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Article

New Generalization of Geodesic Convex Function

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Abstract: As a generalization of a geodesic function, this paper introduces the notion of the geodesic φ_E-convex function. Some properties of the φ_E-convex function and geodesic φ_E-convex function are established. The concepts of a geodesic φ_E-convex set and φ_E-epigraph are also given. The characterization of geodesic φ_E-convex functions in terms of their φ_E-epigraphs, are also obtained.

Keywords: Riemannian manifolds; geodesic; E-convex sets; φ_E-convex function; geodesic φ_E-convex function

1. Introduction

Convexity is an essential concept in pure and applied mathematics, serving as a potent instrument for analyzing functions and sets, establishing inequalities, and modeling and solving real-world problems. This concept is crucial for estimating integrals and establishing bounds in numerous mathematical fields and beyond [1–7]. Thus, the convex function can be defined as follows:

A function $h : U \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is convex if

$$h(\eta u_1 + (1-\eta)u_2) \leq \eta h(u_1) + (1-\eta)h(u_2), \ \ \forall u_1, u_2 \in U, \ \ \eta \in [0,1]. \tag{1}$$

If the inequality sign in (1) is reversed, then h is called a concave function on the set U.

For example, in economics, for a production function $u = h(L)$, the concavity of h is expressed by saying that h exhibits diminishing returns. If h is convex, then it exhibits increasing returns. On the other hand, many new problems in applied mathematics are encountered where the notion of convexity is not enough to describe them, in order to reach favorable results. For this reason, the concept of convexity has been extended and generalized in several studies, see [8–13]. Curvature and torsion of Riemannian manifolds lead to a high level of nonlinearity when examining the convexity of such manifolds. A geodesic, is a locally length-minimizing curve, and the notion of a geodesic convex function occurs naturally in a complete Riemannian manifold, which has been studied in [14,15]. The geodesic bifurcation has equally been studied by many authors [16,17].

In 1999, an important generalization of the convex function, called the E-convex function, was defined by Youness [18]. This type of function has some applications in various branches of mathematical sciences [19,20]. On the other hand, Yang [21] showed that some results given by Youness [18] seem to be incorrect. Following these developments, Duca and Lupşa [22] fixed the mistakes in both Youness [18] and Yang [21]. Therefore, Chen [23] extended E-convexity to a semi E-convexity and discussed some of its properties. For more results on the E-convex function and semi E-convex function, one should consult the following references [22,24–27]. The geodesic convexity involving sets was first studied by [28], who extended the existing concept of geodesic convexity defined by [29]. Geodesic E-convex sets and geodesic E-convex functions on Riemannian manifolds, are a new class of convex sets and functions, that Iqbal et al. introduced and researched in [26], these were extended to geodesic strongly E-convex sets and geodesic strongly E-convex functions in

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2015, by Adem and Saleh [30]. In addition, Iqbal et al. [25] introduced geodesic semi E-convex functions. Following these developments, Adem and Saleh [4] introduced geodesic semi E-b-vex (GSEB) functions, of which some properties were discussed.

Other developments include the work of Eshaghi Gordji et al. [31], who introduced the notion of a φ-convex function, in 2016. They equally studied Jensen and Hermite–Hadamard type inequalities related to this function. Moreover, the notion of φ_E-convex functions was defined as the generalization of φ-convex functions. Absos et al. further introduced the notion of a geodesic φ-convex function, through which some basic properties of this function were studied [32].

The structure of this article is as follows. Basic information about convex functions and convex sets is covered in Section 2. The evaluation of the properties of φ_E-convex functions is covered in Section 3. In Section 4, we discuss a new class of functions on Riemannian manifolds, called the geodesic φ_E-convex function. Some of the properties of this function are also studied. In Section 5, the characterization of geodesic φ_E-convex functions, through their corresponding φ_E-epigraphs, is reported.

2. Preliminaries

This section provides some definitions and properties that can later be used in the study, to report our results. Several definitions and properties of real number sets and the Riemannian manifold can be found in many different geometry books and papers [15]. Throughout this paper, we consider an interval $U = [u_1, u_2]$ in $\mathbb{R}$ and $\varphi\colon \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is a bifunction.

Definition 1. *Ref. [31]. A function $h : U \longrightarrow \mathbb{R}$ is called φ-convex if*

$$h(tu_1 + (1-t)u_2) \leq h(u_2) + t\varphi(h(u_1), h(u_2)), \tag{2}$$

for all $u_1, u_2 \in U, t \in [0,1]$

In the above definition, if $\varphi(h(u_1), h(u_2)) = h(u_1) - h(u_2)$, then inequality (2) becomes inequality (1).

Definition 2. *Ref. [31]. The function $\varphi\colon \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$, is called*

1. *additive if*

$$\varphi(u_1 + v_1, u_2 + v_2) = \varphi(u_1, u_2) + \varphi(v_1, v_2), \forall u_1, u_2, v_1, v_2 \in \mathbb{R}.$$

2. *non-negatively homogeneous if*

$$\varphi(tu_1, tu_2) = t\varphi(u_1, u_2), \forall u_1, u_2 \in \mathbb{R}, t \geqslant 0.$$

3. *non-negatively linear if φ is both non-negatively homogeneous and additive.*

Definition 3. *Ref. [18]. A set $U \subset \mathbb{R}^n$, is said to be an E-convex set if there is a mapping $E : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, such that*

$$t(E(u_1)) + (1-t)E(u_2) \in U, \ \ \forall u_1, u_2 \in U, t \in [0,1]$$

Definition 4. *Refs. [18,31]. Consider $U \subset \mathbb{R}^n$ to be an E-convex set, then the function $h : U \longrightarrow \mathbb{R}$ is said to be*

1. *an E-convex function, if*

$$h(tE(u_1) + (1-t)E(u_2) \leq th(E(u_1)) + (1-t)h(E(u_2)), \tag{3}$$

$\forall u_1, u_2 \in U, t \in [0,1]$

2. *a φ_E-convex function, if*

$$h(tE(u_1) + (1-t)E(u_2) \leq h(E(u_2)) + t\varphi(h(E(u_1)), h(E(u_2)), \quad (4)$$

$\forall u_1, u_2 \in U, t \in [0,1]$

If $\varphi(h(E(u_1)), h(E(u_2)) = h(E(u_1)) - h(E(u_2))$ in inequality (4), then we obtain the E-convex function.

Now, let (N, g) be a complete m-dimensional Riemannian manifold, with Riemannian connection $\bigtriangledown$. If a_1 and a_2 are two points on N, and $\gamma : [\mu_1, \mu_2] \longrightarrow N$ is a piecewise smooth curve joining $\gamma(\mu_1) = a_1$ to $\gamma(\mu_2) = a_2$ and its length, $L(\gamma)$, is defined by

$$L(\gamma) = \int_{\mu_1}^{\mu_2} \| \frac{d\gamma(\lambda)}{d\lambda} \| d\lambda.$$

For any two points $a_1, a_2 \in N$, we define $d(a_1, a_2) = \inf\{L(\gamma) : \gamma$ a piecewise smooth curve connecting the points a_1 to $a_2\}$.

Then d is a metric, which induces the original topology on N.

For every Riemannian manifold, there is a unique determined Riemannian connection, called a Levi–Civita connection, denoted by $\bigtriangledown_X Y$, for any vector fields $X, Y \in N$. In addition, a smooth path γ, is a geodesic if and only if its tangent vector is a parallel vector field along the path γ, i.e., γ satisfies the equation $\bigtriangledown_{\frac{d\gamma(t)}{dt}} \frac{d\gamma(t)}{dt} = 0$. Any path γ joining μ_1 and μ_2 in N, such that $L(\gamma) = d(\mu_1, \mu_2)$, is a geodesic and is called a minimal geodesic. Let N be a C^∞ complete n-dimensional Riemannian manifold, with metric g and Levi–Civita connection $\bigtriangledown$. Moreover, considering that the points $\mu_1, \mu_2 \in N$ and γ: $[0,1] \longrightarrow N$ is a geodesic joining μ_1, μ_2, i.e., $\gamma_{\mu_1,\mu_2}(0) = \mu_2$ and $\gamma_{\mu_1,\mu_2}(1) = \mu_1$.

Definition 5. *Ref. [33]. Assume that N_1, N_2 are smooth manifolds. A map $h : N_1 \longrightarrow N_2$ is a diffeomorphism if it is smooth, bijective, and the inverse h^{-1} is smooth.*

Definition 6. *Ref. [15]. A subset $U \subseteq N$, is called t-convex if and only if U contains every geodesic γ_{μ_1,μ_2} of N whose endpoints μ_1 and μ_2 are in U.*

Remark 1. *If U_1 and U_2 are t-convex sets, then $U_1 \cap U_2$ is a t-convex set, but $U_1 \cup U_2$ is not necessarily a t-convex set.*

Definition 7. *Ref. [15]. A function $h : U \subset N \longrightarrow \mathbb{R}$ is called geodesic convex if and only if for all geodesic arcs γ_{μ_1,μ_2}, then*

$$h(\gamma_{\mu_1,\mu_2}(t)) \leq th(\mu_1) + (1-t)h(\mu_2)$$

for each $\mu_1, \mu_2 \in U$ and $t \in [0,1]$.

Definition 8. *Ref. [26]. A set $U \subset N$, is geodesic E-convex, where $E : N \longrightarrow N$, if and only if there exists a unique geodesic $\gamma_{E(\mu_1),E(\mu_2)}(t)$of length $d(\mu_1, \mu_2)$, which belongs to U for every $\mu_1, \mu_2 \in U$ and $t \in [0,1]$.*

Definition 9. *Refs. [26,32]. A function $h : U \longrightarrow \mathbb{R}$ is said to be*

1. *geodesic E-convex if U is a geodesic E-convex set and*

$$h(\gamma_{E(\mu_1),E(\mu_2)}(t)) \leq th(E(\mu_1)) + (1-t)h(E(\mu_2)), \forall \mu_1, \ \mu_2 \in U, \ t \in [0,1].$$

2. *geodesic φ-convex if U is a t-convex set and*

$$h(\gamma_{\mu_1,\mu_2}(t)) \leq h(\mu_2) + t\varphi(h(\mu_1), h(\mu_2)), \forall \mu_1, \ \mu_2 \in U, \ t \in [0,1].$$

3. Some Properties of φ_E-Convex Functions

This part of the work deals with some properties of φ_E-convex functions. Considering that $h : B \longrightarrow \mathbb{R}$ is a φ_E-convex function and $E : \mathbb{R} \longrightarrow \mathbb{R}$, we present the following. For any two points $E(\mu_1), E(\mu_2) \in B$ with $E(\mu_1) < E(\mu_2)$ and for each point $E(\mu) \in (E(\mu_1), E(\mu_2))$ can be expressed as

$$E(\mu) = tE(\mu_1) + (1-t)E(\mu_1), \;\; t = \frac{E(\mu_2) - E(\mu)}{E(\mu_2) - E(\mu_1)}.$$

Also, since a function h is φ_E-convex function if

$$h(E(\mu)) \leq h(E(\mu_2)) + \frac{E(\mu_2) - E(\mu)}{E(\mu_2) - E(\mu_1)}\varphi(h(E(\mu_1)), h(E(\mu_2))),$$

then

$$\frac{h(E(\mu_2)) - h(E(\mu))}{E(\mu_2) - E(\mu)} \geqslant \frac{\varphi(h(E(\mu_1)), h(E(\mu_2)))}{E(\mu_1) - E(\mu_2)}, \tag{5}$$

$\forall E(\mu) \in (E(\mu_2), E(\mu_1))$.

Hence, we can say that a function h is a φ_E-convex function if it satisfies the inequality (5).

The next example shows that a φ_E-convex function is not necessarily a φ-convex function.

Example 1. *Consider*

$$h(u_1) \;\; = \;\; \begin{cases} 1; u_1 \geqslant 0, \\ -u_1^2; u_1 < 0, \end{cases}$$

with $E(u_1) = -a$ where $a \in \mathbb{R}^+$ and $\varphi(u_1, u_2) = u_1 - 2u_2$. Then $h(tE(u_1) + (1-t)E(u_2)) = -a^2$ while $h(E(u_2)) + t\varphi(h(E(u_1)), h(E(u_2))) = (t-1)a^2$, which means that h is a φ_E-convex function. On the other hand, if we take $u_1 > 0$ and $u_2 > 0$, then h is not a φ-convex function.

Theorem 1. *If $h : B \subset E(\mathbb{R}) \longrightarrow \mathbb{R}$ is differentiable and a φ_E-convex function in B, and $h(E(u_1)) \neq h(E(u_2))$, then there are $E(\alpha), E(\beta) \in (E(u_2), E(u_1)) \subset B$, such that*

$$h'(E(\alpha)) \geqslant \frac{\varphi(h(E(u_1)), h(E(u_2)))}{h(E(u_1)) - h(E(u_2))}h'(E(\beta)) \geqslant h'(E(\beta)).$$

Proof. Since h is a φ_E-convex function, then

$$\begin{aligned} \frac{h(E(u_2)) - h(E(u))}{E(u_2) - E(u)} \;\; &\geqslant \;\; \frac{\varphi(h(E(u_1)), h(E(u_2)))}{E(u_1) - E(u_2)} \\ &= \frac{\varphi(h(E(u_1)), h(E(u_2)))}{h(E(u_1)) - h(E(u_2))} \times \frac{h(E(u_1)) - h(E(u_2))}{E(u_1) - E(u_2)}. \end{aligned} \tag{6}$$

Now, applying the mean value theorem, then the inequality (6) can be written as

$$h'(E(\alpha)) \geqslant \frac{\varphi(h(E(u_1)), h(E(u_2)))}{h(E(u_1)) - h(E(u_2))}h'(E(\beta)), \tag{7}$$

for some $E(\alpha) \in (E(u_1), E(u)) \subset (E(u_1), E(u_2))$ and $E(\beta) \in (E(u_1), E(u_2))$.

Since $\varphi(h(E(u_1)), h(E(u_2))) \geqslant h(E(u_1)) - h(E(u_2))$, then the inequality (7) yields

$$h'(E(\alpha)) \geqslant \frac{\varphi(h(E(u_1)), h(E(u_2)))}{h(E(u_1)) - h(E(u_2))}h'(E(\beta)) \geqslant h'(E(\beta)).$$

□

Theorem 2. *Assume that $h : B \longrightarrow \mathbb{R}$ is a differentiable φ_E-convex function. Then, for all $E(\mu_i) \in B,\ \ i = 1, 2, 3$, such that $E(\mu_1) < E(\mu_2) < E(\mu_3)$, the following inequality holds*

$$h'(E(\mu_2)) + h'(E(\mu_3)) \leq \frac{\varphi(h(E(\mu_1)), h(E(\mu_2))) + \varphi(h(E(\mu_2)), h(E(\mu_3)))}{E(\mu_1), E(\mu_3)}.$$

Proof. Since h is φ_E-convex in each interval $W_1 = [E(\mu_1), E(\mu_2)]$ and $W_2 = [E(\mu_2), E(\mu_3)]$, hence

$$h(tE(\mu_1) + (1-t)E(\mu_2)) \leq h(E(\mu_2)) + t\varphi(h(E(\mu_1)), h(E(\mu_2))) \tag{8}$$

and

$$h(tE(\mu_2) + (1-t)E(\mu_3)) \leq h(E(\mu_3)) + t\varphi(h(E(\mu_2)), h(E(\mu_3))). \tag{9}$$

From inequalities (8) and (9), we get

$$\begin{aligned}&\frac{h(tE(\mu_1) + (1-t)E(\mu_2)) - h(E(\mu_2)) + h(tE(\mu_2) + (1-t)E(\mu_3)) - h(E(\mu_3))}{t}\\ &\qquad \leq \varphi(h(E(\mu_1)), h(E(\mu_2))) + \varphi(h(E(\mu_2)), h(E(\mu_3))).\end{aligned}$$

Now, setting $t \longrightarrow 0$, we get

$$\begin{aligned}&h'(E(\mu_2))(E(\mu_1) - E(\mu_2)) + h'(E(\mu_3))(E(\mu_2) - E(\mu_3))\\ &\leq \varphi(h(E(\mu_1)), h(E(\mu_2))) + \varphi(h(E(\mu_2)), h(E(\mu_3))).\end{aligned} \tag{10}$$

Also, $E(\mu_3) > E(\mu_2)$ and $E(\mu_2) > E(\mu_1)$, which means that $E(\mu_1) - E(\mu_3) < E(\mu_1) - E(\mu_2)$ and $E(\mu_1) - E(\mu_3) < E(\mu_2) - E(\mu_3)$, then

$$\begin{aligned}&(E(\mu_1) - E(\mu_3))(h'(E(\mu_2)) + h'(E(\mu_3)))\\ &\leq (E(\mu_1) - E(\mu_2))h'(E(\mu_2)) + (E(\mu_2) - E(\mu_3))h'(E(\mu_3)).\end{aligned} \tag{11}$$

Hence, from inequalities (10) and (11), we get the required result. □

4. Properties of Geodesic φ_E-Convex Functions

This section makes the assumption that $\mu_1, \mu_2 \in N$ and $\gamma\colon [0, 1] \longrightarrow N$ is a geodesic joining μ_1, μ_2, i.e., $\gamma_{\mu_1,\mu_2}(0) = \mu 2$ and $\gamma_{\mu_1,\mu_2}(1) = \mu_1$, and E is a mapping, such that $E : N \longrightarrow N$, where N is a C^∞ complete n-dimensional Riemannian manifold, with Riemannian connection $\bigtriangledown$. In addition, we define the geodesic φ_E-convex function in N and examine some of its characteristics.

Definition 10. *A function $h : B \longrightarrow \mathbb{R}$ is geodesic φ_E-convex if B is also a geodesic E-convex set and*

$$h(\gamma_{E(\mu_1),E(\mu_2)}(t)) \leq h(E(\mu_2)) + t\varphi(h(E(\mu_1)), h(E(\mu_2))),$$

for all $\mu_1,\ \ \mu_2 \in B,\ \ t \in [0, 1]$.

If the above inequality strictly holds for all $\mu_1, \mu_2 \in B, E(\mu_1) \neq E(\mu_2), t \in [0, 1]$, then h is called a strictly geodesic φ_E-convex function.

Remark 2. *If E is identity mapped in the above definition, then we have a geodesic φ-convex function. Moreover, if*

$$\varphi(h(E(\mu_1)), h(E(\mu_2))) = (h(E(\mu_1)) - h(E(\mu_2))),$$

then we have a geodesic E-convex function.

Example 2. *This example shows that the geodesic φ_E-convex function on N does not necessarily have to be geodesic convex. Let $N = \mathbb{R} \times \mathbb{S}^1$ and $h : N \longrightarrow \mathbb{R}$ is defined as $h(\mu, a) = \mu^3$, then h is not geodesic convex in N. Now, by taking a function $\varphi : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ as$\varphi(\mu_1, \mu_2) = \mu_1^3 - \mu_2^3$ and $E : \mathbb{R} \longrightarrow \mathbb{R}^+$, then for any two points (μ_1, a) and (μ_2, b), the geodesic joining them is a portion of a helix of the form $\gamma(t) = (t\mu_1 + (1-t)\mu_2, \exp^{i[t\omega_1 + (1-t)\omega_2]})$ for $t \in [0, 1]$ and $\exp^{i\omega_1} = a, \exp^{i\omega_2} = b$ for $\omega_1, \omega_2 \in [0, 2\pi]$. Hence,*

$$\begin{aligned} h(\gamma_{E(\mu_1),E(\mu_2)}) &= (tE(\mu_1) + (1-t)E(\mu_2))^3 \\ &= t^3(E(\mu_1) - E(\mu_2))^3 + t^2(3E^2(\mu_1)E(\mu_2) - 6E(\mu_1)E^2(\mu_2)) \\ &+ 3E^3(\mu_2) + t[3E(\mu_1)E^2(\mu_2) - 3E^3(\mu_2)] + E^3(\mu_2) \\ &\le E^3(\mu_2) + t[E^3(\mu_1) - E^3(\mu_2)] \\ &= h(E(\mu_2), b) + t\varphi(h(E(\mu_1), a), hE(\mu_2), b). \end{aligned} \tag{12}$$

Then h is a geodesic φ_E-convex function.

Theorem 3. *Considering that $B \subset N$ is an E-convex set, then a function $h : B \longrightarrow \mathbb{R}$ is a geodesic φ_E-convex if and only if for each $u_1, u_2 \in B$ the function $K = h \circ \gamma_{E(u_1),E(u_2)}$ is φ_E-convex on $[0, 1]$.*

Proof. Let K be φ_E-convex on $[0, 1]$, then

$$K(tE(\mu_1) + (1-t)E(\mu_2)) \le K(E(\mu_2)) + t\varphi(K(E(\mu_1)), K(E(\mu_2))) \tag{13}$$

holds.

Also, let $E(\mu_1) = 1, E(\mu_2) = 0$, then $K(t) \le K(0) + t\varphi(K(1), K(0))$. Hence

$$h(\gamma_{E(\mu_1),E(\mu_2)}(t)) \le h(E(\mu_2)) + t\varphi(h(E(\mu_1)), h(E(\mu_2))).$$

Conversely, assume that h is a geodesic φ_E-convex function. By restricting the domain of $\gamma_{E(\mu_1),E(\mu_2)}$ to $[\eta_1, \eta_2]$, and hence the parametrized form of this restriction can be rewritten as

$$\alpha(t) = \gamma_{E(\mu_1),E(\mu_2)}(tE(\mu_1) + (1-t)E(\mu_2))$$
$$\alpha(0) = \gamma_{E(\mu_1),E(\mu_2)}(E(\mu_2)).$$

Since $h(\alpha(t)) \le h(\alpha(0)) + t\varphi(h(\alpha(1)), h(\alpha(0)))$. That means

$$\begin{aligned} &h(\gamma_{E(\mu_1),E(\mu_2)}(tE(\mu_1) + (1-t)E(\mu_2))) \\ &\le h(\gamma_{E(\mu_1),E(\mu_2)}(E(\mu_2))) + t\varphi(h(\gamma(E(\mu_1))), h(\gamma(E(\mu_2)))) \end{aligned}$$

It follows that

$$\begin{aligned} &K(tE(\mu_1) + (1-t)E(\mu_2))) \\ &\le K(E(\mu_2)) + t\varphi(K(E(\mu_1)), K(E(\mu_2))) \end{aligned}$$

Hence, K is φ_E-convex on $[0, 1]$. □

Proposition 1. *1. If $h : B \longrightarrow \mathbb{R}$ is a geodesic φ_E-convex function, where φ is non-negative linear, then $xh : B \longrightarrow \mathbb{R}, \ \forall x \geqslant 0$ is also geodesic φ_E-convex.*

2. Let $h_i : B \longrightarrow \mathbb{R}, \ i = 1, 2$ be two geodesic φ_E-convex functions, where φ is additive, then $h_1 + h_2$ is also a geodesic φ_E-convex function.

Theorem 4. *Suppose that $B \subset N$ is a geodesic E-convex set, $h_1 : B \longrightarrow \mathbb{R}$ is a geodesic E-convex function, and $h_2 : U \longrightarrow \mathbb{R}$ is a non-decreasing φ_E-convex function, such that $Rang(h_1) \subseteq U$. Then, $h_2 o h_1$ is also a geodesic φ_E-convex.*

Proof. The above theorem can be proved in the following way

$$\begin{aligned} h_2 \circ h_1(\gamma_{E(\mu_1),E(\mu_2)}) &= h_2\Big(h_1(\gamma_{E(\mu_1),E(\mu_2)})\Big) \\ &\leq h_2(h_1(E(\mu_2)) + t\varphi(h_1(E(\mu_1)), h_1(E(\mu_2)))) \\ &\leq h_2(h_1(E(\mu_2))) + t\varphi(h_2(h_1(E(\mu_1))), h_2(h_1(E(\mu_2)))) \\ &= h_2 o h_1(E(\mu_2)) + t\varphi(h_2 o h_1(E(\mu_1)), h_2 o h_1(E(\mu_2))). \end{aligned}$$

Thus, $h_2 \circ h_1$ is a geodesic φ_E-convex function. □

Theorem 5. *Suppose that, $h_i : B \subset N \longrightarrow \mathbb{R}$, $i = 1, 2, \cdots, n$ are geodesic φ_E-convex functions and φ is non-negatively linear. Then the function $h = \sum_{i=1}^{n} x_i h_i$ is also a geodesic φ_E-convex, for all $x_i \in \mathbb{R}$ and $x_i \geqslant 0$.*

Proof. Considering $\mu_1, \mu_2 \in B$, and since $h_i, i = 1, 2, \cdots, n$ are geodesic φ_E-convex functions, then

$$h_i(\gamma_{E(\mu_1),E(\mu_2)}) \leq h_i(E(\mu_2)) + t\varphi(h_i(E(\mu_1)), h_i(E(\mu_2))).$$

Also,

$$x_i h_i(\gamma_{E(\mu_1),E(\mu_2)}) \leq x_i h_i(E(\mu_2)) + t\varphi(x_i h_i(E(\mu_1)), x_i h_i(E(\mu_2))).$$

Hence,

$$\sum_{i=1}^{n} x_i h_i(\gamma_{E(\mu_1),E(\mu_2)}) \leq \sum_{i=1}^{n} [x_i h_i(E(\mu_2)) + t\varphi(x_i h_i(E(\mu_1)), x_i h_i(E(\mu_2)))],$$

which means that

$$h(\gamma_{E(\mu_1),E(\mu_2)}) \leq h(E(\mu_2)) + t\varphi(h(E(\mu_1)), h(E(\mu_2))).$$

□

Now we consider that N_1 and N_2 are two complete Riemannian manifolds, and $\bigtriangledown$ is the Levi–Civita connection on N_1. If $H : N_1 \longrightarrow N_2$ is a diffeomorphism, then $H * \bigtriangledown = \bigtriangledown^*$ is an affine connection of N_2. Moreover, let γ be a geodesic in $(N_1, \bigtriangledown)$, then $Ho\gamma$ is also a geodesic in $(N_2, \bigtriangledown^*)$, see [15].

Theorem 6. *Suppose that $h : B \longrightarrow \mathbb{R}$ is a geodesic φ_E-convex function and $H : N_1 \longrightarrow N_2$, then a sufficient condition for $hoH^{-1} : H(B) \longrightarrow \mathbb{R}$ to be a geodesic φ_E-convex function, is H must be a diffeomorphism.*

Proof. Assume that $\gamma_{E(\mu_1),E(\mu_2)}$ is a geodesic joining $E(\mu_1)$ and $E(\mu_2)$, where $\mu_1, \mu_2 \in B$. Since H is a diffeomorphism, then $H(B)$ is totally geodesic, and $Ho\gamma_{E(\mu_1),E(\mu_2)}$ is geodesic joining $H(E(\mu_1))$ and $H(E(\mu_2))$. Then

$$\begin{aligned} &(hoH^{-1})\Big(Ho\gamma_{E(\mu_1),E(\mu_2)}(t)\Big) \\ &\quad = h(\gamma_{E(\mu_1),E(\mu_2)}(t)) \\ &\quad \leq h(E(\mu_2)) + t\varphi(h(E(\mu_1)), h(E(\mu_2))) \\ &\quad = (hoH^{-})(H(E(\mu_2))) + t\varphi\big((hoH^{-})(E(\mu_1)), (hoH^{-})(E(\mu_2))\big). \end{aligned}$$

□

Theorem 7. *Assume that $h : B \longrightarrow \mathbb{R}$ is a geodesic φ_E-convex function, and φ bounded from above on $h(B) \times h(B)$, with an upper bound k. Then h is continuous on $Int(B)$.*

Proof. Assume that $E(\mu^*) \in Int(B)$, then there exists an open ball $B(E(\mu^*), r) \subset Int(B)$ for some $r > 0$. Let us choose s, where $(0 < s < r)$, such that the closed ball $\bar{B}(E(\mu^*), s + \varepsilon) \subset B(E(\mu^*), r)$ for some arbitrary small $\varepsilon > 0$. Choose any $E(\mu_1), E(\mu_2) \in \bar{B}(E(\mu^*), s)$. Put $E(\mu_3) = E(\mu_2) + \frac{\varepsilon}{\|\mu_2 - \mu_1\|}(E(\mu_2) - E(\mu_1))$ and $t = \frac{\|\mu_2 - \mu_1\|}{\varepsilon + \|\mu_2 - \mu_1\|}$. Then it is obvious that $E(\mu_3) \in \bar{B}(E(\mu^*), s + \varepsilon)$ and $E(\mu_2) = tE(\mu_3) + (1 - t)E(\mu_1)$. Thus,

$$h(E(\mu_2)) \leq h(E(\mu_1)) + t\varphi(h(E(\mu_3)), h(E(\mu_1))) \leq h(E(\mu_1)) + tk.$$

Then, the above inequality can be written as

$$h(E(\mu_2)) - h(E(\mu_1)) \leq tk \leq \frac{\|\mu_2 - \mu_1\|}{\varepsilon} k = L\|E(\mu_2) - E(\mu_3)\|,$$

where $L = \frac{k}{\varepsilon}$.

Moreover,

$$h(E(\mu_1)) - h(E(\mu_2)) \leq L\|E(\mu_2) - E(\mu_3)\|,$$

Then

$$|h(E(\mu_1)) - h(E(\mu_2))\| \leq L\|E(\mu_2) - E(\mu_3)\|,$$

and since $\bar{B}(E(\mu^*), s)$ is arbitrary, then h is continuous on $Int(B)$. □

Definition 11. *A bifunction $\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}$, is called sequentially upper bounded with respect to E if*

$$\sup_i \varphi(E(u_i) - E(v_i)) \leq \varphi\left(\sup_i E(u_i), \sup_i E(v_i)\right)$$

for any two bounded real sequences $\{E(u_i)\}, \{E(v_i)\}$.

Remark 3. *If E is an identity mapping in Definition 11, then a bifunction $\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is called sequentially upper bounded [32].*

Proposition 2. *Suppose that $B \subset N$ is a geodesic E-convex set, and $\{h_i\}_{i \in \mathbb{N}}$ are a non-empty family of geodesic φ_E-convex functions on B, where φ_E is sequentially upper bounded with respect to E. If $\sup_i h_i(u)$ exist for each $u \in B$, then $h(u) = \sup_i h_i(u)$ are also geodesic φ_E-convex functions.*

Proof. Let $E(u_1), E(u_2) \in B$, then

$$\begin{aligned} h\left(\gamma_{E(u_1),E(u_2)}(t)\right) &= \sup_i h_i(\gamma_{E(u_1),E(u_2)}(t)) \\ &\leq \sup_i h_i(E(u_2)) + t \sup_i \varphi(h_i(E(u_1)), h_i(E(u_2))) \\ &\leq \sup_i h_i(E(u_2)) + t\varphi\left(\sup_i h_i(E(u_1)), \sup_i h_i(E(u_2))\right) \\ &\leq h(E(u_2)) + t\varphi(h(E(u_1)), (E(u_2))). \end{aligned}$$

This implies that h is a geodesic φ_E-convex function. □

Theorem 8. *The function $h : C \longrightarrow \mathbb{R}$ is geodesic φ_E-convex, where C is a geodesic E-convex set. The inequality $\varphi(h(E(\mu), h(E(\mu^*)) \geqslant 0, \forall E(\mu) \in C$ is necessary for h to have a local minimum at $E(\mu^*) \in Int(C)$.*

Proof. Due to the fact that C is a geodesic E-convex set and $E(\mu^*) \in Int(C)$, then $B(E(\mu^*), S) \subset C$ for some $S > 0$. Let $E(\mu) \in C$, then

$$h\Big(\gamma_{E(\mu),E(\mu^*)}(t)\Big) \leq h(E(\mu^*)) + t\varphi(h(E(\mu)), (E(\mu^*))).$$

Since h attains its local minimum at $E(\mu^*)$, then

$$h(E(\mu^*)) \leq h\Big(\gamma_{E(\mu),E(\mu^*)}(\zeta)\Big), \tag{14}$$

where $\zeta \in (0, 1]$ such that $h\Big(\gamma_{E(\mu),E(\mu^*)}(t)\Big) \in B(E(\mu^*), S),$ for all $t \in [0, \zeta]$.

Also,

$$h\Big(\gamma_{E(\mu),E(\mu^*)}(\zeta)\Big) \leq h(E(\mu^*)) + \zeta\varphi(h(E(\mu)), h(E(\mu^*)), \tag{15}$$

then from (14) and (15), we obtain $\varphi(h(E(\mu)), h(E(\mu^*)) \geqslant 0,$ for all $E(\mu) \in C$. □

Theorem 9. *The function $h : B \longrightarrow \mathbb{R}$ is geodesic φ_E-convex, where B is a geodesic E-convex set and φ is bounded from above on $h(B) \times h(B)$, with an upper bound K, with respect to E. Then h is continuous on $Int(B)$.*

Proof. Assume that $E(u) \in Int(B)$ and (U, ψ), is a chart containing $E(u)$. Since ψ is a diffeomorphism, and by using Theorems 6 and 7, we get $ho\psi^{-1} : \psi(U \cap Int(B)) \longrightarrow \mathbb{R}$ as also geodesic φ_E-convex and then it is continuous. Hence, $h = ho\psi^{-1}o\psi : (U \cap Int(B)) \longrightarrow \mathbb{R}$ is continuous.

Also, since $E(u)$ is arbitrary, then h is continuous on $Int(B)$. □

From the definition of geodesic φ_E-convex, we obtain the following proposition.

Proposition 3. *Assume that $\{\varphi^i : i \in \mathbb{N}\}$ is a collection of bifunctions, such that $h : B \longrightarrow \mathbb{R}$ is a geodesic φ^i_E-convex function for each i. If $\varphi^i \longrightarrow \varphi$ as $i \longrightarrow \infty$, then h is also a geodesic φ_E-convex function.*

As a special case in the above proposition, we have the following proposition.

Proposition 4. *Assume that $\{\varphi^i : i \in \mathbb{N}\}$ is a collection of bifunctions, such that $h : B \longrightarrow \mathbb{R}$ is a geodesic φ^*_E-convex function, where $\varphi^*_E = \sum_{l=1}^{i} \varphi^l_E$. If φ^*_E converges to φ_E, then h is also a geodesic φ_E-convex function.*

Theorem 10. *Consider $h : B \longrightarrow \mathbb{R}$ to be strictly geodesic φ_E-convex, where B is a geodesic E-convex set, φ is an antisymmetric function with respect to E $\dot{\gamma}$ and stands for the derivative of γ with respect t. Then*

$$dh_{E(\mu_1)}\dot{\gamma}_{E(\mu_1),E(\mu_2)} \neq dh_{E(\mu_2)}\dot{\gamma}_{E(\mu_1),E(\mu_2)},$$

for all $E(\mu_1), E(\mu_2) \in B$ and $E(\mu_1) \neq E(\mu_2)$.

Proof. Since $\gamma_{E(\mu_2),E(\mu_1)}(t) = \gamma_{E(\mu_1),E(\mu_2)}(1-t),\ \ \forall t \in [0, 1]$, then

$$dh_{E(\mu_2)}\dot{\gamma}_{E(\mu_2),E(\mu_1)} = -dh_{E(\mu_2)}\dot{\gamma}_{E(\mu_1),E(\mu_2)}.$$

By contradiction, let

$$dh_{E(\mu_1)}\dot{\gamma}_{E(\mu_1),E(\mu_2)} = dh_{E(\mu_2)}\dot{\gamma}_{E(\mu_1),E(\mu_2)},$$

but if h is a geodesic φ_E-convex function, then

$$dh_{E(\mu_1)}\dot{\gamma}_{E(\mu_1),E(\mu_2)} < \varphi(h(E(\mu_1)), h(E(\mu_2)). \tag{16}$$

Also,

$$dh_{E(\mu_2)}\dot{\gamma}_{E(\mu_2),E(\mu_1)} < \varphi(h(E(\mu_2)), h(E(\mu_1)).$$

On the other hand,

$$dh_{E(\mu_2)}\dot{\gamma}_{E(\mu_2),E(\mu_1)} = -dh_{E(\mu_2)}\dot{\gamma}_{E(\mu_1),E(\mu_2)},$$

then

$$-dh_{E(\mu_2)}\dot{\gamma}_{E(\mu_1),E(\mu_2)} < \varphi(h(E(\mu_2)), h(E(\mu_1)). \tag{17}$$

Moreover, since φ is an antisymmetry function, then (17) becomes

$$dh_{E(\mu_2)}\dot{\gamma}_{E(\mu_1),E(\mu_2)} > \varphi(h(E(\mu_1)), h(E(\mu_2)),$$

hence,

$$dh_{E(\mu_1)}\dot{\gamma}_{E(\mu_1),E(\mu_2)} > \varphi(h(E(\mu_1)), h(E(\mu_2)). \tag{18}$$

From (16) and (18), we obtain a contradiction, then $dh_{E(\mu_1)}\dot{\gamma}_{E(\mu_1),E(\mu_2)} \neq dh_{E(\mu_2)} \dot{\gamma}_{E(\mu_1),E(\mu_2)}$. □

5. φ_E-Epigraphs

In this section, φ_E-epigraphs are introduced on complete Riemannian manifolds, and a characterization of geodesic φ_E-convex functions in terms of their φ_E-epigraphs is obtained.

Definition 12. *A set $B \subset N \times \mathbb{R}$ is called a geodesic φ_E-convex set if*

$$\Big(\gamma_{E(u_1),E(u_2)}(t), v_2 + t\varphi(v_1, v_2)\Big) \in B,$$

for all $(u_i, v_i) \in B, t \in [0,1]$.

Therefore, a φ_E- epigraph of function h is defined by

$$epi_{\varphi_E}(h) = \{(u,v) \in E(N) \times \mathbb{R} : h(u) \leq v\}.$$

Theorem 11. *Consider $B \subset N$ to be a geodesic E-convex set, and φ is non-decreasing. The set $epi_{\varphi_E}(h)$ is geodesic φ_E-convex, if and only if $h : B \longrightarrow \mathbb{R}$ is a geodesic φ_E-convex function.*

Proof. Let $u_1, u_2 \in B$ and $t \in [0,1]$, and since B is an E-convex set, then $E(u_1), E(u_2) \in E(B) \subseteq B$. Hence,

$$\Big((E(u_1), h(E(u_1))), (E(u_2), h(E(u_2)))\Big) \in epi_{\varphi_E}(h).$$

Due to the fact that $epi_{\varphi_E}(h)$ is a geodesic φ_E-convex set, then

$$\Big(\gamma_{E(u_1),E(u_2)}(t), h(E(u_2)) + t\varphi(h(E(u_1)), h(E(u_2)))\Big) \in epi_{\varphi_E}(h).$$

This implies that $h\Big(\gamma_{E(u_1),E(u_2)}(t)\Big) \leq h(E(u_2)) + t\varphi(h(E(u_1)), h(E(u_2)))$. Consequently, h is a geodesic φ_E-convex function.

Now, let us consider that $(u_1^*, v_1), (u_2^*, v_2) \in epi_{\varphi_E}(h)$, then $u_1^*, u_1^* \in E(B)$, which means that there are $u_1, u_2 \in B$ such that $E(u_1) = u_1^*$ and $E(u_2) = u_2^*$. Hence, $h(E(u_1)) \leq v_1, h(E(u_2)) \leq v_2$ and, since h is a geodesic φ_E-convex function, then

$$\begin{aligned} h\left(\gamma_{E(u_1),E(u_2)}(t)\right) &\leq h(E(u_2)) + t\varphi(h(E(u_1)), h(E(u_2))) \\ &\leq v_2 + t\varphi(v_1, v_2), \end{aligned}$$

which implies that $\left(\gamma_{E(u_1),E(u_2)}(t), v_2 + t\varphi(v_1, v_2)\right) \in epi_{\varphi_E}(h)$, for all $t \in [0,1]$.

That is, $epi_{\varphi_E}(h)$ is a geodesic φ_E-convex set. □

Theorem 12. *Consider $\{B_i, i \in I\}$ to be a family of geodesic φ_E-convex sets, then $B = \cap_{i\in I} B_i$ is also a geodesic φ_E-convex set.*

Proof. Let $(\mu_1, \nu_1), (\mu_2, \nu_2) \in \cap_{i\in I} B_i$, then $(\mu_1, \nu_1), (\mu_2, \nu_2) \in B_i$, for all $i \in I$. Hence,

$$\left(\gamma_{E(\mu_1),E(\mu_2)}(t), \nu_2 + t\varphi(\nu_1, \nu_2)\right) \in B_i$$

Then,

$$\left(\gamma_{E(\mu_1),E(\mu_2)}(t), \nu_2 + t\varphi(\nu_1, \nu_2)\right) \in \cap_{i\in I} B_i$$

for all $t \in [0,1]$.

This implies $\cap_{i\in I} B_i$ is a geodesic φ_E-convex function. □

By using the above theorem, we can obtain the following corollary

Corollary 1. *Let $\{h_i, i \in I\}$ be a family of geodesic φ_E-convex functions defined on a geodesic E-convex set $B \subset N$, which is bounded above, and φ is non-decreasing. If the E-epigraphs $epi_{\varphi_E}(h_i)$ are geodesic φ_E-convex sets, then $h = \sup_{i\in I} h_i$ is also a geodesic φ_E-convex function on B.*

6. Conclusions

Some important properties of geodesic φ_E-convex functions are established in this study. A new class of function on Riemannian manifold—together with its properties—is also studied here. We also reported how the characterization of the geodesic φ_E-convex function can be obtained through their φ_E-epigraph counterparts. The results presented in this paper can be used for future research on the Riemannian manifold. The ideas and techniques of this paper may motivate further research, for example, in fractional manifolds.

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References

1. Bolty nski, V.; Martini, H.; Soltan, P.S. *Excursions into Combinatorial Geometry*; Springer: Berlin/Heildeberg, Germany, 1997.
2. Danzer, L.; Grunbaum, B.; Klee, V. Helly's theorem and its relatives. *Proc. Sympos Pure Math.* **1963**, *7*, 101–180.
3. Kılıçman, A.; Saleh, W. Notions of generalized s-convex functions on fractal sets. *J. Inequal. Appl.* **2015**, *1*, 312. [CrossRef]
4. Kılıçman, A.; Saleh, W. Some properties of geodesic semi Eb-vex functions. *Open Math.* **2015**, *13*, 795–804. [CrossRef]

5. Kolwankar, K.M.; Gangal, A.D. Local fractional calculus: A calculus for fractal space-time. In *Fractals: Theory and Applications in Engineering*; Springer: London, UK, 1999; pp. 171–181.
6. Özdemir, M.E.; Yıldız, Ç.; Akdemir, A.O.; Set, E. On some inequalities for s-convex functions and applications. *J. Inequal. Appl.* **2013**, *1*, 1–11. [CrossRef]
7. Zhao, Y.; Cheng, D.F.; Yang, X.J. Approximation solutions for local fractional Schrödinger equation in the one-dimensional Cantorian system. *Adv. Math. Physics* **2013**, *2013*, 5. [CrossRef]
8. Almutairi, O.; Kılıçman, A. Generalized Fejèr- Hermite-Hadamard type via generalized (h- m)-convexity on fractal sets and applications. *Chaos Solitons Fractals* **2021**, *147*, 110938. [CrossRef]
9. Grinalatt, M.; Linnainmaa, J.T. Jensen's inquality, parameter uncertainty, and multiperiod investment. *Rev. Asset Pricing Stud.* **2011**, *1*, 1–34. [CrossRef]
10. Kılıçman, A.; Saleh, W. Some Inequalities For Generalized s-Convex Functions. *JP J. Geom. Topol.* **2015**, *17*, 63–82.
11. Ruel, J.J.; Ayres, M.P. Jensen's inequality predicts effects of environmental variation. *Trends Ecol. Evol.* **1999**, *14*, 361–366. [CrossRef]
12. Syau, Y.R.; Jia, L.; Lee, E.S. Generalizations of E-convex and B-vex functions. *Comput. Math. Appl.* **2009**, *58*, 711–716. [CrossRef]
13. Awan, M.U.; Noor, M.A.; Noor, K.I.; Safdar, F. On strongly generalized convex functions. *Filomat* **2017**, *47*, 5783–5790. [CrossRef]
14. Greene, R.E.; Shiohama, G. Convex functions on complete noncompact manifolds:Topological structure. *Invent. Math* **1981**, *63*, 129–157. [CrossRef]
15. Udrişte, C. *Convex Functions and Optimization Methods on Riemannian Manifolds*; Kluwer Academic Publishers: Amsterdam, The Netherlands, 1994.
16. Thielhelm, H.; Vais, A.; Wolter, F.E. Geodesic bifurcation on smooth surfaces. *Vis. Comput.* **2015**, *31*, 187–204. [CrossRef]
17. Rýparová, L.; Mikeš, J.; Sabykanov, A. On geodesic bifurcations of product spaces. *J. Math. Sci.* **2019**, *239*, 86–91. [CrossRef]
18. Youness, E.A. E-convex set, E-convex functions and E-convex programming. *J. Optim. Theory Appl.* **1999**, *102*, 439–450. [CrossRef]
19. Abou-Tair, I.; Sulaiman, W.T. Inequalities via convex functions. *J. Math. Math. Sci.* **1999**, *22*, 543–546. [CrossRef]
20. Noor, M.A. Fuzzy preinvex functions. *Fuzzy Sets Syst.* **1994**, *64*, 95–104. [CrossRef]
21. Yang, X.M. On E-convex set, E-convex functions and E-convex programming. *J. Optim. Theory Appl.* **2001**, *109*, 699–703. [CrossRef]
22. Duca, D.I.; Lupșa, L. On the E-epigraph of an E-convex function. *J. Optim. Theory Appl.* **2006**, *129*, 341–348. [CrossRef]
23. Chen, X. Some properties of semi-E-convex functions. *J. Math. Anal. Appl.* **2002**, *275*, 251–262. [CrossRef]
24. Fulga, C.; Preda, V. Nonlinear programming with E-preinvex and local E-preinvex functions. *Eur. J. Oper. Res.* **2009**, *192*, 737–743. [CrossRef]
25. Iqbal, A.; Ahmad, I.; Ali, S. Some properties of geodesic semi-E-convex functions. *Nonlinear Anal. Theory Method Appl.* **2011**, *74*, 6805–6813. [CrossRef]
26. Iqbal, A.; Ahmad, I.; Ali, S. On geodesic E-convex sets, geodesic E-convex functions and E-epigraphs. *J. Optim. Theory Appl.* **2012**, *55*, 239–251. [CrossRef]
27. Syau, Y.R.; Lee, E.S. Some properties of E-convex functions. *Appl. Math. Lett.* **2005**, *18*, 1074–1080. [CrossRef]
28. Rapcsák, T. Geodesic convexity in nonlinear optimization. *J. Optim. Theory Appl.* **1991**, *69*, 169–183. [CrossRef]
29. Hicks, N.J. *Notes on Differential Geometry*; Van Nostrand Publishing Company: Princeton, NJ, USA, 1965; Volume 1.
30. Kılıçman, A.; Saleh, W. On Geodesic Strongly E-convex Sets and Geodesic Strongly E-convex Functions. *J. Inequal. Appl.* **2015**, *1*, 1–10. [CrossRef]
31. Eshaghi Gordji, M.; Rostamian Delavar, M.; de La Sen, M. On φ-convex functions. *J. Math. Inequal* **2016**, *10*, 173–183. [CrossRef]
32. Shaikh, A.A.; Iqbal, A.; Mondal, C.K. Some results on φ–convex functions and geodesic φ-convex functions. *Differ. Geom. Dyn. Syst.* **2018**, *20*, 159–169.
33. Nicolaescu, L.I. *Lectures on the Geometry of Manifolds*; World Scientific: Singapore, 2020.