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Higher-Order Nabla Difference Equations of Arbitrary Order with Forcing, Positive and Negative Terms: Non-Oscillatory Solutions

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Abstract: This work provides new adequate conditions for difference equations with forcing, positive and negative terms to have non-oscillatory solutions. A few mathematical inequalities and the properties of discrete fractional calculus serve as the fundamental foundation to our approach. To help establish the main results, an analogous representation for the main equation, called a Volterra-type summation equation, is constructed. Two numerical examples are provided to demonstrate the validity of the theoretical findings; no earlier publications have been able to comment on their solutions' non-oscillatory behavior.

Keywords: non-oscillatory solutions; asymptotic behavior; caputo nabla fractional difference; nabla fractional difference equations

MSC: 34K11; 34N05

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1. Introduction

Fractional order differential equations (FDEs) are generalized, non-integer order differential equations that can be obtained in time and space with a power law memory kernel of the nonlocal relationships; they offer an effective means of describing the memory of various substances and the characteristics of inheritance. The authors, who have shown a great deal of interest in studying the qualitative characteristics of the solution of FDEs, such as existence, uniqueness, oscillation, stability, and control, have provided details of significant findings in this area; see some of the illustrious monographs [1–3] and recent papers [4–10]. In particular, the oscillation of solutions was a subject that was taken into account for FDEs; the review paper in [11] is available to readers.

In recent years, academics have started to pay significant attention to discrete fractional calculus. The arbitrary order difference and summation features have considerably demonstrated their utility and validity due to their long memory nature and their flexible capability in carrying out mathematical computations [12]. As a result, numerous studies that investigate the qualitative traits of fractional difference equation solutions have been published, including their oscillation properties [13–16].

Let $\mathbb{N}_\chi = \{\chi, \chi+1, \chi+2, \ldots\}$ for any $\chi \in \mathbb{R}$. Research on the oscillation of solutions of nabla fractional difference equations was started by Alzabut et al. [15] with the following problems:

$$\begin{cases} \nabla^{\eta}_{\sigma+\varkappa-2}\varphi(\varpi) + \xi_1(\varpi,\varphi(\varpi)) = \zeta(\varpi) + \xi_2(\varpi,\varphi(\varpi)), \quad \varpi \in \mathbb{N}_{\eta+\sigma-1}, \\ \nabla^{-(1-\eta)}_{\sigma+\varkappa-2}\varphi(\varpi)\Big|_{\varpi=\sigma+\varkappa-1} = \chi, \quad \chi \in \mathbb{R}, \end{cases} \tag{1}$$

and

$$\begin{cases} \nabla^{\eta}_{\sigma+\varkappa-1*}\varphi(\varpi) + \xi_1(\varpi,\varphi(\varpi)) = \zeta(\varpi) + \xi_2(\varpi,\varphi(\varpi)), \quad \varpi \in \mathbb{N}_{\sigma+\varkappa-1}, \\ \nabla^m y(\sigma+\varkappa-1) = \chi_m, \quad \chi_m \in \mathbb{R}, \quad m = 0,1,2,\cdots,\varkappa-1, \end{cases} \tag{2}$$

where $\eta > 0$ and $\varkappa \in \mathbb{N}_1$ such that $\varkappa - 1 < \eta < \varkappa$; ξ_1, $\xi_2 : \mathbb{N}_{\sigma+\varkappa-1} \times \mathbb{R} \to \mathbb{R}$ and $\zeta : \mathbb{N}_{\sigma+\varkappa-1} \to \mathbb{R}$.

Then, Abdalla et al. [13,14] continued to study the oscillation of solutions of different types of mixed nonlinear nabla fractional difference equations:

$$\begin{cases} \nabla^{\eta}_{\sigma+\varkappa-2}\varphi(\varpi) - b(\varpi)\varphi(\varpi) + \sum_{j=1}^{k} b_j(\varpi)|\varphi(\varpi)|^{\alpha_j-1} = \zeta(\varpi), \quad \varpi \in \mathbb{N}_{\sigma+\varkappa}, \\ \nabla^{-(\varkappa-\eta)}_{\sigma+\varkappa-2}\varphi(\varpi)\Big|_{\varpi=\sigma+\varkappa-1} = \chi, \quad \chi \in \mathbb{R}, \end{cases} \tag{3}$$

and

$$\begin{cases} \nabla^{\eta}_{\sigma+\varkappa-1*}\varphi(\varpi) - b(\varpi)\varphi(\varpi) + \sum_{j=1}^{k} b_j(\varpi)|\varphi(\varpi)|^{\alpha_j-1} = \zeta(\varpi), \quad \varpi \in \mathbb{N}_{\sigma+\varkappa-1}, \\ \nabla^m \varphi(\sigma+\varkappa-1) = \chi_m, \quad \chi_m \in \mathbb{R}, \quad m = 0,1,2,\cdots,\varkappa-1, \end{cases} \tag{4}$$

where b, $b_j : \mathbb{N}_{\sigma+\varkappa-1} \to \mathbb{R}$, $j = 1,2,\cdots k$; α_1, α_2, $\cdots$, and α_k are the ratios of odd natural numbers with $\alpha_1 > \cdots > \alpha_i > 1 > \alpha_{i+1} > \cdots > \alpha_k$.

In this vein, Alzabut et al. [16] derived the conditions for the oscillation of solutions of a forced and damped nabla fractional difference equation:

$$\begin{cases} (1-p(\varpi))\nabla\nabla^{\eta}_{0}\varphi(\varpi) + p(\varpi)\nabla^{\eta}_{0}\varphi(\varpi) + p_2(\varpi)\xi(\varphi(\varpi)) = p_1(\varpi), \quad \varpi \in \mathbb{N}_1, \\ \nabla_0^{-(1-\eta)}\varphi(\varpi)\Big|_{\varpi=1} = \chi, \quad \chi \in \mathbb{R}, \end{cases} \tag{5}$$

where $0 < \mu < 1$; $\xi : \mathbb{R} \to \mathbb{R}$; p, $p_1 : \mathbb{N}_1 \to \mathbb{R}$ and $p_2 : \mathbb{N}_1 \to \mathbb{R}^+$.

Motivated by the above studies, which concentrated on oscillation discussion, and for the sake of giving an affirmative response about the behavior of non-oscillatory solutions, in this work, we consider the higher-order forced nabla fractional difference equation with positive and negative terms of the following form:

$$\nabla^{x}_{c*}z(\varpi) + \phi(\varpi, y(\varpi)) = \eta(\varpi) + \zeta(\varpi)y^{\beta}(\varpi) + \Phi(\varpi, y(\varpi)), \quad \varpi \in \mathbb{N}_{c+1}, \tag{6}$$

where

$$z(\varpi) = \nabla^{n-1}\Big[d(\varpi)(\nabla y(\varpi))^{\beta}\Big], \quad \varpi \in \mathbb{N}_c, \quad n \in \mathbb{N}_1, \tag{7}$$

where $0 < x < 1$, β is the ratio of two odd natural numbers, $c \in \mathbb{N}_1$, and $\nabla^{x}_{c*}z$ denotes the xth Caputo nabla fractional difference of z. Throughout this work, we need the following conditions in the sequel.

(i) ζ, $d : \mathbb{N}_c \to (0,\infty)$, $\eta : \mathbb{N}_c \to \mathbb{R}$ and Φ, $\phi : \mathbb{N}_c \times \mathbb{R} \to \mathbb{R}$ are real valued continuous functions;

(ii) There exist two continuous functions Θ_1 and $\Theta_2 : \mathbb{N}_c \to (0,\infty)$, and positive real numbers λ and γ, where $\lambda > \gamma$ such that

$$y\phi(\varpi, y) \geq \Theta_1(\varpi)|y|^{\lambda+1}, \quad 0 \leq y\Phi(\varpi, y) \leq \Theta_2(\varpi)|y|^{\gamma+1}$$

for $y \neq 0$ and $\varpi \in \mathbb{N}_c$.

Unlike most existing results, which often discuss the oscillation of solutions, the asymptotic behavior of the Equation (6)'s non-oscillatory solutions is examined in this study. Our method is essentially based on some mathematical inequalities and the properties of discrete fractional calculus. A Volterra-type summation equation is built as an analogous representation for Equation (6) to aid in establishing the key conclusions. In order to demonstrate the validity of the theoretical findings, we offer numerical examples.

2. Essential Preliminaries

The results in this section are adopted from the two main monographs [12,17].

Definition 1 (See [12]). *For $\varpi \in \mathbb{R} \setminus \{\ldots, -2, -1, 0\}$ and $\theta \in \mathbb{R}$ such that $(\varpi + \theta) \in \mathbb{R} \setminus \{\ldots, -2, -1, 0\}$, we define the generalized rising function by*

$$\varpi^{\overline{\theta}} = \frac{\Gamma(\varpi + \theta)}{\Gamma(\varpi)}.$$

Furthermore, if $\varpi \in \{\ldots, -2, -1, 0\}$ and $\theta \in \mathbb{R}$ such that $(\varpi + \theta) \in \mathbb{R} \setminus \{\ldots, -2, -1, 0\}$, then $\varpi^{\overline{\theta}} = 0$.

Definition 2 (See [12]). *Let κ be a real valued function defined on $\mathbb{N}_\chi$. The first nabla difference of κ is given by*

$$\nabla \kappa(\varpi) = \kappa(\varpi) - \kappa(\varpi - 1), \quad \varpi \in \mathbb{N}_{\chi+1}.$$

Definition 3 (See [12]). *Let κ be a real valued function defined on $\mathbb{N}_{\chi+1}$ and $x > 0$. The xth nabla fractional sum of κ based at χ is given by*

$$\nabla_\chi^{-x} \kappa(\varpi) = \frac{1}{\Gamma(x)} \sum_{\varpi_1 = \chi+1}^{\varpi} (\varpi - \varpi_1 + 1)^{\overline{x-1}} \kappa(\varpi_1), \quad \varpi \in \mathbb{N}_\chi,$$

where, by convention, $\nabla_\chi^{-x} \kappa(\chi) = 0$.

Definition 4 (See [3]). *Let $0 < x < 1$ and κ be a real valued function defined on $\mathbb{N}_\chi$. The xth Caputo nabla fractional difference of κ based at χ is given by*

$$\nabla_{\chi *}^{x} \kappa(\varpi) = \nabla_\chi^{-(1-x)} \nabla \kappa(\varpi), \quad \varpi \in \mathbb{N}_{\chi+1}.$$

Theorem 1. *The initial value problem*

$$\begin{cases} \nabla_{a*}^{x} \kappa(\varpi) = \omega(\varpi), & \varpi \in \mathbb{N}_{a+1}, \\ \kappa(a) = \kappa_0, \end{cases} \tag{8}$$

has the unique solution

$$\kappa(\varpi) = \kappa_0 + \frac{1}{\Gamma(x)} \sum_{\varpi_1 = a+1}^{\varpi} (\varpi - \varpi_1 + 1)^{\overline{x-1}} \omega(\varpi_1), \quad \varpi \in \mathbb{N}_a \tag{9}$$

where $0 < x < 1$ and $\omega : \mathbb{N}_{a+1} \to \mathbb{R}$.

Lemma 1. *The following properties hold well.*

1. *If $r_3 < \varpi \leq \varpi_1$, then $\varpi_1^{\overline{-r_3}} \leq \varpi^{\overline{-r_3}}$;*
2. *$\varpi^{\overline{r_1}} (\varpi + r_1)^{\overline{r_2}} = \varpi^{\overline{r_1 + r_2}}$;*

3. *If* $0 < r_3 < 1$ *and* $\vartheta > 1$, *then*

$$\left[\varpi^{\overline{-r_3}}\right]^{\vartheta} \leq \frac{\Gamma(1+r_3\vartheta)}{[\Gamma(1+r_3)]^{\vartheta}}\varpi^{\overline{-r_3\vartheta}}, \quad \varpi > r_3\vartheta.$$

Lemma 2. *Under the assumption that* b, x *and* p *are positive constants with* $b > 1$ *and* $p(x-1)+1 > 0$, *we obtain*

$$\sum_{\varpi_1=1}^{\varpi}(\varpi-\varpi_1+1)^{\overline{p(x-1)}}b^{p\varpi_1} \leq Qb^{pt}, \quad \varpi \in \mathbb{N}_1,$$

where

$$Q = \left(\frac{b^p}{b^p-1}\right)^{p(x-1)+1}\Gamma(p(x-1)+1).$$

Lemma 3. *If* R *and* S *are nonnegative,* $\frac{1}{\gamma}+\frac{1}{v}=1$, *and* $\gamma > 1$, *then*

$$RS \leq \frac{1}{\gamma}P^{\gamma} + \frac{1}{v}S^{v}, \tag{10}$$

where equality holds if and only if $S = R^{\gamma-1}$.

We denote

$$m(\varpi) = \left[\frac{\Theta_2^{\lambda}(\varpi)}{\Theta_1^{\gamma}(\varpi)}\right]^{\left(\frac{1}{\lambda-\gamma}\right)}, \tag{11}$$

and

$$A(\varpi,c) = \sum_{\varpi_1=c+1}^{\varpi} d^{-\frac{1}{\beta}}(\varpi_1). \tag{12}$$

3. Main Results

In this section, we provide sufficient conditions for which any non-oscillatory solution of (6) satisfies

$$|y(\varpi)| = O\left(\left[\varpi^{\overline{n-1}}\right]^{\frac{1}{\beta}} b^{\frac{\varpi}{\beta}} A(\varpi,c)\right) \text{ as } \varpi \to \infty.$$

Theorem 2. *Under the assumptions that* (i)–(ii), $0 < x < 1$, $p(x-1)+1 > 0$ *for* $p > 1$ *and*

$$\sum_{\varpi_1=c+1}^{\infty}\zeta^{q}(\varpi_1)\left[\varpi_1^{\overline{n-1}}\right]^{q}A^{\beta q}(\varpi_1,c) < \infty, \quad q = \frac{p}{p-1}, \tag{13}$$

$$\lim_{\varpi\to\infty}\left[\frac{1}{\Gamma(x)}\sum_{\varpi_1=c+1}^{\varpi}(\varpi-\varpi_1+1)^{\overline{x-1}}|\eta(\varpi_1)|\right] < \infty, \tag{14}$$

$$\lim_{\varpi\to\infty}\left[\frac{1}{\Gamma(x)}\sum_{\varpi_1=c+1}^{\varpi}(\varpi-\varpi_1+1)^{\overline{x-1}}m(\varpi_1)\right] < \infty, \tag{15}$$

every non-oscillatory solution of (6) *satisfies*

$$\limsup_{\varpi\to\infty}\frac{|y(\varpi)|}{\left[\varpi^{\overline{n-1}}\right]^{\frac{1}{\beta}} b^{\frac{\varpi}{\beta}} A(\varpi,c)} < \infty. \tag{16}$$

Proof. Let y be a non-oscillatory solution of (6), say $y(\varpi) > 0$ for $\varpi \in \mathbb{N}_{\varpi_1}$ for some $\varpi_1 \in \mathbb{N}_{c+1}$. Take $z(c) = c_0$. Letting $F(\varpi) = \Phi(\varpi, y(\varpi)) - \phi(\varpi, y(\varpi))$, it follows from (6) and (i)–(ii) that, for $\varpi \in \mathbb{N}_c$,

$$\begin{aligned}
&\nabla^{n-1}\left[d(\varpi)(\nabla y(\varpi))^{\beta}\right] \\
&= c_0 + \frac{1}{\Gamma(x)} \sum_{\varpi_1=c+1}^{\varpi} (\varpi - \varpi_1 + 1)^{\overline{x-1}} \left[\eta(\varpi_1) + \zeta(\varpi_1) y^{\beta}(\varpi_1) + F(\varpi_1)\right] \\
&\leq |c_0| + \frac{1}{\Gamma(x)} \sum_{\varpi_1=c+1}^{\varpi_1} (\varpi - \varpi_1 + 1)^{\overline{x-1}} |F(\varpi_1)| + \frac{1}{\Gamma(x)} \sum_{\varpi_1=c+1}^{\varpi} (\varpi - \varpi_1 + 1)^{\overline{x-1}} |\eta(\varpi_1)| \\
&\quad + \frac{1}{\Gamma(x)} \sum_{\varpi_1=\varpi_1+1}^{\varpi} (\varpi - \varpi_1 + 1)^{\overline{x-1}} \left[\Theta_2(\varpi_1) y^{\gamma}(\varpi_1) - \Theta_1(\varpi_1) y^{\lambda}(\varpi_1)\right] \\
&\quad + \frac{1}{\Gamma(x)} \sum_{\varpi_1=c+1}^{\varpi_1} (\varpi - \varpi_1 + 1)^{\overline{x-1}} \zeta(\varpi_1) \left| y^{\beta}(\varpi_1) \right| \\
&\quad + \frac{1}{\Gamma(x)} \sum_{\varpi_1=\varpi_1+1}^{\varpi} (\varpi - \varpi_1 + 1)^{\overline{x-1}} \zeta(\varpi_1) y^{\beta}(\varpi_1).
\end{aligned} \tag{17}$$

Applying Lemma 3 to $\left[\Theta_2(\varpi) y^{\gamma}(\varpi) - \Theta_1(\varpi) y^{\lambda}(\varpi)\right]$ with

$$\delta = \frac{\lambda}{\gamma} > 1, \quad X = y^{\gamma}(\varpi), \quad Y = \frac{\gamma}{\lambda} \frac{\Theta_2(\varpi)}{\Theta_1(\varpi)}, \quad \eta = \frac{\lambda}{\lambda - \gamma},$$

we obtain

$$\begin{aligned}
\Theta_2(\varpi) y^{\gamma}(\varpi) - \Theta_1(\varpi) y^{\lambda}(\varpi) &= \frac{\lambda}{\gamma} \Theta_1(\varpi) \left[y^{\gamma}(\varpi) \frac{\gamma}{\lambda} \frac{\Theta_2(\varpi)}{\Theta_1(\varpi)} - \frac{\gamma}{\lambda} (y^{\gamma}(\varpi))^{\frac{\lambda}{\gamma}} \right] \\
&= \frac{\lambda}{\gamma} \Theta_1(\varpi) \left[XY - \frac{1}{\delta} X^{\delta} \right] \\
&\leq \frac{\lambda}{\gamma} \Theta_1(\varpi) \left[\frac{1}{\eta} Y^{\eta} \right] \\
&= \left(\frac{\lambda - \gamma}{\gamma} \right) \Theta_1(\varpi) \left[\frac{\gamma}{\lambda} \frac{\Theta_2(\varpi)}{\Theta_1(\varpi)} \right]^{\frac{\lambda}{\lambda - \gamma}} \\
&= (\lambda - \gamma) \left[\frac{\gamma^{\gamma}}{\lambda^{\lambda}} \right]^{\left(\frac{1}{\lambda - \gamma}\right)} m(\varpi).
\end{aligned} \tag{18}$$

Substituting (18) into (17) and applying Lemma 1, for $\varpi \in \mathbb{N}_c$, we obtain

$$\begin{aligned}
&\nabla^{n-1}\left[d(\varpi)(\nabla y(\varpi))^{\beta}\right] \\
&\leq |c_0| + \frac{1}{\Gamma(x)} \sum_{\varpi_1=c+1}^{\varpi_1} (\varpi_1 - \varpi_1 + 1)^{\overline{x-1}} |F(\varpi_1)| + \frac{1}{\Gamma(x)} \sum_{\varpi_1=c+1}^{\varpi} (\varpi - \varpi_1 + 1)^{\overline{x-1}} |\eta(\varpi_1)| \\
&\quad + \frac{1}{\Gamma(x)} (\lambda - \gamma) \left[\frac{\gamma^{\gamma}}{\lambda^{\lambda}} \right]^{\left(\frac{1}{\lambda - \gamma}\right)} \sum_{\varpi_1=\varpi_1+1}^{\varpi} (\varpi - \varpi_1 + 1)^{\overline{x-1}} m(\varpi_1) \\
&\quad + \frac{1}{\Gamma(x)} \sum_{\varpi_1=c+1}^{\varpi_1} (\varpi_1 - \varpi_1 + 1)^{\overline{x-1}} \zeta(\varpi_1) \left| y^{\beta}(\varpi_1) \right| \\
&\quad + \frac{1}{\Gamma(x)} \sum_{\varpi_1=c+1}^{\varpi} (\varpi - \varpi_1 + 1)^{\overline{x-1}} \zeta(\varpi_1) y^{\beta}(\varpi_1).
\end{aligned} \tag{19}$$

In view of (14) and (15), we see from (19) that, for $\varpi \in \mathbb{N}_c$,

$$\nabla^{n-1}\Big[d(\varpi)(\nabla y(\varpi))^{\beta}\Big] \leq C_{n-1} + \frac{1}{\Gamma(x)} \sum_{\varpi_1=\varpi_1+1}^{\varpi} (\varpi - \varpi_1 + 1)^{\overline{x-1}} \zeta(\varpi_1) y^{\beta}(\varpi_1), \tag{20}$$

where $C_{n-1} > 0$ is defined by

$$\begin{aligned}
C_{n-1} = {} & |c_0| + \frac{1}{\Gamma(x)} \sum_{\varpi_1=c+1}^{\varpi_1} (\varpi_1 - \varpi_1 + 1)^{\overline{x-1}} |F(\varpi_1)| + \frac{1}{\Gamma(x)} \sum_{\varpi_1=c+1}^{\varpi} (\varpi - \varpi_1 + 1)^{\overline{x-1}} |\eta(\varpi_1)| \\
& + \frac{1}{\Gamma(x)} (\lambda - \gamma) \left[\frac{\gamma^{\gamma}}{\lambda^{\lambda}}\right]^{\left(\frac{1}{\lambda-\gamma}\right)} \sum_{\varpi_1=\varpi_1+1}^{\varpi} (\varpi - \varpi_1 + 1)^{\overline{x-1}} m(\varpi_1) \\
& + \frac{1}{\Gamma(x)} \sum_{\varpi_1=c+1}^{\varpi_1} (\varpi - \varpi_1 + 1)^{\overline{x-1}} \zeta(\varpi_1) \Big| y^{\beta}(\varpi_1) \Big|.
\end{aligned}$$

By the integer order variation of constants formula, it follows from (20) that

$$\begin{aligned}
& d(\varpi)(\nabla y(\varpi))^{\beta} \\
& \leq \sum_{k=0}^{n-2} \Big(\nabla^{k}\Big[d(\varpi)(\nabla y(\varpi))^{\beta}\Big]\Big)_{\varpi=\varpi_1-1} \frac{(\varpi - \varpi_1 + 1)^{\overline{k}}}{\Gamma(k+1)} \\
& \quad + \sum_{r=\varpi_1}^{\varpi} \frac{(\varpi - r + 1)^{\overline{n-2}}}{\Gamma(n-1)} \left[C_{n-1} + \frac{1}{\Gamma(x)} \sum_{\varpi_1=\varpi_1+1}^{r} (r - \varpi_1 + 1)^{\overline{x-1}} \zeta(\varpi_1) y^{\beta}(\varpi_1)\right] \\
& \leq \sum_{k=0}^{n-2} \left| \Big(\nabla^{k}\Big[d(\varpi)(\nabla y(\varpi))^{\beta}\Big]\Big)_{\varpi=\varpi_1-1} \right| \frac{(\varpi - \varpi_1 + 1)^{\overline{k}}}{\Gamma(k+1)} \\
& \quad + C_{n-1} \sum_{r=\varpi_1}^{\varpi} \frac{(\varpi - r + 1)^{\overline{n-2}}}{\Gamma(n-1)} \\
& \quad + \sum_{r=\varpi_1+1}^{\varpi} \frac{(\varpi - r + 1)^{\overline{n-2}}}{\Gamma(n-1)} \left[\frac{1}{\Gamma(x)} \sum_{\varpi_1=\varpi_1+1}^{r} (r - \varpi_1 + 1)^{\overline{x-1}} \zeta(\varpi_1) y^{\beta}(\varpi_1)\right] \\
& = \sum_{k=0}^{n-2} \left| \Big(\nabla^{k}\Big[d(\varpi)(\nabla y(\varpi))^{\beta}\Big]\Big)_{\varpi=\varpi_1-1} \right| \frac{(\varpi - \varpi_1 + 1)^{\overline{k}}}{\Gamma(k+1)} \\
& \quad + C_{n-1} \frac{(\varpi - \varpi_1 + 1)^{\overline{n-1}}}{\Gamma(n)} \\
& \quad + \sum_{\varpi_1=\varpi_1+1}^{\varpi} \left[\sum_{r=\varpi_1}^{\varpi} \frac{(\varpi - r + 1)^{\overline{n-2}}}{\Gamma(n-1)} \frac{(r - \varpi_1 + 1)^{\overline{x-1}}}{\Gamma(x)}\right] \zeta(\varpi_1) y^{\beta}(\varpi_1) \\
& = \sum_{k=0}^{n-1} C_k \frac{(\varpi - \varpi_1 + 1)^{\overline{k}}}{\Gamma(k+1)} + \sum_{\varpi_1=\varpi_1+1}^{\varpi} \frac{(\varpi - \varpi_1 + 1)^{\overline{x+n-2}}}{\Gamma(x+n-1)} \zeta(\varpi_1) y^{\beta}(\varpi_1).
\end{aligned} \tag{21}$$

where

$$C_k = \left| \Big(\nabla^{k}\Big[d(\varpi)(\nabla y(\varpi))^{\beta}\Big]\Big)_{\varpi=\varpi_1-1} \right| > 0, \quad k = 0, 1, 2, \cdots, n-2.$$

Note that (21) holds for $n = 1$. Hence, (21) holds for all $n \in \mathbb{N}_1$ and for all $\varpi \in \mathbb{N}_{\varpi_1}$. Next, we proceed to estimate (21) as

$$d(\varpi)(\nabla y(\varpi))^{\beta} \leq \sum_{k=0}^{n-1} C_k \frac{\varpi^{\overline{k}}}{\Gamma(k+1)} + \sum_{\varpi_1=\varpi_1+1}^{\varpi} \frac{(\varpi-\varpi_1)^{\overline{n-1}}(\varpi-\varpi_1+n)^{\overline{x-1}}}{\Gamma(x+n-1)} \zeta(\varpi_1) y^{\beta}(\varpi_1)$$
$$\leq \varpi^{\overline{n-1}}\left[\sum_{k=0}^{n-1}\frac{C_k}{k!} + \frac{1}{\Gamma(x+n-1)}\sum_{\varpi_1=\varpi_1+1}^{\varpi}(\varpi-\varpi_1+1)^{\overline{x-1}}\zeta(\varpi_1)y^{\beta}(\varpi_1)\right],$$

implying that

$$d(\varpi)(\nabla y(\varpi))^{\beta} \leq \varpi^{\overline{n-1}}\left[\Theta_1 + \Theta_2 \sum_{\varpi_1=\varpi_1+1}^{\varpi}(\varpi-\varpi_1+1)^{\overline{x-1}}\zeta(\varpi_1)y^{\beta}(\varpi_1)\right], \tag{22}$$

where

$$\Theta_1 = \sum_{k=0}^{n-1}\frac{C_k}{k!} > 0, \quad \Theta_2 = \frac{1}{\Gamma(x+n-1)} > 0.$$

Applying Lemmas 1 and 2, and Holder's inequality to the sum on the far right in (22), we have

$$\begin{aligned}
&\sum_{\varpi_1=\varpi_1+1}^{\varpi}(\varpi-\varpi_1+1)^{\overline{x-1}}\zeta(\varpi_1)y^{\beta}(\varpi_1) \\
&= \sum_{\varpi_1=\varpi_1+1}^{\varpi}\left[(\varpi-\varpi_1+1)^{\overline{x-1}}b^{\varpi_1}\right]\left[b^{-\varpi_1}\zeta(\varpi_1)y^{\beta}(\varpi_1)\right] \\
&\leq \left(\sum_{\varpi_1=\varpi_1+1}^{\varpi}\left[(\varpi-\varpi_1+1)^{\overline{x-1}}\right]^{p}b^{p\varpi_1}\right)^{1/p}\left(\sum_{\varpi_1=\varpi_1+1}^{\varpi}b^{-q\varpi_1}\zeta^{q}(\varpi_1)y^{\beta q}(\varpi_1)\right)^{1/q} \\
&\leq \left(A\sum_{\varpi_1=\varpi_1+1}^{\varpi}(\varpi-\varpi_1+1)^{\overline{p(x-1)}}b^{p\varpi_1}\right)^{1/p}\left(\sum_{\varpi_1=\varpi_1+1}^{\varpi}b^{-q\varpi_1}\zeta^{q}(\varpi_1)y^{\beta q}(\varpi_1)\right)^{1/q} \qquad (23)\\
&\leq \left(AQb^{p\varpi}\right)^{1/p}\left(\sum_{\varpi_1=\varpi_1+1}^{\varpi}b^{-q\varpi_1}\zeta^{q}(\varpi_1)y^{\beta q}(\varpi_1)\right)^{1/q} \\
&= (AQ)^{1/p}b^{\varpi}\left(\sum_{\varpi_1=\varpi_1+1}^{\varpi}b^{-q\varpi_1}\zeta^{q}(\varpi_1)y^{\beta q}(\varpi_1)\right)^{1/q},
\end{aligned}$$

where

$$A = \frac{\Gamma(1+(1-x)p)}{[\Gamma(2-x)]^{p}}.$$

Using (23) in (22), we obtain from (22) that

$$d(\varpi)(\nabla y(\varpi))^{\beta} \leq \varpi^{\overline{n-1}}b^{\varpi}\omega(\varpi), \tag{24}$$

where

$$\omega(\varpi) = \Theta_1 + M_3\left(\sum_{\varpi_1=\varpi_1+1}^{\varpi}b^{-q\varpi_1}\zeta^{q}(\varpi_1)y^{\beta q}(\varpi_1)\right)^{1/q},$$

with

$$M_3 = \Theta_2(AQ)^{1/p} > 0.$$

We rewrite (24) as

$$\nabla y(\varpi) \leq \left(\frac{\varpi^{\overline{n-1}}b^{\varpi}\omega(\varpi)}{d(\varpi)}\right)^{\frac{1}{\beta}}, \quad \varpi \in \mathbb{N}_{\varpi_1}. \tag{25}$$

Noting that $\varpi^{\overline{n-1}}$, b^{ϖ}, and $\omega(\varpi)$ are all increasing, summing (25) from $\varpi_1 + 1$ to ϖ yields that

$$\begin{aligned}
y(\varpi) &\leq y(\varpi_1) + \sum_{\varpi_1=\varpi_1+1}^{\varpi} \left[\varpi_1^{\overline{n-1}}\right]^{\frac{1}{\beta}} b^{\frac{\varpi_1}{\beta}} \omega^{\frac{1}{\beta}}(\varpi_1) d^{-\frac{1}{\beta}}(\varpi_1) \\
&\leq y(\varpi_1) + \left[\varpi^{\overline{n-1}}\right]^{\frac{1}{\beta}} b^{\frac{\varpi}{\beta}} \omega^{\frac{1}{\beta}}(\varpi) \sum_{\varpi_1=\varpi_1+1}^{\varpi} d^{-\frac{1}{\beta}}(\varpi_1) \\
&= y(\varpi_1) + \left[\varpi^{\overline{n-1}}\right]^{\frac{1}{\beta}} b^{\frac{\varpi}{\beta}} \omega^{\frac{1}{\beta}}(\varpi) A(\varpi, \varpi_1) \\
&= \left(\frac{y(\varpi_1)}{\left[\varpi^{\overline{n-1}}\right]^{\frac{1}{\beta}} b^{\frac{\varpi}{\beta}} A(\varpi, \varpi_1)} + \omega^{\frac{1}{\beta}}(\varpi) \right) \left[\varpi^{\overline{n-1}}\right]^{\frac{1}{\beta}} b^{\frac{\varpi}{\beta}} A(\varpi, \varpi_1) \\
&\leq \left(\frac{y(\varpi_1)}{\left[\varpi_2^{\overline{n-1}}\right]^{\frac{1}{\beta}} b^{\frac{\varpi_2}{\beta}} A(\varpi_2, \varpi_1)} + \omega^{\frac{1}{\beta}}(\varpi) \right) \left[\varpi^{\overline{n-1}}\right]^{\frac{1}{\beta}} b^{\frac{\varpi}{\beta}} A(\varpi, \varpi_1),
\end{aligned}$$

holds for $\varpi \in \mathbb{N}_{\varpi_2}$ with $\varpi_2 > \varpi_1$. Thus,

$$\frac{y(\varpi)}{\left[\varpi^{\overline{n-1}}\right]^{\frac{1}{\beta}} b^{\frac{\varpi}{\beta}} A(\varpi, \varpi_1)} \leq M_4 + \omega^{\frac{1}{\beta}}(\varpi), \quad \varpi \in \mathbb{N}_{\varpi_2}, \tag{26}$$

where

$$M_4 = \frac{y(\varpi_1)}{\left[\varpi_2^{\overline{n-1}}\right]^{\frac{1}{\beta}} b^{\frac{\varpi_2}{\beta}} A(\varpi_2, \varpi_1)}.$$

Applying one of the elementary inequalities

$$(y+z)^q \leq \begin{cases} 2^{q-1}(y^q + z^q), & q \geq 1, \\ y^q + z^q, & 0 < q < 1, \end{cases} \tag{27}$$

with $y, z \geq 0$, to (26) gives

$$\left(\frac{y(\varpi)}{\left[\varpi^{\overline{n-1}}\right]^{\frac{1}{\beta}} b^{\frac{\varpi}{\beta}} A(\varpi, \varpi_1)} \right)^{\beta} \leq M_5 + M_6 \omega(\varpi), \quad \varpi \in \mathbb{N}_{\varpi_2}, \tag{28}$$

where M_5 and $M_6 > 0$ are defined by

$$M_5 = \begin{cases} 2^{\beta-1} M_4^{\beta}, & q \geq 1, \\ M_4^{\beta}, & 0 < q < 1, \end{cases} \tag{29}$$

and

$$M_6 = \begin{cases} 2^{\beta-1}, & q \geq 1, \\ 1, & 0 < q < 1. \end{cases} \tag{30}$$

Recalling the definition of $\omega(\varpi)$, from (28), we have that

$$\left(\frac{y(\varpi)}{\left[\overline{\varpi^{n-1}}\right]^{\frac{1}{\beta}} b^{\frac{\varpi}{\beta}} A(\varpi,\varpi_1)}\right)^{\beta} \leq M_7 + M_8 \left(\sum_{\varpi_1=\varpi_1+1}^{\varpi} b^{-q\varpi_1} \zeta^q(\varpi_1) y^{\beta q}(\varpi_1)\right)^{1/q}, \tag{31}$$

holds for $\varpi \in \mathbb{N}_{\varpi_2}$, where

$$M_7 = M_5 + \Theta_1 M_6 > 0, \quad M_8 = M_3 M_6 > 0.$$

Applying the inequality (27) to (31) gives that

$$\left(\frac{y(\varpi)}{\left[\overline{\varpi^{n-1}}\right]^{\frac{1}{\beta}} b^{\frac{\varpi}{\beta}} A(\varpi,\varpi_1)}\right)^{\beta q} \leq M_9 + M_{10} \sum_{r=\varpi_1+1}^{\varpi} b^{-qr} \zeta^q(r) y^{\beta q}(r), \tag{32}$$

holds for $\varpi \in \mathbb{N}_{\varpi_2}$, where

$$M_9 = 2^{\beta-1} M_7^q > 0, \quad M_{10} = 2^{\beta-1} M_8^q > 0.$$

Denoting the left-hand side of (32) by $w(\varpi)$, (32) yields that

$$w(\varpi) \leq M_9 + M_{10} \sum_{\varpi_1=\varpi_1+1}^{\varpi} \left[\overline{\varpi_1^{n-1}}\right]^q A^{\beta q}(\varpi_1,\varpi_1) \zeta^q(\varpi_1) w(\varpi_1), \tag{33}$$

holds for $\varpi \in \mathbb{N}_{\varpi_2}$, and this can be rewritten as

$$w(\varpi) \leq M_{11} + M_{10} \sum_{\varpi_1=\varpi_2+1}^{\varpi} \left[\overline{\varpi_1^{n-1}}\right]^q A^{\beta q}(\varpi_1,\varpi_1) \zeta^q(\varpi_1) w(\varpi_1), \tag{34}$$

which holds for $\varpi \in \mathbb{N}_{\varpi_2}$, where

$$M_{11} = M_9 + M_{10} \sum_{\varpi_1=\varpi_1+1}^{\varpi_2} \left[\overline{\varpi_1^{n-1}}\right]^q A^{\beta q}(\varpi_1,\varpi_1) \zeta^q(\varpi_1) w(\varpi_1) > 0.$$

Using (13) and Gronwall's inequality, we have the conclusion to the theorem. The proof for an eventually negative solution is similar. So, we omit it here. Thus, the theorem is proved. □

Next, we consider $\beta = 1$ and we provide sufficient conditions for which any non-oscillatory solution of (6) is bounded.

Theorem 3. *Assume that $(i) - (ii)$, $0 < x < 1$, $p(x-1)+1 > 0$ for $p > 1$ and that* (14) *and* (15) *hold. Furthermore, assume that there exist real numbers $S > 0$ and $\tau > 1$ such that*

$$\left(\frac{\overline{\varpi^{n-1}}}{d(\varpi)}\right) \leq \varpi_1 b^{-\tau\varpi} \tag{35}$$

and

$$\sum_{\varpi_1=c+1}^{\infty} b^{-q\varpi_1} \zeta^q(\varpi_1) < \infty, \quad q = \frac{p}{p-1}, \tag{36}$$

hold; then, all non-oscillatory solutions of (6) *are bounded.*

Proof. Let y be a non-oscillatory solution of (6), say $y(\varpi) > 0$ for $\varpi \in \mathbb{N}_{\varpi_1}$ for some $\varpi_1 \in \mathbb{N}_{c+1}$. Proceeding as in the proof of Theorem 2, we obtain (25) when $\beta = 1$. Since ω is increasing, summing (25) from $\varpi_1 + 1$ to ϖ yields

$$\begin{aligned}
y(\varpi) &\leq y(\varpi_1) + \sum_{\varpi_1=\varpi_1+1}^{\varpi} \frac{\varpi_1^{\overline{n-1}} b^{\varpi_1} \omega(\varpi_1)}{d(\varpi_1)} \\
&\leq y(\varpi_1) + \sum_{\varpi_1=\varpi_1+1}^{\varpi} S b^{(1-\tau)\varpi_1} \omega(\varpi_1) \\
&\leq y(\varpi_1) + S\omega(t) \sum_{\varpi_1=\varpi_1+1}^{\varpi} b^{(1-\tau)\varpi_1} \\
&\leq y(\varpi_1) + S\omega(t) \sum_{\varpi_1=\varpi_1+1}^{\varpi} \left(\frac{1}{b^{(\tau-1)}}\right)^{s} \\
&= y(\varpi_1) + S\omega(t) \left(\frac{b^{(\tau-1)}}{b^{(\tau-1)} - 1}\right) \left[\left(\frac{1}{b^{(\tau-1)}}\right)^{\varpi_1+1} - \left(\frac{1}{b^{(\tau-1)}}\right)^{\varpi+1}\right] \\
&= y(\varpi_1) + S\omega(t) \left(\frac{1}{b^{(\tau-1)} - 1}\right) \left[\left(\frac{1}{b^{(\tau-1)}}\right)^{\varpi_1} - \left(\frac{1}{b^{(\tau-1)}}\right)^{\varpi}\right] \\
&\leq y(\varpi_1) + S\omega(t) \left(\frac{1}{b^{(\tau-1)} - 1}\right) \left(\frac{1}{b^{(\tau-1)}}\right)^{\varpi_1}.
\end{aligned}$$

Using the definition of ω, we obtain

$$y(\varpi) \leq M_{12} + M_{13} \left(\sum_{\varpi_1=\varpi_1+1}^{\varpi} b^{-q\varpi_1} \zeta^q(\varpi_1) y^q(\varpi_1)\right)^{1/q}, \tag{37}$$

for $\varpi \in \mathbb{N}_{\varpi_2}$, where

$$M_{12} = y(\varpi_1) + \Theta_1 S \left(\frac{1}{b^{(\tau-1)} - 1}\right) \left(\frac{1}{b^{(\tau-1)}}\right)^{\varpi_1} > 0,$$

and

$$M_{13} = M_3 S \left(\frac{1}{b^{(\tau-1)} - 1}\right) \left(\frac{1}{b^{(\tau-1)}}\right)^{\varpi_1} > 0.$$

Using the inequality (27) to (37), we have

$$y^q(\varpi) \leq M_{14} + M_{15} \sum_{\varpi_1=\varpi_1+1}^{\varpi} b^{-q\varpi_1} \zeta^q(\varpi_1) y^q(\varpi_1), \tag{38}$$

for $\varpi \in \mathbb{N}_{\varpi_1}$, where

$$M_{14} = 2^{q-1} M_{12}^q > 0, \quad M_{15} = 2^{q-1} M_{13}^q > 0.$$

Now, using (36) and Gronwall's inequality, we have the conclusion to the theorem. The proof for an eventually negative solution is similar. So, we omit it here. The theorem is proved. □

4. Examples

We conclude this paper with the following examples to illustrate our main results.

Example 1. *Consider the equation*

$$\nabla_{1*}^{0.75}\left(\nabla^3\left(e^{3\varpi}(\nabla y(\varpi))^3\right)\right)+\phi(\varpi,y(\varpi))$$
$$=(\varpi-1)^{\overline{-0.9}}+\frac{y(\varpi)}{\varpi(\varpi+1)(\varpi+2)e^{\varpi/2}}+\Phi(\varpi,y(\varpi)),\quad \varpi\in\mathbb{N}_2. \quad (39)$$

Here, we have $z(\varpi)=\nabla^3\left(e^{3\varpi}(\nabla y(\varpi))^3\right)$, $n=4$, $x=0.75$, $c=1$, $\beta=3$, $d(\varpi)=e^{3\varpi}$, $\eta(\varpi)=(\varpi-1)^{\overline{-0.9}}$, $\zeta(\varpi)=\frac{1}{\varpi(\varpi+1)(\varpi+2)e^{\varpi/2}}$, and

$$A(\varpi,c)=A(\varpi,1)=\sum_{\varpi_1=2}^{\varpi}d^{-\frac{1}{3}}(\varpi_1)=\sum_{\varpi_1=2}^{\varpi}e^{-\varpi_1}=\frac{1}{e(e-1)}\left[1-\left(\frac{1}{e}\right)^{\varpi-1}\right]\leq\frac{1}{e(e-1)}.$$

Clearly, condition (i) holds. Let $b=e$ and $p=2$. Clearly, $p(x-1)+1>0$. Additionally, we have $q=2$, and

$$\sum_{\varpi_1=c+1}^{\infty}\zeta^q(\varpi_1)\left[\varpi_1^{\overline{n-1}}\right]^q A^{\beta q}(\varpi_1,c)\leq\frac{1}{e^2(e-1)^2}\sum_{\varpi_1=2}^{\infty}e^{-\varpi_1}<\infty,$$

implying that (13) holds. Considering $\phi(\varpi,y(\varpi))=\Theta_1(\varpi)|y(\varpi)|^{\lambda-1}y(\varpi)$ and $\Phi(\varpi,y(\varpi))$ = $\Theta_2(\varpi)|y(\varpi)|^{\gamma-1}y(\varpi)$ with $\lambda>\gamma$, $\Theta_1(\varpi)=\Theta_2(\varpi)=(\varpi-1)^{\overline{-0.9}}$, we see that (ii) holds. To check (14), we assume

$$\begin{aligned}\frac{1}{\Gamma(0.75)}\sum_{\varpi_1=1+1}^{\varpi}(\varpi-\varpi_1+1)^{\overline{0.75-1}}|\eta(\varpi_1)| &= \frac{1}{\Gamma(0.75)}\sum_{\varpi_1=2}^{\varpi}(\varpi-\varpi_1+1)^{\overline{0.75-1}}\left|(\varpi_1-1)^{\overline{-0.9}}\right| \\ &= \frac{1}{\Gamma(0.75)}\sum_{\varpi_1=2}^{\varpi}(\varpi-\varpi_1+1)^{\overline{0.75-1}}(\varpi_1-1)^{\overline{-0.9}} \\ &= \nabla_1^{-0.75}(\varpi-1)^{\overline{-0.9}} \\ &= \frac{\Gamma(1-0.9)}{\Gamma(1-0.9+0.75)}(\varpi-1)^{\overline{-0.9+0.75}} \\ &= \frac{\Gamma(0.1)}{\Gamma(0.85)}(\varpi-1)^{\overline{-0.15}} \\ &\leq \frac{\Gamma(0.1)}{\Gamma(0.85)}1^{\overline{-0.15}} \\ &= \Gamma(0.1),\end{aligned}$$

that is,

$$\lim_{\varpi\to\infty}\left[\frac{1}{\Gamma(0.75)}\sum_{\varpi_1=1+1}^{\varpi}(\varpi-\varpi_1+1)^{\overline{0.75-1}}|e(\varpi_1)|\right]<\infty.$$

Similarly, it is easy to verify that (15) holds. Therefore, all conditions of Theorem 2 are satisfied. Thus, every non-oscillatory solution of (6) satisfies

$$\limsup_{\varpi\to\infty}\frac{|y(\varpi)|}{\left[\varpi^{\overline{3}}\right]^{\frac{1}{2}}e^{\frac{\varpi}{2}}A(\varpi,1)}<\infty. \quad (40)$$

Example 2. *Consider the equation*

$$\nabla_{1*}^{0.5}\left(\nabla^2\left(\varpi(\varpi+1)e^{5\varpi}(\nabla v(\varpi))\right)\right)+\phi(\varpi,y(\varpi))$$
$$=(\varpi-1)^{\overline{-0.75}}+e^{2\varpi/3}y(\varpi)+\Phi(\varpi,y(\varpi)),\quad \varpi\in\mathbb{N}_2. \quad (41)$$

Here, we have $z(\varpi)=\nabla^2(\varpi(\varpi+1)e^{5\varpi}(\nabla v(\varpi)))$, $c=1$, $x=0.5$, $n=3$, $d(\varpi)=\varpi(\varpi+1)e^{5\varpi}$, $e(\varpi)=(\varpi-1)^{\overline{-0.75}}$, and $\zeta(\varpi)=e^{2\varpi/3}$. Hence, condition (i) holds. Assuming $b=e$, $\varpi_1=1$, and $\tau=5$, we find

$$\left(\frac{\varpi^{\overline{2}}}{d(\varpi)}\right)=e^{-5\varpi}.$$

Therefore, (35) holds. Now, if we take $p=3/2$, then we have $q=3$, and

$$\sum_{\varpi_1=c+1}^{\infty}b^{-q\varpi_1}\zeta^q(\varpi_1)=\sum_{\varpi_1=2}^{\infty}e^{-3\varpi_1}e^{2\varpi_1}=\sum_{\varpi_1=2}^{\infty}e^{-\varpi_1}=\frac{1}{e(e-1)}<\infty,$$

that is, (36) holds. Again, if

$$\phi(\varpi,y(\varpi))=\Theta_1(\varpi)|y(\varpi)|^{\lambda-1}y(\varpi)\ \text{ and }\ \Phi(\varpi,y(\varpi))=\Theta_2(\varpi)|y(\varpi)|^{\gamma-1}y(\varpi)$$

with $\lambda>\gamma$, $\Theta_1(\varpi)=\Theta_2(\varpi)=(\varpi-1)^{\overline{-0.75}}$, then it is easy to verify that condition (ii) holds. To check that (14) holds, we assume

$$\begin{aligned}\frac{1}{\Gamma(0.5)}\sum_{\varpi_1=1+1}^{\varpi}(\varpi-\varpi_1+1)^{\overline{0.5-1}}|\eta(\varpi_1)| &= \frac{1}{\Gamma(0.5)}\sum_{\varpi_1=2}^{\varpi}(\varpi-\varpi_1+1)^{\overline{0.5-1}}\left|(\varpi_1-1)^{\overline{-0.75}}\right|\\ &= \frac{1}{\Gamma(0.5)}\sum_{\varpi_1=2}^{\varpi}(\varpi-\varpi_1+1)^{\overline{0.5-1}}(\varpi_1-1)^{\overline{-0.75}}\\ &= \nabla_1^{-0.5}(\varpi-1)^{\overline{-0.75}}\\ &= \frac{\Gamma(1-0.75)}{\Gamma(1-0.75+0.5)}(\varpi-1)^{\overline{-0.75+0.5}}\\ &= \frac{\Gamma(0.25)}{\Gamma(0.75)}(\varpi-1)^{\overline{-0.25}}\\ &\le \frac{\Gamma(0.25)}{\Gamma(0.75)}1^{\overline{-0.25}}\\ &= \Gamma(0.25),\end{aligned}$$

that is,

$$\lim_{\varpi\to\infty}\left[\frac{1}{\Gamma(0.5)}\sum_{\varpi_1=1+1}^{\varpi}(\varpi-\varpi_1+1)^{\overline{0.5-1}}|e(\varpi_1)|\right]<\infty.$$

Similarly, it is easy to verify that (15) holds. Therefore, all conditions of Theorem 3 are satisfied. Thus, every non-oscillatory solution of (41) is bounded.

5. Concluding Remarks

Unlike most existing results in the literature that have been dedicated to oscillation criteria, we introduced a number of additional necessary conditions for non-oscillatory solutions to forced nabla difference equations with positive and negative terms. The main equation is of a general nature, and it covers many particular cases. By creating an equivalent representation of the primary equation in the form of a summation equation similar to Volterra and using some mathematical inequalities, the results are stated and proved. Some earlier findings in the literature were enhanced by the results. In fact, we give two brandnew cases, the non-oscillatory behavior of whose solutions has never been discussed in earlier studies. The existing methodology can be used in the future to produce comparable outcomes for higher order dynamic equations with forcing, positive and negative terms.

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References

1. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. Theory and Applications of Fractional Differential Equations. In *North-Holland Mathematics Studies*; Elsevier Science B.V.: Amsterdam, The Netherlands, 2006; Volume 204.
2. Podlubny, I. Fractional differential equations. An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications. In *Mathematics in Science and Engineering*; Academic Press, Inc.: San Diego, CA, USA, 1999; Volume 198.
3. Anastassiou, G.A. Nabla discrete fractional calculus and nabla inequalities. *Math. Comput. Model.* **2010**, *51*, 562–571. [CrossRef]
4. Shah, K.; Khan, R.A. Existence and uniqueness results to a coupled system of fractional order boundary value problems by topological degree theory. *Numer. Funct. Anal. Optim.* **2016**, *37*, 887–899. [CrossRef]
5. Shatanawi, W.; Boutiara, A.; Abdo, M.S.; Jeelani, M.B.; Abodayeh, K. Nonlocal and multiple-point fractional boundary value problem in the frame of a generalized Hilfer derivative. *Adv. Differ. Equ.* **2021**, 1–19. [CrossRef]
6. Wang, J.; Zhou, Y.; Wei, W. Study in fractional differential equations by means of topological degree methods. *Numer. Funct. Anal. Optim.* **2012**, *33*, 216–238. [CrossRef]
7. Khan, Z.A.; Khan, A.; Abdeljawad, T.; Khan, H. *Computational Analysis of Fractional Order Imperfect Testing Infection Disease Model*; World Scientific Publishing Co. Pte Ltd.: Singapore, 2022. [CrossRef]
8. Alzabut, J.; Grace, S.R.; Santra, S.S.; Chhatria, G.N. Asymptotic and oscillatory behaviour of third order non-linear differential equations with canonical operator and mixed neutral terms. *Qual. Theory Dyn. Syst.* **2023**, *22*, 1–15. [CrossRef]
9. Palanisamy, A.; Alzabut, J.; Muthulakshmi, V.; Santra, S.S.; Nonlaopon, K. Oscillation results for a fractional partial differential system with damping and forcing terms. *AIMS Math.* **2023**, *8*, 4261–4279. [CrossRef]
10. Altanji, M.; Chhatria, G.N.; Santra, S.S.; Scapellato, A. Oscillation criteria for sublinear and superlinear first-order difference equations of neutral type with several delays. *AIMS Math.* **2022**, *7*, 17670–17684. [CrossRef]
11. Alzabut, J.; Agarwal, R.P.; Grace, S.R.; Jonnalagadda, J.M.; Selvam, A.G.M.; Wang, C. A Survey on the Oscillation of Solutions for Fractional Difference Equations. *Mathematics* **2022**, *10*, 894. [CrossRef]
12. Goodrich, C.; Peterson, A.C. *Discrete Fractional Calculus*; Springer: Cham, Switzerland, 2015.
13. Abdalla, B.; Abodayeh, K.; Abdeljawad, T.; Alzabut, J. New oscillation criteria for forced nonlinear fractional difference equations. *Vietnam J. Math.* **2017**, *45*, 609–618. [CrossRef]
14. Abdalla, B.; Alzabut, J.; Abdeljawad, T. On the oscillation of higher order fractional difference equations with mixed nonlinearities. *Hacet. J. Math. Stat.* **2018**, *47*, 207–217. [CrossRef]
15. Alzabut, J.O.; Abdeljawad, T. Sufficient conditions for the oscillation of nonlinear fractional difference equations. *J. Fract. Calc. Appl.* **2014**, *5*, 177–187.

16. Alzabut, J.; Abdeljawad, T.; Alrabaiah, H. Oscillation criteria for forced and damped nabla fractional difference equations. *J. Comput. Anal. Appl.* **2018**, *24*, 1387–1394.
17. Hardy, G.H.; Littlewood, J.E.; Polya, G. *Inequalities*; Reprint of the 1952 Edition; Cambridge Mathematical Library, Cambridge University Press: Cambridge, UK, 1988.