Linguistic Complex Fuzzy Sets

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Abstract: Complex fuzzy sets (CFSs) are a suitable tool to manage spatial directional information which includes distance and direction. However, spatial directional information is given by linguistic values. It is very awkward for the CFS to describe this type of spatial directional information. To overcome this limitation, we first propose a novel concept called a linguistic complex fuzzy set (LCFS) to serve as an extension of the CFS. Then we put forward some basic operational laws for LCFSs. After that, we define three operators for LCFSs: the linguistic complex fuzzy weighted averaging (LCFWA) operator, the linguistic amplitude max (Amax) operator and the linguistic amplitude min (Amin) operator. In actual application, we use the LCFWA operator to deal with group decision making when the importance weights of experts are known. For the situation in which the weights of experts are unknown, we develop an Amax-Amin method for group decision making.

Keywords: complex fuzzy sets; linguistic fuzzy set; linguistic complex fuzzy set; decision making

MSC: 03E72

1. Introduction

Ramot et al. [1] introduced the concept of complex fuzzy sets (CFSs) as an extension of fuzzy sets [2]. A CFS and its extensions have wide applications in practice, such as signal processing [3–5], image processing [6], time series forecasting [7–9] and decision making [10–13].

Although many studies have paid attention to CFSs and their applications, some scholars have the question: why complex fuzzy sets? As we know, there are many extensions of FSs, such as interval-valued fuzzy sets (IVFSs) [14], intuitionistic fuzzy sets (IFSs) [15], interval-valued intuitionistic fuzzy sets (IVIFSs) [16], Pythagorean fuzzy set [17], q-rung orthopair fuzzy sets [18], Plithogenic Set [19] and hesitant fuzzy sets (HFSs) [20]. Among these sets, CFSs are a special extension of FSs due to their complex-valued membership. But complex-valued membership remains a puzzle from the intuitive viewpoint.

In order to answer the question: *How to use CFSs in our real life?* we use the following example to show the particularity of CFSs, which is introduced by Dai et al. [21]. In real life, we often ask passers-by for directions to the nearest market. If there are two markets and their distance and direction are given by two people and they both think market P is about 0.98 km from you and market Q is about 1 km from you, they have different views on the direction of market Q. One person thinks market P is located at *A* and market *Q* is located at *B*, another person thinks market P is located at A but market *Q* is located at C, as shown in Figure 1. Combining the opinions of these two people, we think market *Q* is located at the center of *B* and *C*. Then, the distance to market *Q* is about 0.95 km. Therefore, we decided to go to market *Q*. Interestingly, it is a preference reversal phenomenon. Both people think market *P* is nearer than market *Q*, but our result is that *Q* is nearer than market *P*. Distance and direction are two parameters used in this example. The center-based aggregation method is reasonable. The ranking which is only based on the distance is also reasonable. CFSs theory can perfectly describe the above phenomenon. In many other extensions of FSs, all parameters are used in the ranking. For example, both membership degree and non-membership degree are used in the ranking of IFSs.
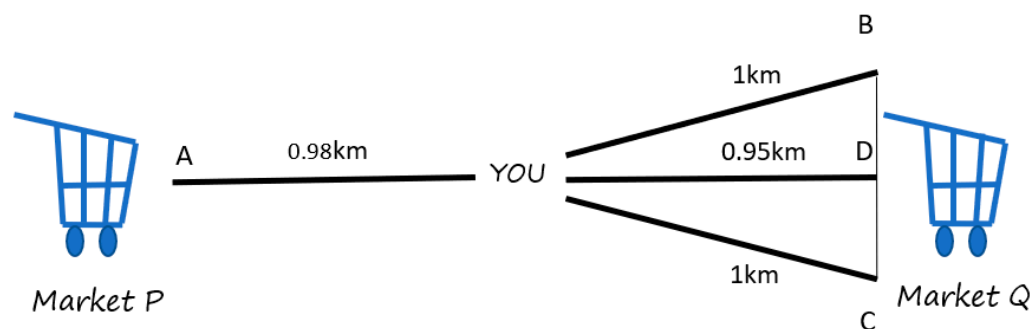
Figure 1. Distances and directions of two markets.

In decision-making problems under CFSs or their extension environment, CFSs were analyzed from two angles: (1) a real part and an imaginary part, (2) an amplitude term and a phase term. This paper considers the decision-making problem from the second angle of CFSs. In this situation, both amplitude term and phase term play important roles in the ranking [10–13]. However, there is a very special situation, as the above example illustrates. The phase term does not play a role in the ranking, but it still cannot be ignored in the decision-making process because it plays a role in the aggregation process and then affects the final sorting result. In some decision-making problems under CFS or its extension environment [10–13], the phase term plays a role in the ranking. However, as the above example illustrates, the phase term does not play a role in the ranking, but it still cannot be ignored because it plays a role in the aggregation and then affects the final sorting result. Moreover, in some literature [22–24], the phase term plays an important role in the operations and algebraic structures of the CFS.

In this paper, we develop the linguistic complex fuzzy sets (LCFSs) to improve the modeling ability of linguistic approaches motivated by CFSs. On the one hand, as far as we know, nowadays, there are no corresponding discussions to propose the LCFS, although the CFS has been extended to complex intuitionistic fuzzy sets [25], complex Pythagorean fuzzy sets [26], complex hesitant fuzzy sets [27] and complex neutrosophic sets [28]. On the other hand, in our daily life, "very far, south" and "far, northeast" often appear. We need a new fuzzy approach of dealing with this type of linguistic information. Comparing with other fuzzy linguistic approaches, LCFSs are particular suitable for these linguistic assessments in decision making. We also study the aggregation operators to aggregate LCFSs arguments and use these operators to deal with group decision-making problems. However, when enjoying the benefits of the presented LCFS, we also have to face the ensuing challenges:

1. The operators of LCFSs should be reduced to that of the LFS when the phases of the CHFSs are zero.
2. As mentioned in [1], phase is relative. Although the order of CFSs only relies on their amplitudes, the phase of the CFS should have a role in decision making under the CFS environment.

Motivated by the above challenges and keeping the advantages of the CFS, the main contributions of this article are as follows:

1. We establish the LCFS, which is the combination of the LFS and the CFS, to manage linguistic variables of spatial directional information in real-decision theory.
2. We define the LCFWA operator, which is used to deal with group decision making when the importance weights of experts are known.
3. We define the Amax and Amin operators which do not consider the phase terms. Based on these two operators, we establish the Amax-Amin method for group decision making when the importance weights of experts are unknown.

The rest of the study is organized as follows: Section 1 reviews the theories of CFSs and linguistic fuzzy sets (LFSs). Section 2 defines LCFSs and their operators. Section 3 provides some decision-making methods to deal with group decision-making problems

under LCFS information. Section 4 provides a case study. Conclusions are given in the last section.

2. Preliminaries

Our proposal is based on the LFS and CFS. In this section, we briefly review their concepts.

2.1. Complex Fuzzy Set

Ramot et al. [1] introduced the complex fuzzy sets (CFSs) as follows:

Definition 1. *[1]. A CFS E on the universal set X is defined as*

$$E = \{(x, r_E(x)e^{i\omega_E(x)})|x \in X\} \tag{1}$$

where $\sqrt{-1} = i$, $r_E(x) \in [0, 1]$ *is the amplitude term and* $\omega_E(x) \in [0, 2\pi)$ *is the phase term. For convenience,* $\alpha_E = r_E e^{i\omega_E}$ *is called a complex fuzzy number (CFN).*

For two CFNs $\alpha_E = r_E e^{i\omega_E}$ and $\alpha_F = r_F e^{i\omega_F}$, Zhang et al. [18] gave some operations on them, shown as follows:

(1) $Neg(\alpha_E) = (1 - r_E)e^{i(2\pi - \omega_E)}$;
(2) $max(\alpha_E, \alpha_F) = max(r_E, r_F)e^{max(\omega_E, \omega_F)i}$;
(3) $min(\alpha_E, \alpha_F) = min(r_E, r_F)e^{min(\omega_E, \omega_F)i}$.

2.2. Linguistic Fuzzy Set

Definition 2. *[11]. Let* $S = \{s_0, s_1, \cdots, s_t\}$ *be a finite linguistic term set, where* s_i *owns the following characteristics:*

(1) *The set is ordered:* $s_j \geq s_k$ *iff* $j \geq k$;
(2) *Negation operator:* $Neg(s_j) = s_{t-j}$;
(3) *Max operator:* $max(s_j, s_k) = s_{max(j,k)}$;
(4) *Min operator:* $min(s_j, s_k) = s_{min(j,k)}$.

Example 1. *A set of seven linguistic terms is given as:*

$$S = \{s_0 = very\ near, s_1 = near, s_2 = slightly\ near, s_3 = not\ far\ not\ near,$$
$$s_4 = slightly\ far, s_5 = far, s_6 = very\ far\}.$$

3. Linguistic Complex Fuzzy Set

In this section, we explore the concept of LCFSs and their basic operational laws. The established work is also verified with the help of some numerical examples. We also introduce the LCFWA operator in LCFSs environment.

The concept of the LCFS is given as follows:

Definition 3. *A LCFS is defined on the universal set X with the form*

$$S = \{s_a \big| a = r_a e^{i\omega_a}, r \in [0, t], w \in [0, 2\pi)\} \tag{2}$$

where s_a *owns the following characteristics:*
  The set is ordered: $s_a \geq s_b$ *iff* $|a| \geq |b|$;
  Negation operator: $Neg(s_a) = s_b$, *where* $r_b = t - r_a$ *and* $\omega_b = 2\pi - \omega_a$;
  Max operator: $max(s_a, s_b) = s_c$, *where* $r_c = max(r_a, r_b)$ *and* $\omega_c = max(\omega_a, \omega_b)$;
  Min operator: $min(s_a, s_b) = s_c$, *where* $r_c = min(r_a, r_b)$ *and* $\omega_c = min(\omega_a, \omega_b)$.

We denote s_a as the linguistic complex fuzzy value (LCFV) or the linguistic complex fuzzy number (LCFN).

For the two LCFNs s_a and s_b, we use $s_a \leq s_b$ if and only if $|a| \leq |b|$, i.e., $r_a \leq r_b$, obviously.

Proposition 1. *Let s_a, s_b, s_c, s_d be four LCFSs. Then:*

1) $Neg(Neg(s_a)) = s_a$;
2) $Neg(\max(s_a, s_b)) = \min(Neg(s_a), Neg(s_b))$;
3) $Neg(\min(s_a, s_b)) = \max(Neg(s_a), Neg(s_b))$;
4) $\max(\max(s_a, s_b), s_c) = \max(s_a, \max(s_b, s_c))$;
5) $\min(\min(s_a, s_b), s_c) = \min(s_a, \min(s_b, s_c))$;
6) $\max(\min(s_a, s_b), s_c) = \min(\max(s_a, s_b), \max(s_b, s_c))$;
7) $\min(\max(s_a, s_b), s_c) = \max(\min(s_a, s_c), \min(s_b, s_c))$.

Proof. 1)

$$Neg(Neg(s_a)) = Neg((t - r_a)e^{i(2\pi - \omega_a)}) = (t - (t - r_a))e^{i(2\pi - (2\pi - \omega_a))} = r_a e^{i\omega_a} = s_a.$$

2) $Neg(\max(s_a, s_b))$
$= Neg(\max(r_a, r_b)e^{\max(\omega_a, \omega_b)i})$
$= (t - \max(r_a, r_b))e^{i(2\pi - \max(\omega_a, \omega_b))}$
$= \min(t - r_a, t - r_b)e^{\min(2\pi - \omega_a, 2\pi - \omega_b)i}$
$= \min(Neg(s_a), Neg(s_b))$.

3) $Neg(\min(s_a, s_b))$
$= Neg(\min(r_a, r_b)e^{\min(\omega_a, \omega_b)i})$
$= (t - \min(r_a, r_b))e^{i(2\pi - \min(\omega_a, \omega_b))}$
$= \max(t - r_a, t - r_b)e^{\max(2\pi - \omega_a, 2\pi - \omega_b)i}$
$= \max(Neg(s_a), Neg(s_b))$.

4) $\max(\max(s_a, s_b), s_c)$
$= \max(\max(r_a, r_b)e^{\max(\omega_a, \omega_b)i}, s_c)$
$= \max(\max(r_a, r_b), r_c)e^{\max(\max(\omega_a, \omega_b), \omega_c)i}$
$= \max(r_a, \max(r_b, r_c))e^{\max(\omega_a, \max(\omega_b, \omega_c))i}$
$= \max(s_a, \max(s_b, s_c))$.

5) $\min(\min(s_a, s_b), s_c)$
$= \min(\min(r_a, r_b)e^{\min(\omega_a, \omega_b)i}, s_c)$
$= \min(\min(r_a, r_b), r_c)e^{\min(\min(\omega_a, \omega_b), \omega_c)i}$
$= \min(r_a, \min(r_b, r_c))e^{\min(\omega_a, \min(\omega_b, \omega_c))i}$
$= \min(s_a, \min(s_b, s_c))$.

6) $\max(\min(s_a, s_b), s_c)$
$= \max(\min(r_a, r_b)e^{\min(\omega_a, \omega_b)i}, s_c)$
$= \max(\min(r_a, r_b), r_c)e^{\max(\min(\omega_a, \omega_b), \omega_c)i}$
$= \min(\max(r_a, r_c), \max(r_b, r_c))e^{\min(\max(\omega_a, \omega_b), \max(\omega_a, \omega_c))i}$
$= \min(\max(s_a, s_b), \max(s_b, s_c))$.

7) $\min(\max(s_a, s_b), s_c)$
$= \min(\max(r_a, r_b)e^{\max(\omega_a, \omega_b)i}, s_c)$
$= \min(\max(r_a, r_b), r_c)e^{\min(\max(\omega_a, \omega_b), \omega_c)i}$
$= \max(\min(r_a, r_c), \min(r_b, r_c))e^{\max(\min(\omega_a, \omega_b), \min(\omega_a, \omega_c))i}$
$= \max(\min(s_a, s_b), \min(s_b, s_c))$.
□

Definition 4. *Let s_{a_j} ($j = 1, 2, \cdots, n$) be a collection of LCFNs, the linguistic complex fuzzy weighted averaging (LCFWA) operator is defined as*

$$s_c = LCFWA(s_{a_1}, s_{a_2}, \cdots, s_{a_n}) \tag{3}$$

where

$$c = w_1 a_1 + w_2 a_2 + \cdots + w_n a_n$$

where $w_j \in [0, 1]$, $\sum_{j=1}^{n} w_j = 1$.

If $w_j = \frac{1}{n}$ for all j, then the LCFWA operator is the arithmetic average of s_{a_j} (j = 1, 2, $\cdots$, n), denoted by the linguistic complex fuzzy arithmetic average (LCFAA) operator, i.e.,

$$s_c = \text{LCFAA}(s_{a_1}, s_{a_2}, \cdots, s_{a_n}) \tag{4}$$

where

$$c = \frac{a_1 + a_2 + \cdots + a_n}{n}$$

Proposition 2. *Let s_{a_j} (j = 1, 2, $\cdots$, n) be a collection of LCFNs, then*

(1)   *Idempotency: If $s_{a_1} = s_{a_2} = \cdots = s_{a_n}$, then $\text{LCFWA}(s_{a_1}, s_{a_2}, \cdots, s_{a_n}) = s_{a_1}$;*
(2)   *Upper boundedness: $\text{LCFAA}(s_{a_1}, s_{a_2}, \cdots, s_{a_n}) \leq s_b$,*
      where $r_b = max(r_{a_1}, r_{a_2}, \cdots, r_{a_n})$.

Proof. (1) Since for any $\alpha \in C$, $w_1\alpha + w_2\alpha + \cdots + w_n\alpha = (w_1 + w_2 + \cdots + w_n)\alpha = \alpha$.
(2) Since $r_b = max(r_{a_1}, r_{a_2}, \cdots, r_{a_n})$, $|w_1a_1 + w_2a_2 + \cdots + w_na_n| \leq w_1|a_1| + w_2|a_2| + \cdots + w_n|a_n| \leq |b|$. Thus $\text{LCFAA}(s_{a_1}, s_{a_2}, \cdots, s_{a_n}) \leq s_b$. $\square$

However, the LCFWA operator does not satisfy the property of (amplitude) monotonicity.

Example 2. *We have $|s_{2e^{i\pi/12}}| > |s_{1.9e^{i\pi}}|$ and $|s_{2e^{i\pi5/12}}| > |s_{1.9e^{i\pi}}|$, but*

$$|\text{LCFAA}(s_{2e^{i\pi/12}}, s_{2e^{i\pi5/12}})| = s_{1.732e^{i\pi/4}} < |s_{1.9e^{i\pi}}| = |\text{LCFAA}(s_{1.9e^{i\pi}}, s_{1.9e^{i\pi}})|.$$

If we only have one direction, we can use the subset of eight linguistic terms as follows:

$$S = \{s_0 = here, s_1 = very\ near, s_2 = near, s_3 = slightly\ near, s_4 = not\ far\ not\ near, s_5 = slightly\ far, s_6 = far, s_7 = very\ far\}.$$

If we have the positive and negative directions, we can use the subset of eight linguistic terms as follows:

$$S = \{s_{-7} = (very\ far, west), s_{-6} = (far, west), s_{-5} = (slightly\ far, west), s_{-4} = (not\ far\ not\ near, west), s_{-3} = (slightly\ near, west), s_{-2} = (near, west), s_{-1} = (very\ near, west), s_0 = here, s_1 = (very\ near, east), s_2 = (near, east), s_3 = (slightly\ near, east), s_4 = (not\ far\ not\ near, east), s_5 = (slightly\ far, east), s_6 = (far, east), s_7 = (very\ far, east)\}.$$

Where east can be used as the positive direction, west can be used as the negative direction. Therefore, LCFNs are more flexible to manage spatial directions than LFSs.
Here, we introduce two linguistic operators.

Definition 5. *Let s_{a_j} (j = 1, 2, $\cdots$, n) be a collection of LCFNs, the linguistic amplitude max (Amax) operator is defined as*

$$Amax(s_{a_1}, s_{a_2}, \cdots, s_{a_n}) = \{s_{c_1}, s_{c_2}, \cdots, s_{c_m}\}, \tag{5}$$

where $\{s_{c_1}, s_{c_2}, \cdots, s_{c_m}\} \subseteq \{s_{a_1}, s_{a_2}, \cdots, s_{a_n}\}$ and $r_{c_k} = max(r_{a_1}, r_{a_2}, \cdots, r_{a_n})$ for all $k = 1, 2, \cdots, m$.
The linguistic amplitude min (Amin) operator is defined as

$$Amin(s_{a_1}, s_{a_2}, \cdots, s_{a_n}) = \{s_{c_1}, s_{c_2}, \cdots, s_{c_m}\}, \tag{6}$$

where $\{s_{c_1}, s_{c_2}, \cdots, s_{c_m}\} \subseteq \{s_{a_1}, s_{a_2}, \cdots, s_{a_n}\}$ and $r_{c_k} = min(r_{a_1}, r_{a_2}, \cdots, r_{a_n})$ for all $k = 1, 2, \cdots, m$.

Example 3. *Consider the following sets of LCFNs,*

$$G = \{s_{2e^{i\pi/4}} = (near, northeast), s_{2e^{i\pi 5/12}} = (near, north\ by\ east),$$
$$s_{5e^{i\pi/2}} = (slightly\ far, north), s_{2e^{i\pi/3}} = (slightly\ near, north - northeast),$$
$$s_3 = (not\ far\ not\ near, east)$$
$$s_{4e^{i\pi 3/2}} = (slightly\ far, south), s_5 = (far, east),$$
$$s_{6e^{i\pi 17/12}} = (very\ far, south\ by\ west), s_{6e^{i\pi/4}} = (very\ far, northeast)\}$$

Then

$$Amax(G) = \left\{ s_{6e^{\frac{i\pi 17}{12}}} = (very\ far, south\ by\ west), s_{6e^{\frac{i\pi}{4}}} = (very\ far, northeast) \right\},$$

and

$$Amax(G) = \{ s_{2e^{\frac{i\pi}{4}}} = (near, northeast), s_{2e^{\frac{i\pi 5}{12}}} = (near, north\ by\ east),$$
$$s_{2e^{i\pi/3}} = (slightly\ near, north - northeast)\}.$$

In real life, you try to find a bank near you. Then, you may get the answer such as follows: "Bank A is near in the east, Bank B is near in the west, but Bank C is near in the south, so you can go to Bank A or Bank B".

Proposition 3. *Let s_{a_j} ($j = 1, 2, \cdots, n$) and s_b be LCFNs, then*

(1) $Amax(s_{a_1}, \cdots, s_{a_{n_1}}, s_b) = Amax(Amax(s_{a_1}, \cdots, s_{a_{n_1}}), s_b),$

(2) $Amin(s_{a_1}, \cdots, s_{a_{n_1}}, s_b) = Amin(Amin(s_{a_1}, \cdots, s_{a_{n_1}}), s_b).$

Proof. (1) Assume that $Amax(s_{a_1}, s_{a_2}, \cdots, s_{a_n}) = \{s_{c_1}, s_{c_2}, \cdots, s_{c_m}\}$, if $r_{c_1} > r_b$, then

$Amax(s_{a_1}, \cdots, s_{a_{n_1}}, s_b)$
$= \{s_{c_1}, s_{c_2}, \cdots, s_{c_m}\}$
$= Amax(s_{c_1}, s_{c_2}, \cdots, s_{c_m}, s_b)$
$= Amax(Amax(s_{a_1}, \cdots, s_{a_{n_1}}), s_b);$
　if $r_{c_1} < r_b$, then
　$Amax(s_{a_1}, \cdots, s_{a_{n_1}}, s_b)$
$= \{s_b\}$
$= Amax(s_{c_1}, s_{c_2}, \cdots, s_{c_m}, s_b)$
$= Amax(Amax(s_{a_1}, \cdots, s_{a_{n_1}}), s_b);$
　if $r_{c_1} = r_b$ and $s_b = s_{c_j}$ for some $j \in \{1, 2, \cdots, m\}$, then
　$Amax(s_{a_1}, \cdots, s_{a_{n_1}}, s_b)$
$= \{s_{c_1}, s_{c_2}, \cdots, s_{c_m}\}$
$= Amax(s_{c_1}, s_{c_2}, \cdots, s_{c_m}, s_b)$
$= Amax(Amax(s_{a_1}, \cdots, s_{a_{n_1}}), s_b);$
　if $r_{c_1} = r_b$ and $s_b \neq s_{c_j}$ for all $j \in \{1, 2, \cdots, m\}$, then
　$Amax(s_{a_1}, \cdots, s_{a_{n_1}}, s_b)$
$= \{s_{c_1}, s_{c_2}, \cdots, s_{c_m}, s_b\}$
$= Amax(s_{c_1}, s_{c_2}, \cdots, s_{c_m}, s_b)$
$= Amax(Amax(s_{a_1}, \cdots, s_{a_{n_1}}), s_b).$
　(2) Assume that $Amin(s_{a_1}, s_{a_2}, \cdots, s_{a_n}) = \{s_{c_1}, s_{c_2}, \cdots, s_{c_m}\}$, if $r_{c_1} > r_b$, then
　$Amin(s_{a_1}, \cdots, s_{a_{n_1}}, s_b)$
$= \{s_b\}$
$= Amin(s_{c_1}, s_{c_2}, \cdots, s_{c_m}, s_b)$
$= Amin(Amin(s_{a_1}, \cdots, s_{a_{n_1}}), s_b);$
　if $r_{c_1} < r_b$, then
　$Amin(s_{a_1}, \cdots, s_{a_{n_1}}, s_b)$
$= \{s_{c_1}, s_{c_2}, \cdots, s_{c_m}\}$
$= Amin(s_{c_1}, s_{c_2}, \cdots, s_{c_m}, s_b)$
$= Amin(Amin(s_{a_1}, \cdots, s_{a_{n_1}}), s_b);$
　If $r_{c_1} = r_b$ and $s_b = s_{c_j}$ for some $j \in \{1, 2, \cdots, m\}$, then

$$\text{Amin}(s_{a_1}, \cdots, s_{a_{n_1}}, s_b)$$
$$= \{s_{c_1}, s_{c_2}, \cdots, s_{c_m}\}$$
$$= \text{Amin}(s_{c_1}, s_{c_2}, \cdots, s_{c_m}, s_b)$$
$$= \text{Amin}(\text{Amin}(s_{a_1}, \cdots, s_{a_{n_1}}), s_b);$$

if $r_{c_1} = r_b$ and $s_b \neq s_{c_j}$ for all $j \in \{1, 2, \cdots, m\}$, then

$$\text{Amin}(s_{a_1}, \cdots, s_{a_{n_1}}, s_b)$$
$$= \{s_{c_1}, s_{c_2}, \cdots, s_{c_m}, s_b\}$$
$$= \text{Amin}(s_{c_1}, s_{c_2}, \cdots, s_{c_m}, s_b)$$
$$= \text{Amin}(\text{Amin}(s_{a_1}, \cdots, s_{a_{n_1}}), s_b).$$

□

Proposition 4. *Let s_{a_j} ($j = 1, 2, \cdots, n_1$) and s_{b_k} ($k = 1, 2, \cdots, n_2$) be two collections of LCFNs, then*

(1) $Amax(s_{a_1}, \cdots, s_{a_{n_1}}, s_{b_1}, \cdots, s_{b_{n_2}}) = Amax(Amax(s_{a_1}, \cdots, s_{a_{n_1}}), Amax(s_{b_1}, \cdots, s_{b_{n_2}}));$

(2) $Amin(s_{a_1}, \cdots, s_{a_{n_1}}, s_{b_1}, \cdots, s_{b_{n_2}}) = Amin(Amin(s_{a_1}, \cdots, s_{a_{n_1}}), Amin(s_{b_1}, \cdots, s_{b_{n_2}})).$

Proof. Trivial from Proposition 3. □

Definition 6. *Let $G_1 = \{s_{a_{1j_1}} (j_1 = 1, 2, \cdots, n_1)\}$, $G_2 = \{s_{a_{2j_2}} (j_2 = 1, 2, \cdots, n_2)\}$, ..., $G_k = \{s_{a_{kj_l}} (j_l = 1, 2, \cdots, n_k)\}$ be collections of LCFNs, the Amax-Amin operator defined as*

$$Amax\text{-}Amin\{G_1, G_2, \cdots, G_k\} = Amax(Amin(G_1), Amin(G_2), \cdots, Amin(G_k)). \tag{7}$$

The Amin-Amax operator defined as

$$Amin\text{-}Amax\{G_1, G_2, \cdots, G_k\} = Amin(Amax(G_1), Amax(G_2), \cdots, Amax(G_k)). \tag{8}$$

The goal of the Amax-Amin operator is to maximize the minimum LCFNs, and the goal of the Amin-Amax operator is to minimize the maximum LCFNs.

Proposition 5. *Let $G_1 = \{s_{a_{1j_1}}(j_1 = 1, 2, \cdots, n_1)\}$, $G_2 = \{s_{a_{2j_2}}(j_2 = 1, 2, \cdots, n_2)\}$, ..., $G_k = \{s_{a_{kj_l}}(j_l = 1, 2, \cdots, n_k)\}$ be collections of LCFNs, then*

(1) $Amax\text{-}Amin\{ G_1, G_2, \cdots, G_k\} \geq Amax\text{-}Amin\{G_1, G_2, \cdots, G_{k-1}\};$

(2) $Amin\text{-}Amax\{G_1, G_2, \cdots, G_{k-1}\} \geq Amin\text{-}Amax\{G_1, G_2, \cdots, G_k\}.$

Proof. Trivial. □

4. Group Decision-Making Method Based on the Amin-Amax and LCFWA Operators

In this section, the Amin-Amax and LCFWA operators are applied to decision making. We also show the consistency and superiority of the proposed methods by comparing with some conventional methods.

A decision-making problem considered can be described as follows: let $X = \{x_1, x_2, \cdots, x_n\}$ be the set of alternatives and $S = \{s_a | a = r_a e^{i\omega_a}, r \in [0, t], w \in [0, 2\pi)\}$ the linguistic complex fuzzy set. Assume that $D = \{d_1, d_2, \cdots, d_p\}$ is the set of decision makers and $R = \left(s_{a_{jk}}\right)_{p \times n}$ is their linguistic complex fuzzy decision matrix, where each $s_{a_{jk}}$ is a LCFN on S and represents the linguistic assessment of the alternative $x_k \in X$ obtained by the decision maker $d_j \in D$.

Applying the Amin-Amax operator or LCFWA operator on LCFNs, our selection method of the alternatives is given as follows:

Seep 1: If the important weights of decision makers are unknown, then we utilize the Amax operator to aggregate all $s_{a_{jk}}$ ($j = 1, 2, \cdots, p$) for each alternative $x_k \in X$,

$$s_{a_k} = \text{Amax}(s_{a_{1k}}, s_{a_{2k}}, \cdots, s_{a_{pk}}).$$

If the important weights of decision makers are given as $w = \{w_1, w_2, \cdots, w_p\}$, then we utilize the LCFWA operator to aggregate all $s_{a_{jk}}$ ($j = 1, 2, \cdots, p$) for each alternative $x_k \in X$,

$$s'_{a_k} = \text{LCFWA}(s_{a_{1k}}, s_{a_{2k}}, \cdots, s_{a_{pk}}).$$

Seep 2: Rank all the alternatives x_k in accordance with s_{a_k} or s'_{a_k} ($k = 1, 2, \cdots, n$).

Note that in the case of that the important weights of decision makers are unknown, we can use utilize the Amin-Amax operator to combine setp step 1 and step 2, i.e.,

$$\text{Amin}(\text{Amax}(s_{a_{11}}, s_{a_{21}}, \cdots, s_{a_{p1}}), \ \text{Amax}(s_{a_{12}}, s_{a_{22}}, \cdots, s_{a_{p2}}), \cdots, \text{Amax}(s_{a_{1n}}, s_{a_{2n}}, \cdots, s_{a_{pn}})).$$

If the result is denoted by $\{s_{a_{j_1 k_1}}, s_{a_{j_2 k_2}}, \cdots, s_{a_{j_q k_q}}\}$, then $\{j_1, j_2, \cdots, j_q\}$ is the set of the best alternatives.

Example 4. *In real life, we ask strangers for directions, assume that $X = \{x_1, x_2, x_3\}$ is the set of alternatives, $D = \{d_1, d_2, d_3, d_4\}$ is the set of strangers, the linguistic complex fuzzy set given in Table 1 is used to generate the linguistic expressions. The assessments given by strangers to the four alternatives are shown in Table 2.*

Table 1. Representation of spatial orientation.

	Very Near	Near	Slightly Near	Not Far Not Near	Slightly Far	Far	Very Far
East	s_1	s_2	s_3	s_4	s_5	s_6	s_7
South by east	$s_{e^{i\pi 23/12}}$	$s_{2e^{i\pi 23/12}}$	$s_{3e^{i\pi 23/12}}$	$s_{4e^{i\pi 23/12}}$	$s_{5e^{i\pi 23/12}}$	$s_{6e^{i\pi 23/12}}$	$s_{7e^{i\pi 23/12}}$
Southeast	$s_{e^{i\pi 11/6}}$	$s_{2e^{i\pi 11/6}}$	$s_{3e^{i\pi 11/6}}$	$s_{4e^{i\pi 11/6}}$	$s_{5e^{i\pi 11/6}}$	$s_{6e^{i\pi 11/6}}$	$s_{7e^{i\pi 11/6}}$
East-southeast	$s_{e^{i\pi 7/4}}$	$s_{2e^{i\pi 7/4}}$	$s_{3e^{i\pi 7/4}}$	$s_{4e^{i\pi 7/4}}$	$s_{5e^{i\pi 7/4}}$	$s_{6e^{i\pi 7/4}}$	$s_{7e^{i\pi 7/4}}$
Southeast	$s_{e^{i\pi 5/3}}$	$s_{2e^{i\pi 5/3}}$	$s_{3e^{i\pi 5/3}}$	$s_{4e^{i\pi 5/3}}$	$s_{5e^{i\pi 5/3}}$	$s_{6e^{i\pi 5/3}}$	$s_{7e^{i\pi 5/3}}$
South-southeast	$s_{e^{i\pi 19/12}}$	$s_{2e^{i\pi 19/12}}$	$s_{3e^{i\pi 19/12}}$	$s_{4e^{i\pi 19/12}}$	$s_{5e^{i\pi 19/12}}$	$s_{6e^{i\pi 19/12}}$	$s_{7e^{i\pi 19/12}}$
South	$s_{e^{i\pi 3/2}}$	$s_{2e^{i\pi 3/2}}$	$s_{3e^{i\pi 3/2}}$	$s_{4e^{i\pi 3/2}}$	$s_{5e^{i\pi 3/2}}$	$s_{6e^{i\pi 3/2}}$	$s_{7e^{i\pi 3/2}}$
South by west	$s_{e^{i\pi 17/12}}$	$s_{2e^{i\pi 17/12}}$	$s_{3e^{i\pi 17/12}}$	$s_{4e^{i\pi 17/12}}$	$s_{5e^{i\pi 17/12}}$	$s_{6e^{i\pi 17/12}}$	$s_{7e^{i\pi 17/12}}$
Southwest by south	$s_{e^{i\pi 4/3}}$	$s_{2e^{i\pi 4/3}}$	$s_{3e^{i\pi 4/3}}$	$s_{4e^{i\pi 4/3}}$	$s_{5e^{i\pi 4/3}}$	$s_{6e^{i\pi 4/3}}$	$s_{7e^{i\pi 4/3}}$
Southwest	$s_{e^{i\pi 5/4}}$	$s_{2e^{i\pi 5/4}}$	$s_{3e^{i\pi 5/4}}$	$s_{4e^{i\pi 5/4}}$	$s_{5e^{i\pi 5/4}}$	$s_{6e^{i\pi 5/4}}$	$s_{7e^{i\pi 5/4}}$
Southwest by west	$s_{e^{i\pi 7/6}}$	$s_{2e^{i\pi 7/6}}$	$s_{3e^{i\pi 7/6}}$	$s_{4e^{i\pi 7/6}}$	$s_{5e^{i\pi 7/6}}$	$s_{6e^{i\pi 7/6}}$	$s_{7e^{i\pi 7/6}}$
West by south	$s_{e^{i\pi 13/12}}$	$s_{2e^{i\pi 13/12}}$	$s_{3e^{i\pi 13/12}}$	$s_{4e^{i\pi 13/12}}$	$s_{5e^{i\pi 13/12}}$	$s_{6e^{i\pi 13/12}}$	$s_{7e^{i\pi 13/12}}$
West	$s_{e^{i\pi}}$	$s_{2e^{i\pi}}$	$s_{3e^{i\pi}}$	$s_{4e^{i\pi}}$	$s_{5e^{i\pi}}$	$s_{6e^{i\pi}}$	$s_{7e^{i\pi}}$
West by north	$s_{e^{i\pi 11/12}}$	$s_{2e^{i\pi 11/12}}$	$s_{3e^{i\pi 11/12}}$	$s_{4e^{i\pi 11/12}}$	$s_{5e^{i\pi 11/12}}$	$s_{6e^{i\pi 11/12}}$	$s_{7e^{i\pi 11/12}}$
Northwest by west	$s_{e^{i\pi 5/6}}$	$s_{2e^{i\pi 5/6}}$	$s_{3e^{i\pi 5/6}}$	$s_{4e^{i\pi 5/6}}$	$s_{5e^{i\pi 5/6}}$	$s_{6e^{i\pi 5/6}}$	$s_{7e^{i\pi 5/6}}$
Northwest	$s_{e^{i\pi 3/4}}$	$s_{2e^{i\pi 3/4}}$	$s_{3e^{i\pi 3/4}}$	$s_{4e^{i\pi 3/4}}$	$s_{5e^{i\pi 3/4}}$	$s_{6e^{i\pi 3/4}}$	$s_{7e^{i\pi 3/4}}$
North-northwest	$s_{e^{i\pi 2/3}}$	$s_{2e^{i\pi 2/3}}$	$s_{3e^{i\pi 2/3}}$	$s_{4e^{i\pi 2/3}}$	$s_{5e^{i\pi 2/3}}$	$s_{6e^{i\pi 2/3}}$	$s_{7e^{i\pi 2/3}}$
North by west	$s_{e^{i\pi 7/12}}$	$s_{2e^{i\pi 7/12}}$	$s_{3e^{i\pi 7/12}}$	$s_{4e^{i\pi 7/12}}$	$s_{5e^{i\pi 7/12}}$	$s_{6e^{i\pi 7/12}}$	$s_{7e^{i\pi 7/12}}$
North	$s_{e^{i\pi/2}}$	$s_{2e^{i\pi/2}}$	$s_{3e^{i\pi/2}}$	$s_{4e^{i\pi/2}}$	$s_{5e^{i\pi/2}}$	$s_{6e^{i\pi/2}}$	$s_{7e^{i\pi/2}}$
North by east	$s_{e^{i\pi 5/12}}$	$s_{2e^{i\pi 5/12}}$	$s_{3e^{i\pi 5/12}}$	$s_{4e^{i\pi 5/12}}$	$s_{5e^{i\pi 5/12}}$	$s_{6e^{i\pi 5/12}}$	$s_{7e^{i\pi 5/12}}$
North-northeast	$s_{e^{i\pi/3}}$	$s_{2e^{i\pi/3}}$	$s_{3e^{i\pi/3}}$	$s_{4e^{i\pi/3}}$	$s_{5e^{i\pi/3}}$	$s_{6e^{i\pi/3}}$	$s_{7e^{i\pi/3}}$
Northeast	$s_{e^{i\pi/4}}$	$s_{2e^{i\pi/4}}$	$s_{3e^{i\pi/4}}$	$s_{4e^{i\pi/4}}$	$s_{5e^{i\pi/4}}$	$s_{6e^{i\pi/4}}$	$s_{7e^{i\pi/4}}$
East-northeast	$s_{e^{i\pi/6}}$	$s_{2e^{i\pi/6}}$	$s_{3e^{i\pi/6}}$	$s_{4e^{i\pi/6}}$	$s_{5e^{i\pi/6}}$	$s_{6e^{i\pi/6}}$	$s_{7e^{i\pi/6}}$
East by north	$s_{e^{i\pi/12}}$	$s_{2e^{i\pi/12}}$	$s_{3e^{i\pi/12}}$	$s_{4e^{i\pi/12}}$	$s_{5e^{i\pi/12}}$	$s_{6e^{i\pi/12}}$	$s_{7e^{i\pi/12}}$

Table 2. Assessments.

	x_1	x_2	x_3
d_1	Far, South-southeast	Very far, North-northeast	Very far, Northwest by west
d_2	Slightly far, East-southeast	Slightly near, Northeast	Not far not near, Northwest
d_3	Far, Southeast	Slightly far, East by north	Slightly far, West by north
d_4	Not far not near, Southeast	Slightly near, East-northeast	Slightly far, Northwest

By using Table 1, we transform the linguistic expressions into LCFNs, which are shown in Table 3. By using the Amax operator, we get

$$
\begin{aligned}
s_{a_1} &= \mathrm{Amax}(s_{6e^{i\pi 19/12}}, s_{5e^{i\pi 7/4}}, s_{6e^{i\pi 5/3}}, s_{4e^{i\pi 5/3}}) = \{s_{6e^{i\pi 19/12}}, s_{6e^{i\pi 5/3}}\}; \\
s_{a_2} &= \mathrm{Amax}(s_{7e^{i\pi/3}}, s_{3e^{i\pi/4}}, s_{5e^{i\pi/12}}, s_{3e^{i\pi/6}}) = \{s_{7e^{i\pi/3}}\}; \\
s_{a_3} &= \mathrm{Amax}(s_{7e^{i\pi 5/6}}, s_{4e^{i\pi 3/4}}, s_{5e^{i\pi 11/12}}, s_{5e^{i\pi 3/4}}) = \{s_{7e^{i\pi 5/6}}\}.
\end{aligned}
$$

Table 3. Assessments transformed into LCFNs.

	x_1	x_2	x_3
d_1	$s_{6e^{i\pi 19/12}}$	$s_{7e^{i\pi/3}}$	$s_{7e^{i\pi 5/6}}$
d_2	$s_{5e^{i\pi 7/4}}$	$s_{3e^{i\pi/4}}$	$s_{4e^{i\pi 3/4}}$
d_3	$s_{6e^{i\pi 5/3}}$	$s_{5e^{i\pi/12}}$	$s_{5e^{i\pi 11/12}}$
d_4	$s_{4e^{i\pi 5/3}}$	$s_{3e^{i\pi/6}}$	$s_{5e^{i\pi 3/4}}$

Clearly, we have $s_{a_1} < s_{a_2} = s_{a_3}$ since $|6e^{i\pi 19/12}| = |6e^{i\pi 5/3}| < |s_{7e^{i\pi/3}}| = |7e^{i\pi 5/6}|$. In this case, the decision maker is pessimistic, x_1 is the nearest alternative.

If the decision maker thinks the strangers are the same weight, i.e., $w = \{\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\}$. Then, by using the LCFAA operator to aggregate all $s_{a_{jk}}$ (j = 1, 2, 3, 4) for each alternative $x_k \in X$,

$$
\begin{aligned}
s'_{a_1} &= \mathrm{LCFAA}(s_{6e^{i\pi 19/12}}, s_{5e^{i\pi 7/4}}, s_{6e^{i\pi 5/3}}, s_{4e^{i\pi 5/3}}) \\
&= s_{\frac{6e^{i\pi 19/12}+5e^{i\pi 7/4}+6e^{i\pi 5/3}+4e^{i\pi 5/3}}{4}} \\
&= s_{5.1567e^{-1.0597i}} \\
s'_{a_2} &= \mathrm{LCFAA}(s_{7e^{i\pi/3}}, s_{3e^{i\pi/4}}, s_{5e^{i\pi/12}}, s_{3e^{i\pi/6}}) \\
&= s_{\frac{7e^{i\pi/3}+7e^{i\pi/4}+7e^{i\pi/12}+7e^{i\pi/6}}{4}} \\
&= s_{4.2631e^{0.6994i}} \\
s'_{a_3} &= \mathrm{LCFAA}(s_{7e^{i\pi 5/6}}, s_{4e^{i\pi 3/4}}, s_{5e^{i\pi 11/12}}, s_{5e^{i\pi 3/4}}) \\
&= s_{\frac{7e^{i\pi 5/6}+4e^{i\pi 3/4}+5e^{i\pi 11/12}+5e^{i\pi 3/4}}{4}} \\
&= s_{3.2093e^{2.6987i}}.
\end{aligned}
$$

Therefore, the ranking of alternatives $s'_{a_1} > s'_{a_2} > s'_{a_3}$. Therefore, x_3 is the nearest alternative.

In our methods, the phase term plays a role in decision making. First, we show the consistency of the Amin-Amax method by comparing it with the conventional Min-Max method. Since for any collection of LCFNs s_{a_j} (j = 1, 2, $\cdots$, n), we have

$$
\begin{aligned}
|\mathrm{Amin}(s_{a_1}, s_{a_2}, \cdots, s_{a_n})| &= \mathrm{Min}(|s_{a_1}|, |s_{a_2}|, \cdots, |s_{a_n}|), \\
|\mathrm{Amax}(s_{a_1}, s_{a_2}, \cdots, s_{a_n})| &= \mathrm{Max}(|s_{a_1}|, |s_{a_2}|, \cdots, |s_{a_n}|).
\end{aligned}
$$

Then the final sorting result of the Amin-Amax method is consistent with that of the conventional Min-Max method.

If we only consider the amplitude term of LCFNs in aggregation by using the conventional linguistic fuzzy weighted averaging (FWA) operator,

$$s''_{a_1} = \text{FWA}\left(|s_{6e^{i\pi 19/12}}|, |s_{5e^{\frac{i\pi 7}{4}}}|, |s_{6e^{\frac{i\pi 5}{3}}}|, |s_{4e^{\frac{i\pi 5}{3}}}|\right) = s_{\frac{6+5+6+4}{4}} = s_{5.25};$$

$$s''_{a_2} = \text{FWA}\left(|s_{7e^{\frac{i\pi}{3}}}|, |s_{3e^{\frac{i\pi}{4}}}|, |s_{5e^{\frac{i\pi}{12}}}|, |s_{3e^{\frac{i\pi}{6}}}|\right) = s_{\frac{7+7+7+7}{4}} = s_7;$$

$$s'_{a_3} = \text{FWA}\left(|s_{7e^{\frac{i\pi 5}{6}}}|, |s_{4e^{\frac{i\pi 3}{4}}}|, |s_{5e^{\frac{i\pi 11}{12}}}|, |s_{5e^{\frac{i\pi 3}{4}}}|\right) = s_{\frac{7+4+5+5}{4}} = s_{5.25}.$$

Therefore, the ranking of alternatives is $s''_{a_1} > s''_{a_2} = s''_{a_3}$. Therefore, both x_2 and x_3 are the nearest alternative.

Although the ranking method relies only on the amplitude term of LCFNs, the phase term can not be neglected in the aggregation of LCFSs. LCFAA is a generalization of conventional linguistic FWA. Compared with the conventional linguistic FWA operator, the phase term in the LCFAA-based method is considered to make the result more accurate and complete.

5. Case Study

In the following, we further illustrate the practicality of LCFSs by utilizing a practical example.

We assume that a person (decision maker) likes to take the train from Hangzhou to one of three cities (Beijing, Xiamen and Changsha) on 1 April 2022. The linguistic expressions of travel times are shown in Table 4.

Table 4. LFSs of travel times.

LFSs	Very Near	Near	Slightly Near	Not Far Not Near	Slightly Far	Far	Very Far
Travel time	<1 h	1–2 h	2–3 h	3–4 h	4–5 h	5–6 h	>6 h

The departure times are divided into three intervals, $t_1 = [8:00–11:00]$, $t_2 = [11:00–14:00]$ and $t_3 = [14:00–20:00]$. Since the possibility of departure time belonging to these intervals are different, each interval is given a weight, as shown in Table 5.

Table 5. Relative weights.

	$t_1 = [8:00–11:00]$	$t_2 = [11:00–14:00]$	$t_3 = [14:00–20:00]$
w_j	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{2}$

From the China Railway's official website (https://12306.cn, accessed on 26 February 2023), the number of trains from Hangzhou to these cities on 1 April 2022 are given in Tables 6–8, respectively. Here, we only consider the high-speed trains.

Table 6. Trains from Hangzhou to Beijing on 1 April 2022.

Departure Time	<4 h	4–5 h	5–6 h	>6 h
$t_1 = [8:00–11:00]$	0	0	0	1
$t_2 = [11:00–14:00]$	0	0	0	3
$t_3 = [14:00–20:00]$	0	1	0	1

Table 7. Trains from Hangzhou to Xiamen on 1 April 2022.

Departure Time	<4 h	4–5 h	5–6 h	>6 h
$t_1 = [8{:}00{-}11{:}00]$	0	0	1	2
$t_2 = [11{:}00{-}14{:}00]$	0	2	0	2
$t_3 = [14{:}00{-}20{:}00]$	0	0	0	3

Table 8. Trains from Hangzhou to Changsha on 1 April 2022.

Departure Time	<4 h	4–5 h	5–6 h	>6 h
$t_1 = [8{:}00{-}11{:}00]$	0	3	0	0
$t_2 = [11{:}00{-}14{:}00]$	0	2	0	0
$t_3 = [14{:}00{-}20{:}00]$	0	4	0	0

Based on above three tables, the linguistic assessments of these cities for different departure times are shown in Table 9.

Table 9. Linguistic assessments.

Departure Time	Beijing	Xiamen	Changsha
$t_1 = [8{:}00{-}11{:}00]$	Very far, north	Far, south	Slightly far, southwest
$t_2 = [11{:}00{-}14{:}00]$	Far, north	Slightly far, south	Slightly far, southwest
$t_3 = [14{:}00{-}20{:}00]$	Slightly far, north	Very far, south	Slightly far, southwest

By using Table 1, we transform the linguistic assessments into LCFNs, which are shown in Table 10.

Table 10. Linguistic assessments transformed into LCFNs.

Departure Time	Beijing	Xiamen	Changsha
$t_1 = [8{:}00{-}11{:}00]$	$s_{7e^{i\pi/2}}$	$s_{6e^{i\pi 3/2}}$	$s_{5e^{i\pi 5/4}}$
$t_2 = [11{:}00{-}14{:}00]$	$s_{6e^{i\pi/2}}$	$s_{5e^{i\pi 3/2}}$	$s_{5e^{i\pi 5/4}}$
$t_3 = [14{:}00{-}20{:}00]$	$s_{5e^{i\pi/2}}$	$s_{7e^{i\pi 3/2}}$	$s_{5e^{i\pi 5/4}}$

Case 1: If we do not consider the possibility of the departure time belonging to different intervals, by using Amax operator, we get

$$s_{Beijing} = \mathrm{Amax}\left(s_{7e^{i\pi/2}}, s_{6e^{i\pi/2}}, s_{5e^{i\pi/2}}\right) = \left\{s_{7e^{i\pi/2}}\right\}.$$

$$s_{Xiamen} = Amax\left(s_{6e^{\frac{i\pi 3}{2}}}, s_{5e^{\frac{i\pi 3}{2}}}, s_{7e^{\frac{i\pi 3}{2}}}\right) = \left\{s_{7e^{\frac{i\pi}{3}}}\right\}.$$

$$s_{Changsha} = \mathrm{Amax}\left(s_{5e^{i\pi 5/4}}, s_{5e^{i\pi 5/4}}, s_{5e^{i\pi 5/4}}\right) = \left\{s_{5e^{i\pi 5/4}}\right\}.$$

Clearly, we have $s_{\mathrm{Changsha}} < s_{\mathrm{Xiamen}} = s_{\mathrm{Beijing}}$ since $|5e^{i\pi 5/4}| < |7e^{i\pi/3}| = |7e^{i\pi/2}|$. In this case, the decision maker is pessimistic, Changsha is the nearest alternative.

Case 2: If we consider the possibility of the departure time belonging to different intervals, i.e., the weight vector of intervals for the departure times is w $= \{\frac{1}{3}, \frac{1}{6}, \frac{1}{2}\}$, then by using the LCFWA operator to aggregate all $s_{a_{jk}}$ (j $= 1, 2, 3$) for each alternative,

$$s'_{Beijing} = \text{LCFWA}(s_{7e^{i\pi/2}}, s_{6e^{i\pi/2}}, s_{5e^{i\pi/2}}) = s_{\frac{2*7e^{i\pi/2}+6e^{i\pi/2}+3*5e^{i\pi5/3}}{6}} \cdot = s_{5.83e^{i\pi/2}}$$

$$s'_{Xiamen} = \text{LCFWA}(s_{6e^{i\pi3/2}}, s_{5e^{i\pi3/2}}, s_{7e^{i\pi3/2}}) = s_{\frac{2*6e^{i\pi3/2}+5e^{i\pi3/2}+3*7e^{i\pi3/2}}{6}} = s_{6.33e^{0.6994i}}$$

$$s'_{Changsha} = \text{LCFWA}(s_{5e^{i\pi5/4}}, s_{5e^{i\pi5/4}}, s_{5e^{i\pi5/4}}) = s_{\frac{2*5e^{i\pi5/4}+5e^{i\pi5/4}+3*5e^{i\pi5/4}}{6}} = s_{5e^{i\pi5/4}}$$

Therefore, the ranking of alternatives becomes $s'_{Xiamen} > s'_{Beijing} > s'_{Changsha}$. Therefore, Changsha also is the nearest alternative. Note that Beijing is nearer than Xiamen in this way.

In this example, Case 1 only considered the amplitude term of LCFNs, s_{Xiamen} and $s_{Beijing}$ are not comparable. Case 2 considered both the amplitude and phase terms of LCFNs, which makes the result more accurate and complete.

6. Conclusions

The objective of this work is to establish the LCFS, which is the combination of the CFS and LFS to manage spatial directional information in real-decision theory. Furthermore, Amax, Amin, LCFWA and LCFAA operators of LCFS have been presented. After this, these operators are utilized in decision making under the LCFS environment to examine the feasibility and validity of the explored operators. Finally, the numerical example for established operators is solved to express the validity of the explored work.

Of course, many works should be done on the theory of LCFSs. We give some possible topics for future consideration.

1. We do not consider the weight determination method of LCFNs. This is a problem left for further investigation. In this paper, the numerical weight is used in aggregation. In future work, we will consider the linguistic weights and complex weights for LCFNs aggregation.
2. We will also study the distance and entropy measures of LCFSs and develop more aggregation operators for group decision-making problems with LCFSs information.
3. There exists LCFNs that are incomparable. For any two LCFNs s_a, s_b with $|a| = |b|$ but $a \neq b$, they are incomparable. Naturally, a more detailed discussion of the comparison of LCFNs will be both necessary and interesting.

Data Availability Statement: The train information can be found in the China Railway's official website (https://12306.cn, accessed on 26 February 2023).

Conflicts of Interest: The author declares no conflict of interest.

Abbreviations

The following abbreviations are used in this manuscript:

CFS	Complex fuzzy set
LFS	Linguistic fuzzy set
LCFS	Linguistic complex fuzzy set
LCFWA	Linguistic complex fuzzy weighted averaging
Amax	Amplitude max
Amin	Amplitude min
IVFS	Interval-valued fuzzy set
IFS	Intuitionistic fuzzy set
IVIFS	Interval-valued intuitionistic fuzzy set

PFS	Pythagorean fuzzy set
HFS	Hesitant fuzzy set
CFN	Complex fuzzy number
LCFN	Linguistic complex fuzzy number
LCFV	Linguistic complex fuzzy value
FWA	Fuzzy weighted averaging

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