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Article

Inequalities and Reverse Inequalities for the Joint A-Numerical Radius of Operators

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Abstract: In this paper, we aim to establish several estimates concerning the generalized Euclidean operator radius of d-tuples of A-bounded linear operators acting on a complex Hilbert space $\mathscr{H}$, which leads to the special case of the well-known A-numerical radius for $d = 1$. Here, A is a positive operator on $\mathscr{H}$. Some inequalities related to the Euclidean operator A-seminorm of d-tuples of A-bounded operators are proved. In addition, under appropriate conditions, several reverse bounds for the A-numerical radius in single and multivariable settings are also stated.

Keywords: positive operator; joint A-numerical radius; Euclidean operator A-seminorm; joint operator A-seminorm

MSC: 47B65; 47A12; 47A13; 47A30

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1. Introduction

The theory of inequalities remains a very attractive area of research in the last few decades. In particular, the investigation of numerical radius inequalities in Hilbert and semi-Hilbert spaces has occupied an important and central role in the theory of operator inequalities. For further details, interested readers are referred to the very recent book by Bhunia et al. [1].

Throughout the present article, $\mathscr{H}$ stands for a non-trivial complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and the corresponding norm $\| \cdot \|$. By $\mathbb{B}(\mathscr{H})$, we denote the C^*-algebra of all bounded linear operators acting on $\mathscr{H}$. The identity operator on $\mathscr{H}$ will be simply written as I. Let $T \in \mathbb{B}(\mathscr{H})$. The range and the adjoint of T will be denoted by $\mathcal{R}(T)$ and T^*, respectively. An operator $T \in \mathbb{B}(\mathscr{H})$ is called positive and we write $T \geq 0$ if $\langle Tx, x \rangle \geq 0$ for all $x \in \mathscr{H}$. If $T \geq 0$, then $T^{1/2}$ denotes the square root of T.

If $\mathcal{S}$ is a subspace of $\mathscr{H}$, then we mean by $\overline{\mathcal{S}}$ the closure of $\mathcal{S}$ in the norm topology of $\mathscr{H}$. Let $\mathcal{C}$ be a closed subspace of $\mathscr{H}$. We denote by $P_{\mathcal{C}}$ the orthogonal projection onto $\mathcal{C}$.

For the rest of this work, by an operator, we mean a bounded linear operator acting on $\mathscr{H}$. We also assume that $A \in \mathbb{B}(\mathscr{H})$ is a non-zero, positive operator. Such an A defines the following semi-inner product on $\mathscr{H}$:

$$\langle x, y \rangle_A = \langle Ax, y \rangle = \langle A^{1/2}x, A^{1/2}y \rangle,$$

for all $x, y \in \mathscr{H}$. The seminorm on $\mathscr{H}$ induced by $\langle \cdot, \cdot \rangle_A$ is stated as: $\|x\|_A = \|A^{1/2}x\|$ for every $x \in \mathscr{H}$. Hence, we see that the above seminorm is a norm on $\mathscr{H}$ if and only if A is a one-to-one operator. Furthermore, one can prove that the semi-Hilbert space $(\mathscr{H}, \|\cdot\|_A)$ is a complete space if and only if $\overline{\mathcal{R}(A)} = \mathcal{R}(A)$. The A-unit sphere of $\mathscr{H}$ is defined as

$$\mathbb{S}_A^1 = \{y \in \mathscr{H}\,;\ \|y\|_A = 1\}.$$

We refer the reader to the following list of recent works on the theory of semi-Hilbert spaces [1–6].

Let $T \in \mathbb{B}(\mathscr{H})$. We recall from [7] that an operator $R \in \mathbb{B}(\mathscr{H})$ is called an A-adjoint of T if the equality

$$\langle Ty, z\rangle_A = \langle y, Rz\rangle_A$$

holds for all $y, z \in \mathscr{H}$, that is, $AR = T^*A$. In general, the existence and the uniqueness of an A-adjoint of an arbitrary bounded operator T are not guaranteed. By using a famous theorem due to Douglas [8], we see that the sets of all operators that admit A-adjoint and $A^{1/2}$-adjoint operators are, respectively, given by

$$\mathbb{B}_A(\mathscr{H}) = \{S \in \mathbb{B}(\mathscr{H})\,;\ \mathcal{R}(S^*A) \subset \mathcal{R}(A)\},$$

and

$$\mathbb{B}_{A^{1/2}}(\mathscr{H}) = \left\{S \in \mathbb{B}(\mathscr{H})\,;\ \exists\,\zeta > 0 \ \text{ such that } \ \|Sx\|_A \leq \zeta\|x\|_A,\ \forall\, x \in \mathscr{H}\right\}.$$

When an operator S belongs to $\mathbb{B}_{A^{1/2}}(\mathscr{H})$, we say that S is A-bounded. It is not difficult to check that $\mathbb{B}_A(\mathscr{H})$ and $\mathbb{B}_{A^{1/2}}(\mathscr{H})$ represent two subalgebras of $\mathbb{B}(\mathscr{H})$. Moreover, the following inclusions

$$\mathbb{B}_A(\mathscr{H}) \subseteq \mathbb{B}_{A^{1/2}}(\mathscr{H}) \subseteq \mathbb{B}(\mathscr{H})$$

hold and are, in general, proper. For more details, we refer to [7,9–11] and the references therein. We recall now that an operator $S \in \mathbb{B}(\mathscr{H})$ is called A-self-adjoint if AS is self-adjoint. Clearly the fact that S is A-self-adjoint implies that $S \in \mathbb{B}_A(\mathscr{H})$. Furthermore, we say that an operator S is called A-positive (and we write $S \geq_A 0$) if $AS \geq 0$. Obviously, A-positive operators are A-self-adjoint. For $S \in \mathbb{B}_{A^{1/2}}(\mathscr{H})$, the operator A-seminorm and the A-numerical radius of S are given, respectively, by

$$\|S\|_A = \sup_{x\in\mathbb{S}_A^1} \|Sx\|_A \quad \text{and} \quad \omega_A(S) = \sup_{x\in\mathbb{S}_A^1} \left|\langle Sx, x\rangle_A\right|. \tag{1}$$

The quantities in (1) are also intensively studied when $A = I$, and the reader is referred to [12–22] as a recent list of references treating the numerical radius and operator norm of operators on complex Hilbert spaces.

If $S \in \mathbb{B}_A(\mathscr{H})$, then by the Douglas theorem [8], there exists a unique solution, denoted by $S^{\dagger_A}$, of the problem: $AX = S^*A$ and $\mathcal{R}(X) \subseteq \overline{\mathcal{R}(A)}$. We emphasize here that if $S \in \mathbb{B}_A(\mathscr{H})$, then $S^{\dagger_A} \in \mathbb{B}_A(\mathscr{H})$ and $(S^{\dagger_A})^{\dagger_A} = P_{\overline{\mathcal{R}(A)}} S P_{\overline{\mathcal{R}(A)}}$.

Now, let $\mathcal{T} = (T_1, \ldots, T_d) \in \mathbb{B}_{A^{1/2}}(\mathscr{H})^d$ be a d-tuple of operators. According to [23], the following two quantities

$$\omega_A(\mathcal{T}) := \sup_{y\in\mathbb{S}_A^1} \sqrt{\sum_{k=1}^{d} |\langle T_k y, y\rangle_A|^2},$$

and

$$\|\mathcal{T}\|_A := \sup_{y\in\mathbb{S}_A^1} \sqrt{\sum_{k=1}^{d} \|T_k y\|_A^2}$$

generalize the notions in (1) and define two equivalent norms on $\mathbb{B}_{A^{1/2}}(\mathscr{H})^d$. Namely, we have

$$\frac{1}{2\sqrt{d}}\|\mathcal{T}\|_A \leq \omega_A(\mathcal{T}) \leq \|\mathcal{T}\|_A, \tag{2}$$

for every operator tuple $\mathcal{T} = (T_1, \ldots, T_d) \in \mathbb{B}_{A^{1/2}}(\mathscr{H})^d$. Note that $\omega_A(\mathcal{T})$ and $\|\mathcal{T}\|_A$ are called the joint A-numerical radius and joint operator A-seminorm of $\mathcal{T}$, respectively. The above two quantities have been investigated by several authors when $A = I$ (see for instance [24–27]). Another joint A-seminorm of A-bounded operators has been recently introduced [28]. Namely, the Euclidean A-seminorm of an operator tuple $\mathcal{T} = (T_1, \ldots, T_d) \in \mathbb{B}_{A^{1/2}}(\mathscr{H})^d$ is given by

$$\|\mathcal{T}\|_{e,A} = \sup_{(\nu_1,\ldots,\nu_d)\in\overline{\mathcal{B}}_d} \|\nu_1 T_1 + \ldots + \nu_d T_d\|_A, \tag{3}$$

where $\overline{\mathcal{B}}_d$ denotes the closed unit ball of $\mathbb{C}^d$, i.e.,

$$\overline{\mathcal{B}}_d := \left\{ \nu = (\nu_1, \ldots, \nu_d) \in \mathbb{C}^d\, ;\; \|\nu\|_2^2 := \sum_{k=1}^{d} |\nu_k|^2 \leq 1 \right\},$$

where $\mathbb{C}$ denotes the set of all complex numbers. It is important to note that the following inequalities,

$$\frac{1}{\sqrt{d}}\|\mathcal{T}\|_A \leq \|\mathcal{T}\|_{e,A} \leq \|\mathcal{T}\|_A,$$

hold for any d-tuple of operators $\mathcal{T} = (T_1, \ldots, T_d) \in \mathbb{B}_{A^{1/2}}(\mathscr{H})^d$ (see [28]).

Our aim in the present article is to establish several estimates involving the quantities $\omega_A(\mathcal{T})$, $\|\mathcal{T}\|_A$ and $\|\mathcal{T}\|_{e,A}$, where $\mathcal{T} = (T_1, \ldots, T_d)$ is a d-tuple of A-bounded operators. Some inequalities connecting the A-numerical radius and operator A-seminorm for A-bounded operators are established. One main target of this work is to derive, under appropriate conditions, several reverse bounds for $\omega_A(\mathcal{T})$ in both single and multivariable settings. In particular, for $T \in \mathbb{B}_{A^{1/2}}(\mathscr{H})$, $\nu \in \mathbb{C}$ and $r > 0$, we will demonstrate under appropriate conditions on T, ν and r that

$$\|T\|_A^2 \leq \omega_A^2(T) + \frac{2r^2}{|\nu| + \sqrt{|\nu|^2 - r^2}}\omega_A(T).$$

2. Results

This section is devoted to present our contributions. By $\Re e z$, we will denote the real part of any complex number $z \in \mathbb{C}$. In the next theorem, we state our first result.

Theorem 1. *Let $T \in \mathbb{B}_{A^{1/2}}(\mathscr{H})$ and $\rho, \sigma \in \mathbb{C}$ with $\rho \neq \sigma$. If*

$$\Re e\langle \rho x - Tx, Tx - \sigma x\rangle_A \geq 0 \quad \text{for any} \quad x \in \mathbb{S}_A^1 \tag{4}$$

or, equivalently

$$\left\| Tx - \frac{\rho + \sigma}{2}x \right\|_A \leq \frac{1}{2}|\rho - \sigma| \text{ for any} \quad x \in \mathbb{S}_A^1, \tag{5}$$

then

$$\|T\|_A^2 \leq \omega_A^2(T) + \frac{1}{4}|\rho - \sigma|^2. \tag{6}$$

Proof. Notice first that the following assertions,

(i) $\Re e\langle u - y, y - z\rangle_A \geq 0$,
(ii) $\left\| y - \frac{z+u}{2} \right\|_A \leq \frac{1}{2}\|u - z\|_A$,

are equivalent for every $y, z, u \in \mathscr{H}$. Indeed, one can see that

$$\begin{aligned}
\frac{1}{4}\|u-z\|_A^2 - \left\|y - \frac{z+u}{2}\right\|_A^2 &= \frac{1}{4}\|u-y+y-z\|_A^2 - \frac{1}{4}\|y-z+y-u\|_A^2 \\
&= \frac{1}{4}\Big(\|u-y\|_A^2 + 2\Re e\langle u-y, y-z\rangle_A + \|y-z\|_A^2\Big) \\
&- \frac{1}{4}\Big(\|y-z\|_A^2 + 2\Re e\langle y-z, y-u\rangle_A + \|u-y\|_A^2\Big) \\
&= \frac{1}{2}\Big(\Re e\langle u-y, y-z\rangle_A - \Re e\langle y-z, y-u\rangle_A\Big) \\
&= \frac{1}{2}\Big(\Re e\langle u-y, y-z\rangle_A - \Re e\overline{\langle y-u, y-z\rangle_A}\Big) \\
&= \Re e\langle u-y, y-z\rangle_A.
\end{aligned}$$

Hence, the equivalence is proved.

By taking $u = \rho x$, $z = \sigma x$ and $y = Tx$ in the statements (i) and (ii), we deduce that (4) and (5) are equivalent.

Now, for $x \in \mathbb{S}_A^1$, we define

$$I_1 := \Re e\Big[\Big(\rho - \langle Tx, x\rangle_A\Big)\Big(\overline{\langle Tx, x\rangle_A} - \overline{\sigma}\Big)\Big]$$

and

$$I_2 := \Re e\langle \rho x - Tx, Tx - \sigma x\rangle_A.$$

Then,

$$I_1 = \Re e\Big[\rho\overline{\langle Tx, x\rangle_A} + \overline{\sigma}\langle Tx, x\rangle_A\Big] - \Big|\langle Tx, x\rangle_A\Big|^2 - \Re e(\rho\overline{\sigma})$$

and

$$I_2 = \Re e\Big[\rho\overline{\langle Tx, x\rangle_A} + \overline{\sigma}\langle Tx, x\rangle_A\Big] - \|Tx\|_A^2 - \Re e(\rho\overline{\sigma}).$$

This gives

$$I_1 - I_2 = \|Tx\|_A^2 - \Big|\langle Tx, x\rangle_A\Big|^2,$$

for any $x \in \mathbb{S}_A^1$ and $\sigma, \rho \in \mathbb{C}$. This is an interesting identity itself as well.

If (4) holds, then $I_2 \geq 0$ and thus

$$\|Tx\|_A^2 - \Big|\langle Tx, x\rangle_A\Big|^2 \leq \Re e\Big[\Big(\rho - \langle Tx, x\rangle_A\Big)\Big(\overline{\langle Tx, x\rangle_A} - \overline{\sigma}\Big)\Big]. \tag{7}$$

Furthermore, it can be checked that for every $u, v \in \mathbb{C}$, we have

$$\Re e(u\overline{v}) \leq \frac{1}{4}|u+v|^2.$$

By letting

$$u := \rho - \langle Tx, x\rangle_A, \;\; v := \langle Tx, x\rangle_A - \sigma$$

in the above elementary inequality, we obtain

$$\Re e\Big[\Big(\rho - \langle Tx, x\rangle_A\Big)\Big(\overline{\langle Tx, x\rangle_A} - \overline{\sigma}\Big)\Big] \leq \frac{1}{4}|\rho - \sigma|^2. \tag{8}$$

Making use of the inequalities (7) and (8), we deduce that

$$\|Tx\|_A^2 \leq \Big|\langle Tx, x\rangle_A\Big|^2 + \frac{1}{4}|\rho - \sigma|^2 \tag{9}$$

and by taking the supremum over all $x \in \mathbb{S}_A^1$ in (9), we obtain the required result (6). □

Remark 1. *Let $S \in \mathbb{B}(\mathscr{H})$. We say that S is an A-accretive operator, if*

$$\Re e\langle Sx, x\rangle_A \geq 0, \quad \text{for all } x \in \mathscr{H}.$$

Now, let $T \in \mathbb{B}_A(\mathscr{H})$. If $\theta \geq \mu > 0$ are such that either $(T^{\dagger_A} - \mu I)(\theta I - T)$ is A-accretive or $(T^{\dagger_A} - \mu I)(\theta I - T) \geq_A 0$, then by (6), we obtain

$$\|T\|_A^2 \leq \omega_A^2(T) + \frac{1}{4}(\theta - \mu)^2,$$

which gives

$$\|T\|_A \leq \sqrt{\omega_A^2(T) + \frac{1}{4}(\theta - \mu)^2}.$$

As an application of Theorem 1, we state the following result.

Corollary 1. *Let $\mathcal{T} = (T_1, \ldots, T_d) \in \mathbb{B}_{A^{1/2}}(\mathscr{H})^d$ and $\rho, \sigma \in \mathbb{C}$ be such that $\rho \neq \sigma$ and*

$$\left\| T_i x - \frac{\rho + \sigma}{2} x \right\|_A \leq \frac{1}{2}|\rho - \sigma|,$$

for any $x \in \mathbb{S}_A^1$ and every $i \in \{1, \ldots, d\}$. Then,

$$\|\mathcal{T}\|_{e,A}^2 \leq d\left(\max_{k \in \{1,\ldots,d\}} \omega_A^2(T_k) + \frac{1}{4}|\rho - \sigma|^2 \right). \tag{10}$$

Proof. Let $(\nu_1, \ldots, \nu_d) \in \overline{\mathcal{B}}_d$. From Theorem 1, we have

$$\|T_i\|_A^2 \leq \omega_A^2(T_i) + \frac{1}{4}|\rho - \sigma|^2$$

for $i \in \{1, \ldots, d\}$. This gives,

$$\sum_{i=1}^d |\nu_i|^2 \|T_i\|_A^2 \leq \sum_{i=1}^d |\nu_i|^2 \omega_A^2(T_i) + \frac{1}{4}|\rho - \sigma|^2 \sum_{i=1}^d |\nu_i|^2. \tag{11}$$

By using the triangle and Cauchy–Schwarz inequalities, we have

$$\frac{1}{d}\left\| \sum_{i=1}^d \nu_i T_i \right\|_A^2 \leq \frac{1}{d}\left(\sum_{i=1}^d \|\nu_i T_i\|_A \right)^2 \leq \sum_{i=1}^d |\nu_i|^2 \|T_i\|_A^2. \tag{12}$$

Moreover, since

$$\sum_{i=1}^d |\nu_i|^2 \omega_A^2(T_i) \leq \max_{k \in \{1,\ldots,d\}} \omega_A^2(T_k) \sum_{i=1}^d |\nu_i|^2,$$

then, by applying (11) and (12), we obtain

$$\frac{1}{d}\left\| \sum_{i=1}^d \nu_i T_i \right\|_A^2 \leq \max_{k \in \{1,\ldots,d\}} \omega_A^2(T_k) \sum_{i=1}^d |\nu_i|^2 + \frac{1}{4}|\rho - \sigma|^2 \sum_{i=1}^d |\nu_i|^2$$

for all $(\nu_1, \ldots, \nu_d) \in \overline{\mathcal{B}}_d$.

By taking the supremum over all $(\nu_1, \ldots, \nu_d) \in \overline{\mathcal{B}}_d$ in the last inequality and then using the identity in (3), we reach (10) as desired. □

An important application of the inequality (9) can be stated as follows.

Corollary 2. *Let $\mathcal{T} = (T_1, \ldots, T_d) \in \mathbb{B}_{A^{1/2}}(\mathscr{H})^d$ and $\rho_i, \sigma_i \in \mathbb{C}$ with $\rho_i \neq \sigma_i$ for $i \in \{1, \ldots, d\}$. Assume that for every $x \in \mathbb{S}_A^1$, we have*

$$\left\| T_i x - \frac{\rho_i + \sigma_i}{2} x \right\|_A \leq \frac{1}{2}|\rho_i - \sigma_i|, \quad \forall\, i \in \{1, \ldots, d\}. \tag{13}$$

Then,

$$\|\mathcal{T}\|_A \leq \sqrt{\omega_A^2(\mathcal{T}) + \frac{1}{4}\sum_{i=1}^{d}|\rho_i - \sigma_i|^2}. \tag{14}$$

Proof. Let $x \in \mathbb{S}_A^1$. By applying (9), we obtain

$$\|Tx_i\|_A^2 \leq \left|\langle T_i x, x\rangle_A\right|^2 + \frac{1}{4}|\rho_i - \sigma_i|^2$$

for $i \in \{1, \ldots, d\}$.

By summing over $i = 1, \ldots, d$, we obtain

$$\sum_{i=1}^{d}\|Tx_i\|_A^2 \leq \sum_{i=1}^{d}\left|\langle T_i x, x\rangle_A\right|^2 + \frac{1}{4}\sum_{i=1}^{d}|\rho_i - \sigma_i|^2$$

Finally, by taking the supremum over $x \in \mathbb{S}_A^1$, we obtain

$$\|\mathcal{T}\|_A^2 \leq \omega_A^2(\mathcal{T}) + \frac{1}{4}\sum_{i=1}^{d}|\rho_i - \sigma_i|^2.$$

This establishes (14). □

The following lemma is needed for the sequel.

Lemma 1 ([29] p. 9)**.** *Let $\sigma, \rho \in \mathbb{C}$ and $\zeta_j \in \mathbb{C}$ be such that*

$$\left|\zeta_j - \frac{\sigma + \rho}{2}\right| \leq \frac{1}{2}|\rho - \sigma|$$

for all $j \in \{1, \ldots, d\}$. Then,

$$d\sum_{j=1}^{d}|\zeta_j|^2 - \left|\sum_{j=1}^{d}\zeta_j\right|^2 \leq \frac{1}{4}d^2|\rho - \sigma|^2. \tag{15}$$

We can now prove the next proposition.

Proposition 1. *Let $\mathcal{T} = (T_1, \ldots, T_d) \in \mathbb{B}_{A^{1/2}}(\mathscr{H})^d$ and $\rho, \sigma \in \mathbb{C}$ with $\rho \neq \sigma$. Assume that*

$$\omega_A\left(T_j - \frac{\sigma + \rho}{2} I\right) \leq \frac{1}{2}|\rho - \sigma| \text{ for any } j \in \{1, \ldots, d\}. \tag{16}$$

Then,

$$\omega_A^2(\mathcal{T}) \leq \frac{1}{d}\omega_A^2\left(\sum_{j=1}^{d} T_j\right) + \frac{1}{4}d|\rho - \sigma|^2. \tag{17}$$

Proof. Assume that (16) is valid. Let $x \in \mathbb{S}_A^1$ and take $\zeta_j = \langle T_j x, x\rangle_A$ for all $j \in \{1,\dots,d\}$. Then, we see that

$$\begin{aligned}\left|\zeta_j - \frac{\sigma+\rho}{2}\right| &= \left|\langle T_j x, x\rangle_A - \frac{\sigma+\rho}{2}\langle x, x\rangle_A\right| \\ &= \left|\left\langle \left(T_j - \frac{\sigma+\rho}{2} I\right)x, x\right\rangle_A\right| \\ &\leq \sup_{x\in\mathbb{S}_A^1}\left|\left\langle \left(T_j - \frac{\sigma+\rho}{2} I\right)x, x\right\rangle_A\right| \\ &= \omega_A\left(T_j - \frac{\sigma+\rho}{2}\right) \leq \frac{1}{2}|\rho-\sigma|,\end{aligned}$$

for any $j \in \{1,\dots,d\}$.

By using (15), we obtain

$$\sum_{j=1}^{d}\left|\langle T_j x, x\rangle_A\right|^2 \leq \frac{1}{d}\left|\left\langle \sum_{j=1}^{d} T_j x, x\right\rangle_A\right|^2 + \frac{1}{4}d|\rho-\sigma|^2.$$

So, by taking the supremum over all $x \in \mathbb{S}_A^1$, we obtain (17) as desired. □

We now have the following result.

Theorem 2. *Let* $T \in \mathbb{B}_{A^{1/2}}(\mathscr{H})$. *If* $\nu \in \mathbb{C}\backslash\{0\}$ *and* $r > 0$ *are such that*

$$\|T - \nu I\|_A \leq r. \tag{18}$$

Then,

$$\|T\|_A \leq \omega_A(T) + \frac{1}{2}\cdot\frac{r^2}{|\nu|}.$$

Proof. Let $x \in \mathbb{S}_A^1$. It follows from (18) that

$$\|Tx - \nu x\|_A \leq \|T - \nu I\|_A \leq r.$$

This implies that

$$\|Tx\|_A^2 + |\nu|^2 \leq 2\Re e\left[\overline{\nu}\langle Tx, x\rangle_A\right] + r^2 \leq 2|\nu|\left|\langle Tx, x\rangle_A\right| + r^2.$$

Taking the supremum over $x \in \mathbb{S}_A^1$ in the last inequality, we obtain

$$\|T\|_A^2 + |\nu|^2 \leq 2\omega_A(T)|\nu| + r^2. \tag{19}$$

Moreover, it is clear that

$$2\|T\|_A|\nu| \leq \|T\|_A^2 + |\nu|^2, \tag{20}$$

thus, by applying (19) and (20), we infer that

$$2\|T\|_A|\nu| \leq 2\omega_A(T)|\nu| + r^2.$$

So, we immediately obtain the desired result. □

The following corollary is now in order.

Corollary 3. *Let $T \in \mathbb{B}_{A^{1/2}}(\mathscr{H})$ and $\alpha, \beta \in \mathbb{C}$ with $\alpha \notin \{-\beta, \beta\}$. Assume that*

$$\Re e\langle \alpha x - Tx, Tx - \beta x\rangle_A \geq 0 \quad \forall\, x \in \mathbb{S}_A^1. \tag{21}$$

Then,

$$\|T\|_A \leq \omega_A(T) + \frac{1}{4}\frac{|\alpha - \beta|^2}{|\alpha + \beta|}. \tag{22}$$

Proof. According to the proof of Theorem 1, we observe that (21) is equivalent to

$$\left\|Tx - \frac{\alpha+\beta}{2}x\right\|_A \leq \frac{1}{2}|\alpha - \beta| \text{ for any } \quad x \in \mathbb{S}_A^1, \tag{23}$$

which is, in turn, equivalent to the following operator norm inequality:

$$\left\|T - \frac{\alpha+\beta}{2}I\right\|_A \leq \frac{1}{2}|\alpha - \beta|.$$

Now, applying Theorem 2 for $\nu = \frac{\alpha+\beta}{2}$ and $r = \frac{1}{2}|\alpha - \beta|$, we deduce the desired result. □

Another sufficient condition under which the inequality (22) hold is presented in terms of A-positive operators and reads as follows.

Corollary 4. *Let $\alpha, \beta \in \mathbb{C}$ with $\alpha \notin \{-\beta, \beta\}$ and $T \in \mathbb{B}_A(\mathscr{H})$. If*

$$\left(T^{\dagger_A} - \bar{\beta}I\right)(\alpha I - T) \geq_A 0,$$

then

$$\|T\|_A \leq \omega_A(T) + \frac{1}{4}\frac{|\alpha - \beta|^2}{|\alpha + \beta|}.$$

Corollary 5. *Suppose that T, ν and r are as in Theorem 2. If, in addition,*

$$\Big||\nu| - \omega_A(T)\Big| \geq \rho, \tag{24}$$

for some $\rho > 0$, then

$$(0 \leq)\|T\|_A^2 - \omega_A^2(T) \leq r^2 - \rho^2.$$

Proof. From the inequality (19), we see that

$$\begin{aligned}\|T\|_A^2 - \omega_A^2(T) &\leq r^2 - \omega_A^2(T) + 2\omega_A(T)|\nu| - |\nu|^2 \\ &= r^2 - (|\nu| - \omega_A(T))^2.\end{aligned}$$

Hence, an application of (24) leads to the desired inequality. □

Remark 2. *If, in particular, $\|T - \nu I\|_A \leq r$ with $|\nu| = \omega_A(T)$, $\nu \in \mathbb{C}$, then*

$$(0 \leq)\|T\|_A^2 - \omega_A^2(T) \leq r^2.$$

Our next result reads as follows.

Theorem 3. *Let $\mathcal{T} = (T_1, \ldots, T_d) \in \mathbb{B}_{A^{1/2}}(\mathscr{H})^d$ and $\alpha_i, \beta_i \in \mathbb{C}$ with $\alpha_i \notin \{-\beta_i, \beta_i\}$ for $i \in \{1, \ldots, d\}$. If*

$$\left\|T_i - \frac{\alpha_i + \beta_i}{2}I\right\|_A \leq \frac{1}{2}|\alpha_i - \beta_i|, \tag{25}$$

for $i \in \{1, \dots, d\}$, *then*

$$\|\mathcal{T}\|_{e,A} \le \left(\sum_{i=1}^{d} \omega_A^2(T_i)\right)^{\frac{1}{2}} + \frac{1}{4}\left(\sum_{i=1}^{d} \frac{|\alpha_i - \beta_i|^4}{|\alpha_i + \beta_i|^2}\right)^{\frac{1}{2}} \tag{26}$$

and

$$\|\mathcal{T}\|_A \le \omega_A(\mathcal{T}) + \frac{1}{4}\frac{\sum_{i=1}^{d}|\alpha_i - \beta_i|^2}{\left(\sum_{i=1}^{d}|\alpha_i + \beta_i|^2\right)^{\frac{1}{2}}}. \tag{27}$$

Proof. Using Corollary 3, we have

$$\|T_i\|_A \le \omega_A(T_i) + \frac{1}{4}\frac{|\alpha_i - \beta_i|^2}{|\alpha_i + \beta_i|}$$

for $i \in \{1, \dots, d\}$.

Let $(\nu_1, \dots, \nu_d) \in \overline{\mathcal{B}}_d$, multiply by $|\nu_i|$ and sum to obtain

$$\sum_{i=1}^{d}\|\nu_i T_i\|_A \le \sum_{i=1}^{d}|\nu_i|\omega_A(T_i) + \frac{1}{4}\sum_{i=1}^{d}|\nu_i|\frac{|\alpha_i - \beta_i|^2}{|\alpha_i + \beta_i|}. \tag{28}$$

By the triangle inequality, we have

$$\left\|\sum_{i=1}^{d}\nu_i T_i\right\|_A \le \sum_{i=1}^{d}\|\nu_i T_i\|_A,$$

while by the Cauchy–Schwarz inequality, we obtain

$$\sum_{i=1}^{d}|\nu_i|\omega_A(T_i) \le \left(\sum_{i=1}^{d}|\nu_i|^2\right)^{\frac{1}{2}}\left(\sum_{i=1}^{d}\omega_A^2(T_i)\right)^{\frac{1}{2}} \le \left(\sum_{i=1}^{d}\omega_A^2(T_i)\right)^{\frac{1}{2}}$$

and

$$\begin{aligned}\sum_{i=1}^{d}|\nu_i|\frac{|\alpha_i - \beta_i|^2}{|\alpha_i + \beta_i|} &\le \left(\sum_{i=1}^{d}|\nu_i|^2\right)^{\frac{1}{2}}\left(\sum_{i=1}^{d}\frac{|\alpha_i - \beta_i|^4}{|\alpha_i + \beta_i|^2}\right)^{\frac{1}{2}} \\ &\le \left(\sum_{i=1}^{d}\frac{|\alpha_i - \beta_i|^4}{|\alpha_i + \beta_i|^2}\right)^{\frac{1}{2}}.\end{aligned}$$

From (28), we then obtain

$$\left\|\sum_{i=1}^{d}\nu_i T_i\right\|_A \le \left(\sum_{i=1}^{d}\omega_A^2(T_i)\right)^{\frac{1}{2}} + \frac{1}{4}\left(\sum_{i=1}^{d}\frac{|\alpha_i - \beta_i|^4}{|\alpha_i + \beta_i|^2}\right)^{\frac{1}{2}}$$

for all $(\nu_1, \dots, \nu_d) \in \overline{\mathcal{B}}_d$.

By taking the supremum over $(\nu_1, \dots, \nu_d) \in \overline{\mathcal{B}}_d$ and using the representation (3), we obtain (26).

The inequality (25) is equivalent for $x \in \mathbb{S}_A^1$ to

$$\|T_i x\|_A^2 - 2\Re e\left[\overline{\frac{\alpha_i + \beta_i}{2}}\langle T_i x, x\rangle_A\right] + \frac{1}{4}|\alpha_i + \beta_i|^2 \le \frac{1}{4}|\alpha_i - \beta_i|^2$$

for $i \in \{1, \ldots, d\}$. Therefore,

$$\|T_i x\|_A^2 + \frac{1}{4}|\alpha_i + \beta_i|^2 \leq \frac{1}{4}|\alpha_i - \beta_i|^2 + 2\Re e\left[\overline{\frac{\alpha_i + \beta_i}{2}}\langle T_i x, x\rangle_A\right] \tag{29}$$
$$\leq \frac{1}{4}|\alpha_i - \beta_i|^2 + |\alpha_i + \beta_i|\left|\langle T_i x, x\rangle_A\right|$$

for $i \in \{1, \ldots, d\}$.

If we sum and apply the Cauchy–Schwarz inequality, we then obtain

$$\begin{aligned}
&\sum_{i=1}^{d}\|T_i x\|_A^2 + \frac{1}{4}\sum_{i=1}^{d}|\alpha_i + \beta_i|^2 \\
&\leq \frac{1}{4}\sum_{i=1}^{d}|\alpha_i - \beta_i|^2 + \sum_{i=1}^{d}|\alpha_i + \beta_i|\left|\langle T_i x, x\rangle_A\right| \\
&\leq \frac{1}{4}\sum_{i=1}^{d}|\alpha_i - \beta_i|^2 + \left(\sum_{i=1}^{d}|\alpha_i + \beta_i|^2\right)^{\frac{1}{2}}\left(\sum_{i=1}^{d}\left|\langle T_i x, x\rangle_A\right|^2\right)^{\frac{1}{2}}.
\end{aligned}$$

On the other hand, an application of the arithmetic-geometric mean inequality shows that

$$\left(\sum_{i=1}^{d}\|T_i x\|_A^2\right)^{\frac{1}{2}}\left(\sum_{i=1}^{d}|\alpha_i + \beta_i|^2\right)^{\frac{1}{2}} \leq \sum_{i=1}^{d}\|T_i x\|_A^2 + \frac{1}{4}\sum_{i=1}^{d}|\alpha_i + \beta_i|^2.$$

Therefore, we deduce that

$$\begin{aligned}
&\left(\sum_{i=1}^{d}\|T_i x\|_A^2\right)^{\frac{1}{2}}\left(\sum_{i=1}^{d}|\alpha_i + \beta_i|^2\right)^{\frac{1}{2}} \\
&\leq \frac{1}{4}\sum_{i=1}^{d}|\alpha_i - \beta_i|^2 + \left(\sum_{i=1}^{d}|\alpha_i + \beta_i|^2\right)^{\frac{1}{2}}\left(\sum_{i=1}^{d}\left|\langle T_i x, x\rangle_A\right|^2\right)^{\frac{1}{2}}.
\end{aligned}$$

If we take the supremum over all $x \in \mathbb{S}_A^1$, we obtain

$$\|\mathcal{T}\|_A\left(\sum_{i=1}^{d}|\alpha_i + \beta_i|^2\right)^{\frac{1}{2}} \leq \frac{1}{4}\sum_{i=1}^{d}|\alpha_i - \beta_i|^2 + \left(\sum_{i=1}^{d}|\alpha_i + \beta_i|^2\right)^{\frac{1}{2}}\omega_A(\mathcal{T}),$$

which gives (27). Hence, the proof is complete. □

An immediate application of Theorem 3 is derived in the next corollary.

Corollary 6. *Let $\mathcal{T} = (T_1, \ldots, T_d) \in \mathbb{B}_{A^{1/2}}(\mathscr{H})^d$ and $\sigma, \rho \in \mathbb{C}$ with $\rho \neq \pm\sigma$. Assume that*

$$\left\|T_j - \frac{\sigma + \rho}{2}I\right\|_A \leq \frac{1}{2}|\rho - \sigma| \tag{30}$$

for $i \in \{1, \ldots, d\}$. Then,

$$\|\mathcal{T}\|_{e,A} \leq \left(\sum_{i=1}^{d}\omega_A^2(T_i)\right)^{\frac{1}{2}} + \frac{1}{4}\sqrt{d}\frac{|\rho - \sigma|^2}{|\sigma + \rho|}$$

and

$$\|\mathcal{T}\|_A \leq \omega_A(\mathcal{T}) + \frac{1}{4}\sqrt{d}\frac{|\rho - \sigma|^2}{|\sigma + \rho|}.$$

Now, we state in the next lemma a reverse of the Cauchy–Schwarz inequality (see for instance ([29] p. 32) for a more general result).

Lemma 2. *Under the same assumptions of Lemma 1, we have*

$$\left(\sum_{j=1}^{d}|\zeta_j|^2\right)^{\frac{1}{2}} \leq \frac{1}{\sqrt{d}}\left(\left|\sum_{j=1}^{d}\zeta_j\right| + \frac{1}{4}d\frac{|\rho-\sigma|^2}{|\rho+\sigma|}\right). \tag{31}$$

We state our next result as follows.

Theorem 4. *Let $\mathcal{T} = (T_1, \ldots, T_d) \in \mathbb{B}_{A^{1/2}}(\mathcal{H})^d$ and $\sigma, \rho \in \mathbb{C}$ with $\rho \neq \pm\sigma$. Assume that*

$$\omega_A\left(T_j - \frac{\sigma+\rho}{2}I\right) \leq \frac{1}{2}|\rho-\sigma| \text{ for any } j \in \{1, \ldots, d\}. \tag{32}$$

Then,

$$\omega_A(\mathcal{T}) \leq \frac{1}{\sqrt{d}}\omega_A\left(\sum_{j=1}^{d}T_j\right) + \frac{1}{4}\sqrt{d}\frac{|\rho-\sigma|^2}{|\rho+\sigma|}.$$

Proof. Let $x \in \mathbb{S}_A^1$ and $\mathcal{T} = (T_1, \ldots, T_d) \in \mathbb{B}_{A^{1/2}}(\mathscr{H})^d$ with the property (32). By letting $\zeta_j = \langle T_j x, x\rangle_A$ and then proceeding as in the proof of Proposition 1, we see that

$$\left|\zeta_j - \frac{\sigma+\rho}{2}\right| \leq \omega_A\left(T_j - \frac{\sigma+\rho}{2}\right) \leq \frac{1}{2}|\rho-\sigma|,$$

for any $j \in \{1, \ldots, d\}$. So, by employing (31), we obtain

$$\begin{aligned}\left(\sum_{j=1}^{d}\left|\langle T_j x, x\rangle_A\right|^2\right)^{\frac{1}{2}} &\leq \frac{1}{\sqrt{d}}\left(\left|\sum_{j=1}^{d}\langle T_j x, x\rangle_A\right| + \frac{1}{4}d\frac{|\rho-\sigma|^2}{|\rho+\sigma|}\right)\\ &= \frac{1}{\sqrt{d}}\left(\left|\langle \sum_{j=1}^{d}T_j x, x\rangle_A\right| + \frac{1}{4}d\frac{|\rho-\sigma|^2}{|\rho+\sigma|}\right)\end{aligned}$$

for every $x \in \mathbb{S}_A^1$. By taking the supremum over all $x \in \mathbb{S}_A^1$ in the last inequality, we reach the desired result. □

Remark 3. *Since $\omega_A(\mathcal{T}) \leq \|\mathcal{T}\|_A$, then (30) implies (32).*

Now, we aim to establish several reverse inequalities for the A-numerical radius of operators acting on semi-Hilbert spaces in both single and multivariable settings under some boundedness conditions for the operators. Our first new result in this context may be stated as follows.

Theorem 5. *Let $T \in \mathbb{B}_{A^{1/2}}(\mathscr{H})$ be such that $AT \neq 0$. If $\nu \in \mathbb{C}\backslash\{0\}$ and $r > 0$ are such that $|\nu| > r$ and*

$$\|T - \nu I\|_A \leq r,$$

then

$$\sqrt{1 - \frac{r^2}{|\nu|^2}} \leq \frac{\omega_A(T)}{\|T\|_A} \quad (\leq 1). \tag{33}$$

Proof. By (19), we have

$$\|T\|_A^2 + |\nu|^2 - r^2 \leq 2|\nu|\omega_A(T).$$

Dividing by $\sqrt{|\nu|^2 - r^2} > 0$, we obtain

$$\frac{\|T\|_A^2}{\sqrt{|\nu|^2 - r^2}} + \sqrt{|\nu|^2 - r^2} \leq \frac{2|\nu|\omega_A(T)}{\sqrt{|\nu|^2 - r^2}}. \tag{34}$$

Further, it is easy to verify that

$$2\|T\|_A \leq \frac{\|T\|_A^2}{\sqrt{|\nu|^2 - r^2}} + \sqrt{|\nu|^2 - r^2}.$$

So, by using (34), we deduce

$$\|T\|_A \leq \frac{\omega_A(T)|\nu|}{\sqrt{|\nu|^2 - r^2}},$$

which is immediately equivalent to (33). □

Remark 4. *(1) Squaring the inequality (33), we obtain the following inequality:*

$$(0 \leq)\|T\|_A^2 - \omega_A^2(T) \leq \frac{r^2}{|\nu|^2}\|T\|_A^2.$$

(2) For every operator $T \in \mathbb{B}_{A^{1/2}}(\mathscr{H})$, we have the relation $\omega_A(T) \geq \frac{1}{2}\|T\|_A$ (see [23]). Inequality (33) would produce an improvement of the above classic fact only in the case when

$$\frac{1}{2} \leq \left(1 - \frac{r^2}{|\nu|^2}\right)^{\frac{1}{2}},$$

which is, in turn, equivalent to $\frac{r}{|\nu|} \leq \frac{\sqrt{3}}{2}$.

The next corollary holds.

Corollary 7. *Let $\alpha, \beta \in \mathbb{C}$ with $\Re e(\alpha\bar{\beta}) > 0$. Additionally, let $T \in \mathbb{B}_{A^{1/2}}(\mathscr{H})$ be such that $AT \neq 0$. Assume that either (21) or (23) holds. Then, we have*

$$\frac{2\sqrt{\Re e(\alpha\bar{\beta})}}{|\alpha + \beta|} \leq \frac{\omega_A(T)}{\|T\|_A}(\leq 1) \tag{35}$$

and

$$(0 \leq)\|T\|_A^2 - \omega_A^2(T) \leq \left|\frac{\alpha - \beta}{\alpha + \beta}\right|^2 \|T\|_A^2.$$

Proof. If we consider $\nu = \frac{\alpha+\beta}{2}$ and $r = \frac{1}{2}|\alpha - \beta|$, then

$$|\nu|^2 - r^2 = \left|\frac{\alpha + \beta}{2}\right|^2 - \left|\frac{\alpha - \beta}{2}\right|^2 = \Re e(\alpha\bar{\beta}) > 0.$$

Now, by applying Theorem 5, we deduce the desired result. □

Remark 5. *If $|\alpha - \beta| \leq \frac{\sqrt{3}}{2}|\alpha + \beta|$ and $\Re e(\alpha\bar{\beta}) > 0$, then (35) is a refinement of the inequality $\omega_A(T) \geq \frac{1}{2}\|T\|_A$.*

Corollary 8. *Let $\alpha, \beta \in \mathbb{C}$ with $\Re e(\alpha\bar{\beta}) > 0$. Additionally, let $\mathcal{T} = (T_1, \ldots, T_d) \in \mathbb{B}_{A^{1/2}}(\mathscr{H})^d$ be such that the condition*

$$\left\| T_j - \frac{\alpha+\beta}{2} I \right\|_A \leq \frac{1}{2}|\alpha - \beta| \tag{36}$$

is true for $i \in \{1, \ldots, d\}$. Then,

$$\|\mathcal{T}\|_{e,A} \leq \frac{|\alpha+\beta|}{2\sqrt{\Re e(\alpha\bar{\beta})}} \left(\sum_{i=1}^{d} \omega_A^2(T_i) \right)^{\frac{1}{2}}. \tag{37}$$

Proof. Notice, first, that since (36) holds, then we infer that

$$\left\| T_i x - \frac{\alpha+\beta}{2} x \right\|_A \leq \frac{1}{2}|\alpha - \beta|,$$

for any $x \in \mathbb{S}_A^1$ and all $i \in \{1, \ldots, d\}$. Therefore, it follows from (35) that

$$\|T_i\|_A \leq \frac{|\alpha+\beta|}{2\sqrt{\Re e(\alpha\bar{\beta})}} \omega_A(T_i)$$

for $i \in \{1, \ldots, d\}$.

Let $(\nu_1, \ldots, \nu_d) \in \overline{\mathcal{B}}_d$, multiply by $|\nu_i|$ and sum to obtain

$$\sum_{i=1}^{d} \|\nu_i T_i\|_A \leq \frac{|\alpha+\beta|}{2\sqrt{\Re e(\alpha\bar{\beta})}} \sum_{i=1}^{d} |\nu_i| \omega_A(T_i).$$

Therefore, we see that

$$\begin{aligned}
\left\| \sum_{i=1}^{d} \nu_i T_i \right\|_A &\leq \sum_{i=1}^{d} \|\nu_i T_i\|_A \\
&\leq \frac{|\alpha+\beta|}{2\sqrt{\Re e(\alpha\bar{\beta})}} \sum_{i=1}^{d} |\nu_i| \omega_A(T_i). \\
&\leq \frac{|\alpha+\beta|}{2\sqrt{\Re e(\alpha\bar{\beta})}} \left(\sum_{i=1}^{d} |\nu_i|^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^{d} \omega_A^2(T_i) \right)^{\frac{1}{2}}.
\end{aligned}$$

By taking the supremum over $(\nu_1, \ldots, \nu_d) \in \overline{B}_d$ and using the representation (3), we obtain (37). □

In the next result, we prove under appropriate conditions a new relation connecting the joint A-seminorms $\| \cdot \|_A$ and $\omega_A(\cdot)$.

Proposition 2. *Let $\alpha_i, \beta_i \in \mathbb{C}$ with $\Re e(\alpha_i\bar{\beta}_i) > 0$ for all $i \in \{1, \ldots, d\}$. Additionally, let $\mathcal{T} = (T_1, \ldots, T_d) \in \mathbb{B}_{A^{1/2}}(\mathcal{H})^d$ be such that (25) is valid for $i \in \{1, \ldots, d\}$. Then,*

$$\|\mathcal{T}\|_A \leq \frac{1}{2} \frac{\left(\sum_{i=1}^{d} |\alpha_i + \beta_i|^2 \right)^{\frac{1}{2}}}{\left(\sum_{i=1}^{d} \Re e(\alpha_i\bar{\beta}_i) \right)^{\frac{1}{2}}} \omega_A(\mathcal{T}). \tag{38}$$

Proof. From (29), we obtain

$$\|T_i x\|_A^2 + \frac{1}{4}|\alpha_i + \beta_i|^2 - \frac{1}{4}|\alpha_i - \beta_i|^2 \leq |\alpha_i + \beta_i| \left| \langle T_i x, x \rangle_A \right|$$

for $i \in \{1, \ldots, d\}$. This is equivalent to

$$\|T_i x\|_A^2 + \Re e(\alpha_i \bar{\beta}_i) \leq |\alpha_i + \beta_i| \left| \langle T_i x, x \rangle_A \right|$$

for $i \in \{1, \ldots, d\}$.

If we sum and then apply the Cauchy–Schwarz inequality, we then obtain

$$\begin{aligned} \sum_{i=1}^d \|T_i x\|_A^2 + \sum_{i=1}^d \Re e(\alpha_i \bar{\beta}_i) &\leq \sum_{i=1}^d |\alpha_i + \beta_i| \left| \langle T_i x, x \rangle_A \right| \\ &\leq \left(\sum_{i=1}^d |\alpha_i + \beta_i|^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^d \left| \langle T_i x, x \rangle_A \right|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

By applying the famous arithmetic–geometric mean inequality, we observe that

$$2 \left(\sum_{i=1}^d \|T_i x\|_A^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^d \Re e(\alpha_i \bar{\beta}_i) \right)^{\frac{1}{2}} \leq \sum_{i=1}^d \|T_i x\|_A^2 + \sum_{i=1}^d \Re e(\alpha_i \bar{\beta}_i).$$

Therefore,

$$\left(\sum_{i=1}^d \|T_i x\|_A^2 \right)^{\frac{1}{2}} \leq \frac{1}{2} \frac{\left(\sum_{i=1}^d |\alpha_i + \beta_i|^2 \right)^{\frac{1}{2}}}{\left(\sum_{i=1}^d \Re e(\alpha_i \bar{\beta}_i) \right)^{\frac{1}{2}}} \left(\sum_{i=1}^d \left| \langle T_i x, x \rangle_A \right|^2 \right)^{\frac{1}{2}}$$

and by taking the supremum over $x \in \mathbb{S}_A^1$, we obtain (38). □

Remark 6. *With the assumptions of Corollary 8, we can prove that*

$$\|\mathcal{T}\|_A \leq \frac{1}{2} \frac{|\alpha + \beta|}{\sqrt{\Re e(\alpha \bar{\beta})}} \omega_A(\mathcal{T}).$$

The following lemma plays a fundamental role in the proof of our next proposition.

Lemma 3 ([29] p. 26)**.** *If $\sigma, \rho \in \mathbb{C}$ and $\zeta_j \in \mathbb{C}$, $j \in \{1, \ldots, d\}$ with the property that $\Re e(\rho \bar{\sigma}) > 0$ and*

$$\left| \zeta_j - \frac{\sigma + \rho}{2} \right| \leq \frac{1}{2}|\rho - \sigma|$$

for each $j \in \{1, \ldots, d\}$, then

$$\sum_{j=1}^d |\zeta_j|^2 \leq \frac{1}{4d} \frac{|\rho + \sigma|^2}{\Re e(\rho \bar{\sigma})} \left| \sum_{j=1}^d \zeta_j \right|^2. \tag{39}$$

By proceeding as in the proof of Theorem 4 and using Lemma 3, we state without proof the following result.

Proposition 3. *Let $\mathcal{T} = (T_1, \ldots, T_d) \in \mathbb{B}_{A^{1/2}}(\mathscr{H})^d$, $\sigma, \rho \in \mathbb{C}$ with $\Re e(\rho\bar{\sigma}) > 0$. Suppose that (32) is satisfied. Then,*

$$\omega_A(\mathcal{T}) \leq \frac{|\alpha+\beta|}{2d\sqrt{\Re e(\alpha\bar{\beta})}}\omega_A\left(\sum_{j=1}^{d} T_j\right).$$

The following result also holds.

Theorem 6. *Let $T \in \mathbb{B}_{A^{1/2}}(\mathscr{H})$ and $\nu \in \mathbb{C}\backslash\{0\}$, $r > 0$ with $|\nu| > r$. If*

$$\|T - \nu I\|_A \leq r, \tag{40}$$

then

$$\|T\|_A^2 \leq \omega_A^2(T) + \frac{2r^2}{|\nu| + \sqrt{|\nu|^2 - r^2}}\omega_A(T). \tag{41}$$

Proof. Let $x \in \mathbb{S}_A^1$. It follows from (40) that

$$\|Tx - \nu x\|_A \leq \|T - \nu I\|_A \leq r,$$

which yields that

$$\|Tx\|_A^2 + |\nu|^2 \leq 2\Re e\left[\overline{\nu}\langle Tx, x\rangle_A\right] + r^2. \tag{42}$$

By using (42), it can be seen that $|\nu|\left|\langle Tx, x\rangle_A\right| \neq 0$. So, by taking (42) into account, we obtain

$$\frac{\|Tx\|_A^2}{|\nu|\left|\langle Tx, x\rangle_A\right|} \leq \frac{2\Re e\left[\overline{\nu}\langle Tx, x\rangle_A\right]}{|\nu|\left|\langle Tx, x\rangle_A\right|} + \frac{r^2}{|\nu|\left|\langle Tx, x\rangle_A\right|} - \frac{|\nu|}{\left|\langle Tx, x\rangle_A\right|}.$$

Moreover, we see that

$$\begin{aligned}
&\frac{\|Tx\|_A^2}{|\nu|\left|\langle Tx, x\rangle_A\right|} - \frac{\left|\langle Tx, x\rangle_A\right|}{|\nu|} \\
&\leq \frac{2\Re e\left[\overline{\nu}\langle Tx, x\rangle_A\right]}{|\nu|\left|\langle Tx, x\rangle_A\right|} + \frac{r^2}{|\nu|\left|\langle Tx, x\rangle_A\right|} - \frac{\left|\langle Tx, x\rangle_A\right|}{|\nu|} - \frac{|\nu|}{\left|\langle Tx, x\rangle_A\right|} \\
&= \frac{2\Re e\left[\overline{\nu}\langle Tx, x\rangle_A\right]}{|\nu|\left|\langle Tx, x\rangle_A\right|} - \frac{|\nu|^2 - r^2}{|\nu|\left|\langle Tx, x\rangle_A\right|} - \frac{\left|\langle Tx, x\rangle_A\right|}{|\nu|} \\
&= \frac{2\Re e\left[\overline{\nu}\langle Tx, x\rangle_A\right]}{|\nu|\left|\langle Tx, x\rangle_A\right|} - \left(\frac{\sqrt{|\nu|^2 - r^2}}{\sqrt{|\nu|\left|\langle Tx, x\rangle_A\right|}} - \frac{\sqrt{\left|\langle Tx, x\rangle_A\right|}}{\sqrt{|\nu|}}\right)^2 - 2\frac{\sqrt{|\nu|^2 - r^2}}{|\nu|}.
\end{aligned}$$

Since

$$\Re e\left[\overline{\nu}\langle Tx, x\rangle_A\right] \leq |\nu|\left|\langle Tx, x\rangle_A\right|$$

and

$$\left(\frac{\sqrt{|\nu|^2 - r^2}}{\sqrt{|\nu|\left|\langle Tx, x\rangle_A\right|}} - \frac{\sqrt{\left|\langle Tx, x\rangle_A\right|}}{\sqrt{|\nu|}}\right)^2 \geq 0,$$

then, we deduce that

$$\frac{\|Tx\|_A^2}{|\nu|\left|\langle Tx,x\rangle_A\right|}-\frac{\left|\langle Tx,x\rangle_A\right|}{|\nu|}\leq\frac{2\left(|\nu|-\sqrt{|\nu|^2-r^2}\right)}{|\nu|},$$

which gives the inequality

$$\|Tx\|_A^2\leq\left|\langle Tx,x\rangle_A\right|^2+2\left|\langle Tx,x\rangle_A\right|\left(|\nu|-\sqrt{|\nu|^2-r^2}\right). \tag{43}$$

By taking the supremum over $x\in\mathbb{S}_A^1$ in (43), we obtain

$$\|T\|_A^2\leq\omega_A^2(T)+2\omega_A(T)\left(|\nu|-\sqrt{|\nu|^2-r^2}\right). \tag{44}$$

So, we immediately obtain (41). □

By making use of the inequalities (44) and (43), we are ready to establish the next two corollaries as applications of our previous result.

Corollary 9. *Let $\rho,\sigma\in\mathbb{C}$ be such that $\rho\neq\sigma$ and $\Re e(\rho\overline{\sigma})\geq 0$. Additionally, let $T\in\mathbb{B}_{A^{1/2}}(\mathscr{H})$ be such that either (4) or (5) holds. Then:*

$$\|T\|_A^2\leq\omega_A^2(T)+\left[|\rho+\sigma|-2\sqrt{\Re e(\rho\overline{\sigma})}\right]\omega_A(T). \tag{45}$$

Proof. Set $\nu:=\frac{\rho+\sigma}{2}$ and $r:=\frac{|\rho-\sigma|}{2}$. Clearly, $|\nu|>r$. Moreover, since (5) holds, then so is (40). So, the desired result follows by applying (44) and then observing that

$$|\nu|^2-r^2=\left|\frac{\rho+\sigma}{2}\right|^2-\left|\frac{\rho-\sigma}{2}\right|^2=\Re e(\rho\overline{\sigma}). \tag{46}$$

□

Remark 7. *Assume that $T\in\mathbb{B}_A(\mathscr{H})$. If $\theta\geq\mu>0$ are such that either $(T^{\dagger_A}-\mu I)(\theta I-T)$ is A-accretive or*

$$\left(T^{\dagger_A}-\mu I\right)(\theta I-T)\geq_A 0$$

then, by applying (45), we infer that

$$\|T\|_A^2\leq\omega_A^2(T)+\left(\sqrt{\theta}-\sqrt{\mu}\right)^2\omega_A(T).$$

Corollary 10. *Let $\mathcal{T}=(T_1,\ldots,T_d)\in\mathbb{B}_{A^{1/2}}(\mathscr{H})^d$ and $\rho_i,\sigma_i\in\mathbb{C}$ with $\rho_i\neq\sigma_i$, $\Re e(\rho_i\overline{\sigma_i})\geq 0$ for $i\in\{1,\ldots,d\}$. Assume that*

$$\left\|T_i-\frac{\rho_i+\sigma_i}{2}I\right\|_A\leq\frac{1}{2}|\rho_i-\sigma_i|, \tag{47}$$

for all $i\in\{1,\ldots,d\}$. Then,

$$\|\mathcal{T}\|_A^2\leq\omega_A^2(\mathcal{T})\left[\sum_{i=1}^d\left(|\rho_i+\sigma_i|-2\sqrt{\Re e(\rho_i\overline{\sigma_i})}\right)^2\right]^{\frac{1}{2}}\omega_A(\mathcal{T}).$$

Proof. Let $x \in \mathbb{S}_A^1$. Set $\nu_i := \frac{\rho_i+\sigma_i}{2}$ and $r_i = \frac{|\rho_i-\sigma_i|}{2}$ for all $i \in \{1, \ldots, d\}$. Clearly, we have $|\nu_i| > r_i$ and $\|T_i - \nu_i I\|_A \leq r_i$ for every i. Thus, an application of (43) shows that

$$\|T_i x\|_A^2 \leq \left|\langle T_i x, x\rangle_A\right|^2 + 2\left|\langle T_i x, x\rangle_A\right|\left(|\nu_i| - \sqrt{|\nu_i|^2 - r_i^2}\right).$$

This yields, through (46), that

$$\|T_i x\|_A^2 \leq \left|\langle T_i x, x\rangle_A\right|^2 + \left(|\rho_i + \sigma_i| - 2\sqrt{\Re e(\rho_i\overline{\sigma_i})}\right)\left|\langle T_i x, x\rangle_A\right|$$

for $i \in \{1, \ldots, d\}$.

If we sum and then apply the Cauchy–Schwarz inequality, we then obtain

$$\begin{aligned}
&\sum_{i=1}^{d}\|T_i x\|_A^2 \\
&\leq \sum_{i=1}^{d}\left|\langle T_i x, x\rangle_A\right|^2 + \sum_{i=1}^{d}\left(|\rho_i + \sigma_i| - 2\sqrt{\Re e(\rho_i\overline{\sigma_i})}\right)\left|\langle T_i x, x\rangle_A\right| \\
&\leq \sum_{i=1}^{d}\left|\langle T_i x, x\rangle_A\right|^2 + \left[\sum_{i=1}^{d}\left(|\rho_i + \sigma_i| - 2\sqrt{\Re e(\rho_i\overline{\sigma_i})}\right)^2\right]^{\frac{1}{2}}\left(\sum_{i=1}^{d}\left|\langle T_i x, x\rangle_A\right|^2\right)^{\frac{1}{2}}.
\end{aligned}$$

By taking the supremum over this inequality, we derive the desired result. □

Another application of the inequality (45) provides an upper bound for the Euclidean operator A-seminorm of d-tuples of operators in $\mathbb{B}_{A^{1/2}}(\mathscr{H})^d$ and stated in the next proposition.

Proposition 4. *Let $\mathcal{T} = (T_1, \ldots, T_d) \in \mathbb{B}_{A^{1/2}}(\mathscr{H})^d$. Let also $\rho, \sigma \in \mathbb{C}$ with $\rho \neq \sigma$ and $\Re e(\rho\overline{\sigma}) \geq 0$. Suppose that*

$$\left\|T_i x - \frac{\rho+\sigma}{2}x\right\|_A \leq \frac{1}{2}|\rho - \sigma|, \tag{48}$$

for any $x \in \mathbb{S}_A^1$ and all $i \in \{1, \ldots, d\}$. Then,

$$\|\mathcal{T}\|_{e,A}^2 \leq d \max_{k\in\{1,\ldots,d\}} \omega_A(T_k)\left\{\max_{k\in\{1,\ldots,d\}} \omega_A(T_k) + \left[|\rho+\sigma| - 2\sqrt{\Re e(\rho\overline{\sigma})}\right]\right\}.$$

Proof. From (45), we see that

$$\|T_i\|_A^2 \leq \omega_A^2(T_i) + \left[|\rho+\sigma| - 2\sqrt{\Re e(\rho\overline{\sigma})}\right]\omega_A(T_i)$$

for $i \in \{1, \ldots, d\}$.

Let $(\nu_1, \ldots, \nu_d) \in \overline{\mathcal{B}}_d$, multiply by $|\nu_i|^2$ and sum to obtain

$$\begin{aligned}
\sum_{i=1}^{d}|\nu_i|^2\|T_i\|_A^2 &\leq \sum_{i=1}^{d}|\nu_i|^2\omega_A^2(T_i) + \left[|\rho+\sigma| - 2\sqrt{\Re e(\rho\overline{\sigma})}\right]\sum_{i=1}^{d}|\nu_i|^2\omega_A(T_i) \\
&\leq \left(\sum_{i=1}^{d}|\nu_i|^2\right)\max_{k\in\{1,\ldots,d\}}\omega_A^2(T_k) \\
&\quad + \left(\sum_{i=1}^{d}|\nu_i|^2\right)\max_{k\in\{1,\ldots,d\}}\omega_A(T_k)\left[|\rho+\sigma| - 2\sqrt{\Re e(\rho\overline{\sigma})}\right] \\
&\leq \max_{k\in\{1,\ldots,d\}}\omega_A^2(T_k) + \left[|\rho+\sigma| - 2\sqrt{\Re e(\rho\overline{\sigma})}\right]\max_{k\in\{1,\ldots,d\}}\omega_A(T_k).
\end{aligned}$$

Moreover, since

$$\frac{1}{d}\left\|\sum_{i=1}^{d}\nu_i T_i\right\|_A^2 \leq \sum_{i=1}^{d}|\nu_i|^2\|T_i\|_A^2,$$

hence

$$\frac{1}{d}\left\|\sum_{i=1}^{d}\nu_i T_i\right\|_A^2 \leq \max_{k\in\{1,\ldots,d\}}\omega_A^2(T_k)+\left[|\rho+\sigma|-2\sqrt{\Re e(\rho\bar{\sigma})}\right]\max_{k\in\{1,\ldots,d\}}\omega_A(T_k).$$

By taking the supremum over $(\nu_1,\ldots,\nu_d)\in\overline{B}_d$ and using the representation (3), we obtain the desired result. □

The next lemma plays a crucial role in establishing our final result in this paper.

Lemma 4 ([30]). *If $\sigma,\rho,\zeta_j\in\mathbb{C}$ are such that $\Re e(\rho\bar{\sigma})>0$ and*

$$\left|\zeta_j-\frac{\sigma+\rho}{2}\right|\leq\frac{1}{2}|\rho-\sigma|$$

for each $j\in\{1,\ldots,d\}$, then we have

$$\sum_{j=1}^{d}|\zeta_j|^2\leq\left(\frac{1}{d}\left|\sum_{j=1}^{d}\zeta_j\right|+|\rho+\sigma|-2\sqrt{\Re e(\rho\bar{\sigma})}\right)\left|\sum_{j=1}^{d}\zeta_j\right|.$$

Now, we are ready to state our final proposition.

Proposition 5. *Let $\mathcal{T}=(T_1,\ldots,T_d)\in\mathbb{B}_{A^{1/2}}(\mathscr{H})^d$ and let $\rho,\sigma\in\mathbb{C}$ be such that $\rho\neq\sigma,\Re e(\rho\bar{\sigma})>0$. Assume that the condition (16) is valid. Then,*

$$\omega_A^2(\mathcal{T})\leq\left[\frac{1}{d}\omega_A\left(\sum_{j=1}^{d}T_j\right)+|\rho+\sigma|-2\sqrt{\Re e(\rho\bar{\sigma})}\right]\omega_A\left(\sum_{j=1}^{d}T_j\right).$$

Proof. The proof follows by proceeding as in the proof of Proposition 2 and then taking Lemma 4 into consideration. □

3. Conclusions

In this paper, we established several inequalities involving the generalized Euclidean operator radius of d-tuples of A-bounded linear operators acting on a complex Hilbert space $\mathscr{H}$. The obtained bounds lead to the special case of the classical A-numerical radius of semi-Hilbert space operators. We proved also some estimates related to the Euclidean operator A-seminorm of d-tuples of A-bounded operators. In addition, we stated, under appropriate conditions, several reverse inequalities for the A-numerical radius in single and multivariable setting.

These inequalities can be further utilized to provide reverse triangle inequalities for the operator A-seminorm and A-numerical radius of semi-Hilbert space operators that play an important role in the geometrical structure of the A-inner product space under consideration.

Additionally, the techniques and ideas of this article can be useful for future investigations in this area of research. In future papers, we aim to investigate the connections between the joint A-numerical radius and joint operator A-seminorm of some special classes of multivariable operators such that the class of jointly A-hyponormal operators in semi-Hilbert spaces.

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