators on the IFSs can be defined for the TIFSs. However, the opposite is also true: each TIFS $A(T)$ is a standard IFS, but over universe $E \times T$. For this reason, we can re-define all operations, relations, and operators defined over standard IFSs, now over TIFSs. For example, if we have two TIFSs:

$$A(T') = \{\langle x, \mu_A(x,t), \nu_A(x,t)\rangle | \langle x,t\rangle \in E \times T'\},$$

and

$$B(T'') = \{\langle x, \mu_B(x,t), \nu_B(x,t)\rangle | \langle x,t\rangle \in E \times T''\},$$

where T' and T'' are temporal scales, then

$$A(T') \cup B(T'') = \{\langle x, \mu_{A(T')\cup B(T'')}(x,t), \nu_{A(T')\cup B(T'')}(x,t)\rangle | \langle x,t\rangle \in E \times (T' \cup T'')\},$$

where

$$\langle x, \mu_{A(T')\cup B(T'')}(x,t), \nu_{A(T')\cup B(T'')}(x,t)\rangle$$
$$= \begin{cases} \langle x, \mu_A(x,t'), \nu_A(x,t')\rangle, & \text{if } t = t' \in T' - T'' \\ \langle x, \mu_B(x,t''), \nu_A(x,t'')\rangle, & \text{if } t = t'' \in T'' - T' \\ \langle x, \max(\mu_A(x,t'), \mu_B(x,t'')), \\ \quad \min(\nu_A(x,t'), \nu_A(x,t''))\rangle, & \text{if } t = t' = t'' \in T' \cap\ T'' \\ \langle x, 0, 1\rangle, & \text{otherwise} \end{cases}$$

$$A(T') \cap B(T'') = \{\langle x, \mu_{A(T')\cap B(T'')}(x,t), \nu_{A(T')\cap B(T'')}(x,t)\rangle | \langle x,t\rangle \in E \times (T' \cup T'')\},$$

where

$$\langle x, \mu_{A(T')\cap B(T'')}(x,t), \nu_{A(T')\cap B(T'')}(x,t)\rangle$$
$$= \begin{cases} \langle x, \mu_A(x,t'), \nu_A(x,t')\rangle, & \text{if } t = t' \in T' - T'' \\ \langle x, \mu_B(x,t''), \nu_A(x,t'')\rangle, & \text{if } t = t'' \in T'' - T' \\ \langle x, \min(\mu_A(x,t'), \mu_B(x,t'')), \\ \quad \max(\nu_A(x,t'), \nu_A(x,t''))\rangle, & \text{if } t = t' = t'' \in T' \cap\ T'' \\ \langle x, 0, 1\rangle, & \text{otherwise} \end{cases}$$

For the needs of the present research, we define also operation "(classical) negation" by

$$\neg A(T) = \{\langle x, \nu_A(x,t), \mu_A(x,t)\rangle | \langle x,t\rangle \in E \times T\}$$

and the simplest relations, defined for the case when $T = T' = T''$:

$$\begin{aligned} A(T) \subseteq B(T) \quad &\text{iff} \quad (\forall x \in E)(\forall t \in T)(\mu_{A(T)}(x,t) \le \mu_{B(T)}(x,t) \wedge \nu_{A(T)}(x,t) \ge \nu_{B(T)}(x,t)); \\ A(T) \supseteq B(T) \quad &\text{iff} \quad B(T) \subseteq A(T); \\ A(T) = B(T) \quad &\text{iff} \quad (\forall x \in E)(\forall t \in T)(\mu_{A(T)}(x,t) = \mu_{B(T)}(x,t) \wedge \nu_{A(T)}(x,t) = \nu_{B(T)}(x,t)) \end{aligned}$$

In future research, we will discuss the form of the above relations in the case when $T' \neq T''$.

Since we assume that $T' = T''$ here, we see that the above definitions of the operations $\cup$ and $\cap$ are reduced to the forms:

$$A(T) \cup B(T) = \{\langle x, \max(\mu_A(x,t), \mu_B(x,t)), \min(\nu_A(x,t), \nu_A(x,t))\rangle | \langle x,t\rangle \in E \times T\},$$

$$A(T) \cap B(T) = \{\langle x, \min(\mu_A(x,t), \mu_B(x,t)), \max(\nu_A(x,t), \nu_A(x,t))\rangle | \langle x,t\rangle \in E \times T\}.$$

The first two (simplest) analogues of the topological operators "closure" and "interior" (defined over IFSs) are introduced over TIFSs in [30,31] as follows:

$$\mathcal{C}(A(T)) = \{\langle x, \sup_{y\in E} \mu_{A(T)}(y,t), \inf_{y\in E} \nu_{A(T)}(y,t)\rangle | \langle x,t\rangle \in E \times T\},$$
$$\mathcal{I}(A(T)) = \{\langle x, \inf_{y\in E} \mu_{A(T)}(y,t), \sup_{y\in E} \nu_{A(T)}(y,t)\rangle | \langle x,t\rangle \in E \times T\},$$

while specifically related to temporality are the following two operators over TIFSs (see [30,31]):

$$\mathcal{C}^*(A(T)) = \{\langle x, \sup_{t\in T} \mu_{A(T)}(x,t), \inf_{t\in T} \nu_{A(T)}(x,t)\rangle | \langle x,t\rangle \in E \times T\},$$
$$\mathcal{I}^*(A(T)) = \{\langle x, \inf_{t\in T} \mu_{A(T)}(x,t), \sup_{t\in T} \nu_{A(T)}(x,t)\rangle | \langle x,t\rangle \in E \times T\}.$$

Thus, we say that the operator $\mathcal{C}^*$ is an operator of closure type, while the operator $\mathcal{I}^*$ is an operator of interior type. Notably, the first two operators, $\mathcal{C}$ and $\mathcal{I}$, are related to the elements of the universe, while the second two operators, $\mathcal{C}^*$ and $\mathcal{I}^*$, are related to the elements of the chosen time-scale.

3. Definitions of Four Intuitionistic Fuzzy Temporal Topological Structures

Following [30], let us define

$$O^*(T) = \{\langle x, 0, 1\rangle | \langle x,t\rangle \in E \times T\},$$

$$E^*(T) = \{\langle x, 1, 0\rangle | \langle x,t\rangle \in E \times T\}.$$

Therefore, for each TIFS $A(T)$:

$$O^*(T) \subseteq A(T) \subseteq E^*(T).$$

If for each set X

$$\mathcal{P}(X) = \{Y | Y \subseteq X\},$$

then for each TIFS $A(T)$ over the universe E and time-scale T:

$$\mathcal{P}(O^*(T)) = O^*(T),$$

$$\mathcal{P}(E^*(T)) = \{A(T) | A(T) \subseteq E^*(T)\}.$$

Let $\mathcal{O}$ and $\mathcal{Q}$ be operators such that for each TIFS $A(T) \in \mathcal{P}(E^*(T))$:

$$\mathcal{O}(A(T)) = \neg\mathcal{Q}(\neg A(T)),$$

$$\mathcal{Q}(A(T)) = \neg\mathcal{O}(\neg A(T)).$$

Let $\Delta, \nabla : \mathcal{P}(E^*(T)) \times \mathcal{P}(E^*(T)) \to \mathcal{P}(E^*(T))$ be operations such that for every two TIFSs $A(T), B(T) \in \mathcal{P}(E^*(T))$:

$$A(T)\nabla B(T) = \neg(\neg A(T)\Delta\neg B(T)),$$

$$A(T)\Delta B(T) = \neg(\neg A(T)\nabla\neg B(T)).$$

Let $\circ$ and $\bullet$ be the special operators over TIFS that are related to temporality and such that for each TIFS $A(T) \in \mathcal{P}(E^*(T))$:

$$\circ A(T) = \neg \bullet \neg A(T),$$

$$\bullet A(T) = \neg \circ \neg A(T).$$

By analogy with [22], and extending the definitions from there, we will introduce four types of IFTTSs: a *cl*-*cl*-IFTTS, *cl*-*in*-IFTTS, *in*-*cl*-IFTTS, and *in*-*in*-IFTTS.

3.1. cl-cl-Intuitionistic Fuzzy Temporal Topological Structure

The *cl*-*cl*-IFTTS is the object

$$\langle \mathcal{P}(E^*(T)), \mathcal{O}, \Delta, \circ \rangle,$$

where T is a fixed time-scale; E is a fixed universe; $\mathcal{O} : \mathcal{P}(E^*(T)) \to \mathcal{P}(E^*(T))$ is an operator of a closure type related to operation Δ; $\circ : \mathcal{P}(E^*(T)) \to \mathcal{P}(E^*(T))$ is a temporal operator of a closure type; and for every two TIFSs $A(T), B(T) \in \mathcal{P}(E^*(T))$, the following nine conditions hold:

CC1 $\mathcal{O}(A(T)\Delta B(T)) = \mathcal{O}(A(T))\Delta\mathcal{O}(B(T))$,
CC2 $A(T) \subseteq \mathcal{O}(A(T))$,
CC3 $\mathcal{O}(O^*(T)) = O^*(T)$,
CC4 $\mathcal{O}(\mathcal{O}(A(T))) = \mathcal{O}(A(T))$,
CC5 $\circ(A(T)\nabla B(T)) \subseteq \circ A(T)\nabla \circ B(T)$,
CC6 $A(T) \subseteq \circ A(T)$,
CC7 $\circ E^*(T) = E^*(T)$,
CC8 $\circ \circ A(T) = \circ A(T)$,
CC9 $\circ\, \mathcal{O}(A(T)) = \mathcal{O}(\circ A(T))$.

Now, we see that the first four conditions correspond to the conditions C1–C4 for a topological operator "closure"; the next four conditions correspond to the conditions C1–C4, too, but for a temporal topological operator "closure"; and condition CC9 determines the relation between the two types of operators.

Obviously, when $T = \{t\}$ is a single set, then the IFTTS is reduced to an IFTS. On the other hand, if we have a set of IFTSs for which we know that they are related to fixed time-moments $t_1 < t_2 < \ldots$, respectively, then on their basis we can construct an IFTTS.

Theorem 1. *For each universe E and each time-scale T, $\langle \mathcal{P}(E^*(T)), \mathcal{C}, \cup, \mathcal{C}^* \rangle$ is a cl-cl-IFTTS.*

Proof. Let the IFSs $A(T), B(T) \in \mathcal{P}(E^*(T))$ be given. We will sequentially prove the validity of the nine conditions CC1–CC9.

CC1.

$$\begin{aligned}
&\mathcal{C}(A(T) \cup B(T)) \\
&= \mathcal{C}(\{\langle x, \mu_{A(T)}(x,t), \nu_{A(T)}(x,t)\rangle | \langle x,t\rangle \in E \times T\} \\
&\quad \cup \{\langle x, \mu_{B(T)}(x,t), \nu_{B(T)}(x,t)\rangle | \langle x,t\rangle \in E \times T\}) \\
&= \mathcal{C}(\{\langle x, \max(\mu_{A(T)}(x,t), \mu_{B(T)}(x,t)), \\
&\qquad \min(\nu_{A(T)}(x,t), \nu_{B(T)}(x,t))\rangle | \langle x,t\rangle \in E \times T\}) \\
&= \{\langle x, \sup_{y\in E} \max(\mu_{A(T)}(y,t), \mu_{B(T)}(y,t)), \\
&\qquad \inf_{y\in E} \min(\nu_{A(T)}(y,t), \nu_{B(T)}(y,t))\rangle | \langle x,t\rangle \in E \times T\} \\
&= \{\langle x, \max(\sup_{y\in E} \mu_{A(T)}(y,t), \sup_{y\in E} \mu_{B(T)}(y,t)), \\
&\qquad \min(\inf_{y\in E} \nu_{A(T)}(y,t), \inf_{y\in E} \nu_{B(T)}(y,t))\rangle | \langle x,t\rangle \in E \times T\} \\
&= \mathcal{C}(A(T)) \cup \mathcal{C}(B(T));
\end{aligned}$$

CC2.

$$\begin{aligned} A(T) &= \{\langle x, \mu_{A(T)}(x,t), \nu_{A(T)}(x,t)\rangle | \langle x,t\rangle \in E\times T\} \\ &\subseteq \{\langle x, \sup_{y\in E}\mu_{A(T)}(y,t), \inf_{y\in E}\nu_{A(T)}(y,t)\rangle | \langle x,t\rangle \in E\times T\} \\ &= \mathcal{C}(A(T)); \end{aligned}$$

CC3.

$$\begin{aligned} \mathcal{C}(O^*(T)) &= \mathcal{C}(\{\langle x,0,1\rangle | \langle x,t\rangle \in E\times T\}) \\ &= \{\langle x, \sup_{y\in E} 0, \inf_{y\in E} 1\rangle | \langle x,t\rangle \in E\times T\} \\ &= \{\langle x,0,1\rangle | \langle x,t\rangle \in E\times T\} \\ &= O^*(T); \end{aligned}$$

CC4. Bearing in mind that for each fixed $t \in T$: $\sup_{y\in E}\mu_{A(T)}(y,t)$ and $\inf_{y\in E}\nu_{A(T)}(y,t)$ are constants, we obtain that:

$$\begin{aligned} \mathcal{C}(\mathcal{C}(A(T))) &= \mathcal{C}(\{\langle x, \sup_{y\in E}\mu_{A(T)}(y,t), \inf_{y\in E}\nu_{A(T)}(y,t)\rangle | \langle x,t\rangle \in E\times T\}) \\ &= \{\langle x, \sup_{z\in E}\sup_{y\in E}\mu_{A(T)}(y,t), \inf_{z\in E}\inf_{y\in E}\nu_{A(T)}(y,t)\rangle | \langle x,t\rangle \in E\times T\} \\ &= \{\langle x, \sup_{y\in E}\mu_{A(T)}(y,t), \inf_{y\in E}\nu_{A(T)}(y,t)\rangle | \langle x,t\rangle \in E\times T\} \\ &= \mathcal{C}(A(T)). \end{aligned}$$

CC5. To check condition CC5, we will first prove that for each $x \in E$:

$$\sup_{u\in T}\min(\mu_{A(T)}(x,u), \mu_{B(T)}(x,u)) \le \min(\sup_{u\in T}\mu_{A(T)}(x,u), \sup_{u\in T}\mu_{B(T)}(x,u)), \tag{1}$$

$$\inf_{u\in T}\max(\nu_{A(T)}(x,u), \nu_{B(T)}(x,u)) \ge \max(\inf_{t\in T}\nu_{A(T)}(x,t), \inf_{t\in T}\nu_{B(T)}(x,t)). \tag{2}$$

Let for a fixed $x \in E$:

$$\min(\mu_{A(T)}(x,u), \mu_{B(T)}(x,u)) = g(u).$$

Therefore, for each $x \in E$:

$$\begin{aligned} g(u) &\le \mu_{A(T)}(x,u), \\ g(u) &\le \mu_{B(T)}(x,u). \end{aligned}$$

Therefore,

$$\begin{aligned} \sup_{u\in T} g(u) &\le \sup_{u\in T}\mu_{A(T)}(x,u), \\ \sup_{u\in T} g(u) &\le \sup_{u\in T}\mu_{B(T)}(x,u) \end{aligned}$$

hence,

$$\sup_{u\in T} g(u) \le \min(\sup_{u\in T}\mu_{A(T)}(x,u), \sup_{u\in T}\mu_{B(T)}(x,u)),$$

i.e., (1) is valid. (2) is proved in a similar manner.

Back to the proof of CC5, using (1) and (2), we see that:

$$\begin{aligned}
&\mathcal{C}^*(A(T) \cap B(T)) \\
&= \mathcal{C}^*(\{\langle x, \min(\mu_{A(T)}(x,t), \mu_{B(T)}(x,t)), \\
&\qquad \max(\nu_{A(T)}(x,t), \nu_{B(T)}(x,t))|\rangle \langle x,t\rangle \in E \times T\}) \\
&= \{\langle x, \sup_{u \in T} \min(\mu_{A(T)}(x,u), \mu_{B(T)}(x,u))), \\
&\qquad \inf_{u \in T} \max(\nu_{A(T)}(x,u), \nu_{B(T)}(x,u)))\rangle | \langle x,t\rangle \in E \times T\} \\
&\subseteq \{\langle x, \min(\sup_{t \in T} \mu_{A(T)}(x,t), \sup_{t \in T} \mu_{B(T)}(x,t)), \\
&\qquad \max(\inf_{t \in T} \nu_{A(T)}(x,t), \inf_{t \in T} \nu_{B(T)}(x,t)\rangle | \langle x,t\rangle \in E \times T\} \\
&= \{\langle x, \sup_{t \in T} \mu_{A(T)}(x,t), \inf_{t \in T} \nu_{A(T)}(x,t)\rangle | \langle x,t\rangle \in E \times T\} \\
&\qquad \cap \{\langle x, \sup_{t \in T} \mu_{B(T)}(x,t), \inf_{t \in T} \nu_{B(T)}(x,t)\rangle | \langle x,t\rangle \in E \times T\} \\
&= \mathcal{C}^*(A(T)) \cap \mathcal{C}^*(B(T)).
\end{aligned}$$

CC6.

$$\begin{aligned}
A(T) &= \{\langle x, \mu_{A(T)}(x,t), \nu_{A(T)}(x,t)\rangle | \langle x,t\rangle \in E \times T\} \\
&\subseteq \{\langle x, \sup_{t \in T} \mu_{A(T)}(x,t), \inf_{t \in T} \nu_{A(T)}(x,t)\rangle | \langle x,t\rangle \in E \times T\} \\
&= \mathcal{C}^*(A(T));
\end{aligned}$$

CC7.

$$\begin{aligned}
\mathcal{C}^*(E^*(T)) &= \mathcal{C}^*(\{\langle x, 1, 0\rangle\} | \langle x,t\rangle \in E \times T\}) \\
&= (\{\langle x, 1, 0\rangle\} | \langle x,t\rangle \in E \times T\}) \\
&= E^*(T)
\end{aligned}$$

CC8. Bearing in mind that for each fixed $x \in T$: $\sup_{t \in T} \mu_{A(T)}(x,t)$ and $\inf_{t \in T} \nu_{A(T)}(x,t)$ are constants, we obtain that:

$$\begin{aligned}
\mathcal{C}^*(\mathcal{C}^*(T)) &= \mathcal{C}^*(\{\langle x, \sup_{t \in T} \mu_{A(T)}(x,t), \inf_{t \in T} \nu_{A(T)}(x,t)\rangle | \langle x,t\rangle \in E \times T\} \\
&= \{\langle x, \sup_{u \in T} \sup_{t \in T} \mu_{A(T)}(x,t), \inf_{u \in T} \inf_{t \in T} \nu_{A(T)}(x,t)\rangle | \langle x,t\rangle \in E \times T\} \\
&= \{\langle x, \sup_{t \in T} \mu_{A(T)}(x,t), \inf_{t \in T} \nu_{A(T)}(x,t)\rangle | \langle x,t\rangle \in E \times T\} \\
&= \mathcal{C}^*(A(T));
\end{aligned}$$

CC9.

$$\begin{aligned}
\mathcal{C}^*(\mathcal{C}(A(T))) &= \mathcal{C}^*(\{\langle x, \sup_{y \in E} \mu_{A(T)}(y), \inf_{y \in E} \nu_{A(T)}(y)\rangle | \langle x,t\rangle \in E \times T\}) \\
&= \{\langle x, \sup_{u \in T} \sup_{y \in E} \mu_{A(T)}(y,u), \inf_{u \in T} \inf_{y \in E} \nu_{A(T)}(y,u)\rangle | \langle x,t\rangle \in E \times T\}) \\
&= \{\langle x, \sup_{y \in E} \sup_{u \in T} \mu_{A(T)}(y,u), \inf_{y \in E} \inf_{u \in T} \nu_{A(T)}(y,u)\rangle | \langle x,t\rangle \in E \times T\} \\
&= \mathcal{C}(\{\langle x, \sup_{t \in T} \mu_{A(T)}(x,t), \inf_{t \in T} \nu_{A(T)}(x,t)\rangle | \langle x,t\rangle \in E \times T\}) \\
&= \mathcal{C}(\mathcal{C}^*(A(T))).
\end{aligned}$$

This completes the proof. $\square$

3.2. in-in-Intuitionistic Fuzzy Temporal Topological Structure

The *in-in*-IFTTS is the object

$$\langle \mathcal{P}(E^*(T)), \mathcal{Q}, \nabla, \bullet \rangle,$$

where T is a fixed time-scale; E is a fixed universe; $\mathcal{Q} : \mathcal{P}(E^*(T)) \to \mathcal{P}(E^*(T))$ is an operator of interior type, related to the operation ∇; $\bullet : \mathcal{P}(E^*(T)) \to \mathcal{P}(E^*(T))$ is a temporal operator from interior type; and for every two TIFSs $A(T), B(T) \in \mathcal{P}(E^*(T))$, the following nine conditions hold:

II1 $\mathcal{Q}(A(T) \nabla B(T)) = \mathcal{Q}(A(T)) \nabla \mathcal{Q}(B(T))$,
II2 $\mathcal{Q}(A(T)) \subseteq A(T)$,
II3 $\mathcal{Q}(E^*(T)) = E^*(T)$,
II4 $\mathcal{Q}(\mathcal{Q}(A(T))) = \mathcal{Q}(A(T))$,
II5 $\bullet(A(T) \Delta B(T)) \supseteq \bullet A(T) \Delta \bullet B(T)$,
II6 $\bullet A(T) \subseteq A(T)$,
II7 $\bullet O^*(T) = O^*(T)$,
II8 $\bullet \bullet A(T) = \bullet A(T)$,
II9 $\bullet\ \mathcal{Q}(A(T)) = \mathcal{Q}(\bullet A(T))$.

We see that the first four conditions correspond to the conditions I1–I4 for a topological operator "interior"; the next four conditions correspond to the conditions I1–I4, too, but for a temporal topological operator the "interior" and condition CC9 determine the relation between the two types of operators.

Theorem 2. *For each universe E and each time-scale T, $\langle \mathcal{P}(E^*(T)), \mathcal{I}, \cap, \mathcal{I}^* \rangle$ is an in-in-IFTTS.*

Proof. Let the IFSs $A(T), B(T) \in \mathcal{P}(E^*(T))$ be given. We will sequentially prove the validity of the nine conditions II1–II9.

II1.

$$\begin{aligned}
&\mathcal{I}(A(T) \cap B(T)) \\
&= \mathcal{I}(\{\langle x, \mu_{A(T)}(x,t), \nu_{A(T)}(x,t)\rangle | \langle x,t\rangle \in E \times T\} \\
&\qquad \cap \{\langle x, \mu_{B(T)}(x,t), \nu_{B(T)}(x,t)\rangle | \langle x,t\rangle \in E \times T\}) \\
&= \mathcal{C}(\{\langle x, \min(\mu_{A(T)}(x,t), \mu_{B(T)}(x,t)), \\
&\qquad \max(\nu_{A(T)}(x,t), \nu_{B(T)}(x,t))\rangle | \langle x,t\rangle \in E \times T\}) \\
&= \{\langle x, \inf_{y\in E} \min(\mu_{A(T)}(y,t), \mu_{B(T)}(y,t)), \\
&\qquad \sup_{y\in E} \max(\nu_{A(T)}(y,t), \nu_{B(T)}(y,t))\rangle | \langle x,t\rangle \in E \times T\} \\
&= \{\langle x, \min(\inf_{y\in E} \mu_{A(T)}(y,t), \inf_{y\in E} \mu_{B(T)}(y,t)), \\
&\qquad \max(\sup_{y\in E} \nu_{A(T)}(y,t), \sup_{y\in E} \nu_{B(T)}(y,t))\rangle | \langle x,t\rangle \in E \times T\} \\
&= \mathcal{I}(A(T)) \cap \mathcal{I}(B(T));
\end{aligned}$$

II2.

$$\begin{aligned}
\mathcal{I}(A(T)) &= \{\langle x, \inf_{y\in E} \mu_{A(T)}(y,t), \sup_{y\in E} \nu_{A(T)}(y,t)\rangle | \langle x,t\rangle \in E \times T\} \\
&\subseteq \{\langle x, \mu_{A(T)}(x,t), \nu_{A(T)}(x,t)\rangle | \langle x,t\rangle \in E \times T\} \\
&= A(T);
\end{aligned}$$

II3.

$$\begin{aligned}\mathcal{I}(E^*(T)) &= \mathcal{I}(\{\langle x,1,0\rangle | \langle x,t\rangle \in E\times T\})\\ &= \{\langle x, \inf_{y\in E} 1, \sup_{y\in E} 0\rangle | \langle x,t\rangle \in E\times T\}\\ &= \{\langle x,1,0\rangle | \langle x,t\rangle \in E\times T\}\\ &= E^*(T);\end{aligned}$$

II4. Similarly to the above, bearing in mind that for each fixed $t\in T$: $\inf\limits_{y\in E}\mu_{A(T)}(y,t)$ and $\sup\limits_{y\in E}\nu_{A(T)}(y,t)$ are constants, we obtain that:

$$\begin{aligned}\mathcal{I}(\mathcal{I}(A(T))) &= \mathcal{I}(\{\langle x, \inf_{y\in E}\mu_{A(T)}(y,t), \sup_{y\in E}\nu_{A(T)}(y,t)\rangle | \langle x,t\rangle \in E\times T\})\\ &= \{\langle x, \inf_{z\in E}\inf_{y\in E}\mu_{A(T)}(y,t), \sup_{z\in E}\sup_{y\in E}\nu_{A(T)}(y,t)\rangle | \langle x,t\rangle \in E\times T\}\\ &= \{\langle x, \inf_{y\in E}\mu_{A(T)}(y,t), \sup_{y\in E}\nu_{A(T)}(y,t)\rangle | \langle x,t\rangle \in E\times T\}\\ &= \mathcal{I}(A(T)).\end{aligned}$$

II5. To check condition II5, as above, we will first prove that for each $x\in E$:

$$\inf_{u\in T}\max(\mu_{A(T)}(x,u),\mu_{B(T)}(x,u)) \geq \max(\inf_{u\in T}\mu_{A(T)}(x,u), \inf_{u\in T}\mu_{B(T)}(x,u)), \quad (3)$$

$$\sup_{u\in T}\min(\nu_{A(T)}(x,u),\nu_{B(T)}(x,u)) \leq \min(\sup_{t\in T}\nu_{A(T)}(x,t), \sup_{t\in T}\nu_{B(T)}(x,t)). \quad (4)$$

Let for a fixed $x\in E$:

$$\max(\mu_{A(T)}(x,u),\mu_{B(T)}(x,u)) = h(u).$$

Therefore, for each $x\in E$:

$$\begin{aligned}h(u) &\geq \mu_{A(T)}(x,u),\\ h(u) &\geq \mu_{B(T)}(x,u).\end{aligned}$$

Therefore,

$$\begin{aligned}\inf_{u\in T} h(u) &\geq \inf_{u\in T}\mu_{A(T)}(x,u),\\ \inf_{u\in T} h(u) &\geq \inf_{u\in T}\mu_{B(T)}(x,u)\end{aligned}$$

and hence

$$\inf_{u\in T} h(x) \geq \max(\inf_{u\in T}\mu_{A(T)}(x,u), \inf_{u\in T}\mu_{B(T)}(x,u)),$$

i.e., (3) is valid. Proving (4) is done in a similar manner.

Back to the proof of II5, using (3) and (4), we see that:

$$\begin{aligned}
&\mathcal{I}^*(A(T) \cup (B(T)) \\
&= \mathcal{I}^*(\{\langle x, \mu_{A(T)}(x,t), \nu_{A(T)}(x,t)\rangle | \langle x,t\rangle \in E \times T\} \\
&\qquad \cup \{\langle x, \mu_{B(T)}(x,t), \nu_{B(T)}(x,t)\rangle | \langle x,t\rangle \in E \times T\}) \\
&= \mathcal{I}^*(\{\langle x, \max(\mu_{A(T)}(x,t), \mu_{B(T)}(x,t)), \\
&\qquad \min(\nu_{A(T)}(x,t), \nu_{B(T)}(x,t))\rangle | \langle x,t\rangle \in E \times T\}) \\
&= \{\langle x, \inf_{u\in T} \max(\mu_{A(T)}(x,u), \mu_{B(T)}(x,u))), \\
&\qquad \sup_{u\in T} \min(\nu_{A(T)}(x,u), \nu_{B(T)}(x,u)))\rangle | \langle x,t\rangle \in E \times T\} \\
&\supseteq \{\langle x, \max(\inf_{t\in T} \mu_{A(T)}(x,t), \inf_{t\in T} \mu_{B(T)}(x,t)), \\
&\qquad \min(\sup_{t\in T} \nu_{A(T)}(x,t), \sup_{t\in T} \nu_{B(T)}(x,t))\rangle | \langle x,t\rangle \in E \times T\} \\
&= \{\langle x, \inf_{t\in T} \mu_{A(T)}(x,t), \sup_{t\in T} \nu_{A(T)}(x,t)\rangle | \langle x,t\rangle \in E \times T\} \\
&\qquad \cup \{\langle x, \inf_{t\in T} \mu_{B(T)}(x,t), \sup_{t\in T} \nu_{B(T)}(x,t)\rangle | \langle x,t\rangle \in E \times T\} \\
&= \mathcal{I}^*(A(T) \cup \mathcal{I}^* B(T)).
\end{aligned}$$

II6.

$$\begin{aligned}
\mathcal{I}^*(A(T)) &= \{\langle x, \inf_{t\in T} \mu_{A(T)}(x,t), \sup_{t\in T} \nu_{A(T)}(x,t)\rangle | \langle x,t\rangle \in E \times T\} \\
&\subseteq \{\langle x, \mu_{A(T)}(x,t), \nu_{A(T)}(x,t)\rangle | \langle x,t\rangle \in E \times T\} \\
&= A(T);
\end{aligned}$$

II7.

$$\begin{aligned}
\mathcal{I}^*(O^*(T)) &= \mathcal{I}^*(\{\langle x, 0, 1\rangle\} | \langle x,t\rangle \in E \times T\}) \\
&= (\{\langle x, 0, 1\rangle\} \langle x,t\rangle \in E \times T\}) \\
&= O^*(T)
\end{aligned}$$

II8. As above, bearing in mind that for each fixed $x \in T$: $\inf_{t\in T} \mu_{A(T)}(x,t)$ and $\sup_{t\in T} \nu_{A(T)}(x,t)$ are constants, we obtain that:

$$\begin{aligned}
\mathcal{I}^*(\mathcal{I}^*(T)) &= \mathcal{I}^*(\{\langle x, \inf_{t\in T} \mu_{A(T)}(x,t), \sup_{t\in T} \nu_{A(T)}(x,t)\rangle | \langle x,t\rangle \in E \times T\} \\
&= \{\langle x, \inf_{u\in T} \inf_{t\in T} \mu_{A(T)}(x,t), \sup_{u\in T} \sup_{t\in T} \nu_{A(T)}(x,t)\rangle | \langle x,t\rangle \in E \times T\} \\
&= \{\langle x, \inf_{t\in T} \mu_{A(T)}(x,t), \sup_{t\in T} \nu_{A(T)}(x,t)\rangle | \langle x,t\rangle \in E \times T\} \\
&= \mathcal{I}^*(A(T));
\end{aligned}$$

II9.

$$\begin{aligned}
\mathcal{I}^*(\mathcal{I}(A(T))) &= \mathcal{I}^*(\{\langle x, \inf_{y\in E} \mu_{A(T)}(y), \sup_{y\in E} \nu_{A(T)}(y)\rangle | \langle x,t\rangle \in E \times T\}) \\
&= \{\langle x, \inf_{u\in T} \inf_{y\in E} \mu_{A(T)}(y,u), \sup_{u\in T} \sup_{y\in E} \nu_{A(T)}(y,u)\rangle | \langle x,t\rangle \in E \times T\}) \\
&= \{\langle x, \inf_{y\in E} \inf_{u\in T} \mu_{A(T)}(y,u), \sup_{y\in E} \sup_{u\in T} \nu_{A(T)}(y,u)\rangle | \langle x,t\rangle \in E \times T\}
\end{aligned}$$

$$= \mathcal{I}(\{\langle x, \inf_{t\in T} \mu_{A(T)}(x,t), \sup_{t\in T} \nu_{A(T)}(x,t)\rangle | \langle x,t\rangle \in E \times T\})$$
$$= \mathcal{I}(\mathcal{I}^*(A(T))).$$

This completes the proof. □

3.3. *cl-in-Intuitionistic Fuzzy Temporal Topological Structure*

The *cl-in*-IFTTS is the object

$$\langle \mathcal{P}(E^*(T)), \mathcal{O}, \Delta, \bullet\rangle,$$

where T is a fixed time-scale; E is a fixed universe; $\mathcal{O} : \mathcal{P}(E^*(T)) \to \mathcal{P}(E^*(T))$ is an operator of a closure type related to operation Δ; $\bullet : \mathcal{P}(E^*(T)) \to \mathcal{P}(E^*(T))$ is a temporal operator from an interior type; and for every two TIFSs $A(T), B(T) \in \mathcal{P}(E^*(T))$, the following nine conditions hold:

CI1 $\mathcal{O}(A(T)\Delta B(T)) = \mathcal{O}(A(T))\Delta\mathcal{O}(B(T))$,
CI2 $A(T) \subseteq \mathcal{O}(A(T))$,
CI3 $\mathcal{O}(O^*(T)) = O^*(T)$,
CI4 $\mathcal{O}(\mathcal{O}(A(T))) = \mathcal{O}(A(T))$,
CI5 $\bullet(A(T)\nabla B(T)) = \bullet A(T)\nabla \bullet B(T)$,
CI6 $\bullet A(T) \subseteq A(T)$,
CI7 $\bullet O^*(T) = O^*(T)$,
CI8 $\bullet\bullet A(T) = \bullet A(T)$,
CI9 $\bullet\, \mathcal{O}(A(T)) \supseteq \mathcal{O}(\bullet A(T))$.

Now, we see that the first four conditions correspond to the conditions C1–C4 for a topological operator "closure"; the next four conditions correspond to the conditions I1–I4, too, but for a temporal topological operator "interior", and condition CC9 determines the relation between the two types of operators.

Theorem 3. *For each universe E and each time-scale T, $\langle \mathcal{P}(E^*(T)), \mathcal{C}, \cup, \mathcal{I}^*\rangle$ is a cl-in-IFTTS.*

Proof. Let the TIFS $A(T) \in \mathcal{P}(E^*(T))$ be given.

The checks of the conditions CI1–CI4 coincide with the proofs of conditions CC1–CC4 in Theorem 1. The checks of the conditions CI6–CI8 coincide with the proofs of conditions II6–II8.

Hence, it remains that we check only the validity of the conditions CI5 and CI9.

For the validity of condition (CI5), we obtain:

$$\begin{aligned}
\mathcal{I}^*(A(T) \cap B(T)) &= \mathcal{I}^*(\{\langle x, \min(\mu_A(x,t), \mu_B(x,t)),\\
&\qquad \max(\nu_A(x,t), \nu_A(x,t))\rangle | \langle x,t\rangle \in E \times T\}\\
&= \{\langle x, \inf_{u\in T} \min(\mu_A(x,u), \mu_B(x,u)),\\
&\qquad \sup_{u\in T} \max(\nu_A(x,u), \nu_B(x,u))\rangle \langle x,t\rangle \in E \times T\}\\
&= \{\langle x, \min(\inf_{u\in T} \mu_A(x,u), \inf_{u\in T} \mu_B(x,u)),\\
&\qquad \max(\sup_{u\in T} \nu_A(x,u), \sup_{u\in T} \nu_B(x,u))\rangle | \langle x,t\rangle \in E \times T\}\\
&= \mathcal{I}^*(A(T)) \cap \mathcal{I}^*(B(T)).
\end{aligned}$$

For the validity of condition CI9, we will first prove that for each TIFS $A(T)$:

$$\sup_{u\in T} \inf_{y\in E} \nu_{A(T)}(y,u) \le \inf_{y\in E} \sup_{u\in T} \nu_{A(T)}(y,u), \tag{5}$$

and

$$\inf_{u\in T}\sup_{y\in E}\mu_{A(T)}(y,u) \geq \sup_{y\in E}\inf_{u\in T}\mu_{A(T)}(y,u). \tag{6}$$

Let for a fixed $u \in T$,

$$h(u) = \inf_{y\in E}\nu_{A(T)}(y,u).$$

Therefore, for each $y \in E$,

$$h(u) \leq \nu_{A(T)}(y,u)$$

and hence

$$\sup_{u\in T} h(u) \leq \sup_{u\in T}\nu_{A(T)}(y,u).$$

Since $\sup_{u\in T} h(u)$ is a constant for each $y \in E$ and

$$\sup_{u\in T} h(u) \leq \inf_{y\in E}\sup_{u\in T}\nu_{A(T)}(y,u),$$

i.e., (5) is valid. The check of (6) is similar.

Now, using (5) and (6), we obtain:

$$\begin{aligned}
\mathcal{I}^*(\mathcal{C}(A(T))) &= \mathcal{I}^*(\{\langle x, \sup_{y\in E}\mu_{A(T)}(y), \inf_{y\in E}\nu_{A(T)}(y)\rangle | \langle x,t\rangle \in E\times T\}) \\
&= \{\langle x, \inf_{u\in T}\sup_{y\in E}\mu_{A(T)}(y,u), \sup_{u\in T}\inf_{y\in E}\nu_{A(T)}(y,u)\rangle | \langle x,t\rangle \in E\times T\}) \\
&\supseteq \{\langle x, \sup_{y\in E}\inf_{u\in T}\mu_{A(T)}(y,u), \inf_{y\in E}\sup_{u\in T}\nu_{A(T)}(y,u)\rangle | \langle x,t\rangle \in E\times T\} \\
&= \mathcal{C}(\{\langle x, \inf_{t\in T}\mu_{A(T)}(x,t), \sup_{t\in T}\nu_{A(T)}(x,t)\rangle | \langle x,t\rangle \in E\times T\}) \\
&= \mathcal{C}(\mathcal{I}^*(A(T))).
\end{aligned}$$

This completes the proof. □

3.4. *in-cl-Intuitionistic Fuzzy Temporal Topological Structure*

The *in-cl*-IFTTS is the object

$$\langle \mathcal{P}(E^*(T)), \mathcal{Q}, \nabla, \circ\rangle,$$

where T is a fixed time-scale; E is a fixed universe; $\mathcal{Q} : \mathcal{P}(E^*(T)) \to \mathcal{P}(E^*(T))$ is an operator of interior type, related to the operation ∇; $\circ : \mathcal{P}(E^*(T)) \to \mathcal{P}(E^*(T))$ is a temporal operator of closure type; and for every two TIFSs $A(T), B(T) \in \mathcal{P}(E^*(T))$, the following nine conditions hold:

IC1 $\mathcal{Q}(A(T)\nabla B(T)) = \mathcal{Q}(A(T))\nabla\mathcal{Q}(B(T))$,
IC2 $\mathcal{Q}(A(T)) \subseteq A(T)$,
IC3 $\mathcal{Q}(E^*(T)) = E^*(T)$,
IC4 $\mathcal{Q}(\mathcal{Q}(A(T))) = \mathcal{Q}(A(T))$,
IC5 $\circ(A(T)\Delta B(T)) = \circ A(T)\Delta \circ B(T)$,
IC6 $A(T) \subseteq \circ A(T)$,
IC7 $\circ E^*(T) = E^*(T)$,
IC8 $\circ\circ A(T) = \circ A(T)$,
IC9 $\circ\, \mathcal{Q}(A(T)) \subseteq \mathcal{Q}(\circ A(T))$.

Now, we see that the first four conditions correspond to the conditions I1–I4 for a topological operator "interior" and the next four conditions correspond to the conditions C1–C4, too, but for a temporal topological operator, "closure" and condition CC9 determine the relation between the two types of operators.

Theorem 4. *For each universe E and each time-scale T*, $\langle \mathcal{P}(E^*(T)), \mathcal{I}, \cap, \mathcal{C}^* \rangle$ *is an in-cl-IFTTS.*

Proof. Let the TIFS $A(T) \in \mathcal{P}(E^*(T))$ be given.

The checks of the conditions IC1–IC4 are similar to the proofs of conditions II1–II4 in Theorem 2. The checks of the conditions IC6–IC8 coincide with the proofs of conditions CC6–CC8.

Hence, it remains that we check only the validity of the conditions IC5 and IC9.

For the validity of condition (IC5), we obtain:

$$\begin{aligned}
\mathcal{C}^*(A(T) \cap B(T)) &= \mathcal{I}^*(\{\langle x, \max(\mu_A(x,t), \mu_B(x,t)), \\
&\qquad \min(\nu_A(x,t), \nu_A(x,t))\rangle | \langle x,t\rangle \in E \times T\} \\
&= \{\langle x, \sup_{u \in T} \max(\mu_A(x,u), \mu_B(x,u)), \\
&\qquad \inf_{u \in T} \min(\nu_A(x,u), \nu_B(x,u))\rangle \langle x,t\rangle \in E \times T\} \\
&= \{\langle x, \max(\sup_{u \in T} \mu_A(x,u), \sup_{u \in T} \mu_B(x,u)), \\
&\qquad \min(\inf_{u \in T} \nu_A(x,u), \inf_{u \in T} \nu_B(x,u))\rangle | \langle x,t\rangle \in E \times T\} \\
&= \mathcal{C}^*(A(T)) \cap \mathcal{C}^*(B(T)).
\end{aligned}$$

Now, using (5) and (6), we obtain:

$$\begin{aligned}
\mathcal{C}^*(\mathcal{I}(A(T))) &= \mathcal{C}^*(\{\langle x, \inf_{y \in E} \mu_{A(T)}(y), \sup_{y \in E} \nu_{A(T)}(y)\rangle | \langle x,t\rangle \in E \times T\}) \\
&= \{\langle x, \sup_{u \in T} \inf_{y \in E} \mu_{A(T)}(y,u), \inf_{u \in T} \sup_{y \in E} \nu_{A(T)}(y,u)\rangle | \langle x,t\rangle \in E \times T\}) \\
&\subseteq \{\langle x, \inf_{y \in E} \sup_{u \in T} \mu_{A(T)}(y,u), \sup_{y \in E} \inf_{u \in T} \nu_{A(T)}(y,u)\rangle | \langle x,t\rangle \in E \times T\} \\
&= \mathcal{I}(\{\langle x, \sup_{t \in T} \mu_{A(T)}(x,t), \inf_{t \in T} \nu_{A(T)}(x,t)\rangle | \langle x,t\rangle \in E \times T\}) \\
&= \mathcal{I}(\mathcal{C}^*(A(T))).
\end{aligned}$$

This completes the proof. □

4. A Short Example of the Application of the IFTTSs in a Multi-Criteria Decision Making Procedure

Let us have e experts $E_1, E_2, \ldots, E_e$ and c criteria $C_1, C_2, \ldots, C_c$ and let the experts evaluate some object O using the criteria in the subsequent time-moments $t_1 < t_2 < \cdots < t_s$.

Let the universe be $C = \{C_1, C_2, \ldots, C_c\}$ and let the time-scale be $T = \{t_1, t_2, \ldots, t_s\}$. So, to each expert E_i we juxtapose the TIFS

$$F_i = \{\langle C_j, \mu_i(C_j, t_k), \nu_i(C_j, t_k)\rangle | C_j \in C, t_k \in T, 1 \leq j \leq c\}$$

that corresponds to the expert evaluations of the object O against the subsequent criteria in the different time-moments, where $1 \leq i \leq e$. Here, $0 \leq \mu_i(C_j, t_k), \nu_i(C_j, t_k), \mu_i(C_j, t_k) + \nu_i(C_j, t_k) \leq 1$, where $\mu_i(C_j, t_k)$ and $\nu_i(C_j, t_k)$ are the degrees of validity and of non-validity of criterion C_j with respect to the object O at the time-moment t_k according to expert E_i.

Now, we can construct the set

$$\mathcal{F} = \{F_i | 1 \leq i \leq e\}$$

that can be the basis of the *cl-cl-*, *cl-in-*, *in-cl-* or *in-in-*IFTTSs $\langle \mathcal{P}(\mathcal{F}), \mathcal{C}, \cup, \mathcal{C}^* \rangle$, $\langle \mathcal{P}(\mathcal{F}), \mathcal{C}, \cup, \mathcal{I}^* \rangle$, $\langle \mathcal{P}(\mathcal{F}), \mathcal{I}, \cap, \mathcal{C}^* \rangle$, $\langle \mathcal{P}(\mathcal{F}), \mathcal{I}, \cap, \mathcal{I}^* \rangle$, respectively.

In these structures, the operators $\mathcal{C}$ and $\mathcal{I}$ will determine the maximal and minimal evaluations, given by the experts for object O in the separate time-moments, while the operators $\mathcal{C}^*$ and $\mathcal{I}^*$ will determine the maximal and minimal evaluations given by each one of the experts for object O in the whole time-period.

The more interesting case is this one, when we have more than one object for evaluation by the experts. This case will be discussed in future with respect to a more complex topological structure.

5. Conclusions and Ideas for Future Research

In [22], the author wrote: *"The described above idea opens some directions for future research..."*. Nevertheless, the present direction was not discussed there. On the other hand, the directions for the development of the IFMTS discussed in [22] are valid for the IFTTS, too. For example, one of the possible directions is related to the use of the extended topological operators, discussed in [31]. These operators will be applied over the elements of the universe E, as well as over the elements of the time-scale T.

Of course, as it has turned out with the IFMTSs, a lot of them will be (using the notation of [27–29]) feeble structures because they will not satisfy some of the conditions introduced above.

All of these directions for the development of IFTTSs will be an object of future research of the author.

Funding: This research was realized in the frames of project KP-06-N22-1/2018 "Theoretical research and applications of InterCriteria Analysis", Bulgarian National Science Fund.

Data Availability Statement: Not applicable.

Acknowledgments: The author is very thankful to his reviewers for their valuable comments and corrections.

Conflicts of Interest: The author declares no conflict of interest.

References

1. Bourbaki, N. *Éléments De Mathématique, Livre III: Topologie Générale, Chapitre 1: Structures Topologiques, Chapitre 2: Structures Uniformes*; Herman: Paris, France, 1960. (In French)
2. Kuratowski, K. *Topology*; Academic Press: New York, NY, USA, 1966; Volume 1.
3. Munkres, J. *Topology*; Prentice Hall Inc.: New Jersey, NJ, USA, 2000.
4. Çoker, D. On topological structures using intuitionistic fuzzy sets. *Notes Intuitionistic Fuzzy Sets* **1997**, *3*, 138–142.
5. Çoker, D. An introduction to intuitionistic fuzzy topological spaces. *Fuzzy Sets Syst.* **1997**, *88*, 81–89. [CrossRef]
6. Lupiañez, F.G. Separation in intuitionistic fuzzy topological spaces. *Int. Pure Appl. Math.* **2004**, *17*, 29–34.
7. Lupiañez, F.G. On intuitionistic fuzzy topological spaces. *Kybernetes* **2006**, *35*, 743–747. [CrossRef]
8. El-Latif, A.; Khalaf, M. Connectedness in Intuitionistic Fuzzy Topological Spaces in Sostak's Sense. *Ital. J. Pure Appl. Math.* **2015**, *35*, 649–668.
9. Haydar Eş, A.; Çoker, D. More on fuzzy compactness in intuitionistic fuzzy topological spaces. *Notes Intuitionistic Fuzzy Sets* **1996**, *2*, 4–10.
10. Kim, Y.C.; Abbas, S.E. Connectedness in Intuitionistic Fuzzy Topological Spaces. *Commun. Korean Math. Soc.* **2005**, *20*, 117–134. [CrossRef]
11. Lee, S.J.; Lee, E.P. The category of intuitionistic fuzzy topological spaces. *Bull. Korean Math. Soc.* **2000**, *37*, 63–76.
12. Milles, S. The Lattice of Intuitionistic Fuzzy Topologies Generated by Intuitionistic Fuzzy Relations. *Appl. Appl. Math.* **2020**, *15*, 942–956.
13. Mondal, K.; Samanta, S.K. A study on intuitionistic fuzzy topological spaces. *Notes Intuitionistic Fuzzy Sets* **2003**, *9*, 1–32.
14. Özbakir, O.; Çoker, D. Fuzzy multifunctions in intuitionistic fuzzy topological spaces. *Notes Intuitionistic Fuzzy Sets* **1999**, *5*, 1–5.
15. Rajarajeswari, P.; Krishna Moorthy, R. Intuitionistic fuzzy completely weakly generalized continuous mappings. *Notes Intuitionistic Fuzzy Sets* **2012**, *18*, 25–36.
16. Roopkumar, R.; Kalaivani, C. Continuity of intuitionistic fuzzy proper functions on intuitionistic smooth fuzzy topological spaces. *Notes Intuitionistic Fuzzy Sets* **2010**, *16*, 1–21.
17. Saadati, R.; Park, J.H. On the intuitionistic fuzzy topological spaces. *Chaos Solitons Fractals* **2006**, *27*, 331–334. [CrossRef]
18. Thakur, S.; Chaturvedi, R. Generalized continuity in intuitionistic fuzzy topological spaces. *Notes Intuitionistic Fuzzy Sets* **2006**, *12*, 38–44.
19. Tiwari, S. On relationships among intuitionistic fuzzy approximation operators, intuitionistic fuzzy topology and intuitionistic fuzzy automata. *Notes Intuitionistic Fuzzy Sets* **2010**, *16*, 1–9.
20. Yılmaz, S.; Cuvalcıoglu, G. On Level Operators for Temporal intuitionistic fuzzy sets. *Notes Intuitionistic Fuzzy Sets* **2014**, *20*, 6–15.

21. Yongfa, H.; Changjun, J. Some properties of intuitionistic fuzzy metric spaces. *Notes Intuitionistic Fuzzy Sets* **2004**, *10*, 18–26.
22. Atanassov, K. Intuitionistic Fuzzy Modal Topological Structure. *Mathematics* **2022**, *10*, 3313. [CrossRef]
23. De Poidevin, R. Relationism and Temporal Topology: Physics or Metaphysics? *Philos. Q.* **1990**, *40*, 419–432. [CrossRef]
24. Kutlu, F.; Bilgin, T. Temporal Intuitionistic Fuzzy Topology in Sostak's Sense. *Notes Intuitionistic Fuzzy Sets* **2015**, *21*, 63–70.
25. Kutlu, F.; Ramadan, A.; Bilgin, T. On Compactness in Temporal Intuitionistic Fuzzy Sostak Topology. *Notes Intuitionistic Fuzzy Sets* **2016**, *22*, 46–62.
26. Kutlu, F. On Separation Axioms in Temporal Intuitionistic Fuzzy Sostak Topology. *Notes Intuitionistic Fuzzy Sets* **2017**, *23*, 21–30.
27. Atanassov, K. On the Intuitionistic Fuzzy Modal Feeble Topological Structures. *Notes Intuitionistic Fuzzy Sets* **2022**, *28*, 211–222. [CrossRef]
28. Atanassov, K. On four intuitionistic fuzzy feeble topological structures. In Proceedings of the IEEE 11th International Conference on Intelligent Systems, Warsaw, Poland, 12–14 October 2022. [CrossRef]
29. Atanassov, K. On intuitionistic fuzzy modal topological structures with modal operator of second type. *Notes Intuitionistic Fuzzy Sets* **2022**, *28*, 457–463. [CrossRef]
30. Atanassov, K. *Intuitionistic Fuzzy Sets*; Springer: Heidelberg, Germany, 1999.
31. Atanassov, K. *On Intuitionistic Fuzzy Sets Theory*; Springer: Berlin, Germany, 2012.
32. Atanassov, K. Temporal intuitionistic fuzzy sets. *Comptes Rendus L'Academie Bulg. Sci.* **1991**, *44*, 5–7.
33. Zadeh, L. Fuzzy sets. *Inf. Control* **1965**, *8*, 338–353. [CrossRef]