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C-R Immersions and Sub-Riemannian Geometry

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Abstract: On any strictly pseudoconvex CR manifold M, of CR dimension n, equipped with a positively oriented contact form θ, we consider natural ϵ-contractions, i.e., contractions g^M_ϵ of the Levi form G_θ, such that the norm of the Reeb vector field T of (M, θ) is of order $O(\epsilon^{-1})$. We study isopseudohermitian (i.e., $f^*\Theta = \theta$) Cauchy–Riemann immersions $f : M \to (A, \Theta)$ between strictly pseudoconvex CR manifolds M and A, where Θ is a contact form on A. For every contraction g^A_ϵ of the Levi form G_Θ, we write the embedding equations for the immersion $f : M \to (A, g^A_\epsilon)$. A pseudohermitan version of the Gauss equation for an isopseudohermitian C-R immersion is obtained by an elementary asymptotic analysis as $\epsilon \to 0^+$. For every isopseudohermitian immersion $f : M \to S^{2N+1}$ into a sphere $S^{2N+1} \subset \mathbb{C}^{N+1}$, we show that Webster's pseudohermitian scalar curvature R of (M, θ) satisfies the inequality $R \le 2n\left[(f^*g_\Theta)(T, T) + n + 1\right] + \frac{1}{2}\left\{\|H(f)\|^2_{g^f_\Theta} + \|\mathrm{trace}_{G_\theta}\Pi_{H(M)}(\nabla^\top - \nabla)\|^2_{f^*g_\Theta}\right\}$ with equality if and only if $B(f) = 0$ and $\nabla^\top = \nabla$ on $H(M) \otimes H(M)$. This gives a pseudohermitian analog to a classical result by S-S. Chern on minimal isometric immersions into space forms.

Keywords: Levi form; contact form; Tanaka–Webster connection; pseudohermitian scalar curvature; sublaplacian; CR immersion; isopseudohermitian immersion; sub-Riemannian structure; ϵ-contraction; pseudohermitan second fundamental form; pseudohermitian Gauss equation

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1. Introduction

The present paper has two main purposes: a general one, which looks at certain problems originating in complex analysis from the point of view of pseudohermitian geometry, and a more specific purpose, which is contributing to the study of CR immersions between strictly pseudoconvex CR manifolds, from a differential geometric viewpoint. Pseudohermitian geometry was brought into mathematical practice by S.M. Webster [1] and N. Tanaka [2], and the term *pseudohermitian structure* was coined by S.M. Webster himself (see *op. cit.*). Pseudohermitian geometry soon became a popular research area, and its development up to 2006 is reported in the monographs by S. Dragomir and G. Tomassini [3] and by E. Barletta, S. Dragomir, and K.L. Duggal [4]. The further growth of the theory, though confined to the topic of subelliptic harmonic maps and vector fields on pseudohermitian manifolds, is reported in the monograph by S. Dragomir and D. Perrone [5]. The part added to the theory of CR immersions by the present paper, which is deriving a pseudohermitian analog to the Gauss equation (of an isometric immersion between Riemannian manifolds), aims to contribute applications to rigidity theory. The remainder of the Introduction is devoted to a brief parallel between rigidity within Riemannian geometry on one hand and complex analysis on the other, and to a glimpse into the main results. The authors benefit from the (partial) embedding (described in detail in [6] and adopted there for different purposes, i.e., the study of the geometry of Jacobi fields on Sasakian manifolds) of pseudohermitian geometry into sub-Riemannian geometry, and the main novelty from a methodological viewpoint is the use of methods in sub-Riemannian geometry (see [7,8]).

Rigidity in differential geometry has a long history, perhaps starting with rigidity of regular curves $\alpha : I \to \mathbb{R}^3$ of curvature $k(s) > 0$ and torsion $\tau(s)$ ($s \in I$): any other regular curve $\overline{\alpha} : I \to \mathbb{R}^3$ with the same curvature $k(s)$ and torsion $\tau(s)$ differs from α by a *rigid* motion i.e., $\overline{\alpha}(s) = \rho\left[\alpha(s)\right] + c$ for some orthogonal linear map $\rho : \mathbb{R}^3 \to \mathbb{R}^3$ and some vector $c \in \mathbb{R}^3$. See M.P. Do Carmo [9], p. 19.

As a step further, one knows about the rigidity of real hypersurfaces in Euclidean space $\mathbb{R}^{n+1}$, i.e., if $f : M \to \mathbb{R}^{n+1}$ and $\overline{f} : M \to \mathbb{R}^{n+1}$ are two isometric immersions of an n-dimensional orientable Riemannian manifold M, whose second fundamental forms coincide on M, then $\overline{f} = \tau \circ f$ for some isometry $\tau : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$. See Theorem 6.4 in S. Kobayashi and K Nomizu [10], Volume II, p. 45.

A close analog to rigidity in the above sense, occurring in complex analysis of functions of several complex variables, is that of rigidity of CR immersions, and our starting point is S.M. Webster's legacy; see [11]. A *CR immersion* is a map $f : M \to A$ of CR manifolds M and A such that: (i) f is a C^∞ immersion, and (ii) f is a CR map; i.e., it maps the CR structure $T_{1,0}(M)$ onto $T_{1,0}(A)$. Let $(M,\, T_{1,0}(M))$ be a $(2n+1)$-dimensional CR manifold, of CR dimension n. M is a *CR hypersurface* of the sphere S^{2n+3} if $M \subset S^{2n+3}$ is a (codimension two) submanifold and the inclusion $\iota : M \to S^{2n+3}$ is a CR immersion. A CR hypersurface M is *rigid* in S^{2n+3} if for any other CR hypersurface $M' \subset S^{2n+3}$, every CR isomorphism $\phi : M \to M'$ extends to a CR automorphism $\Phi \in \mathrm{Aut}_{\mathrm{CR}}\left(S^{2n+3}\right)$. By a classical result of S.M. Webster (see [11]), if $n \geq 3$, every CR hypersurface $M \subset S^{2n+3}$ is rigid.

The proof (see [10], Volume II, pp. 45–46) of rigidity of real hypersurfaces in Euclidean space relies on the analysis of the Gauss–Codazzi equations for a given isometric immersion, and the treatment of rigidity of CR hypersurfaces in S^{2n+3} exploits (again see [11]) in a rather similar manner CR, or more precisely pseudohermitian, analogs to Gauss–Codazzi equations, where the ambient and intrinsic Levi–Civita connections (at work within the geometry of isometric immersions between Riemannian manifolds) are replaced by the Tanaka–Webster connections. The Tanaka–Webster connection is a canonical connection (similar to the Levi–Civita connection in Riemannian geometry, and to the Chern connection in Hermitian geometry) occurring on any nondegenerate CR manifold, on which a contact form has been fixed (see [1,2]). The Tanaka–Webster connection is also due to S.M. Webster (see [1]), yet was independently discovered by N. Tanaka in a monograph (see [2]) that remained little known to Western scientists up to the end of the 1980s. The pseudohermitian analog to the Gauss equation in Webster's theory (see [11]) is stated as:

$$\begin{aligned} S_{\beta\overline{\alpha}\rho\overline{\sigma}} = {} & -b_{\beta\rho}\, b_{\overline{\alpha}\,\overline{\sigma}} - \tfrac{1}{(n+1)(n+2)}\left[g_{\beta\overline{\alpha}}\, g_{\rho\overline{\sigma}} + g_{\rho\overline{\alpha}}\, g_{\beta\overline{\sigma}}\right] b_{\mu\nu} b^{\mu\nu} \\ & + \tfrac{1}{n+2}\left[g_{\beta\overline{\alpha}}\, b_{\rho\mu}\, b^{\mu}{}_{,\overline{\sigma}} + g_{\rho\overline{\alpha}}\, b_{\beta\mu}\, b^{\mu}{}_{,\overline{\sigma}} + b_{\beta\mu}\, b^{\mu}{}_{,\overline{\alpha}}\, g_{\rho\overline{\sigma}} + b_{\rho\mu}\, b^{\mu}{}_{,\overline{\alpha}}\, g_{\beta\overline{\sigma}}\right]. \end{aligned} \tag{1}$$

Insufficient computational details are furnished in [11], and the derivation of (1) remains rather obscure.

A more recent tentative approach to the (CR analog to) the Gauss–Codazzi–Ricci equations was taken up by P. Ebenfelt, X-J. Huang, and D. Zaitsev (see [12]). They introduced and made use of a CR analog to the second fundamental form (of an isometric immersion), which is naturally associated with a given CR immersion and springs from work in complex analysis by B. Lamel (see [13,14]). Their pseudohermitian (analog to) the Gauss equation

$$R^A(X, Y, Z, W) = R(X, Y, Z, V) + \langle \Pi(X, Z),\, \Pi(Y, V) \rangle,$$

$$X, Y, Z, V \in T_{1,0}(M),$$

for a given CR immersion $f : M \to A$ depends on a particular choice of contact forms θ and Θ, respectively, on the submanifold M and on the ambient space A, such that: (i) $f^*\Theta = \theta$, and (ii) $f(M)$ is tangent to the ambient Reeb vector field T_A (the globally defined nowhere zero tangent vector field on A, transverse to the Levi distribution, uniquely determined by

$\Theta(T_A) = 1$ and $T_A \rfloor \Theta = 0$). However, the proof of the existence of such θ and Θ is purely local and, in general, global contact forms on M and A such that f is isopseudohermitian, and $T_A^\perp = 0$ might not exist at all.

The class of isopseudohermitian immersions between strictly pseudoconvex CR manifolds enjoying the property $T_A^\perp = 0$ was studied independently by S. Dragomir (see [15]). As it turns out, any CR immersion in the class is also isometric with respect to the Webster metrics, i.e., $f^* g_\Theta = g_\theta$, and then a pseudohermitian (analog to the) geometry of the second fundamental form (of an isometric immersion) may be built by closely following its Riemannian counterpart, in a rather trivial manner. Despite the enthusiastic review by K. Spallek (see [16]) and the later development (by S. Dragomir and A. Minor [17,18]) relating the geometry of the second fundamental form (of a CR immersion in the class above) to the Fefferman metrics of (M, θ) and (N, Θ), the built theory of CR immersions is not general enough: it does not suggest a path towards a theory of CR immersions not belonging to the class, within which one may hope to recover Webster's mysterious "Gauss equation" (1). It is our purpose, within the present paper, to adopt an entirely new approach to building a "second fundamental form" based theory of CR immersions, using methods coming from sub-Riemannian geometry (e.g., in the sense of R.S. Strichartz [8]).

That CR geometry (partially) embeds into sub-Riemannian geometry is a rather well-known fact: given a strictly pseudoconvex CR manifold M, endowed with a positively oriented contact form θ, the pair $(H(M), G_\theta)$, consisting of the Levi distribution $H(M) = \mathrm{Re}\{T_{1,0}(M) \oplus T_{0,1}(M)\}$ and the Levi form $G_\theta(X, Y) = (d\theta)(X, JY)$, $X, Y \in H(M)$, is a sub-Riemannian structure on M, and the Webster metric g_θ is a contraction of G_θ (see [6–8,19]).

We adopt the additional assumption that the given CR immersion $f : M \to A$ (between the strictly pseudoconvex CR manifolds M and A) is *isopseudohermitian*, i.e., $f^*\Theta = \theta$ for some choice of contact forms θ and Θ on M and A, respectively, yet we refrain from assuming that $f(M)$ is tangent to the Reeb vector field of the ambient space (A, Θ); rather, T_A will be, relative to $f(M)$, always oblique. $f(M)$ may be looked at as a submanifold in the Riemannian manifold (A, g_Θ), yet, by our assumption $T_A^\perp \neq 0$, the first fundamental form (i.e., the pullback $f^* g_\Theta$ to M of the ambient Webster metric g_Θ) of the given immersion $f : M \to (A, g_\Theta)$ does not coincide with the intrinsic Webster metric g_θ. That is, $f : (M, g_\theta) \to (A, g_\Theta)$ is not an isometric immersion, and the well-established and powerful apparatus based on the Gauss–Codazzi–Mainnardi–Ricci equations cannot be a priori applied to f.

To circumnavigate this obstacle, one endows A with the Riemannian metric g_ϵ^A, the contraction of the Levi form G_Θ associated with each $0 < \epsilon < 1$, given by

$$g_\epsilon^A = g_\Theta + \left(\frac{1}{\epsilon^2} - 1\right) \Theta \otimes \Theta. \tag{2}$$

Our strategy will be to regard $f(M)$ as a submanifold of the Riemannian manifold (A, g_ϵ^A) and derive the Gauss–Weingarten and Gauss–Ricci–Codazzi equations of the immersion $f : M \to (A, g_\epsilon^A)$. In the end, these will lead, as $\epsilon \to 0^+$, to the seek after pseudohermitian analogs to the embedding equations. To illustrate the expected results, we state the pseudohermitian Gauss equation of a CR immersion into a sphere.

Corollary 1. *Let M be a strictly pseudoconvex CR manifold, of CR dimension n, equipped with the positively oriented contact form $\theta \in \mathcal{P}_+(M)$. Let $f : M \to S^{2(n+k)+1}$, $k \geq 1$, be a CR immersion of M into the standard sphere $S^{2(n+k)+1}$ carrying the CR structure induced by the complex structure of $\mathbb{C}^{n+k+1}$. Let $\Theta = \frac{i}{2}(\overline{\partial} - \partial)|Z|^2$ be the canonical contact form on $S^{2(n+k)+1}$. If f is isopseudohermitian (i.e., $f^*\Theta = \theta$), then*

$$
\begin{aligned}
g_\theta\big(R^\nabla(X,Y)Z,\, W\big) &= g_\theta(Y,\, Z)\, g_\theta(X,\, W) - g_\theta(X,\, Z)\, g_\theta(Y,\, W) \\
&+ g^f_\Theta\big(B(f)(X,W),\, B(f)(Y,Z)\big) - g^f_\Theta\big(B(f)(Y,W),\, B(f)(X,Z)\big) \\
&+ (2\lambda - 1)\big\{\Omega(X,W)\,\Omega(Y,Z) - \Omega(Y,W)\,\Omega(X,Z)\big\} - 2\lambda\,\Omega(X,Y)\,\Omega(Z,W) \\
&- \Omega(Y,Z)\,A(X,W) - \Omega(X,W)\,A(Y,Z) + \Omega(X,Z)\,A(Y,W) + \Omega(Y,W)\,A(X,Z) \\
&+ \big(f^* g_\Theta\big)\big(U(f)(X,W),\, U(f)(Y,Z)\big) - \big(f^* g_\Theta\big)\big(U(f)(Y,W),\, U(f)(X,Z)\big) \\
&- \Omega(Y,Z)\,\big(f^* g_\Theta\big)\big(U(f)(X,W),\, T\big) - \Omega(X,W)\,\big(f^* g_\Theta\big)\big(U(f)(Y,Z),\, T\big) \\
&+ \Omega(X,Z)\,\big(f^* g_\Theta\big)\big(U(f)(Y,W),\, T\big) + \Omega(Y,W)\,\big(f^* g_\Theta\big)\big(U(f)(X,Z),\, T\big).
\end{aligned}
\tag{3}
$$

for any $X, Y, Z, W \in H(M)$.

Here, R^∇ is the curvature tensor field of the Tanaka–Webster connection ∇ of $(M,\, \theta)$, and $B(f)$ is the pseudohermitian second fundamental form of the given immersion $f : M \to S^{2(n+k)+1}$. A brief inspection of (3) reveals a strong formal analogy to the ordinary Gauss equation in Riemannian geometry; see B-Y. Chen [20]. At the same time, all obstructions springing from the geometric structure at hand (which is pseudohermitian, rather than Riemannian) are inbuilt in Equation (3). For instance, Equation (3) contains the (eventually nonzero) pseudohermitian torsion tensor field A of $(M,\, \theta)$. Additionally, (3) contains the $(1,2)$ tensor field $U(f)$ expressing the difference between the induced connection $\nabla^\top$ and the Tanaka–Webster connection ∇ (the non-uniqueness of the canonical connection on M is of course tied to the failure of $f : (M,\, g_\theta) \to (S^{2(n+k)+1},\, g_\Theta)$ to be isometric). Our expectation is that an analysis of the pseudohermitian Gauss–Codazzi equations will lead to rigidity theorems for isopseudohermitian CR immersions $f : M \to S^{2n+3}$ and, in particular, $f : M \to S^5$ (focusing on the case $M = S^3$).

The certitude that Riemannian objects on $(A,\, g^A_\epsilon)$ (and their tangential and normal components, relative to $f(M)$) will give, in the limit as $\epsilon \to 0^+$, the "correct" pseudohermitian analogs to the (Riemannian) embedding equations is already acquired from the following early observations: let (M, θ) be endowed with the contraction of G_θ given by $g_\epsilon = g_\theta + \big(\epsilon^{-2} - 1\big)\, \theta \otimes \theta$, and let ∇^ϵ and Δ_ϵ be respectively the gradient and Laplace–Beltrami operators (on functions) of the Riemannian manifold $(M,\, g_\epsilon)$. Then,

$$
\nabla^\epsilon u = \nabla^H u + \epsilon^2\, \theta(\nabla u)\, T, \quad u \in C^1(M),
$$

$$
\Delta_\epsilon u = \Delta_b u - \epsilon^2\, T^2(u), \quad u \in C^2(M),
$$

showing that $\nabla^\epsilon u$ tends, in the limit as $\epsilon \to 0^+$, to the horizontal gradient $\nabla^H u$ (familiar in subelliptic theory; see, e.g., [19]), while Δ_ϵ tends (in an appropriate Banach space topology, where second order elliptic operators such as Δ_ϵ form an open set, one of whose boundary points is Δ_b) to the sublaplacian Δ_b of $(M,\, \theta)$.

As an application of the pseudohermitian Gauss Equation (3) in Corollary 2, we shall establish the following result.

Theorem 1. *Let M be a strictly pseudoconvex CR manifold, of CR dimension n, equipped with the contact form $\theta \in \mathcal{P}_+(M)$. Let $f : M \to S^{2(n+k)+1}$ be an isopseudohermitian immersion of $(M,\, \theta)$ into the sphere $\mathbf{j} : S^{2(n+k)+1} \hookrightarrow \mathbb{C}^{n+k+1}$, endowed with the contact form $\Theta = \mathbf{j}^*\big[\frac{i}{2}\big(\overline{\partial} - \partial\big)|Z|^2\big]$. Then, the pseudohermitian scalar curvature $R = g^{\alpha\overline{\beta}}\, R_{\alpha\overline{\beta}}$ of $(M,\, \theta)$ satisfies the inequality*

$$
R \le 2n\big[(f^* g_\Theta)(T,T) + n + 1\big] + \frac{1}{2}\Big\{\|H(f)\|_{g^f_\Theta} + \big\|\mathrm{trace}_{G_\theta}\, \Pi_{H(M)}\, U(f)\big\|_{f^* g_\Theta}\Big\}
$$

with equality if and only if $B(f) = 0$ and $\nabla^\top = \nabla$ on $H(M) \otimes H(M)$.

Here, $H(f) = \mathrm{trace}_{G_\theta}\, \Pi_{H(M)}\, B(f)$. Theorem 1 generalizes a classical result by S-S. Chern (see [21]) on isometric immersions of Riemannian manifolds into a space form (to the case of isopseudohermitian immersions of strictly pseudoconvex CR manifolds into a sphere).

The definitions of objects used in the present Introduction can be found in Section 2 of the present paper.

2. Sub-Riemannian Techniques in CR Geometry

All basic notions and results used through the paper are described in detail in Section 2, following the monograph by S. Dragomir and G. Tomassini [3]. Specifically, in Section 2.1, we recall the necessary material in Cauchy–Riemann (CR) and pseudohermitian geometry by essentially following monograph [3]. CR geometry is known to (partially) embed into sub-Riemannian geometry, in the sense of R. Strichartz [8]. We therefore recall the basics of sub-Riemannian geometry, at work in the present paper, in Section 2.2 by following J.P. D'Angelo and J.T. Tyson (see [7]) and [6,19], and of course [8].

2.1. CR Structures and Pseudohermitian Geometry

Let M be an orientable real $(2n+1)$-dimensional C^∞ differentiable manifold, and let $T(M)$ be the (total space of the) tangent bundle over M.

Definition 1 ([3], pp. 3–4)**.** A *CR structure* is a complex rank n complex subbundle $T_{1,0}(M) \subset T(M) \otimes \mathbb{C}$ of the complexified tangent bundle such that

$$T_{1,0}(M)_x \cap T_{0,1}(M)_x = (0), \quad x \in M,$$

$$Z, W \in C^\infty(U, T_{1,0}(M)) \Longrightarrow [Z, W] \in C^\infty(U, T_{1,0}(M))$$

for any open set $U \subset M$. A pair $(M, T_{1,0}(M))$ consisting of a $(2n+1)$-dimensional C^∞ manifold M and a CR structure $T_{1,0}(M)$ on M is a *CR manifold*. The integer n is the *CR dimension*.

Here, $T_{0,1}(M) = \overline{T_{1,0}(M)}$ (an overbar denotes complex conjugation). Every real hypersurface $M \subset \mathbb{C}^{n+1}$ may be organized (see e.g., formula 1.12 in [3], p. 5) as a CR manifold of CR dimension n, with the CR structure

$$T_{1,0}(M)_x = \left[T_x(M) \otimes_{\mathbb{R}} \mathbb{C}\right] \cap T'(\mathbb{C}^{n+1})_x, \quad x \in M,$$

induced by the complex structure of the ambient space. Here, $T'(\mathbb{C}^{n+1})$ denotes the holomorphic tangent bundle over $\mathbb{C}^{n+1}$, i.e., the span of $\{\partial/\partial z^j \,:\, 1 \le j \le n+1\}$ where $(z^1, \cdots, z^{n+1})$ are the Cartesian complex coordinates on $\mathbb{C}^{n+1}$.

Definition 2 ([3], p. 4)**.** The real rank $2n$ (hyperplane) distribution

$$H(M) = \mathrm{Re}\{T_{1,0}(M) \oplus T_{0,1}(M)\}$$

is the *Levi* (or *maximally complex*) distribution.

$H(M)$ carries the complex structure

$$J = J_M : H(M) \to H(M), \quad J(Z + \overline{Z}) = i(Z - \overline{Z}), \quad Z \in T_{1,0}(M).$$

Definition 3 ([3], p. 4)**.** A C^∞ map $f : M \to A$ of the CR manifold $(M, T_{1,0}(M))$ into the CR manifold $(A, T_{1,0}(A))$ is a *CR map* if

$$(d_x f) T_{1,0}(M) \subset T_{1,0}(A)_{f(x)}, \quad x \in M.$$

Equivalently, a CR map $f : M \to A$ is characterized by the properties

$$(d_x f)\, H(M)_x \subset H(A)_{f(x)}, \quad (d_x f) \circ J_{M,x} = J_{A,f(x)} \circ (d_x f),$$

for any $x \in M$; see formulas 1.10 and 1.11 in [3], p. 4.

Definition 4 ([3], p. 5)**.** A *CR isomorphism* is a C^∞ diffeomorphism and a CR map. A *CR automorphism* of the CR manifold M is a CR isomorphism of M into itself.

For every CR manifold M, let $\mathrm{Aut}_{\mathrm{CR}}(M)$ be the group of all CR automorphisms of M.

Definition 5 ([3], p. 5)**.** The *conormal bundle* is the real line bundle $\mathbb{R} \to H(M)^\perp \to M$ given by

$$H(M)^\perp_x = \{\omega \in T^*_x(M) : \mathrm{Ker}(\omega) \supset H(M)_x\}, \quad x \in M.$$

As M is orientable, and $H(M)$ is oriented by its complex structure, the quotient bundle $T(M)/H(M)$ is orientable. Moreover, there is a (non-canonical) vector bundle isomorphism $H(M)^\perp \approx T(M)/H(M)$, hence $H(M)^\perp$ is orientable as well. Any orientable real line bundle over a connected manifold is trivial (see, e.g., Remark 11.3 in [22], p. 115). Hence, $H(M)^\perp \approx M \times \mathbb{R}$ (a vector bundle isomorphism). Therefore, $H(M)^\perp$ admits globally defined nowhere zero sections.

Definition 6 ([3], p. 5)**.** A global section $\theta \in C^\infty(H(M)^\perp)$ such that $\theta_x \neq 0$ for every $x \in M$ is called a *pseudohermitian structure* on M.

A pseudohermitian structure is a real valued differential 1-form θ on M such that $\mathrm{Ker}(\theta) = H(M)$ (and in particular $\theta_x \neq 0$ for any $x \in M$).

Definition 7. A pair (M, θ) consisting of a CR manifold M and a pseudohermitian structure θ on M is a *pseudohermitian manifold*.

Let $\mathcal{P} = \mathcal{P}(M)$ be the set of all pseudohermitian structures on M.

Definition 8 ([3], pp. 5–6)**.** Given $\theta \in \mathcal{P}$, the *Levi form* is

$$G_\theta(X, Y) = (d\theta)(X, JY), \quad X, Y \in H(M).$$

Definition 9 ([3], p. 6)**.** The CR structure $T_{1,0}(M)$ is *nondegenerate* if the (symmetric bilinear) form G_θ is nondegenerate for some $\theta \in \mathcal{P}$.

Any other pseudohermitian structure $\hat\theta \in \mathcal{P}$ is related to θ by $\hat\theta = \lambda\,\theta$ for some C^∞ function $\lambda : M \to \mathbb{R} \setminus \{0\}$. Then,

$$d\hat\theta = d\lambda \wedge \theta + \lambda\, d\theta\,,$$

hence the corresponding Levi forms G_θ and $G_{\hat\theta}$ are related by $G_{\hat\theta} = \lambda\, G_\theta$. Consequently, if G_θ is nondegenerate for some $\theta \in \mathcal{P}$, it is nondegenerate for all. That is, nondegeneracy is a CR invariant notion; it does not depend on the choice of pseudohermitian structure. Strictly speaking:

Definition 10. A geometric object, or a notion, on a CR manifold M is *CR invariant* if it is invariant with respect to the action of $\mathrm{Aut}_{\mathrm{CR}}(M)$.

The signature of the Levi form G_θ of a nondegenerate CR manifold M is a CR invariant.

Definition 11 ([3], p. 43)**.** A differential 1-form $\theta \in \Omega^1(M)$ is a *contact form* if $\theta \wedge (d\theta)^n$ is a volume form on M.

If $T_{1,0}(M)$ is nondegenerate, then each $\theta \in \mathcal{P}$ is a contact form; see, e.g., Proposition 1.9 in [3], pp. 43–44.

For any nondegenerate CR manifold, on which a contact form $\theta \in \mathcal{P}$ has been fixed, there is a unique globally defined nowhere zero tangent vector field $T = T_M \in \mathfrak{X}(M)$, transverse to the Levi distribution, determined by the requirements

$$\theta(T) = 1, \quad T \rfloor d\theta = 0.$$

See Proposition 1.2 in [3], pp. 8–9.

Definition 12. T is called the *Reeb vector* field of (M, θ).

Let $\theta \in \mathcal{P}$ be a contact form on M, and let us define the $(0, 2)$ tensor field g_θ on M by setting

$$g_\theta(X, Y) = G_\theta(X, Y), \quad g_\theta(X, T) = 0, \quad g_\theta(T, T) = 1,$$

for any $X, Y \in H(M)$. g_θ is a semi-Riemannian metric on M; see [3], p. 9.

Definition 13 ([3], p. 9)**.** g_θ is called the *Webster metric* of (M, θ).

Definition 14 ([3], p. 6)**.** A CR structure $T_{1,0}(M)$ is *strictly pseudoconvex* (and the pair $(M, T_{1,0}(M))$ is a *strictly pseudoconvex* CR manifold) if the Levi form G_θ is positive definite for some $\theta \in \mathcal{P}$.

Let $\mathcal{P}_+ = \mathcal{P}_+(M)$ be the set of all $\theta \in \mathcal{P}$ such that G_θ is positive definite. If M is strictly pseudoconvex, then $\mathcal{P}_+ \neq \emptyset$. Any strictly pseudoconvex CR manifold is nondegenerate. If $\theta \in \mathcal{P}_+$, then the Webster metric g_θ is a Riemannian metric on M.

Definition 15. A contact form $\theta \in \mathcal{P}_+$ is said to be *positively oriented*.

Quadrics $Q_n = \{(z, \zeta) \in \mathbb{C}^n \times \mathbb{C} : \dfrac{1}{2i}(\zeta - \overline{\zeta}) = |z|^2\}$ and odd dimensional spheres $S^{2N+1} \subset \mathbb{C}^{N+1}$ are organized as CR manifolds, with the CR structures naturally induced by the ambient complex structure. $\mathrm{Aut}_{\mathrm{CR}}(S^{2n+1})$ consists of all fractional linear, or projective, transformations preserving S^{2n+1}; see, e.g., [23].

Definition 16 ([3], p. 11)**.** The *Heisenberg group* is the non-commutative Lie group $\mathbb{H}_n = \mathbb{C}^n \times \mathbb{R} \approx \mathbb{R}^{2n+1}$, with the group law

$$(z,\, t) \cdot (w,\, s) = \big(z + w,\; t + s + 2\,\mathrm{Im}\langle z,\, w\rangle\big),$$

$$z,\; w \in \mathbb{C}^n\,, \quad t,\; s \in \mathbb{R}, \quad \langle z,\, w\rangle = \delta_{jk}\, z^j\, \overline{w^k}\,.$$

The Heisenberg group $\mathbb{H}_n$ is organized as a CR manifold with the CR structure spanned by

$$L_\alpha \equiv \frac{\partial}{\partial z^\alpha} + i\, \overline{z}^\alpha\, \frac{\partial}{\partial t}\,, \quad 1 \le \alpha \le n.$$

See formula (1.24) in [3], p. 12.

Definition 17 ([3], p. 12, and [24])**.** When $n = 1$, the first order differential operator $\overline{L}_1$ is the *Lewy operator*.

The mapping

$$f : \mathbb{H}_n \to Q_n\,, \quad f(z, t) = t + i\,|z|^2\,, \quad (z, t) \in \mathbb{H}_n$$

is a CR isomorphism. Let us set

$$\theta_0 = dt + i \sum_{\alpha=1}^{n} \{ z^\alpha \, d\overline{z}_\alpha - \overline{z}^\alpha \, dz^\alpha \}, \quad \Theta = \frac{i}{2}(\overline{\partial} - \partial)|Z|^2 ,$$

with $|Z|^2 = \sum_{A=0}^{N} Z_A \overline{Z}_A$, $Z = (Z_0 , \cdots , Z_N)$. Both $\mathbb{H}_n$ and S^{2N+1} are strictly pseudoconvex, and $\theta_0 \in \mathcal{P}_+(\mathbb{H}_n)$, $\Theta \in \mathcal{P}_+(S^{2N+1})$.

For any nondegenerate CR manifold M on which a contact form $\theta \in \mathcal{P}$ has been fixed, there is a unique linear connection ∇ on M satisfying the following requirements: (i) $H(M)$ is parallel with respect to ∇, i.e., $X \in \mathfrak{X}(M)$ and $Y \in H(M) \Longrightarrow \nabla_X Y \in H(M)$; (ii) $\nabla g_\theta = 0$ and $\nabla J = 0$; and (iii) the torsion T_∇ of ∇ is *pure*, i.e.,

$$\tau \circ J + J \circ \tau = 0, \quad T_\nabla(Z, W) = 0, \quad T_\nabla(Z, \overline{W}) = 2iG_\theta(Z, \overline{W}),$$

for any $Z, W \in T_{1,0}(M)$. Here, $\tau(X) = T_\nabla(T, X)$ for any $X \in \mathfrak{X}(M)$. See Theorem 1.3 in [3], p. 25.

Definition 18 ([3], p. 26)**.** ∇ is the *Tanaka–Webster connection* of (M, θ). The vector-valued 1-form τ on M is the *pseudohermitian torsion* of ∇.

As a consequence of axioms (i)–(ii) $\tau[T_{1,0}(M)] \subset T_{0,1}(M)$. In particular, the pseudohermitian torsion is trace-less; i.e., $\mathrm{trace}(\tau) = 0$. Moreover, if $A(X, Y) = g_\theta(X, \tau Y)$, then A is symmetric, i.e., $A(X, Y) = A(Y, X)$; see Lemma 1.4 in [3], pp. 38–40.

Definition 19. For every C^1 function $u : M \to \mathbb{R}$, the *horizontal gradient* of u is

$$\nabla^H u = \Pi_H \, \nabla u.$$

Here, $\Pi_H : T(M) \to H(M)$ is the projection associated with the direct sum decomposition $T(M) = H(M) \oplus \mathbb{R}T$. Additionally, ∇u is the gradient of u with respect to the Webster metric, i.e.,

$$g_\theta(\nabla u, \, X) = X(u), \quad X \in \mathfrak{X}(M).$$

Definition 20. For every C^1 vector field X on M, the *divergence* of X is its divergence $\mathrm{div}(X)$ with respect to the contact form $\Psi_\theta = \theta \wedge (d\theta)^n$, i.e.,

$$\mathcal{L}_X \Psi_\theta = \mathrm{div}(X) \, \Psi_\theta \, .$$

Here, $\mathcal{L}_X$ is the Lie derivative at the direction X.

Definition 21 ([3], p. 111)**.** Let $\theta \in \mathcal{P}_+$. The *sublaplacian* of $(M, \, \theta)$ is the second order differential operator Δ_b given by

$$\Delta_b u = -\mathrm{div}\big(\nabla^H u\big)$$

for every C^2 function $u : M \to \mathbb{R}$.

Δ_b is a formally self-adjoint, degenerate elliptic operator (formally similar to the Laplace–Beltrami operator of a Riemannian manifold) naturally occurring on a strictly pseudoconvex CR manifold M, on which a positively oriented contact form θ has been fixed. While Δ_b is not elliptic (ellipticity degenerates in the cotangent directions spanned by θ; see [25]), Δ_b is subelliptic of order $1/2$, and hence it is hypoelliptic; see [3], pp. 114–116, and L. Hörmander [26].

We end the section by briefly recalling a few elements of curvature theory on a nondegenerate CR manifold M, endowed with a contact form θ. Let R^∇ be the curvature tensor field of the Tanaka–Webster connection ∇ of (M, θ). Let $\{T_\alpha : 1 \le \alpha \le n\}$ be a local frame of $T_{1,0}(M)$, defined on the open set $U \subset M$, and let $T_{\overline{\alpha}} \equiv \overline{T_\alpha}$. Then,

$$\{T_A : A \in \{0, 1, \cdots, n, \overline{1}, \cdots, \overline{n}\}\} \equiv \{T, T_\alpha, T_{\overline{\alpha}} : 1 \le \alpha \le n\}, \quad T_0 = T$$

is a local frame of $T(M) \otimes \mathbb{C}$ on U. For all local calculations, one sets

$$g_{\alpha\overline{\beta}} = g(T_\alpha, T_{\overline{\beta}}), \quad [g^{\alpha\overline{\beta}}] = [g_{\alpha\overline{\beta}}]^{-1}.$$

Let us consider the C^∞ functions $R_C{}^D{}_{AB} : U \to \mathbb{C}$ determined by

$$R^\nabla(T_A, T_B)T_C = R_C{}^D{}_{AB}\, T_D\,.$$

Definition 22. The Ricci tensor is

$$\mathrm{Ric}_\nabla(X, Y) = \mathrm{Trace}\{Z \mapsto R^\nabla(Z, Y)X\}.$$

The *pseudohermitian Ricci tensor* is $R_{\alpha\overline{\beta}} = \mathrm{Ric}_\nabla(T_\alpha, T_{\overline{\beta}})$.

Then $R_{\alpha\overline{\beta}} = R_\alpha{}^\gamma{}_{\gamma\overline{\beta}}$; see [3], p. 50.

Definition 23. The *pseudohermitian scalar curvature* is $R = g^{\alpha\overline{\beta}}\, R_{\alpha\overline{\beta}}$.

A pseudohermitian analog to the holomorphic sectional curvature (of a Kählerian manifold) was introduced by S.M. Webster [1] and studied in some detail by E. Barletta [27].

2.2. Sub-Riemannian Geometry

Let $(M, T_{1,0}(M))$ be a CR manifold. Let $S : x \in M \mapsto S_x \subset T_x(M)$ be a C^∞ distribution on M.

Definition 24 ([8] p. 224 and [19] p. 124)**.** S is *bracket generating* if the C^∞ sections in S, together with their commutators, span $T_x(M)$ at each point $x \in M$.

Given $v \in S_x$, let $X \in C^\infty(S)$ such that $X_x = v$. Let $S_x + [v, S_x] \subset T_x(M)$ be the subspace spanned by

$$S_x \cup \{[X, Y]_x : Y \in C^\infty(S)\}.$$

Next, let us inductively define the spaces $\mathcal{D}_k(v) \subset T_x(M)$ by setting

$$\mathcal{D}_2(v) = S_x + [v, S_x], \quad \mathcal{D}_k(v) = S_x + [\mathcal{D}_{k-1}, S_x], \quad k \ge 3.$$

Definition 25 ([8,19])**.** A tangent vector $v \in S_x$ is a *k-step bracket generator* if $\mathcal{D}_k(v) = T_x(M)$. The distribution S is said to satisfy the *strong bracket generating hypothesis* if, for arbitrary $x \in M$, each $v \in S_x$ is a 2-step bracket generator.

Let S be a bracket generating distribution on M.

Definition 26 ([8,19])**.** A *sub-Riemannian metric* on S is a Riemannian bundle metric on S, i.e., a C^∞ positive definite section $Q \in C^\infty(S^* \otimes S^*)$. A pair (S, Q) consisting of a bracket generating distribution S on M and a sub-Riemannian metric Q on S is called a *sub-Riemannian structure* on M.

Definition 27 ([8], p. 229 and [7,19])**.** A piecewise C^1 curve $\gamma : I \to M$ (where $I \subset \mathbb{R}$ is an interval) is *horizontal* if $\dot\gamma(t) \in H(M)_{\gamma(t)}$ for all values of the parameter t (for which $\dot\gamma(t)$ makes sense).

Assume M to be strictly pseudoconvex. Let $\theta \in \mathcal{P}_+$ be a positively oriented contact form on M. Then, $(H(M),\, G_\theta)$ is a sub-Riemannian structure on M; see [19], p. 125.

Definition 28. The *sub-Riemannian length* of a horizontal curve $\gamma : I \to M$ is

$$\ell(\gamma) = \int_I G_\theta(\dot{\gamma}(t)\,,\,\dot{\gamma}(t))^{1/2}\,dt.$$

A piecewise C^1 curve $\gamma : I \to M$ *joins* the points $x, y \in M$ if $I = [a, b]$, $\gamma(a) = x$, and $\gamma(b) = y$. Let $\Omega(x, y)$ (,respectively, $\Omega_H(x, y)$) be the set of all piecewise C^1 (respectively horizontal) curves joining x and y. Let $d(x, y)$ (respectively, $d_H(x, y)$) be the distance between $x, y \in M$ induced by the Riemannian metric g_θ (respectively, the greatest lower bound of $\{\ell(\gamma) : \gamma \in \Omega_H(x, y)\}$). $d_H : M \times M \to [0, +\infty)$ is a distance function on M; see [8], p. 230.

Definition 29. d_H is the *Carnot–Carthéodory distance* function on M, induced by the sub-Riemannian structure $(H(M), G_\theta)$.

Definition 30 ([8], p. 230)**.** A Riemannian metric g on M is said to be a *contraction* of the sub-Riemannian metric G_θ if the distance function $\rho : M \times M \to [0, +\infty)$ associated with g satisfies $\rho(x,\, y) \leq d_H(x,\, y)$ for any $x,\, y \in M$.

As $\Omega_H(x, y) \subset \Omega(x, y)$ (a strict inclusion), one has $d(x, y) \leq d_H(x, y)$ for any $x, y \in M$. Hence, the Webster metric g_θ is a contraction of the Levi form G_θ. The construction of a contraction of G_θ by the requirement that the norm of the Reeb vector T be 1, appearing as quite natural a priori, proves to be rather restrictive later on; i.e., the Riemannian geometry of $(M,\, g_\theta)$ turns out to be insufficiently related to the CR and pseudohermitian geometry on $(M,\, \theta)$. As shown by J. Jost and C-J. Xu (see [28]), the requirement that the norm of T be $O(\epsilon^{-1})$ is far reaching (and related to the notion of homogeneous space in PDEs theory; see [28]). In the next section, we adopt a version of the construction in [28], referred to in the sequel as an ϵ*-contraction* of G_θ.

2.3. ϵ-Contractions

Let $(M,\, T_{1,0}(M))$ be a strictly pseudoconvex CR manifold, and let $\theta \in \mathcal{P}_+(M)$ be a positively oriented contact form on M. Let $T \in \mathfrak{X}(M)$ be the Reeb vector field of (M, θ). Let $0 < \epsilon < 1$, and let $g_\epsilon = g_\epsilon^M$ be the $(0, 2)$ tensor field on M defined by

$$g_\epsilon(T, T) = \epsilon^{-2}, \tag{4}$$

$$g_\epsilon(X, Y) = G_\theta(X, Y), \quad g_\epsilon(X, T) = 0, \tag{5}$$

for any $X, Y \in H(M)$. g_ϵ is a Riemannian metric on M and a contraction of G_θ. The direct sum decomposition

$$T(M) = H(M) \oplus \mathbb{R}T \tag{6}$$

together with (4) and (5) yields

$$g_\epsilon = g_\theta + \left(\frac{1}{\epsilon^2} - 1\right)\theta \otimes \theta. \tag{7}$$

Definition 31. g_ϵ is called the ϵ*-contraction* of G_θ.

The contraction g_ϵ is built such that the g_ϵ norm of T is $O(\epsilon^{-1})$, a property of crucial importance in the further asymptotic analysis as $\epsilon \to 0^+$. For every $0 < \epsilon < 1$, we consider the contact form $\theta_\epsilon = \epsilon^{-1}\theta$. The Reeb vector field $T_\epsilon \in \mathfrak{X}(M)$ of $(M,\, \theta_\epsilon)$ is given by $T_\epsilon = \epsilon T$.

Lemma 1. *The Webster metric g_{θ_ϵ} and the ϵ-contraction g_ϵ of G_θ are related by*

$$g_{\theta_\epsilon}(X, Y) = \epsilon^{-1}\, g_\epsilon(X, Y), \tag{8}$$

$$g_{\theta_\epsilon}(X, T) = g_\epsilon(X, T) = 0, \tag{9}$$

$$g_{\theta_\epsilon}(T, T) = g_\epsilon(T, T) = \epsilon^{-2}, \tag{10}$$

for any $X, Y \in H(M)$. Summing up:

$$g_{\theta_\epsilon} = \epsilon^{-1}\, g_\epsilon + \epsilon^{-2}(1 - \epsilon^{-1})\, \theta \otimes \theta. \tag{11}$$

In particular, none of the metrics $\{g_\epsilon \,:\, \epsilon > 0\}$ is a Webster metric; i.e., there is no $u_\epsilon \in C^\infty(M, \mathbb{R})$ such that $g_\epsilon = g_{\exp(u_\epsilon)\,\theta}$.

The proof of Lemma 1 is straightforward.

Lemma 2. *The Levi–Civita connection ∇^ϵ of the Riemannian manifold $(M,\, g_\epsilon)$ and the Tanaka–Webster connection ∇ of the pseudohermitian manifold $(M,\, \theta)$ are related by*

$$\nabla^\epsilon_X Y = \nabla_X Y + \big\{\Omega(X, Y) - \epsilon^2\, A(X, Y)\big\}\, T, \tag{12}$$

$$\nabla^\epsilon_X T = \tau X + \frac{1}{\epsilon^2}\, JX, \tag{13}$$

$$\nabla^\epsilon_T X = \nabla_T X + \frac{1}{\epsilon^2}\, JX, \tag{14}$$

$$\nabla^\epsilon_T T = 0, \tag{15}$$

for any $X, Y \in H(M)$.

Here, $\Omega(X, Y) = g_\theta(X, JY)$ for any $X, Y \in \mathfrak{X}(M)$. Ω is a pseudohermitian analog to the fundamental 2-form in Hermitian geometry. However, $\Omega = -d\theta$ so that, unlike the (perhaps more familiar) case of Kählerian geometry, Ω and its exterior powers *do not* determine nontrivial de Rham cohomology classes on M.

The remainder of the section is devoted to the proof of Lemma 2. This requires a rather involved calculation, as follows. Given a Riemannian metric g on M, it will be useful to adopt the following:

Definition 32. The *Christoffel mapping* is

$$C_g : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to C^\infty(M, \mathbb{R}),$$

$$\begin{aligned} C_g(X,\, Y,\, Z) = {} & X\big(g(Y, Z)\big) + Y\big(g(Z, X)\big) - Z\big(g(X, Y)\big) \\ & + g\big([X,\, Y],\, Z\big) - g\big([Y,\, Z],\, X\big) + g\big([Z,\, X],\, Y\big), \end{aligned}$$

for any $X, Y, Z \in \mathfrak{X}(M)$.

Let g_θ and ∇ be, respectively, the Webster metric and Tanaka–Webster connection of (M, θ). As $\nabla g_\theta = 0$, one may apply the so-called Christoffel process; i.e., starting from

$$X\big(g_\theta(Y, Z)\big) = g_\theta\big(\nabla_X Y\,,\, Z\big) + g_\theta\big(Y,\, \nabla_X Z\big),$$

we produce other two identities of the sort by circular permutation of X, Y, Z,

$$Y\big(g_\theta(Z, X)\big) = g_\theta\big(\nabla_Y Z\,,\, X\big) + g_\theta\big(Z,\, \nabla_Y X\big),$$

$$Z\big(g_\theta(X,Y)\big) = g_\theta\big(\nabla_Z X\,,\,Y\big) + g_\theta\big(X,\,\nabla_Z Y\big),$$

add the first two and subtract the third, and use $\nabla_X Y - \nabla_Y X - [X,Y] = T_\nabla(X,Y)$ to recognize torsion terms. We obtain

$$\begin{aligned} 2\,g_\theta\big(\nabla_X Y,\,Z\big) = {} & X\big(g_\theta(Y,Z)\big) + Y\big(g_\theta(Z,X)\big) - Z\big(g_\theta(X,Y)\big) \\ & + g_\theta\big([X,\,Y],\,Z\big) - g_\theta\big([Y,\,Z],\,X\big) + g_\theta\big([Z,\,X],\,Y\big) \\ & + g_\theta\big(T_\nabla(X,\,Y),\,Z\big) - g_\theta\big(T_\nabla(Y,\,Z),\,X\big) + g_\theta\big(T_\nabla(Z,\,X),\,Y\big) \end{aligned} \tag{16}$$

for any $X,Y,Z \in \mathfrak{X}(M)$. By the purity axiom, the torsion of the Tanaka–Webster connection satisfies

$$\mathrm{Tor}_\nabla = 2\big\{\theta\wedge\tau - \Omega\otimes T\big\}. \tag{17}$$

Then, by (17),

$$g_\theta\big(\mathrm{Tor}_\nabla(X,\,Y),\,Z\big) - g_\theta\big(\mathrm{Tor}_\nabla(Y,\,Z),\,X\big) + g_\theta\big(\mathrm{Tor}_\nabla(Z,\,X),\,Y\big) \tag{18}$$

$$= 2\,\big\{A(X,Y)\,\theta(Z) - A(X,Z)\,\theta(Y)\big\}$$
$$+\Omega(X,Z)\,\theta(Y) + \Omega(Y,Z)\,\theta(X) - \Omega(X,Y)\,\theta(Z)\big\}$$

for any $X,Y,Z \in \mathfrak{X}(M)$. Substitution from (18) into (16) furnishes

$$\begin{aligned} 2\,g_\theta\big(\nabla_X Y,\,Z\big) = {} & C_{g_\theta}(X,\,Y,\,Z) \\ & +2\,\big\{A(X,Y)\,\theta(Z) - A(X,Z)\,\theta(Y)\big\} \\ & +\Omega(X,Z)\,\theta(Y) + \Omega(Y,Z)\,\theta(X) - \Omega(X,Y)\,\theta(Z)\big\}. \end{aligned} \tag{19}$$

Next, we may exploit (7) (relating the ϵ-contraction g_ϵ to the Webster metric g_θ) to derive

$$\begin{aligned} C_{g_\epsilon}(X,\,Y,\,Z) = {} & C_{g_\theta}(X,\,Y,\,Z) \\ & + \Big(\tfrac{1}{\epsilon^2} - 1\Big)\big\{X\big(\theta(Y)\,\theta(Z)\big) + Y\big(\theta(Z)\,\theta(X)\big) - Z\big(\theta(X)\,\theta(Y)\big) \\ & + \theta\big([X,Y]\big)\,\theta(Z) - \theta\big([Y,Z]\big)\,\theta(X) + \theta\big([Z,X]\big)\,\theta(Y)\big\} \end{aligned}$$

or, by using $2\,(d\theta)(X,Y) = X(\theta(Y)) - Y(\theta(X)) - \theta([X,Y])$,

$$\begin{aligned} C_{g_\epsilon}(X,\,Y,\,Z) = {} & C_{g_\theta}(X,\,Y,\,Z) + 2\Big(\tfrac{1}{\epsilon^2} - 1\Big)\big\{X\big(\theta(Y)\,\theta(Z)\big) \\ & + \theta(X)\,(d\theta)(Y,Z) + \theta(Y)\,(d\theta)(X,Z) - \theta(Z)\,(d\theta)(X,Y)\big\}. \end{aligned} \tag{20}$$

Let ∇^ϵ be the Levi–Civita connection of the Riemannian manifold $(M,\,g_\epsilon)$. As ∇^ϵ is symmetric and $\nabla^\epsilon g_\epsilon = 0$, the Christoffel process yields

$$2\,g_\epsilon\big(\nabla^\epsilon_X Y\,,\,Z\big) = C_{g_\epsilon}(X,\,Y,\,Z). \tag{21}$$

Then, by substitution from (20) into (21),

$$2\,g_\epsilon\big(\nabla^\epsilon_X Y,\,Z\big) = C_{g_\theta}(X,\,Y,\,Z) + 2\Big(\frac{1}{\epsilon^2} - 1\Big)\big\{X\big(\theta(Y)\big)\,\theta(Z)$$

$$+\theta(X)\,(d\theta)(Y,Z) + \theta(Y)\,(d\theta)(X,Z) - \theta(Z)\,(d\theta)(X,Y)\big\}$$

or, by replacing g_ϵ in terms of g_θ from (7), substituting $C_{g_\theta}(X,\,Y,\,Z)$ from (19), and using $\Omega = -d\theta$,

$$
\begin{aligned}
& g_\theta\big(\nabla^\epsilon_X Y,\, Z\big) + \left(\tfrac{1}{\epsilon^2} - 1\right) \theta\big(\nabla^\epsilon_X Y\big)\, \theta(Z) \\
& = g_\theta\big(\nabla_X Y,\, Z\big) + \left(\tfrac{1}{\epsilon^2} - 1\right) X\big(\theta(Y)\big)\, \theta(Z) \\
& + A(X, Z)\, \theta(Y) - A(X, Y)\, \theta(Z) \\
& + \tfrac{1}{\epsilon^2} \big\{ \Omega(X, Y)\, \theta(Z) - \Omega(Y, Z)\, \theta(X) - \Omega(X, Z)\, \theta(Y) \big\}.
\end{aligned} \tag{22}
$$

The rather involved relation (22) holding for any $X, Y, Z \in \mathfrak{X}(M)$ can be greatly simplified by using the decomposition (6). For arbitrary $Z \in H(M)$, the relation (22) yields, $\theta(Z) = 0$,

$$
\Pi_H \nabla^\epsilon_X Y = \Pi_H \nabla_X Y + \theta(Y)\, \tau X + \frac{1}{\epsilon^2} \big\{ \theta(X)\, JY + \theta(Y)\, JX \big\} \tag{23}
$$

where $\Pi_H = \Pi_{H(M)} : T(M) \to H(M)$ is the projection with respect to the decomposition (6). Again, by (22), for $Z = T$, we obtain

$$
\theta\big(\nabla^\epsilon_X Y\big) = X\big(\theta(Y)\big) + \Omega(X, Y) - \epsilon^2 A(X, Y), \tag{24}
$$

which determines the component along T of $\nabla^\epsilon_X Y$, with respect to the decomposition (6). At this point, we may use (23) and (24) to compute $\nabla^\epsilon_X Y$ for any $X, Y \in \mathfrak{X}(M)$. For every $V \in \mathfrak{X}(M)$,

$$
V = \Pi_H V + \theta(V)\, T, \tag{25}
$$

hence, by (23) and (24),

$$
\begin{aligned}
\nabla^\epsilon_X Y &= \Pi_H \nabla^\epsilon_X Y + \theta\big(\nabla^\epsilon_X Y\big)\, T = \\
&= \nabla_X Y + \Omega(X, Y)\, T + \theta(Y)\, \tau X + \frac{2}{\epsilon^2} (\theta \odot J)(X, Y) - \epsilon^2 A(X, Y)\, T,
\end{aligned} \tag{26}
$$

where $\odot$ is the symmetric tensor productl i.e., $\alpha \odot \beta = \frac{1}{2}(\alpha \otimes \beta + \beta \otimes \alpha)$. For $X, Y \in H(M)$, Equation (26) becomes

$$
\nabla^\epsilon_X Y = \nabla_X Y + \big\{ \Omega(X,\, Y) - \epsilon^2 A(X,\, Y) \big\}\, T,
$$

and (12) is proved. The remaining relations (13)–(15) in Lemma 2 follow from (26) for: (i) $X \in H(M)$ and $Y = T$; (ii) $X = T$ and $Y \in H(M)$; and (iii) $X = Y = T$. Q.E.D.

It will be useful to compute the covariant derivative of J with respect to ∇^ϵ, where $J : H(M) \to H(M)$ is extended as customary to a $(1, 1)$-tensor field on M by requiring that $JT = 0$. Note that the extension of J depends on the chosen contact form θ on M.

Lemma 3. *For any $X, Y \in H(M)$*

$$
(\nabla^\epsilon_X J) Y = -\Big\{ g_\theta(X,\, Y) + \epsilon^2 A(X,\, JY) \Big\}\, T, \tag{27}
$$

$$
(\nabla^\epsilon_X J) T = \tau\, J\, X + \frac{1}{\epsilon^2}\, X, \tag{28}
$$

$$
(\nabla^\epsilon_T J) X = 0, \tag{29}
$$

$$
(\nabla^\epsilon_T J) T = 0. \tag{30}
$$

Proof. Lemma 3 follows from (12)–(15) together with $\nabla J = 0$. For instance,

$$
(\nabla^\epsilon_X J) Y = \nabla^\epsilon_X JY - J \nabla^\epsilon_X Y
$$

by (12), and $JT = 0$ and $J^2 = -I + \theta \otimes T$

$$= \nabla_X JY + \{\Omega(X, JY) - \epsilon^2 A(X, JY)\} T - J\nabla_X Y$$

by $\nabla J = 0$, and $\Omega(X, Y) = g_\theta(X, JY)$

$$= -\{g_\theta(X, Y) + \epsilon^2 A(X, JY)\} T,$$

yielding (27). □

2.4. Gradients and the Laplace–Beltrami Operator on (M, g_ϵ)

For every function $u \in C^1(M)$, let $\nabla^\epsilon u$ be the gradient of u with respect to the Riemannian metric g_ϵ, i.e.,

$$g_\epsilon(\nabla^\epsilon u, X) = X(u), \quad X \in \mathfrak{X}(M).$$

Let Δ_ϵ be the Laplace–Beltrami operator of (M, g_ϵ), i.e.,

$$\Delta_\epsilon u = -\mathrm{div}_\epsilon(\nabla^\epsilon u), \quad u \in C^2(M).$$

Here, div_ϵ is the divergence operator with respect to the volume form $\Psi_\epsilon = d\,\mathrm{vol}(g_\epsilon)$, i.e.,

$$\mathcal{L}_X \Psi_\epsilon = \mathrm{div}_\epsilon(X)\, \Psi_\epsilon$$

for every C^1 vector field X tangent to M. Let (U, x^i) be a local coordinate system on M, and let us set

$$\mathfrak{g}_\epsilon = \det\left[g_{ij}(\epsilon)\right], \quad g_{ij}(\epsilon) = g_\epsilon\Big(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\Big), \quad 1 \le i, j \le 2n+1.$$

The Riemannian volume form of (M, g_ϵ) is locally given by

$$\Psi_\epsilon = \sqrt{\mathfrak{g}_\epsilon}\, dx^1 \wedge \cdots \wedge dx^{2n+1}.$$

The volume form Ψ_θ is parallel with respect to the Tanaka–Webster connection (i.e., $\nabla \Psi_\theta = 0$), hence the divergence of a C^1 vector field X may be computed as

$$\mathrm{div}(X) = \mathrm{Trace}\Big\{Y \mapsto \nabla_Y X\Big\}. \tag{31}$$

See [3], p. 112.

Lemma 4. (i) *For every* $u \in C^1(M)$,

$$\nabla^\epsilon u = \nabla^H u + \epsilon^2\, \theta(\nabla u)\, T. \tag{32}$$

(ii) *For every* $u \in C^2(M)$,

$$\Delta_\epsilon u = \Delta_b u - \epsilon^2\, T^2(u). \tag{33}$$

Proof. (i) By (7), for every $X \in \mathfrak{X}(M)$,

$$g_\theta(\nabla u, X) = X(u) = g_\epsilon(\nabla^\epsilon u, X)$$

$$= g_\theta(\nabla^\epsilon u, X) + \Big(\frac{1}{\epsilon^2} - 1\Big)\, \theta(\nabla^\epsilon u)\, \theta(X).$$

In particular, for arbitrary $X \in H(M)$,

$$\Pi_H \nabla^\epsilon u = \nabla^H u. \tag{34}$$

Also, for $X = T$,

$$\theta(\nabla^\epsilon u) = \epsilon^2 \, \theta(\nabla u). \tag{35}$$

Finally, by (34) and (35) and the decomposition (25),

$$\nabla^\epsilon u = \Pi_H \nabla^\epsilon u + \theta(\nabla^\epsilon u)\, T = \nabla^H u + \epsilon^2 \, \theta(\nabla u)\, T,$$

and (32) in Lemma 4 is proved.

(ii) Let $\{X_a \,:\, 1 \le a \le 2n\}$ be a local g_θ-orthonormal

$$G_\theta(X_a\,,\, X_b) = \delta_{ab}\,, \quad 1 \le a,\, b \le 2n,$$

frame of $H(M)$, defined on the open subset $U \subset M$. Then, by (7),

$$\{E_j^\epsilon \,:\, 1 \le j \le 2n+1\} \equiv \{X_a\,,\, \epsilon T \,:\, 1 \le a \le 2n\}$$

is a local g_ϵ-orthonormal frame of $T(M)$. Consequently, the Laplace–Beltrami operator of (M, g_ϵ) on functions can be computed as

$$\Delta_\epsilon u = -\sum_{j=1}^{2n+1} \left\{ E_j^\epsilon\big(E_j^\epsilon(u)\big) - \big(\nabla^\epsilon_{E_j^\epsilon} E_j^\epsilon\big)(u) \right\}$$

$$= -\sum_{a=1}^{2n} \left\{ X_a\big(X_a(u)\big) - \big(\nabla^\epsilon_{X_a} X_a\big)(u) \right\} - \epsilon^2 \left\{ T\big(T(u)\big) - \big(\nabla^\epsilon_T T\big)(u) \right\}$$

by (12) and (15)

$$= -\sum_{a=1}^{2n} \left\{ X_a\big(X_a(u)\big) - \big(\nabla_{X_a} X_a\big)(u) \right\}$$

$$+ \sum_{a=1}^{2n} \left\{ \Omega(X_a\,,\, X_a) - \epsilon^2 \, A(X_a\,,\, X_a) \right\} T(u) - \epsilon^2 \, T^2(u)$$

for every $u \in C^2(M)$. Finally, (33) in Lemma 4 follows from

$$\Delta_b u = -\sum_{a=1}^{2n} \left\{ X_a\big(X_a(u)\big) - \big(\nabla_{X_a} X_a\big)(u) \right\} \tag{36}$$

as Ω is skew-symmetric, $T \rfloor A = 0$, and

$$\sum_{a=1}^{2n} A(X_a\,,\, X_a) = \mathrm{trace}(\tau) = 0.$$

The formula (36) is a consequence of (31). $\square$

Let $\Omega \subset \mathbb{H}_n$ be a bounded domain, with boundary $S = \partial\Omega$ of class C^r, $1 \le r \le \infty$. Let us assume that Ω lies on one side of its boundary; i.e., for every $(z_0\,,\, t_0) \in S$, there is a neighborhood $U \subset \mathbb{H}_n$ and a diffeomorphism $\psi : U \to \mathbb{B}^{2n+1}$ such that $\psi(z_0\,,\, t_0) = 0$ and $\psi(U \cap \Omega) = \{(z,t) \in \mathbb{B}^{2n+1} \,:\, t > 0\}$. Here, $\mathbb{B}^N = \{x \in \mathbb{R}^N \,:\, |x| < 1\}$ is the unit ball. Let $\mathrm{DO}_\ell(\overline{\Omega})$ be the space of differential operators of order ℓ with real valued continuous coefficients

$$Lu \equiv \sum_{|\alpha| \le \ell} a_\alpha(z,t)\, D^\alpha u, \quad a_\alpha \in C\big(\overline{\Omega}, \mathbb{R}\big), \quad |\alpha| \le \ell.$$

$\mathrm{DO}_\ell(\overline{\Omega})$ is a Banach space with the norm

$$\|L\| = \sum_{|\alpha|\le k} \sup_{(z,t)\in\overline{\Omega}} \big|a_\alpha(z,t)\big| .$$

Let $L_\ell(z,\, t,\, \xi) = \sum_{|\alpha|=\ell} a_\alpha(z,t)\xi^\alpha$ be the symbol of $L \in \mathrm{DO}_\ell(\overline{\Omega})$. L is *degenerate elliptic* if

(i) There exist $(z,t) \in \overline{\Omega}$ and $\xi \in \mathbb{R}^{2n+1}$ such that $L_\ell(z,\, t,\, \xi) \neq 0$ and sign $L_\ell(z,\, t,\, \xi)$ is constant on $\overline{\Omega} \times S^{2n}$ [where sign : $\mathbb{R} \to \{\pm 1\}$],
(ii) The set $\{(z,\, t,\, \xi) \in \overline{\Omega} \times S^{2n} \; : \; L_\ell(z,\, t,\, \xi) = 0\}$ is nonempty.

Proposition 1. *For every bounded domain $\Omega \subset \mathbb{H}_n$ in the Heisenberg group, the sublaplacian $\Delta_b \in \mathrm{DO}_2(\overline{\Omega})$ is a degenerate elliptic operator of order $\ell = 2$.*

Proof. If $E = \mathbb{H}_n \times \mathbb{R}$ is the trivial vector bundle, one may compute the symbol $\sigma_2(\Delta_b) \in \mathrm{Hom}(\pi^* E,\, \pi^* E)$ [where $\pi \, : \, T^*(\mathbb{H}_n) \setminus \{0\} \to \mathbb{H}_n$ is the projection] and show that the ellipticity of Δ_b degenerates at the cotangent directions spanned by the canonical contact form θ_0 (see E. Barletta and S. Dragomir [25]). Here we wish to give a "sub-Riemannian proof" to the statement. Let us recall that $L \in \mathrm{DO}_\ell(\overline{\Omega})$ is elliptic in $\overline{\Omega}$ if $L_\ell(z,\, t,\, \xi) \neq 0$ for any $(z,t) \in \overline{\Omega}$ and any $\xi \in \mathbb{R}^{2n+1} \setminus \{0\}$. Let $\mathrm{EO}_\ell(\overline{\Omega})$ be the set of elliptic operators of order k. Then, $\Delta_\epsilon \in \mathrm{EO}_2(\overline{\Omega})$ for every $\epsilon > 0$ and, by (33),

$$\big\|\Delta_\epsilon - \Delta_b\big\| = \epsilon^2 \|T^2\| \to 0, \quad \epsilon \to 0^+ ,$$

hence $\Delta_b \in \partial\, \mathrm{EO}_2(\overline{\Omega})$. However, (see, e.g., N. Shimakura [29], p. 184) $\mathrm{EO}_2(\overline{\Omega})$ is an open subset of the Banach space $\mathrm{DO}_2(\overline{\Omega})$ whose boundary consists precisely of the degenerate (second order) elliptic operators on $\overline{\Omega}$. □

2.5. Curvature Properties

Let $R(\nabla)$ and $R(\nabla^\epsilon)$ be the curvature tensor fields of ∇ (the Tanaka–Webster connection of (M, θ)) and of ∇^ϵ (the Levi–Civita connection of $(M,\, g_\epsilon)$).

Lemma 5. *Let M be a strictly pseudoconvex CR manifold and $\theta \in \mathcal{P}_+$ a positively oriented contact form on M. Then, $R(\nabla^\epsilon)$ and $R(\nabla)$ are related by*

$$\begin{gathered} R(\nabla^\epsilon)(X,Y)Z = R(\nabla)(X,Y)Z \\ -\epsilon^2 \left[(\nabla_X A)(Y,Z) - (\nabla_Y A)(X,Z) \right] T \\ +\Omega(Y,Z)\, \tau X - \Omega(X,Z)\, \tau Y - A(Y,Z)\, JX + A(X,Z)\, JY \\ +\epsilon^2 \left[A(X,Z)\, \tau Y - A(Y,Z)\, \tau X \right] \\ +\tfrac{1}{\epsilon^2} [\Omega(Y,Z)\, JX - \Omega(X,Z)\, JY - 2\, \Omega(X,Y)\, JZ], \end{gathered} \tag{37}$$

$$R(\nabla^\epsilon)(X,Y)T = (\nabla_X \tau)Y - (\nabla_Y \tau)X + 2\, \Omega(X,\, \tau Y)\, T, \tag{38}$$

$$\begin{gathered} R(\nabla^\epsilon)(X,T)Y = R(\nabla)(X,T)Y \\ +\big\{ \epsilon^2 \big[(\nabla_T A)(X,Y) + g_\theta(\tau X\,,\, \tau Y) \big] \\ -\tfrac{1}{\epsilon^2}\, g_\theta(X,Y) - 2\, A(X,JY) - \Omega(\tau X,\, Y) \big\}\, T, \end{gathered} \tag{39}$$

$$R(\nabla^\epsilon)(X,T)T = -\tau^2 X - \frac{1}{\epsilon^2}\, JX + \frac{1}{\epsilon^4}\, X - (\nabla_T \tau)X, \tag{40}$$

for any $X, Y, Z \in H(M)$.

Proof. Let $X, Y, Z \in H(M)$. Then,

$$[X,Y] = \Pi_H\,[X,Y] + \theta\big([X,Y]\big)\,T$$

and

$$\theta\big([X,Y]\big) = -2\,(d\theta)(X,Y) = 2\,\Omega(X,Y),$$

hence

$$[X,Y] = \Pi_H\,[X,Y] + 2\,\Omega(X,Y)\,T. \tag{41}$$

By (12)–(15) relating ∇^ϵ to ∇, one conducts the following calculations:

$$R\big(\nabla^\epsilon\big)(X,Y)Z = \nabla^\epsilon_X\nabla^\epsilon_Y Z - \nabla^\epsilon_Y\nabla^\epsilon_X Z - \nabla^\epsilon_{[X,Y]}Z$$

by substitution from (41)

$$\begin{aligned}
&= \nabla^\epsilon_X\Big\{\nabla_Y Z + \Big[\Omega(Y,Z) - \epsilon^2\,A(Y,Z)\Big]\,T\Big\}\\
&-\nabla^\epsilon_Y\Big\{\nabla_X Z + \Big[\Omega(X,Z) - \epsilon^2\,A(X,Z)\Big]\,T\Big\}\\
&-\nabla_{\Pi_H\,[X,Y]}Z - \Big\{\Omega\big(\Pi_H\,[X,Y],\,Z\big) - \epsilon^2\,A\big(\Pi_H\,[X,Y],\,Z\big)\Big\}\,T\\
&-2\,\Omega(X,Y)\Big[\nabla_T Z + \frac{1}{\epsilon^2}\,JZ\Big]
\end{aligned}$$

again by (41) and $\nabla\Omega = 0$,

$$\begin{aligned}
&= R(\nabla)(X,Y)Z - \frac{2}{\epsilon^2}\,\Omega(X,Y)\,JZ\\
&+\Big[\Omega(Y,Z) - \epsilon^2\,A(Y,Z)\Big]\,\Big[\tau X + \frac{1}{\epsilon^2}\,JX\Big]\\
&-\Big[\Omega(X,Z) - \epsilon^2\,A(X,Z)\Big]\,\Big[\tau Y + \frac{1}{\epsilon^2}\,JY\Big]\\
&+\epsilon^2\{(\nabla_Y A)(X,Z) - (\nabla_X A)(Y,Z)\}\,T,
\end{aligned}$$

thus proving (37). To prove (38), one conducts the following calculation

$$R\big(\nabla^\epsilon\big)(X,Y)T = \nabla^\epsilon_X\nabla^\epsilon_Y T - \nabla^\epsilon_Y\nabla^\epsilon_X T - \nabla^\epsilon_{[X,Y]}T$$

by (13) and (41)

$$\begin{aligned}
&= \nabla^\epsilon_X\Big[\tau Y + \frac{1}{\epsilon^2}\,JY\Big] - \nabla^\epsilon_Y\Big[\tau X + \frac{1}{\epsilon^2}\,JX\Big]\\
&-\nabla^\epsilon_{\Pi_H\,[X,Y]}T - 2\,\Omega(X,Y)\,\nabla^\epsilon_T T
\end{aligned}$$

by $\tau\circ J + J\circ\tau = 0$

$$= (\nabla_X\tau)Y - (\nabla_Y\tau)X + \frac{1}{\epsilon^2}[(\nabla_X J)Y - (\nabla_Y J)X],$$

yielding (38) by $\nabla J = 0$. To prove (39), one conducts the following calculation:

$$R\big(\nabla^\epsilon\big)(X,T)Y = \nabla^\epsilon_X\nabla^\epsilon_T Y - \nabla^\epsilon_T\nabla^\epsilon_X Y - \nabla^\epsilon_{[X,T]}Y$$

by (14) and (12), and by $[X,T] \in H(M)$

$$= \nabla^\epsilon_X\Big\{\nabla_T X + \frac{1}{\epsilon^2}\,JY\Big\}$$

$$-\nabla^{\epsilon}_{T}\Big\{\nabla_X Y + \Big[\Omega(X,Y) - \epsilon^2 A(X,Y)\Big]\, T\Big\}$$
$$-\nabla_{[X,T]} Y - \Big\{\Omega\big([X,T],\, Y\big) - \epsilon^2 A\big([X,T],\, Y\big)\Big\}\, T.$$

Note that, by the very definition of the pseudohermitian torsion τ,

$$[X,T] = \tau X - \nabla_T X, \quad X \in H(M).$$

Then, by $\nabla J = 0$ and $\nabla\Omega = 0$,

$$R(\nabla^{\epsilon})(X,T)Y = R^{\nabla}(X,T)Y$$
$$+\frac{1}{\epsilon^2}\Big[\Omega(X,JY) - \epsilon^2 A(X,JY)\Big]\, T$$
$$-\Omega(\tau X, Y)\, T + \epsilon^2 A(\tau X, Y)\, T + \epsilon^2\, (\nabla_T A)(X,Y)\, T,$$

thus yielding (39). Finally, (40) follows from

$$R(\nabla^{\epsilon})(X,T)T = \nabla^{\epsilon}_X \nabla^{\epsilon}_T T - \nabla^{\epsilon}_T \nabla^{\epsilon}_X T - \nabla^{\epsilon}_{[X,T]} T$$

by $\nabla^{\epsilon}_T T = 0$ and (13)

$$= -\nabla^{\epsilon}_T\Big\{\tau X + \frac{1}{\epsilon^2}\, JX\Big\} - \tau[X,T] - \frac{1}{\epsilon^2}\, J[X,T]$$

by $\nabla J = 0$ and $\tau \circ J + J \circ \tau = 0$

$$= -\tau^2 X + \frac{1}{\epsilon^4}\, X - (\nabla_T \tau) X.$$

□

3. First Fundamental Forms

Let $(M,\, T_{1,0}(M))$ be a strictly pseudoconvex CR manifold of CR dimension n. Through this section, given a positive integer $k \geq 1$ and another strictly pseudoconvex CR manifold $(A,\, T_{1,0}(A))$ of CR dimension $N = n + k$, we study the geometry of the second fundamental form of Cauchy–Riemann (CR) immersions $f : M \to A$.

Definition 33. A *CR immersion* $f : M \to A$ is a C^{∞} immersion of $f : M \to A$, which is a CR map.

Our approach to the study of CR immersions is to establish pseudohermitian analogues to the Gauss–Weingarten formulas and to the Gauss–Ricci–Codazzi equations. Let $\theta \in \mathcal{P}_+(M)$ and $\Theta \in \mathcal{P}_+(A)$ be positively oriented contact forms on M and A, respectively.

Lemma 6. *Let $f : M \to A$ be a CR immersion. There is a unique function $\Lambda \in C^{\infty}(M)$ such that*

$$f^*\Theta = \Lambda\, \theta\,. \tag{42}$$

Consequently,

$$f^* G_{\Theta} = \Lambda\, G_{\theta} \tag{43}$$

and $\Lambda(x) \geq 0$ for any $x \in M$.

Proof. For every $x \in M$ and $X \in H(M)_x$,

$$(f^*\Theta)_x X = \Theta_{f(x)}\, (d_x f) X = 0$$

because of

$$(d_x f) H(M)_x \subset H(A)_{f(x)}\,. \tag{44}$$

Hence,

$$H(M)_x \subset \mathrm{Ker}\big[\big(f^*\Theta\big)_x\big]\,. \tag{45}$$

Let $\{E_a \,:\, 1 \le a \le 2n\}$ be a local frame of $H(M)$, defined on an open neighborhood $U \subset M$ of x. Then,

$$\{E_j \,:\, 0 \le j \le 2n\}, \quad E_0 = T,$$

is a local frame of $T(M)$ on U. Let us set

$$\Lambda = \big(f^*\Theta\big) T \in C^\infty(M)\,. \tag{46}$$

For every $X = a^j\, E_{j,\,x} \in T_x(M)$ [by $\theta(T) = 1$]

$$\big(f^*\Theta\big)_x X = a^0\, \big(f^*\Theta\big)_x T_x = a^0\, \Lambda(x) = \Lambda(x)\, \theta_x(X),$$

yielding (42). As f is a CR map, aside from (44), one has

$$(d_x f) \circ J_x = J^A_{f(x)} \circ (d_x f), \quad x \in M, \tag{47}$$

where J and J^A are the complex structures along the Levi distributions $H(M)$ and $H(A)$. Also, by exterior differentiation of (42),

$$d\Lambda \wedge \theta + \Lambda\, d\theta = f^*\big(d\,\Theta\big).$$

Hence, for any $X, Y \in H(M)$,

$$\Lambda\, G_\theta(X, Y) = \Lambda\, (d\theta)(X, JY)$$

$$= \big(f^*(d\,\Theta)\big)(X,\, JY) = (d\Theta)^f\, \big(f_* X,\, f_*\, JY\big)$$

by (47)

$$= (d\Theta)^f\, \big(f_* X,\, (J^A)^f\, f_* Y\big) = (G_\Theta)^f\, (f_* X,\, f_* Y),$$

proving (43). An upper index f denotes composition with f, e.g., $(J^A)^f = J^A \circ f$, where J^A is thought of as a section $J^A : A \to T^*(A) \otimes T(A)$. Finally, for every $X \in T_x(M)$, $X \ne 0$ (as $d_x f$ is a monomorphism, and $G_{\Theta,\, f(x)}$ and $G_{\theta,\, x}$ are positive definite),

$$0 < G_{\Theta,\, f(x)}\big((d_x f)X,\, (d_x f)X\big) = (f^* G_\Theta)_x(X, X) = \Lambda(x)\, G_{\theta,\, x}(X, X),$$

yielding $\Lambda(x) \ge 0$. $\square$

Definition 34. The function $\Lambda = \Lambda(f) = \Lambda(f;\, \theta,\, \Theta)$ given by (46) is the *dilation* of f relative to the choice of contact forms $(\theta,\, \Theta)$.

Definition 35. A CR immersion $f : M \to A$ of (M, θ) into (A, Θ) is said to be *isopseudohermitian* if $\Lambda(f;\, \theta,\, \Theta) \equiv 1$.

Proposition 2. *Let $f \,:\, M \to A$ be a CR immersion between the strictly pseudoconvex CR manifolds M and A. Let $\Theta \in \mathcal{P}_+(A)$. If*

$$\Theta_{f(x)} \circ (d_x f) \ne 0, \quad x \in M,$$

then there is a contact form $\hat\theta \in \mathcal{P}_+(M)$ such that f is an isopseudohermitian immersion of $(M,\, \hat\theta)$ into $(A,\, \Theta)$.

Proof. Let $\theta \in \mathcal{P}_+(M)$ and let Λ be the dilation of the CR immersion f relative to the pair (θ, Θ). Then, $\Lambda(M) \subset (0, +\infty)$, and we may set $\hat{\theta} = \Lambda\,\theta$. □

For every CR immersion $f : M \to A$, we may look at M as an immersed submanifold of the Riemannian manifold (A, g_Θ). However, in general, the *first fundamental form*, i.e., the pullback $f^* g_\Theta$ of the ambient Webster metric g_Θ, of the given immersion $f : M \to (A, g_\Theta)$ does not coincide with the intrinsic Webster metric g_θ, not even if f is isopseudohermitian. To circumnavigate this obstacle, we endow A with the Riemannian metric g_ϵ^A, the contraction of the Levi form G_Θ associated with every $\epsilon > 0$ as in Section 2, given by

$$g_\epsilon^A = g_\Theta + \left(\frac{1}{\epsilon^2} - 1\right) \Theta \otimes \Theta \tag{48}$$

and derive the Gauss–Codazzi–Ricci equations of the immersion $f : M \to (A, g_\epsilon^A)$. As a consequence of (48),

$$g_\epsilon^A = G_\Theta \quad \text{on} \quad H(A) \otimes H(A), \tag{49}$$

$$g_\epsilon^A(X, T_A) = 0, \quad X \in H(A), \tag{50}$$

$$g_\epsilon^A(T_A\,, T_A) = \epsilon^{-2}\,, \tag{51}$$

where $T_A \in \mathfrak{X}(A)$ is the Reeb vector field of (A, Θ). Let

$$g_\epsilon(f) = f^* g_\epsilon^A \tag{52}$$

be the induced metric, i.e., the *first fundamental form* of the given immersion $f : M \to (A, g_\epsilon^A)$. Then, for any $X, Y \in H(M)$,

$$g_\epsilon(f)(X,Y) = \left(g_\epsilon^A\right)^f (f_* X\,, f_* Y)$$

by (49), as $f_* X,\, f_* Y \in C^\infty\big(f^{-1} H(M)\big)$

$$= G_\Theta^f(f_* X,\, f_* Y) = (f^* G_\Theta)(X,Y) = \Lambda\, G_\theta(X,Y)$$

by (43). Throughout, an upper index f denotes composition with f. Summing up:

$$g_\epsilon(f) = \Lambda\, G_\theta \quad \text{on} \quad H(M) \otimes H(M). \tag{53}$$

For every $x \in M$, let us decompose $(d_x f) T_x \in T_{f(x)}(A)$ with respect to

$$T(A) = H(A) \oplus \mathbb{R} T_A, \tag{54}$$

which is

$$(d_x f) T_x = v + \mu\, T_{A, f(x)} \tag{55}$$

for some $v \in H(A)_{f(x)}$ and $\mu \in \mathbb{R}$. If $X \in H(M)$, then

$$g_\epsilon(f)(X,T)_x = (f^* g_\epsilon^A)(X,T)_x$$

$$= g_{\epsilon, f(x)}^A\big((d_x f) X_x\,, (d_x f) T_x\big) = G_{\Theta, f(x)}\big((d_x f) X_x\,, v\big)$$

by (49) and (50), which can be applied because $(d_x f) X_x \in H(A)_{f(x)}$. We have shown that

$$g_\epsilon(f)(X,T)_x = G_{\Theta, f(x)}\big((d_x f) X_x\,, v\big) \tag{56}$$

for any $X \in H(M)$, where

$$v = \Pi_{H(A), f(x)} (d_x f) T_x\,. \tag{57}$$

Let $E_H(f)_x$ be the orthogonal complement of $(d_x f)H(M)_x$ in the inner product space $\left(H(A)_{f(x)}, G_{\Theta, f(x)}\right)$ so that

$$H(A)_{f(x)} = \left[(d_x f)H(M)_x\right] \oplus E_H(f)_x, \quad x \in M. \tag{58}$$

Lemma 7. *$E_H(f) \to M$ is a J_A-invariant real rank $2k$ subbundle of $f^{-1}H(A) \to M$.*

Proof. Let $V \in E_H(f)$. Then,

$$G_\Theta^f(J_A^f V, f_* X) = -G_\Theta^f(V, J_A^f f_* X)$$

by (47)]

$$= -G_\Theta^f(V, f_* J X) = 0,$$

yielding $J_A^f V \in E_H(f)$. □

Definition 36. The real vector bundle $E_H(f) \to M$ is called the *Levi normal bundle* of the given CR immersion $f : M \to (A, \Theta)$. A section $\xi \in C^\infty\left(E_H(f)\right)$ is a *Levi normal field*.

The tangent vector $v \in H(A)_{f(x)}$, first appearing in the decomposition (55), may be further decomposed, with respect to (58), as

$$v = (d_x f)Y_x + v^\perp \tag{59}$$

for some $Y \in H(M)$ and some $v^\perp \in E_H(f)_x$. The Levi normal vector $v^\perp$ and the value of Y at x (but not Y) are uniquely determined by the decomposition (59). With (53) and (56), we started the calculation of the first fundamental form of $f : M \to (A, g_\epsilon^A)$. Let us substitute from (59) into (56) and take into account (43). We obtain

$$g_\epsilon(f)(X, T) = \Lambda\, G_\theta(X, Y), \quad X \in H(M). \tag{60}$$

Let

$$\tan_{H,x} : H(A)_{f(x)} \to H(M)_x, \quad \mathrm{nor}_{H,x} : H(A)_{f(x)} \to E_H(f)_x,$$

be the projections associated with the direct sum decomposition (58), so that

$$w = (d_x f)\tan_{H,x} w + \mathrm{nor}_{H,x} w, \quad w \in H(A)_{f(x)}. \tag{61}$$

Then, by (59),

$$\tan_{H,x} v = Y_x, \quad \mathrm{nor}_{H,x} v = v^\perp.$$

Moreover, by (25) and (57),

$$Y_x = \tan_{H,x}\left\{\Pi_{H(A), f(x)}\, (d_x f)T_x\right\}$$

$$= \tan_{H,x}\left\{(d_x f)T_x - \Theta_{f(x)}\left[(d_x f)T_x\right] T_{A, f(x)}\right\}$$

or

$$Y_x = \tan_{H,x}\left\{(d_x f)T_x - \Lambda(x)\, T_{A, f(x)}\right\} \tag{62}$$

as, by (42),

$$\Theta_{f(x)}\left[(d_x f)T_x\right] = \left(f^*\Theta\right)_x T_x = \Lambda(x)\, \theta_x(T_x) = \Lambda(x).$$

For further use, let us set

$$\mathfrak{X}_\Theta(x) := (d_x f)T_x - \Lambda(x)\, T_{A, f(x)}, \quad x \in M. \tag{63}$$

As a byproduct of the calculations leading to (62), we have $\mathfrak{X}_{\Theta,x} \in H(A)_{f(x)}$ for any $x \in M$; i.e., (63) defines a section $\mathfrak{X}_\Theta$ in the pullback bundle $f^{-1}H(A) \to M$

$$\mathfrak{X}_\Theta = f_* T - \Lambda\, T_A^f \in C^\infty\big(f^{-1}H(A)\big).$$

Next, by (60) and $Y_x = \tan_{H,x} \mathfrak{X}_{\Theta,x}$, i.e., by (62) and (63),

$$g_\epsilon(f)(X,\,T)_x = \Lambda(x)\, G_\theta(X,Y)_x = (f^* G_\Theta)(X,Y)_x$$

$$= G_{\Theta\,,\,f(x)}\big((d_x f)X_x\,,\,(d_x f)Y_x\big)$$

$$= G_{\Theta\,,\,f(x)}\big((d_x f)X_x\,,\,(d_x f)\tan_{H,x} \mathfrak{X}_{\Theta\,,\,x}\big)$$

by (61) for $w = \mathfrak{X}_{\Theta\,,\,x}$

$$= G_{\Theta\,,\,f(x)}\big((d_x f)X_x\,,\,\mathfrak{X}_{\Theta\,,\,x} - \mathrm{nor}_{H,x} \mathfrak{X}_{\Theta\,,\,x}\big)$$

as $(d_x f)X_x$ and $\mathrm{nor}_{H,x} \mathfrak{X}_{\Theta\,,\,x}$ are $G_{\Theta\,,\,f(x)}$-orthogonal

$$= G_{\Theta\,,\,f(x)}\big((d_x f)X_x\,,\,\mathfrak{X}_{\Theta\,,\,x}\big) = -(d\Theta)_{f(x)}\big(J_{A,\,f(x)}\,(d_x f)\,X_x\,,\,\mathfrak{X}_{\Theta,\,x}\big)$$

as f is a CR map

$$= -(d\Theta)_{f(x)}\big((d_x f)\,J_x\,X_x\,,\,\mathfrak{X}_{\Theta,\,x}\big)$$

by $T_A \rfloor d\Theta = 0$

$$= -(d\Theta)_{f(x)}\big((d_x f)\,J_x\,X_x\,,\,(d_x f)T_x\big).$$

Next, note that, by taking the exterior differential of (42),

$$f^*\,(d\Theta) = d\big(f^*\Theta\big) = d\Lambda \wedge \theta + \Lambda\, d\theta$$

so that

$$g_\epsilon(f)(X,\,T)_x = -\big(d\Lambda \wedge \theta\big)(JX,\,T)_x + \Lambda(x)\,\big(T \rfloor d\theta\big)(JX)_x$$

by $\mathrm{Ker}(\theta) = H(M)$, $\theta(T) = 1$, and $T \rfloor d\theta = 0$

$$= -(JX)(\Lambda)_x\,.$$

Summing up:

$$g_\epsilon(f)(X,T) = -(JX)(\Lambda), \quad X \in H(M). \tag{64}$$

Together with (53), this determines the first fundamental form $g_\epsilon(f) = f^* g_\epsilon^A$ on $H(M) \otimes H(M)$ and $H(M) \otimes \mathbb{R}T$. By taking into account the decomposition (6), to fully determine $g_\epsilon(f)$, we ought to compute

$$g_\epsilon(f)(T,T)_x = (f^* g_\epsilon^A)(T,T)_x = g^A_{\epsilon,\,f(x)}\big((d_x f)T_x\,,\,(d_x f)T_x\big)$$

by substitution from (55)

$$= g^A_{\epsilon,\,f(x)}\big(v + \mu\, T_{A,\,f(x)}\,,\,v + \mu\, T_{A,\,f(x)}\big)$$

by (50) i.e., by $v \in H(A)_{f(x)} \perp T_{A,\,f(x)}$ with respect to $g^A_{\epsilon,\,f(x)}$

$$= g^A_{\epsilon,\,f(x)}(v,\,v) + \mu^2\, g^A_\epsilon(T_A\,,\,T_A)_{f(x)}$$

by (49) i.e., $g_\epsilon^A = G_\Theta$ on $H(A) \otimes H(M)$ and by (51) i.e., $g_\epsilon^A(T_A\,,\,T_A) = \epsilon^{-2}$

$$= \left(\frac{\mu}{\epsilon}\right)^2 + G_{\Theta\,,\,f(x)}(v,\,v).$$

On the other hand, by going back to (55),

$$\mu = \Theta_{f(x)}\,(d_x f)\,T_x = \left(f^*\Theta\right)_x T_x = \Lambda(x)\,\theta(T)_x = \Lambda(x)$$

so that (55) *de facto* reads

$$(d_x f)T_x = v + \Lambda(x)\,T_{A,\,f(x)}$$

or, by (63),

$$v = \mathfrak{X}_{\Theta\,,\,x}\,. \tag{65}$$

Our calculations so far lead to

$$g_\epsilon(f)(T,\,T)_x = \left[\frac{\Lambda(x)}{\epsilon}\right]^2 + \|\mathfrak{X}_{\Theta\,,\,x}\|^2_\Theta\,. \tag{66}$$

Here we have set $\|v\|_\Theta = G_{\Theta\,,\,f(x)}(v,\,v)^{1/2}$. Another useful expression of the norm $\|v\|_\Theta$ may be obtained as follows. Note first that, as a consequence of our key observation $\mathfrak{X}_\Theta \in C^\infty(f^{-1}H(M))$,

$$\begin{aligned} g^f_\Theta(f_*T\,,\,T^f_A) &= g^f_\Theta(f_*T - \Lambda\,T^f_A\,,\,T^f_A) + \Lambda\,g_\Theta(T_A\,,\,T_A)^f \\ &= g^f_\Theta(\mathfrak{X}_\Theta\,,\,T^f_A) + \Lambda \end{aligned}$$

or

$$g^f_\Theta(f_*T,\,T^f_A) = \Lambda. \tag{67}$$

Then, by (65),

$$\begin{aligned} G_{\Theta\,,\,f(x)}(v,\,v) &= g_{\Theta,\,f(x)}\big(\mathfrak{X}_{\Theta,\,x}\,,\,\mathfrak{X}_{\Theta,\,x}\big) \\ &= g_{\Theta,\,f(x)}\big((d_x f)T_x\,,\,(d_x f)T_x\big) + \Lambda(x)^2\,g_{\Theta\,,\,f(x)}\big(T_{A,\,f(x)}\,,\,T_{A,\,f(x)}\big) \\ &\quad -2\,\Lambda(x)\,g_{\Theta,\,f(x)}\Big((d_x f)T_x\,,\,T_{A,\,f(x)}\Big) \end{aligned}$$

or, by (67),

$$G_{\Theta\,,\,f(x)}(v,\,v) = \left(f^* g_\Theta\right)(T,\,T)_x - \Lambda(x)^2\,. \tag{68}$$

Finally, by substitution from (68) into (66),

$$g_\epsilon(f)(T,\,T) = \left(\frac{1}{\epsilon^2} - 1\right)\Lambda^2 + \left(f^* g_\Theta\right)(T,T). \tag{69}$$

The first fundamental form of $f : M \to (A,\,g^A_\epsilon)$ is fully determined. Summing up, we have established:

Proposition 3. *Let us set* $\mathfrak{X}_\Theta = f_*T - \Lambda\,T^f_A$*. Then*
(i) $\mathfrak{X}_\Theta \in C^\infty\big(f^{-1}H(A)\big)$,
(ii) *For any* $X \in H(M)$

$$G^f_\Theta(f_*X\,,\,\mathfrak{X}_\Theta) = -(JX)(\Lambda) \tag{70}$$

or equivalently

$$\Lambda \tan_H \mathfrak{X}_\Theta = J\,\nabla^H \Lambda. \tag{71}$$

In particular, if $\Lambda(x) \neq 0$ *for any* $x \in M$*, then*

$$\tan_H \mathfrak{X}_\Theta = J\,\nabla^H \log|\Lambda|,$$

while if $\Lambda = constant$ *(e.g., f is isopseudohermitian, i.e., $\Lambda \equiv 1$), then $\mathfrak{X}_\Theta$ is a Levi normal vector field on M i.e.,* $\mathfrak{X}_\Theta \in C^\infty\big(E_H(f)\big)$.

(iii) *The norm of vector field* $\mathfrak{X}_\Theta$ *is*

$$\|\mathfrak{X}_\Theta\|_\Theta = \left[(f^* g_\Theta)(T, T) - \Lambda^2 \right]^{1/2}.$$

(iv) *The first fundamental form* $g_\epsilon(f)$ *of the immersion* $f : M \to \left(A, g_\epsilon^A\right)$ *is given by*

$$g_\epsilon(f) = \Lambda\, G_\theta \quad on \quad H(M) \otimes H(M), \tag{72}$$

$$g_\epsilon(f)(X, T) = -(JX)(\Lambda), \quad X \in H(M), \tag{73}$$

$$g_\epsilon(f)(T, T) = \left(\frac{1}{\epsilon^2} - 1 \right) \Lambda^2 + \left(f^* g_\Theta\right)(T, T). \tag{74}$$

Consequently,

$$\begin{gathered} g_\epsilon(f) = \Lambda\, g_\theta + \left\{ \left(\frac{\Lambda}{\epsilon} \right)^2 - \Lambda(\Lambda + 1) + \left(f^* g_\Theta\right)(T, T) \right\} \theta \otimes \theta \\ +2\, \theta \odot g_\theta\left(J\, \nabla^H \Lambda\, , \cdot\right) \end{gathered} \tag{75}$$

and, in particular,

$$\begin{gathered} g_\epsilon(f) = \Lambda\, g_\epsilon^M + 2\, \theta \odot g_\theta\left(J\, \nabla^H \Lambda\, , \cdot\right) \\ + \left\{ \tfrac{1}{\epsilon^2} \left(\Lambda^2 - 1\right) - \Lambda(\Lambda + 1) + 1 + \left(f^* g_\Theta\right)(T, T) \right\} \theta \otimes \theta\, . \end{gathered} \tag{76}$$

Corollary 2. *Let* $f : M \to A$ *be an isopseudohermitian CR immersion of* (M, θ) *into* (A, Θ)*. Then, the first fundamental form* $g_\epsilon(f) = f^* g_\epsilon^A$ *and the Webster metric* g_θ*, respectively, the* ϵ*-contraction* g_ϵ^M *of* G_θ*, are related by*

$$g_\epsilon(f) = g_\theta + \left\{ \frac{1}{\epsilon^2} - 2 + \left(f^* g_\Theta\right)(T, T) \right\} \theta \otimes \theta\, , \tag{77}$$

$$g_\epsilon(f) = g_\epsilon^M + \left\{ \left(f^* g_\Theta\right)(T, T) - 1 \right\} \theta \otimes \theta\, . \tag{78}$$

Let $E(f) \to M$ and $E_\epsilon(f) \to M$ be, respectively, the normal bundles of the immersions $f : M \to (A, g_\Theta)$ and $f : M \to (A, g_\epsilon^A)$, so that

$$T_{f(x)}(A) = \left[(d_x f) T_x(M) \right] \oplus E(f)_x\, , \tag{79}$$

$$T_{f(x)}(A) = \left[(d_x f) T_x(M) \right] \oplus E_\epsilon(f)_x\, , \tag{80}$$

for every $x \in M$.

Lemma 8. *The normal bundle* $E_\epsilon(f) \to M$ *and the Levi normal bundle* $E_H(f) \to M$ *are related by*

$$E_\epsilon(f) \subset E_H(f) \oplus \mathbb{R} T_A^f\, . \tag{81}$$

A dimension count shows that the inclusion is strict.

Proof. Let

$$\xi \in E_\epsilon(f) \subset f^{-1} T(A) = \left[f^{-1} H(M) \right] \oplus \mathbb{R} T_A^f$$

so that

$$\xi = W + \mu\, T_A^f$$

for some $W \in f^{-1}H(A)$ and some $\mu \in C^\infty(M)$. Then, for any $X \in H(M)$

$$G_\Theta(W, f_* X) = g_\epsilon^A(W, f_* X) = g_\epsilon^A(\xi - \mu T_A^f, f_* X)$$

as $T_A^f \perp C^\infty(f^{-1}H(M)) \supset f_* C^\infty(H(M))$

$$= g_\epsilon^A(\xi, f_* X) = 0$$

because of $\xi \perp f_* \mathfrak{X}(M) \supset f_* C^\infty H(M)) \ni f_* X$. This yields $W \in E_H(f)$. □

Let

$$\tan_x : T_{f(x)}(A) \to T_x(M), \quad \mathrm{nor}_x : T_{f(x)}(A) \to E(f)_x ,$$
$$\tan_x^\epsilon : T_{f(x)}(A) \to T_x(M), \quad \mathrm{nor}_x^\epsilon : T_{f(x)}(A) \to E_\epsilon(f)_x ,$$

be the projections associated with the decompositions (79) and (80). Next, let us set

$$T_A^\perp = \mathrm{nor}\big(T_A\big) \in C^\infty(E(f)) \tag{82}$$

so that

$$T_A^f = f_* \tan\big(T_A\big) + T_A^\perp . \tag{83}$$

Lemma 9. *Let $f : M \to A$ be a CR immersion of strictly pseudoconvex CR manifolds, and let $\theta \in \mathcal{P}_+(M)$ and $\Theta \in \mathcal{P}_+(A)$ be positively oriented contact forms on M and A. Then*

$$\Lambda^2 \tan\big(T_A\big) = \{1 - \Theta^f(T_A^\perp)\} \, \{\Lambda\, T - J\, \nabla^H \Lambda\}. \tag{84}$$

In particular, if f is isopseudohermitian, then

$$\tan\big(T_A\big) = \{1 - \Theta^f(T_A^\perp)\}\, T. \tag{85}$$

Proof. As $\Theta(T_A) = 1$

$$1 = \Theta(T_A)^f = \Theta^f\big(T_A^f\big)$$

by (83)

$$= \Theta^f\big(f_* \tan\big(T_A\big)\big) + \Theta^f\big(T_A^\perp\big)$$
$$= \big(f^*\Theta\big) \tan\big(T_A\big) + \Theta^f\big(T_A^\perp\big) = \Lambda\, \theta\big[\tan\big(T_A\big)\big] + \Theta^f\big(T_A^\perp\big)$$

so that

$$\Lambda\, \theta\big[\tan\big(T_A\big)\big] = 1 - \Theta^f\big(T_A^\perp\big). \tag{86}$$

Moreover, for any $X \in H(M)$,

$$\Lambda\, g_\theta(X, \tan(T_A)) = \Lambda\, G_\theta(X, \Pi_H \tan(T_A))$$

$$= (f^* G_\Theta)(X, \Pi_H \tan(T_A)) = G_\Theta^f\big(f_* X, f_* \Pi_H \tan(T_A)\big)$$
$$= g_\Theta^f\big(f_* X, f_* \Pi_H \tan(T_A)\big)$$
$$= g_\Theta^f\big(f_* X, f_* \{\tan(T_A) - \theta(\tan(T_A))\, T\}\big)$$
$$= g_\Theta^f\big(f_* X, f_* \tan(T_A)\big) - \theta(\tan(T_A))\, g_\Theta^f\big(f_* X, f_* T\big).$$

As $f_* X$ is tangential, one has $g_\Theta^f\big(f_* X, T_A^\perp\big) = 0$, hence, by (83),

$$g_\Theta^f\big(f_* X, f_* \tan(T_A)\big) = g_\Theta^f\big(f_* X, T_A^f\big) = 0$$

because $f_* X \in C^\infty(f^{-1}H(A))$ and $H(A) \perp T_A$ with respect to g_Θ. We may conclude that

$$\Lambda\, g_\theta(X,\, \tan(T_A)) = -\theta(\tan(T_A))\, (f^* g_\Theta)(X,\, T). \tag{87}$$

On the other hand,

$$\begin{aligned}(f^* g_\Theta)(X,\, T) &= g^f_\Theta(f_* X,\, f_* T)\\ &= g^f_\Theta(f_* X,\, f_* T - \Lambda\, T^f_A) + \Lambda\, g^f_\Theta(f_* X,\, T^f_A)\\ &= g^f_\Theta(f_* X,\, \mathfrak{X}_\Theta) + \Lambda\, \Theta^f(f_* X) = G^f_\Theta(f_* X,\, \mathfrak{X}_\Theta) + \Lambda\,(f^*\Theta)X\\ &= G^f_\Theta(f_* X,\, f_* \tan_H \mathfrak{X}_\Theta) + \Lambda^2\, \theta(X)\end{aligned}$$

as $X \in H(M) = \operatorname{Ker}(\theta)$

$$= (f^* G_\Theta)(X,\, \tan_H \mathfrak{X}_\Theta) = \Lambda\, G_\theta(X,\, \tan_H \mathfrak{X}_\Theta) = -(JX)(\Lambda)$$

by (70). Then, by (87),

$$\Lambda\, \Pi_H \tan T_A = -\theta(\tan T_A)\, J\nabla^H \Lambda. \tag{88}$$

Finally,

$$\Lambda \tan(T_A) = \Lambda\, \{\Pi_H \tan(T_A) + \theta(\tan T_A)\, T\}$$

by (88)

$$= \theta(\tan(T_A))\{-J\nabla^H\Lambda + \Lambda\, T\}$$

that is, by (86)

$$\Lambda^2 \tan(T_A) = \{1 - \Theta^f(T^\perp_A)\}\, \{\Lambda\, T - J\, \nabla^H \Lambda\},$$

and Lemma 9 is proved. □

4. Pseudohermitian Immersions

Definition 37. Let $(M,\, T_{1,0}(M))$ and $(A,\, T_{1,0}(A))$ be strictly pseudoconvex CR manifolds of CR dimensions n and $N = n + k$, $k \geq 1$. Let $\theta \in \mathcal{P}_+(M)$ and $\Theta \in \mathcal{P}_+(A)$. A CR immersion $f : M \to A$ is said to be a *pseudohermitian immersion* of $(M\, \theta)$ into (A, Θ) if

(i) f is isopseudohermitian, i.e., $f^*\Theta = \theta$,
(ii) $T^\perp_A = 0$.

Proposition 4. *Let $f : M \to A$ be an isopseudohermitian immersion. The following statements are equivalent:*

(i) *f is a pseudohermitian immersion.*
(ii) $f^* g_\Theta = g_\theta$.
(iii) $\mathfrak{X}_\Theta = 0$.
(iv) $(f^* g_\Theta)(T, T) = 1$.

Proof. (i) $\Longrightarrow$ (ii). Let f be a pseudohermitian immersion. Then, for any $X, Y \in H(M)$,

$$\begin{aligned}(f^* g_\Theta)(X, Y) &= g^f_\Theta(f_* X,\, f_* Y) = G^f_\theta(f_* X,\, f_* Y)\\ &= (f^* G_\Theta)(X, Y) = G_\theta(X, Y) = g_\theta(X, Y).\end{aligned}$$

Hence, $f^* g_\Theta = g_\theta$ on $H(M) \otimes H(M)$. Next, $T^\perp_A = 0$ together with Lemma 9 yields

$$\tan(T_A) = T. \tag{89}$$

Then,

$$(f^* g_\Theta)(X, T) = g^f_\Theta(f_* X,\, f_* T)$$

by (89)

$$= g_\Theta^f(f_* X,\, f_* \tan(T_A))$$

by $T_A^\perp = 0$

$$= g_\Theta^f(f_* X,\, T_A^f) = 0 = g_\theta(X,\, T).$$

Hence, $f^* g_\Theta = g_\theta$ on $H(M) \otimes \mathbb{R}T$. It remains necessary that we check (ii) on $\mathbb{R}T \otimes \mathbb{R}T$. Indeed,

$$(f^* g_\Theta)(T, T) = g_\Theta^f(f_* T,\, f_* T)$$

by (89)

$$= g_\Theta^f(f_* \tan(T_A),\, f_* \tan(T_A))$$

by $T_A^\perp = 0$

$$= g_\Theta^f(T_A^f,\, T_A^f) = g_\Theta(T_A\,,\, T_A)^f = 1 = g_\theta(T, T)$$

by the very definition of the Webster metrics. Q.E.D.

(ii) $\Longrightarrow$ (iii). Let $f : M \to A$ be a CR immersion such that (ii) holds, i.e., $f^* g_\Theta = g_\theta$. As f is a CR map, the assumption (ii) implies $f^* G_\Theta = G_\theta$ and then, by (43), $\Lambda \equiv 1$. Moreover,

$$\|\mathfrak{X}_\Theta\|_\Theta^2 = g_\Theta^f(\mathfrak{X}_\Theta\,,\, \mathfrak{X}_\Theta) = g_\Theta^f(f_* T - T_A^f\,,\, f_* T - T_A^f)$$

$$= (f^* g_\Theta)(T, T) + g_\Theta(T_A\,,\, T_A)^f - 2\, g_\Theta^f(f_* T,\, T_A^f)$$

by (67)

$$= g_\theta(T,\, T) - 1 = 0$$

so that $\mathfrak{X}_\Theta = 0$. Q.E.D.

(iii) $\Longrightarrow$ (iv). Let $f : M \to A$ be an isopseudohermitian immersion such that $\mathfrak{X}_\Theta = 0$. Then,

$$\|\mathfrak{X}_\Theta\|_\Theta^2 = (f^* g_\Theta)(T,\, T) - 1 \tag{90}$$

yields (iv). Q.E.D.

(iv) $\Longrightarrow$ (i). Let $f : M \to A$ be an isopseudohermitian immersion such that

$$(f^* g_\Theta)(T, T) = 1.$$

Then, by (90), $\mathfrak{X}_\Theta = 0$ i.e.,

$$T_A^f = f_*\, T \in f_*\, \mathfrak{X}(M) \Longrightarrow T_A^\perp = 0.$$

□

5. Gauss and Weingarten Formulas

Let $f : M \to A$ be a CR immersion of strictly pseudoconvex CR manifolds, and let $\theta \in \mathcal{P}_+(M)$ and $\Theta \in \mathcal{P}_+(A)$. We adopt the following notations for the various linear connections we shall work with:

D^ϵ Levi–Civita connection of $(A,\, g_\epsilon^A)$,

D Tanaka–Webster connection of $(A,\, \Theta)$,

$\nabla^{f,\epsilon}$ Levi–Civita connection of $(M,\, g_\epsilon(f))$,

∇^ϵ Levi–Civita connection of $(M,\, g_\epsilon^M)$,

∇ Tanaka–Webster connection of $(M,\, \theta)$.

The Gauss and Weingarten formulas for the isometric immersion $f : (M, g_\epsilon(f)) \to (A, g_\epsilon^A)$ are:

$$(D^\epsilon)^f_X f_* Y = f_* \nabla^{f,\epsilon}_X Y + B_\epsilon(f)(X,Y), \tag{91}$$

$$(D^\epsilon)^f_X \xi = -f_*\, a^\epsilon_\xi X + D^{\perp_\epsilon}_X \xi\,, \tag{92}$$

for any $X, Y \in \mathfrak{X}(M)$ and any $\xi \in C^\infty(E_\epsilon(f))$. Here, $B_\epsilon(f)$, a^ϵ_ξ and $D^{\perp_\epsilon}$ are, respectively, the second fundamental form, the Weingarten operator (associated with the normal vector field ξ), and the normal connection, a connection in the vector bundle $E_\epsilon(f) \to M$, of the given isometric immersion. The symbol $(D^\epsilon)^f$ in (91) and (92) denotes the connection induced by the Levi–Civita connection of (A, g_ϵ^A) in the pullback bundle $f^{-1}T(A) \to M$; i.e., $(D^\epsilon)^f$ is the pullback connection $f^{-1}D_\epsilon$. One has

$$\nabla^{f,\epsilon}_X Y = \tan_\epsilon \{(D^\epsilon)^f_X f_* Y\}, \quad B_\epsilon(f) = \mathrm{nor}_\epsilon \{(D^\epsilon)^f_X f_* Y\},$$

$$a^\epsilon_\xi X = -\tan_\epsilon \{(D^\epsilon)^f_X \xi\}, \quad D^{\perp_\epsilon}_X \xi = \mathrm{nor}\{(D^\epsilon)^f_X \xi\}.$$

The second fundamental form $B_\epsilon(f)$ and Weingarten operator a^ϵ_ξ are related by

$$(g_\epsilon^A)^f (B_\epsilon(f)(X,Y),\, \xi) = g_\epsilon(f)(a^\epsilon_\xi X,\, Y). \tag{93}$$

The second fundamental form $B_\epsilon(f)$ is symmetric and, merely as a consequence of (93)], the Weingarten operator a^ϵ_ξ is self-adjoint with respect to $g_\epsilon(f)$.

6. Gauss–Ricci–Codazzi Equations

Let $\mathbf{E} \to M$ be a vector bundle and $\mathbf{D} \in \mathcal{C}(\mathbf{E})$ a connection. The curvature form $R^{\mathbf{D}} = R(\mathbf{D})$ is

$$R^{\mathbf{D}}(X,Y) = [\mathbf{D}_X\,, \mathbf{D}_Y] - \mathbf{D}_{[X,Y]}\,, \quad X, Y \in \mathfrak{X}(M).$$

The curvature forms of the connections in the Gauss and Weingarten formulas are

$\mathbf{D}$	$\mathbf{E}$	$R(\mathbf{D})$
$(D^\epsilon)^f$	$f^{-1}T(A) \to M$	$R((D^\epsilon)^f)$
$\nabla^{f,\epsilon}$	$T(M) \to M$	$R(\nabla^{f,\epsilon})$
$D^{\perp_\epsilon}$	$E_\epsilon(f) \to M$	$R(D^{\perp_\epsilon})$

The Gauss–Codazzi equation for the isometric immersion $f : (M, g_\epsilon(f)) \to (A, g_\epsilon^A)$ is (see, e.g., [20]):

$$\begin{gathered} R((D^\epsilon)^f)(X,Y) f_* Z = f_*\, R(\nabla^{f,\epsilon})(X,Y)Z \\ -f_*\, a^\epsilon_{B_\epsilon(f)(Y,Z)} X + f_*\, a^\epsilon_{B_\epsilon(f)(X,Z)} Y + (D^\epsilon_X B_\epsilon(f))(Y,Z) - (D^\epsilon_Y B_\epsilon(f))(X,Z) \end{gathered} \tag{94}$$

for any $X, Y, Z \in \mathfrak{X}(M)$. Here, $D^\epsilon_X B_\epsilon(f)$ is the Van der Waerden–Bortolotti covariant derivative (of the second fundamental form), i.e.,

$$(D^\epsilon_X B_\epsilon(f))(Y,Z) = D^{\perp_\epsilon}_X B_\epsilon(f)(Y,Z) - B_\epsilon(f)(\nabla^{f,\epsilon}_X Y,\, Z) - B_\epsilon(f)(Y,\, \nabla^{f,\epsilon}_X Z).$$

The *Codazzi equation* is obtained by identifying the $E_\epsilon(f)$ components of the Gauss–Codazzi equation (94)

$$\begin{gathered}\operatorname{nor}_\epsilon\left\{R\big((D^\epsilon)^f\big)(X,Y)f_*Z\right\} = \\ \big(D^\epsilon_X B_\epsilon(f)\big)(Y,Z) - \big(D^\epsilon_Y B_\epsilon(f)\big)(X,Z).\end{gathered} \tag{95}$$

Let us take the inner product of (94) with $W \in \mathfrak{X}(M)$ in order to identity the tangential components of (94)

$$\begin{gathered}(g^A_\epsilon)^f\Big(R\big((D^\epsilon)^f\big)(X,Y)f_*Z,\ f_*W\Big) = g_\epsilon(f)\Big(R\big(\nabla^{f,\epsilon}\big)(X,Y)Z,\ W\Big) \\ -g_\epsilon(f)\Big(a^\epsilon_{B_\epsilon(f)(Y,Z)}X,\ W\Big) + g_\epsilon(f)\Big(a^\epsilon_{B_\epsilon(f)(X,Z)}Y,\ W\Big)\end{gathered}$$

and let us substitute from (93) so as to obtain (the *Gauss equation* of the given isometric immersion)

$$\begin{gathered}(g^A_\epsilon)^f\Big(R\big((D^\epsilon)^f\big)(X,Y)f_*Z,\ f_*W\Big) \\ = g_\epsilon(f)\Big(R\big(\nabla^{f,\epsilon}\big)(X,Y)Z,\ W\Big) \\ -(g^A_\epsilon)^f\big(B_\epsilon(f)(X,W),\ B_\epsilon(f)(Y,Z)\big) + (g^A_\epsilon)^f\big(B_\epsilon(f)(Y,W),\ B_\epsilon(f)(X,Z)\big)\end{gathered} \tag{96}$$

for any $X,\ Y,\ Z,\ W \in \mathfrak{X}(M)$. For any $X, Y \in \mathfrak{X}(M)$ and any $\xi \in C^\infty\big(E_\epsilon(f)\big)$ as a consequence of the Gauss and Weingarten formulas (91) and (92),

$$\begin{gathered}R\big((D^\epsilon)^f\big)(X,Y)\xi = R\big(D^{\perp\epsilon}\big)(X,Y)\xi \\ +\nabla^{f,\epsilon}_Y a^\epsilon_\xi X - \nabla^{f,\epsilon}_X a^\epsilon_\xi Y + a^\epsilon_\xi[X,\ Y] - a^\epsilon_{D^{\perp\epsilon}_X\xi}Y + a^\epsilon_{D^{\perp\epsilon}_Y\xi}X \\ +B_\epsilon(f)\big(a^\epsilon_\xi X,\ Y\big) - B_\epsilon(f)\big(X,\ a^\epsilon_\xi Y\big)\end{gathered}$$

and, taking the inner product with $\eta \in C^\infty\big(E_\epsilon(f)\big)$, gives

$$\begin{gathered}(g^A_\epsilon)^f\Big(R\big((D^\epsilon)^f\big)(X,Y)\xi,\ \eta\Big) = (g^A_\epsilon)^f\Big(R\big(D^{\perp\epsilon}\big)(X,Y)\xi,\ \eta\Big) \\ +(g^A_\epsilon)^f\Big(B_\epsilon(f)\big(a^\epsilon_\xi X,\ Y\big),\ \eta\Big) - (g^A_\epsilon)^f\Big(B_\epsilon(f)\big(X,\ a^\epsilon_\xi Y\big),\ \eta\Big)\end{gathered} \tag{97}$$

or, by applying (93) to modify the last two terms in (97),

$$\begin{gathered}(g^A_\epsilon)^f\Big(R\big((D^\epsilon)^f\big)(X,Y)\xi,\ \eta\Big) = (g^A_\epsilon)^f\Big(R\big(D^{\perp\epsilon}\big)(X,Y)\xi,\ \eta\Big) \\ -g_\epsilon(f)\Big(\big[a^\epsilon_\xi,\ a^\epsilon_\eta\big]X,\ Y\Big)\end{gathered} \tag{98}$$

(the *Ricci equation* for the given isometric immersion).

7. The Projections $\tan_\epsilon$ and $\operatorname{nor}_\epsilon$

Our main purpose in the present section is to compute the projection $\tan_\epsilon : f^{-1}T(A) \to T(M)$ in terms of pseudohermitian invariants. One has (at every point of M)

$$E_\epsilon(f) = \big\{\xi \in f^{-1}T(A)\ :\ (g^A_\epsilon)^f(\xi,\ f_*V) = 0,\quad \forall\ V \in T(M)\big\}.$$

Also, for every $V \in T(M)$ by (48),

$$(g^A_\epsilon)^f(\xi,\ f_*V) = g^f_\Theta(\xi,\ f_*V) + \Lambda\left(\frac{1}{\epsilon^2} - 1\right)\Theta^f(\xi)\,\theta(V).$$

Therefore, if we set (again pointwise)

$$S_\epsilon(f) = \Big\{ f_* V + \Lambda \left(\frac{1}{\epsilon^2} - 1 \right) \theta(V)\, T_A^f \;:\; V \in T(M) \Big\},$$

then

$$E_\epsilon(f) = S_\epsilon(f)^\perp . \tag{99}$$

As to the notation adopted in (99), if $S_x \subset T_{f(x)}(A)$ is a linear subspace, then $S_x^\perp$ denotes the orthogonal complement of S_x in $T_{f(x)}(A)$ with respect to the inner product $g_{\Theta, f(x)}$. We shall need the linear operator $L_\epsilon : \mathfrak{X}(M) \to C^\infty(f^{-1}T(A))$ given by

$$L_\epsilon V \equiv f_* V + \Lambda \left(\frac{1}{\epsilon^2} - 1 \right) \theta(V)\, T_A^f , \quad V \in \mathfrak{X}(M). \tag{100}$$

The relation (100) also defines a vector bundle morphism $L_\epsilon : T(M) \to f^{-1}T(A)$, denoted by the same symbol. Then,

$$E_\epsilon(f) = \{ \xi \in f^{-1}T(A) \;:\; g_\Theta^f(L_\epsilon V,\, \xi) = 0, \;\; \forall\, V \in T(M) \}.$$

For arbitrary $W \in f^{-1}T(A)$, we take the inner product of

$$W = f_* \operatorname{tan}_\epsilon W + \operatorname{nor}_\epsilon W$$

with $f_* V$, $V \in T(M)$, with respect to the inner product $(g_\epsilon^A)^f$, so as to obtain

$$\big(g_\epsilon^A\big)^f (W,\, f_* V) = g_\epsilon(f)(\operatorname{tan}_\epsilon W,\, V). \tag{101}$$

Lemma 10. *The function*

$$\lambda \equiv \big(f^* g_\Theta\big)(T,\, T) \in C^\infty(M) \tag{102}$$

is strictly positive; i.e., $\lambda(x) > 0$ for any $x \in M$.

Proof. One has

$$\lambda(x) = g_{\Theta, f(x)}\big((d_x f) T_x\, ,\, (d_x f) T_x\big) \geq 0.$$

If there is $x_0 \in M$ such that $\lambda(x_0) = 0$, then $(d_{x_0} f) T_{x_0} = 0$, yielding $T_{x_0} = 0$ (as f is an immersion), a contradiction. □

At this point, we employ the relations (see (48) and (75))

$$g_\epsilon^A = g_\Theta + \left(\frac{1}{\epsilon^2} - 1 \right) \Theta \otimes \Theta ,$$

$$g_\epsilon(f) = \Lambda\, g_\theta + \left\{ \left(\frac{\Lambda}{\epsilon} \right)^2 - \Lambda(\Lambda + 1) + \lambda \right\} \theta \otimes \theta + 2\, \theta \odot g_\theta(J\, \nabla^H \Lambda\, ,\, \cdot\,),$$

and modify (101) accordingly. We obtain

$$g_\Theta^f(W,\, f_* V) + \Lambda \left(\frac{1}{\epsilon^2} - 1 \right) \Theta^f(W) \theta(V) = \tag{103}$$

$$= \Lambda\, g_\theta(V,\, \operatorname{tan}_\epsilon W) + \left\{ \left(\frac{\Lambda}{\epsilon} \right)^2 - \Lambda(\Lambda + 1) + \lambda \right\} \theta(V)\, \theta(\operatorname{tan}_\epsilon W) +$$

$$+ \theta(V)\, g_\theta(J\, \nabla^H \Lambda\, ,\, \operatorname{tan}_\epsilon W) + \theta(\operatorname{tan}_\epsilon W)\, g_\theta(J\, \nabla^H \Lambda\, ,\, V).$$

We ought to examine a few consequences of (103). First, let us use (103) for $V = X \in H(M)$, i.e., as $\theta(X) = 0$,

$$g_\Theta^f(W,\, f_* X) = \Lambda\, g_\theta(X,\, \operatorname{tan}_\epsilon W) + \theta(\operatorname{tan}_\epsilon W)\, g_\theta(J\, \nabla^H \Lambda,\, X). \tag{104}$$

Let $\{X_a \,:\, 1 \le a \le 2n\}$ be a local G_θ-orthonormal [i.e., $G_\theta(X_a\,,\,X_b) = \delta_{ab}$, $1 \le a,\, b \le 2n$] frame of $H(M)$, defined on the open set $U \subset M$. Then,

$$\Lambda\, \Pi_H \tan_\epsilon W = \Lambda \sum_{a=1}^{2n} g_\theta\big(X_a\,,\,\tan_\epsilon W\big)\, X_a$$

by (104)

$$= \sum_{a=1}^{2n} \Big\{ g^f_\Theta\big(W,\, f_* X_a\big) - g_\theta\big(J\, \nabla^H \Lambda\,,\, X_a\big) \Big\}\, X_a$$

or

$$\Lambda\, \Pi_H \tan_\epsilon W = \sum_{a=1}^{2n} g^f_\Theta\big(W,\, f_* X_a\big)\, X_a - J\, \nabla^H \Lambda \tag{105}$$

everywhere in U. Second, let us use (103) for $V = T$; i.e., as $\theta(T) = 1$ and $g_\theta(T, Y) = \theta(Y)$ for any $Y \in \mathfrak{X}(M)$,

$$\begin{gathered} g^f_\Theta\big(W,\, f_* T\big) + \Lambda \left(\tfrac{1}{\epsilon^2} - 1\right) \Theta^f(W) \\ \left\{ \Lambda^2 \left(\tfrac{1}{\epsilon^2} - 1\right) + \lambda \right\} \theta\big(\tan_\epsilon W\big) + g_\theta\big(J\, \nabla^H \Lambda\,,\, \tan_\epsilon W\big). \end{gathered} \tag{106}$$

We shall conduct an asymptotic analysis of our equations as $\epsilon \to 0^+$, so we consider $0 < \epsilon < 1$ to start with. Consequently, by Lemma 10,

$$\Lambda(x)^2 \left(\frac{1}{\epsilon^2} - 1\right) + \lambda(x) > 0, \quad x \in M.$$

For simplicity, we set

$$u_\epsilon = \theta\big(\tan_\epsilon W\big) \in C^\infty(M), \quad X_\epsilon = \Pi_{H(M)} \tan_\epsilon W \in C^\infty(H(M)),$$

so that $\tan_\epsilon W = X_\epsilon + u_\epsilon\, T$ and Equations (105) and (106) read:

$$\Lambda\, X_\epsilon = \sum_{a=1}^{2n} g^f_\Theta\big(W,\, f_* X_a\big)\, X_a - J\, \nabla^H \Lambda\,, \tag{107}$$

$$\begin{gathered} g^f_\Theta\big(W,\, f_* T\big) + \Lambda \left(\tfrac{1}{\epsilon^2} - 1\right) \Theta^f(W) \\ = \left\{ \Lambda^2 \left(\tfrac{1}{\epsilon^2} - 1\right) + \lambda \right\} u_\epsilon + g_\theta\big(J\, \nabla^H \Lambda\,,\, X_\epsilon\big). \end{gathered} \tag{108}$$

Let us multiply (108) by Λ and substitute $\Lambda\, X_\epsilon$ from (107) into the resulting equation. We obtain

$$\begin{gathered} g^f_\Theta\big(W,\, \Lambda\, f_* T\big) + \Lambda^2 \left(\tfrac{1}{\epsilon^2} - 1\right) \Theta^f(W) \\ = \Lambda \left\{ \Lambda^2 \left(\tfrac{1}{\epsilon^2} - 1\right) + \lambda \right\} u_\epsilon + g^f_\Theta\big(W,\, J \nabla^H \Lambda\big) - \|\nabla^H \Lambda\|^2_\theta\,. \end{gathered} \tag{109}$$

Let $Z(\Lambda) = \{x \in M \,:\, \Lambda(x) = 0\}$ be the zero set of Λ. Note that (107) and (109) determine X_ϵ and u_ϵ on the open set $M \setminus Z(\Lambda)$.

From now on, we confine our calculations to isopseudohermitian (i.e., $\Lambda \equiv 1$) CR immersions $f : M \to A$. Then, (107) and (109) read

$$X_\epsilon = \sum_{a=1}^{2n} g^f_\Theta\big(W,\, f_* X_a\big)\, X_a\,, \tag{110}$$

$$g^f_\Theta\big(W,\, f_* T\big) + \left(\frac{1}{\epsilon^2} - 1\right) \Theta^f(W) = \left(\frac{1}{\epsilon^2} - 1 + \lambda\right) u_\epsilon\,. \tag{111}$$

Summing up, we have proved:

Lemma 11. *For every $W \in f^{-1}T(A)$*

$$\begin{aligned} \tan_\epsilon W = & \sum_{a=1}^{2n} g_\Theta^f(W, f_* X_a)\, X_a \\ & + \frac{1}{1+\epsilon^2(\lambda-1)} \Big\{ \Theta^f(W) + \epsilon^2\, g_\Theta^f(W, \mathfrak{X}_\Theta) \Big\} T \end{aligned} \tag{112}$$

everywhere in U, where $\mathfrak{X}_\Theta = f_* T - T_A^f$.

Finally, for the calculation of the projection nor_ϵ, we shall use

$$\mathrm{nor}_\epsilon W = W - f_* \tan_\epsilon W$$

together with (112).

8. Gauss Formula for $f : (M, g_\epsilon(f)) \to (A, g_\epsilon^A)$

The purpose of the present section is to give an explicit form of the Gauss formula

$$(D^\epsilon)_V^f\, f_* W = f_*\, \nabla_V^{f,\epsilon} W + B_\epsilon(f)(V, W), \quad V, W \in \mathfrak{X}(M). \tag{113}$$

To this end, we shall compute

$$\nabla_V^{f,\epsilon} W = \tan_\epsilon \Big\{ (D^\epsilon)_V^f\, f_* W \Big\}, \quad B_\epsilon(f)(V, W) = \mathrm{nor}_\epsilon \Big\{ (D^\epsilon)_V^f\, f_* W \Big\},$$

by essentially using (112) in Lemma 11. Calculations are considerably simplified by exploiting the decomposition $T(M) = H(M) \oplus \mathbb{R}T$. Let us set $V = X$ and $W = Y$ with $X, Y \in H(M)$ in the Gauss formula (113), i.e.,

$$(D^\epsilon)_X^f\, f_* Y = f_*\, \nabla_X^{f,\epsilon} Y + B_\epsilon(f)(X, Y). \tag{114}$$

On the other hand, by (12)–(15), with ∇^ϵ replaced by D^ϵ,

$$D_{\mathbf{X}}^\epsilon \mathbf{Y} = D_{\mathbf{X}} \mathbf{Y} + \{ g_\Theta(\mathbf{X}, J_A \mathbf{Y}) - \epsilon^2\, g_\Theta(\mathbf{X}, \tau_A \mathbf{Y}) \}\, T_A\,, \tag{115}$$

$$D_{\mathbf{X}}^\epsilon T_A = \tau_A \mathbf{X} + \frac{1}{\epsilon^2}\, J_A \mathbf{X}, \tag{116}$$

$$D_{T_A}^\epsilon \mathbf{X} = D_{T_A} \mathbf{X} + \frac{1}{\epsilon^2}\, J_A \mathbf{X}, \tag{117}$$

$$D_{T_A}^\epsilon T_A = 0, \tag{118}$$

for any $\mathbf{X}, \mathbf{Y} \in H(A)$. We systematically apply our findings in Section 2 to the pseudohermitian manifold (A, Θ) and to the Riemannian metric g_ϵ^A (the ϵ-contraction of the Levi form G_Θ). By (115),

$$\begin{aligned} (D^\epsilon)_X^f f_* Y = & D_X^f f_* Y \\ & + \{ g_\Theta^f(f_* X, J_A^f f_* Y) - \epsilon^2\, g_\Theta^f(f_* X, \tau_A^f f_* Y) \}\, T_A^f\,. \end{aligned} \tag{119}$$

Here, $D^f = f^{-1}D$ is the pullback of the Tanaka–Webster connection D–a connection in the pullback bundle $f^{-1}T(A) \to M$. We shall substitute from (119) into the left-hand side of (114). Our ultimate goal is to relate the pseudohermitian geometry of the ambient space (A, Θ) to that of the submanifold (M, θ). Therefore, to compute the right-hand side of (114), one needs a lemma relating the induced connection $\nabla^{f,\epsilon}$, associated with the isometric immersion $f : (M, g_\epsilon(f)) \to (A, g_\epsilon^A)$, to the Tanaka–Webster connection ∇ of (M, θ).

Lemma 12. *Let $f : M \to A$ be an isopseudohermitian (i.e., $f^*\Theta = \theta$) CR immersion. The Levi–Civita connection $\nabla^{f,\epsilon}$ of $(M, g_\epsilon(f))$ and the Tanaka–Webster connection ∇ of (M, θ) are related by*

$$\nabla^{f,\epsilon}_X Y = \nabla_X Y + \Big\{\Omega(X,Y) - \frac{1}{\mu_\epsilon} A(X,Y)\Big\} T, \tag{120}$$

$$\nabla^{f,\epsilon}_X T = \tau X + \mu_\epsilon JX + \frac{1}{2\mu_\epsilon} X(\lambda)\, T, \tag{121}$$

$$\nabla^{f,\epsilon}_T X = \nabla_T X + \mu_\epsilon JX + \frac{1}{2\mu_\epsilon} X(\lambda)\, T, \tag{122}$$

$$\nabla^{f,\epsilon}_T T = -\frac{1}{2}\nabla^H \lambda + \frac{1}{2\mu_\epsilon} T(\lambda)\, T, \tag{123}$$

for any $X, Y \in H(M)$. Here $\mu_\epsilon = \mu_\epsilon(f;\, \theta,\, \theta) \in C^\infty(M)$ is given by

$$\mu_\epsilon = \frac{1}{\epsilon^2} - 1 + \lambda, \quad \lambda = (f^* g_\Theta)(T, T). \tag{124}$$

Proof. We start from the well-known (see, e.g., Proposition 2.3 in [10], Volume I, p. 160) expression of the Levi–Civita connection $\nabla^{f,\epsilon}$ in terms of the Riemannian metric $g_\epsilon(f)$

$$\begin{aligned} &2\, g_\epsilon(f)(\nabla^{f,\epsilon}_U V,\, W) \\ &= U\big(g_\epsilon(f)(V,W)\big) + V\big(g_\epsilon(f)(W,V)\big) - W\big(g_\epsilon(f)(U,V)\big) \\ &+ g_\epsilon(f)\big([U,V],\, W\big) + g_\epsilon(f)\big([W,U],\, V\big) + g_\epsilon(f)\big(U,\, [W,V]\big) \end{aligned}$$

for any $U, V, W \in \mathfrak{X}(M)$. We adopt the notations in Section 2; i.e., we set

$$\begin{aligned} &C_g(U,\, V,\, W) \\ &= U\big(g(V,W)\big) + V\big(g(W,V)\big) - W\big(g(U,V)\big) \\ &+ g\big([U,V],\, W\big) + g\big([W,U],\, V\big) + g\big(U,\, [W,V]\big) \end{aligned}$$

so that

$$2\, g_\epsilon(f)(\nabla^{f,\epsilon}_U V,\, W) = C_{g_\epsilon(f)}(U,\, V,\, W). \tag{125}$$

By (78) in Corollary 2 (to Proposition 3),

$$g_\epsilon(f) = g^M_\epsilon + (\lambda - 1)\, \theta \otimes \theta, \tag{126}$$

where $\lambda \in C^\infty(M)$ is given by (102). □

Lemma 13. *The Christoffel mappings $C_{g_\epsilon(f)}$ and $C_{g^M_\epsilon}$ are related by*

$$\begin{aligned} C_{g_\epsilon(f)}(U,\, V,\, W) &= C_{g^M_\epsilon}(U,\, V,\, W) \\ &+ U(\lambda)\, \theta(V)\, \theta(W) + V(\lambda)\, \theta(U)\, \theta(W) - W(\lambda)\, \theta(U)\, \theta(V) \\ &+ 2(\lambda - 1)\, \big[\theta(W)\, \theta(\nabla_U V) \\ &+ \Omega(U,V)\, \theta(W) + \Omega(W,U)\, \theta(V) + \Omega(W,V)\, \theta(U)\big] \end{aligned} \tag{127}$$

for any $U,\, V,\, W \in \mathfrak{X}(M)$.

Proof. (127) is a straightforward (yet rather involved) consequence of (126). We give a few details, for didactic reasons, as follows. Let us substitute from (126) into $C_{g_\epsilon(f)}$ and recognize the term $C_{g_\epsilon^M}$. To bring into the picture the Tanaka–Webster connection, we substitute the remaining Lie products from

$$[V, W] = \nabla_V W - \nabla_W V - 2(\theta \wedge \tau)(V, W) + 2\Omega(V, W)\, T \tag{128}$$

and use $\nabla\theta = 0$. At its turn, (128) is a mere consequence of

$$T_\nabla = 2\{\theta \wedge \tau - \Omega \otimes T\}.$$

Let ∇^ϵ be the Levi–Civita connection of (M, g_ϵ^M). Similar to (125),

$$2\, g_\epsilon^M(\nabla^\epsilon_U V,\, W) = C_{g_\epsilon^M}(U,\, V,\, W). \tag{129}$$

Then, (125)–(129) yield

$$\begin{gathered} 2\, g_\epsilon(\nabla^{f,\epsilon}_U V,\, W) + 2(\lambda - 1)\, \theta(\nabla^{f,\epsilon}_U V)\, \theta(W) = 2\, g_\epsilon(\nabla^\epsilon_U V,\, W) \\ +U(\lambda)\, \theta(V)\, \theta(W) + V(\lambda)\, \theta(U)\, \theta(W) - W(\lambda)\, \theta(U)\, \theta(V) \\ +2(\lambda - 1)\, \big[\theta(W)\, \theta(\nabla_U V) \\ +\Omega(U,\, V)\, \theta(W) + \Omega(W,\, U)\, \theta(V) + \Omega(W,\, V)\, \theta(U)\big]. \end{gathered} \tag{130}$$

Let us substitute from $g_\epsilon = g_\theta + (\epsilon^{-2} - 1)\, \theta \otimes \theta$ into (130) and use nondegeneracy of g_θ to "simplify" W. We obtain

$$\begin{gathered} 2\, \nabla^{f,\epsilon}_U V + 2\Big(\tfrac{1}{\epsilon^2} + \lambda - 2\Big)\, \theta(\nabla^{f,\epsilon}_U V)\, T \\ = 2\, \nabla^\epsilon_U V + 2\Big(\tfrac{1}{\epsilon^2} - 1\Big)\, \theta(\nabla^\epsilon_U V)\, T \\ +\big[U(\lambda)\, \theta(V) + V(\lambda)\, \theta(U)\big]\, T - \theta(U)\, \theta(V)\, \nabla\lambda \\ +2(\lambda - 1)\, \big\{\big[\theta(\nabla_U V) + \Omega(U,\, V)\big]\, T + \theta(V)\, JU + \theta(U)\, JV\big\}. \end{gathered} \tag{131}$$

Next, let us apply θ to both sides of (131) and use $\theta(T) = 1$ and $\theta \circ J = 0$ in order to yield

$$\begin{gathered} 2\Big(\tfrac{1}{\epsilon^2} + \lambda - 1\Big)\, \theta(\nabla^{f,\epsilon}_U V) = \tfrac{2}{\epsilon^2}\, \theta(\nabla^\epsilon_U V) \\ +\big[U(\lambda)\, \theta(V) + V(\lambda)\, \theta(U)\big] - \theta(U)\, \theta(V)\, T(\lambda) \\ +2(\lambda - 1)\, \big\{\big[\theta(\nabla_U V) + \Omega(U,\, V)\big]\big\}. \end{gathered} \tag{132}$$

In particular, for $U = X$ and $V = Y$ with $X, Y \in H(M)$, the formulas (131) and (132) become

$$\begin{gathered} \nabla^{f,\epsilon}_X Y + \Big(\tfrac{1}{\epsilon^2} + \lambda - 2\Big)\, \theta(\nabla^{f,\epsilon}_X Y)\, T \\ = \nabla^\epsilon_X Y + \Big\{\Big(\tfrac{1}{\epsilon^2} - 1\Big)\, \theta(\nabla^\epsilon_X Y) + (\lambda - 1)\, \Omega(X,\, Y)\Big\}\, T, \end{gathered} \tag{133}$$

$$\Big(\frac{1}{\epsilon^2} + \lambda - 1\Big)\, \theta(\nabla^{f,\epsilon}_X Y) = \frac{1}{\epsilon^2}\, \theta(\nabla^\epsilon_X Y) + (\lambda - 1)\, \Omega(X,\, Y). \tag{134}$$

Here, one also uses $\theta(\nabla_X Y) = 0$ because ∇ parallelizes $H(M)$. Let us multiply (134) by T and subtract the resulting equation from (133). We obtain

$$\nabla^{f,\epsilon}_X Y = \nabla^\epsilon_X Y + \Big\{\theta(\nabla^{f,\epsilon}_X Y) - \theta(\nabla^\epsilon_X Y)\Big\} T, \tag{135}$$

a simplified form of (133) equivalent to

$$\Pi_{H(M)} \nabla^{f,\epsilon}_X Y = \Pi_{H(M)} \nabla^\epsilon_X Y. \tag{136}$$

At this point, we exploit the relationship between the Levi–Civita connection ∇^ϵ and the Tanaka–Webster connection ∇ as established in Lemma 2. Ssee formulas (12)–(15). For instance, by (12) and (136),

$$\nabla^\epsilon_X Y = \nabla_X Y + \{\Omega(X,Y) - \epsilon^2 A(X,Y)\} T \Longrightarrow$$

$$\Pi_{H(M)} \nabla^\epsilon_X Y = \nabla_X Y \Longrightarrow$$

$$\Pi_{H(M)} \nabla^{f,\epsilon}_X Y = \nabla_X Y \tag{137}$$

for any $X, Y \in H(M)$. Also, by applying θ to both sides of (12),

$$\theta(\nabla^\epsilon_X Y) = \Omega(X,Y) - \epsilon^2 \, A(X,Y) \tag{138}$$

and substitution into (134) furnishes

$$\theta(\nabla^{f,\epsilon}_X Y) = \Omega(X,Y) - \frac{1}{\mu_\epsilon} A(X,Y). \tag{139}$$

Finally, by (137) and (138),

$$\nabla^{f,\epsilon}_X Y = \nabla_X Y + \left\{\Omega(X,Y) - \frac{1}{\mu_\epsilon} A(X,Y)\right\} T,$$

which is (120). Q.E.D.

Similarly, the formulas (131) and (132) for $U = X$ and $V = T$ become, by (13), i.e., $\nabla^\epsilon_X T = \tau X + \epsilon^{-2} JX \in H(M)$,

$$\Pi_H \nabla^{f,\epsilon}_X T = \tau X + \mu_\epsilon \, JX, \quad \theta(\nabla^{f,\epsilon}_X T) = \frac{1}{2\mu_\epsilon} X(\lambda),$$

yielding (121). Q.E.D. The proof of the remaining relations (122) and (123) is similar. □

Let us go back to (119). As $J^f_A \circ f_* = f_* \circ J$ formula (119) reads

$$\begin{aligned} (D^\epsilon)^f_X f_* Y &= D^f_X f_* Y \\ &+ \{(f^* g_\Theta)(X, JY) - \epsilon^2 \, g^f_\Theta(f_* X, \tau^f_A f_* Y)\} \, T^f_A \, . \end{aligned} \tag{140}$$

Let us substitute from (140) and (120) into the Gauss formula (114) in order to obtain:

Proposition 5. *Let M and A be strictly pseudoconvex CR manifolds, and let $\theta \in \mathcal{P}_+(M)$ and $\Theta \in \mathcal{P}_+(A)$. Let $f : M \to A$ be an isopseudohermitian immersion of (M,θ) into (A,Θ). Let g^A_ϵ, $0 < \epsilon < 1$, be the ϵ-contraction of the Levi form G_Θ, and let $g_\epsilon(f) = f^* g^A_\epsilon$. Then*

$$\begin{aligned} & D^f_X f_* Y + \{(f^* g_\Theta)(X, JY) - \epsilon^2 \, g^f_\Theta(f_* X, \tau^f_A f_* Y)\} \, T^f_A \\ &= f_* \nabla_X Y + \left\{\Omega(X,Y) - \tfrac{1}{\mu_\epsilon} A(X,Y)\right\} f_* T + B_\epsilon(f)(X,Y), \\ & X, \ Y \in H(M), \end{aligned} \tag{141}$$

is the Gauss formula for the isometric immersion $f : (M, g_\epsilon(f)) \to (A, g^A_\epsilon)$ along $H(M) \otimes H(M)$.

It should be noted that all terms in the Gauss formula (113), except for the second fundamental form $B_\epsilon(f)$, were expressed in terms of pseudohermitian invariants of (M,θ)

and (A, Θ). The remaining components of (113), respectively, along $H(M) \otimes \mathbb{R}T$, $\mathbb{R}T \otimes H(M)$, and $\mathbb{R}T \otimes \mathbb{R}T$, can be derived by setting

$$(U,\, V) \in \big\{(X,\, T),\; (T,\, X),\; (T,\, T)\big\}, \quad X \in H(M), \tag{142}$$

into (113). We relegate the derivation of the components (142) to further work. For the time being, we seek to further split (141) into tangential and normal parts, with respect to the direct sum decomposition (80). This amounts to decomposing $D^f_X f_* Y$ and the Reeb vector field T^f_A with respect to (80).

We start with the decomposition of T^f_A. Formula (112) for $W = T^f_A$ gives, as $\mathfrak{X}_\Theta \in f^{-1}H(A)$, so that $\mathfrak{X}_\Theta$ and T^f_A are g^f_Θ-orthogonal,

$$\tan_\epsilon(T_A) = \sum_{a=1}^{2n} g^f_\Theta\big(f_* X_a\,,\, T^f_A\big)\, X_a + \frac{1}{\epsilon^2\, \mu_\epsilon}\, T$$

and

$$g^f_\Theta\big(f_* X_a\,,\, T^f_A\big) = \Theta^f\big(f_* X_a\big) = \theta(X_a) = 0$$

so that:

Lemma 14. *Let $f : M \to A$ be an isopseudohermitian immersion of (M, θ) into (A, Θ). The tangential and normal components of the Reeb vector field T^f_A, with respect to the decomposition (80), are*

$$\tan_\epsilon\big(T^f_A\big) = \frac{1}{\epsilon^2\, \mu_\epsilon}\, T, \qquad \mathrm{nor}_\epsilon\big(T^f_A\big) = T^f_A - \frac{1}{\epsilon^2\, \mu_\epsilon}\, f_*\, T. \tag{143}$$

Next, we attack the decomposition of $D^f_X f_* Y$ with respect to (80). To this end, we need to introduce pseudohermitian analogs to familiar objects in the theory of isometric immersions between Riemannian manifolds, such as the induced and normal connections, the second fundamental form, and the Weingarten operator. For any $V, W \in \mathfrak{X}(M)$ and any $\xi \in C^\infty(E(f))$, we set by definition

$$\nabla^\top_V W = \tan\big\{D^f_V f_* W\big\}, \quad B(f)(V, W) = \mathrm{nor}\big\{D^f_V f_* W\big\},$$

$$a_\xi V = -\tan\big\{D^f_V \xi\big\}, \quad \nabla^\perp_V \xi = \mathrm{nor}\big\{D^f_V \xi\big\},$$

where

$$\tan : f^{-1}T(A) \to T(M), \quad \mathrm{nor} : f^{-1}T(A) \to E(f),$$

are the natural projections.

Theorem 2. (i) *$\nabla^\top$ is a linear connection on M.*

(ii) *$B(f)$ is $C^\infty(M, \mathbb{R})$-bilinear.*

(iii) *a is $C^\infty(M, \mathbb{R})$-bilinear.*

(iv) *$\nabla^\perp$ is a connection in the vector bundle $E(f) \to M$.*

(v) *For any $V, W \in \mathfrak{X}(M)$ and any $\xi \in C^\infty(E(f))$.*

$$D^f_V f_* W = f_* \nabla^\top_V W + B(f)(V, W), \tag{144}$$

$$D^f_V \xi = -f_* a_\xi V + \nabla^\perp_V \xi\,. \tag{145}$$

The proof of Theorem 2 is straightforward. We adopt the following pseudohermitian analog to the ordinary terminology in use within the theory of isometric immersions between Riemannian manifolds.

Definition 38. $\nabla^\top$ is the *induced connection* (the connection induced by D *via* f). $\nabla^\perp$ is the *normal Tanaka–Webster connection*. $B(f)$ and a_ξ are, respectively, the *pseudohermitian second fundamental form* and the *pseudohermitian Weingarten operator* (associated with the normal vector field ξ) of the CR immersion $f : M \to (A, \Theta)$. (144) is the *pseudohermitan Gauss formula*. (145) is the *pseudohermitian Weingarten formula*.

The induced connection $\nabla^\top$ and the (intrinsic) Tanaka–Webster connection ∇ of (M, θ) do not coincide, in general, unless, e.g., $f : (M, \theta) \to (A, \Theta)$ is a pseudohermitian immersion. The ambient connection–the Tanaka–Webster connection of (A, Θ)–has torsion so that $B(f)$, unlike its Riemannian counterpart, it is never symmetric. We expect that $B(f)$ is the second fundamental form of f as introduced by P. Ebenfelt, X. Huang and D. Zaitsev (see formula 2.3 in [12], p. 636) by making use of B. Lamel's spaces $E_k(p)$ (actually of $E_1(p)$; see Definition 1 in [13], p. 1). The main properties of $\nabla^\top$, $B(f)$, and a_ξ are collected in the following.

Theorem 3. *Let $f : M \to A$ be a CR immersion, and let $\theta \in \mathcal{P}_+(M)$ and $\Theta \in \mathcal{P}_+(A)$.*

(i) *The induced connection ∇^T has torsion, i.e.,*

$$\begin{aligned} &\mathrm{Tor}_{\nabla^\top}(V, W) = -2\,(f^* g_\Theta)(V,\, JW)\, T_A^\top \\ &+\Lambda\left\{\theta(V)\tan\left[\tau_A(f_* W)\right] - \theta(W)\tan\left[\tau_A(f_* V)\right]\right\}. \end{aligned} \tag{146}$$

In particular, if g_Θ is Sasakian, then $\nabla^\top$ is symmetric $\Longleftrightarrow$ the Reeb vector field of (A, Θ) if g_Θ-orthogonal to $f(M)$.

(ii) *The pseudohermitian second fundamental form $B(f)$ is not symmetric, in general, i.e.,*

$$\begin{aligned} &B(f)(V, W) - B(f)(W, V) = -2\,(f^* g_\Theta)(V,\, JW)\, T_A^\perp \\ &+\Lambda\left\{\theta(V)\,\mathrm{nor}\left[\tau_A(f_* W)\right] - \theta(W)\,\mathrm{nor}\left[\tau_A(f_* V)\right]\right\}. \end{aligned} \tag{147}$$

In particular, if g_Θ is Sasakian, then $B(f)$ is symmetric $\Longleftrightarrow$ T_A is tangent to $f(M)$.

(iii) *The metric $f^* g_\Theta$ is parallel with respect to $\nabla^\top$ i.e., $\nabla^\top f^* g_\Theta = 0$.*

(iv) *For any V, $W \in \mathfrak{X}(M)$ and $\xi \in C^\infty(E(f))$*

$$g^f_\Theta(B(f)(V, W),\, \xi) = (f^* g_\Theta)(a_\xi V,\, W). \tag{148}$$

(v) *For any $V, W \in \mathfrak{X}(M)$*

$$(\nabla^\top_V J)W = \tan\left\{J^f_A\, B(f)(V,\, W)\right\}, \tag{149}$$

$$B(f)(V,\, JW) = \mathrm{nor}\left\{J^f_A\, B(f)(V,\, W)\right\}. \tag{150}$$

(vi) *For any $V \in \mathfrak{X}(M)$*

$$\nabla^\top_V T_A^\top - a_{T_A^\perp} V = 0, \quad B(f)(V,\, T_A^\top) + \nabla^\perp_V T_A^\perp = 0. \tag{151}$$

Proof. (i)–(ii) The torsion of D is

$$\mathrm{Tor}_D = 2\{\Theta \wedge \tau_A - \Omega_A \otimes T_A\}$$

where $\Omega_A(\mathbf{V}, \mathbf{W}) = g_\Theta(\mathbf{V}, J_A\,\mathbf{W})$ for any $\mathbf{V}, \mathbf{W} \in \mathfrak{X}(A)$ see e.g., (17)). Then, for any $V, W \in \mathfrak{X}(M)$

$$\Lambda\,\{\theta(V)\,\tau_A(f_*\,W) - \theta(W)\,\tau_A(f_*\,V)\} - 2\,(f^*g_\Theta)(V,\,JW)T^f_A$$

$$= D^f_V f_*\,W - D^f_W f_*\,W - f_*[V,W] =$$

by the pseudohermitian Gauss formula (144)

$$= f_*\,\mathrm{Tor}_{\nabla^\top}(V,W) + B(V,W) - B(W,V)$$

yielding, by $T^f_A = f_*\,T^\top_A + T^\perp_A$, (146) and (147). Q.E.D.

(iii) For any $U,\,V,\,W \in \mathfrak{X}(M)$

$$0 = (D^f_U g^f_\Theta)(f_*\,V,\,f_*\,W)$$

$$= U((f^*g_\Theta)(V,\,W)) - g^f_\Theta(D^f_U f_*V,\,f_*W) - g^f_\Theta(f_*\,V,\,D^f_U f_*W)$$

again by (144)

$$= (\nabla^\top_U(f^*\,g_\Theta))(V,\,W).$$

Q.E.D.

(iv) By (144)

$$g^f_\Theta(B(f)(V,W),\,\xi) = g^f_\Theta(D^f_V f_*W,\,\xi)$$

by $D^f g^f_\Theta = 0$ and $f_*W \perp \xi$ together with the pseudohermitian Weingarten formula (145)

$$= -g^f_\Theta(f_*W,\,D^f_V\xi) = (f^*g_\Theta)(a_\xi V,\,W).$$

Q.E.D.

(v) By (144)

$$f_*\nabla^\top_V JW + B(f)(V,\,JW) = D^f_V f_*\,JW$$

as f is a CR map, and by $D^f J^f_A = 0$ and again (144)

$$= J^f_A\,D^f_V f_*W = f_*J\nabla^\top_V W + J^f_A\,B(f)(V,W).$$

Q.E.D.

(vi) Follows from $D^f J^f_A = 0$, by (144) and (145). □

Formula (112) for $W = D^f_X f_*Y$ gives

$$\begin{aligned} &\tan_\epsilon(D_X f_*Y) = \textstyle\sum_{a=1}^{2n} g^f_\Theta(D_X f_*Y,\,f_*X_a)\,X_a \\ &+\frac{1}{\epsilon^2\mu_\epsilon}\Big\{\Theta^f(D^f_X f_*Y) + \epsilon^2\,g^f_\Theta(D_X f_*Y,\,\mathfrak{X}_\Theta)\Big\}\,T. \end{aligned} \tag{152}$$

On the other hand, by the pseudohermitian Gauss formula (144) for $(V,W) = (X,Y)$ with $X, Y \in H(M)$,

$$g^f_\Theta(D^f_X f_*Y,\,f_*X_a) = (f^*g_\Theta)(\nabla^\top_X Y,\,X_a). \tag{153}$$

Let us set

$$T^\top_A = \tan(T^f_A), \quad T^\perp_A = \mathrm{nor}(T^f_A),$$

so that

$$T_A^f = f_* T_A^\top + T_A^\perp \tag{154}$$

everywhere on M. Then, by (154), the pseudohermitian Gauss formula (144), and $\mathfrak{X}_\Theta = f_* T - T_A^f$, the functions

$$\Theta^f\big(D_X^f f_* Y\big), \;\; g_\Theta^f\big(D_X^f f_* Y, \, \mathfrak{X}_\Theta\big) \in C^\infty(M, \mathbb{R})$$

can be written as

$$\Theta^f\big(D_X^f f_* Y\big) = (f^* g_\Theta)\big(\nabla_X^\top Y, \, T_A^\top\big) + g_\Theta^f\big(B(f)(X,Y), \, T_A^\perp\big), \tag{155}$$

$$\begin{aligned} g_\Theta^f\big(D_X^f f_* Y, \, \mathfrak{X}_\Theta\big) &= (f^* g_\Theta)\big(\nabla_X^\top Y, \, T - T_A^\top\big) \\ &\quad - g_\Theta^f\big(B(f)(X,Y), \, T_A^\perp\big). \end{aligned} \tag{156}$$

Finally, let us substitute from (153) and from (155) and (156) into (152). We obtain:

Lemma 15. *Let $f : M \to A$ be an isopseudohermitian immersion. The tangential component of $D_X^f f_* Y$ with respect to (80) is*

$$\begin{aligned} \tan_\epsilon(D_X f_* Y) &= \textstyle\sum_{a=1}^{2n} (f^* g_\Theta)\big(\nabla_X^\top Y, \, X_a\big) \, X_a \\ &+ \tfrac{1}{\epsilon^2 \mu_\epsilon}\Big\{ (f^* g_\Theta)\big(\nabla_X^\top Y, \, T_A^\top + \epsilon^2 (T - T_A^\top)\big) \\ &+ (1-\epsilon^2) \, g_\Theta^f\big(B(f)(X,Y), \, T_A^\perp\big) \Big\} \, T, \end{aligned} \tag{157}$$

for any $X, Y \in H(M)$.

At this point, we may go back to (141), the Gauss formula for the isometric immersion $f : (M, \, g_\epsilon(f)) \to (A, \, g_\epsilon^A)$ along $H(M) \otimes H(M)$, and apply the projections $\tan_\epsilon$ and nor_ϵ to both sides. We obtain, by $(f^* g_\Theta)(X, \, JY) = \Omega(X,Y)$,

$$\begin{aligned} &\tan_\epsilon\big(D_X^f f_* Y\big) + \Big\{ \Omega(X,Y) - \epsilon^2 \, g_\Theta^f\big(f_* X, \, \tau_A^f f_* Y\big) \Big\} \tan_\epsilon\big(T_A^f\big) \\ &\quad = \nabla_X Y + \Big\{ \Omega(X,Y) - \tfrac{1}{\mu_\epsilon} \, A(X,Y) \Big\} \, T, \end{aligned} \tag{158}$$

$$\begin{aligned} &\text{nor}_\epsilon\big(D_X^f f_* Y\big) + \Big\{ \Omega(X,Y) - \epsilon^2 \, g_\Theta^f\big(f_* X, \, \tau_A^f f_* Y\big) \Big\} \text{nor}_\epsilon\big(T_A^f\big) \\ &\quad = B_\epsilon(f)(X,Y). \end{aligned} \tag{159}$$

Let us substitute from (143) and (157) into (158). We obtain

$$\begin{aligned} &\textstyle\sum_a (f^* g_\Theta)\big(\nabla_X^T Y, \, X_a\big) \, X_a \\ &+ \tfrac{1}{\epsilon^2 \mu_\epsilon}\Big\{ (f^* g_\Theta)\big(\nabla_X^\top Y, \, \epsilon^2 T + (1-\epsilon^2) T_A^\top\big) + g_\Theta^f\big(B(f)(X, \, Y), \, (1-\epsilon^2) \, T_A^\perp\big) \Big\} \, T \\ &+ \tfrac{1}{\epsilon^2 \mu_\epsilon}\{ \Omega(X,Y) - \epsilon^2 \, g_\Theta^f\big(f_* X, \, \tau_A^f \, f_* \, Y\big) \} \, T = \nabla_X Y + \Big\{ \Omega(X,Y) - \tfrac{1}{\mu_\epsilon} \, A(X,Y) \Big\} \, T. \end{aligned} \tag{160}$$

Moreover, by $D_X^f f_* Y \in f^{-1} H(A) = \text{Ker}\big(\Theta^f\big)$,

$$0 = \Theta^f\big(D_X^f f_* Y\big) = g_\Theta^f\big(D_X^f f_* Y, \, T_A^f\big)$$

by the pseudohermitian Gauss formula (144)

$$= g_\Theta^f\big(f_* \nabla_X^T Y, \, T_A^f\big) + g_\Theta^f\big(B(f)(X,Y), \, T_A^f\big)$$

by (154)

$$= (f^* g_\Theta)\big(\nabla^T_X Y,\, T^\top_A\big) + g^f_\Theta\big(B(f)(X,Y),\, T^\perp_A\big).$$

Summing up, we have proved the identity

$$(f^* g_\Theta)\big(\nabla^T_X Y,\, T^\top_A\big) + g^f_\Theta\big(B(f)(X,Y),\, T^\perp_A\big) = 0. \tag{161}$$

As a consequence of (161), Equation (160) simplifies to

$$\begin{aligned} &\textstyle\sum_a (f^* g_\Theta)\big(\nabla^T_X Y,\, X_a\big)\, X_a \\ &+\tfrac{1}{\mu_\epsilon}\Big\{(f^* g_\Theta)\big(\nabla^\top_X Y,\, T\big) - g^f_\Theta\big(f_* X,\, \tau^f_A\, f_*\, Y\big)\Big\}\, T \\ &= \nabla_X Y + \tfrac{1}{\mu_\epsilon}\Big\{(\lambda - 1)\,\Omega(X,Y) - A(X,Y)\Big\}\, T. \end{aligned} \tag{162}$$

Identifying the $H(M)$ and $\mathbb{R}T$ in (162) leads to:

Proposition 6. *Let $f : M \to A$ be an isopseudohermitian immersion of (M,θ) and (A,Θ). The equations*

$$\sum_a (f^* g_\Theta)\big(\nabla^T_X Y,\, X_a\big)\, X_a = \nabla_X Y, \tag{163}$$

$$\begin{aligned} &(f^* g_\Theta)\big(\nabla^\top_X Y,\, T\big) + A(X,Y) \\ &= (\lambda - 1)\,\Omega(X,Y) + g^f_\Theta\big(f_* X,\, \tau^f_A\, f_*\, Y\big), \end{aligned} \tag{164}$$

are equivalent to the tangential part (158) of the Gauss formula (141) along $H(M) \otimes H(M)$ for the isometric immersion $f : (M, g_\epsilon(f)) \to (A, g^A_\epsilon)$.

A remark is in order. Let us apply Θ^f to both sides of the pseudohermitian Gauss formula (144) for $W = X \in C^\infty(H(M))$, i.e.,

$$D^f_V f_* X = f_* \nabla^\top_V X + B(f)(V,X), \quad V \in \mathfrak{X}(M).$$

As D parallelizes $H(A)$ and $f^*\Theta = \theta$

$$0 = \Theta^f\big(D^f_V f_* X\big) = \theta\big(\nabla^\top_V X\big) + \Theta^f\big(B(f)(V,X)\big)$$

showing that in general $\nabla^\top$ does not parallelize $H(M)$ (unlike the Tanaka–Webster connection ∇).

Let us go back to the Gauss formula (159). We need to compute $\mathrm{nor}_\epsilon(D^f_X f_* Y)$. To this end, let us simplify (157) according to our successive finding (161), i.e.,

$$\begin{aligned} \tan_\epsilon\big(D^f_X f_* Y\big) &= \textstyle\sum_a (f^* g_\Theta)\big(\nabla^\top_X Y,\, X_a\big)\, X_a \\ &+\tfrac{1}{\mu_\epsilon}\,(f^* g_\Theta)\big(\nabla^\top_X Y,\, T\big)\, T. \end{aligned} \tag{165}$$

Then

$$\mathrm{nor}_\epsilon(D^f_X f_* Y) = D^f_X f_* Y - f_*\, \tan_\epsilon(D^f_X f_* Y) \tag{166}$$

by (165)

$$= D^f_X f_* Y - \sum_a (f^* g_\Theta)\big(\nabla^\top_X Y,\, X_a\big)\, f_*\, X_a - \frac{1}{\mu_\epsilon}\,(f^* g_\Theta)\big(\nabla^\top_X Y,\, T\big)\, f_*\, T.$$

Also, let us recall that, by (143),

$$\mathrm{nor}_\epsilon\big(T^f_A\big) = T^f_A - \frac{1}{\epsilon^2 \mu_\epsilon}\, f_*\, T. \tag{167}$$

Next, let us modify the Gauss formula (159) by substitution from (166) and (167). We obtain

$$D^f_X f_* Y - \sum_a (f^* g_\Theta)(\nabla^\top_X Y,\, X_a)\, f_*\, X_a - \frac{1}{\mu_\epsilon}\,(f^* g_\Theta)(\nabla^\top_X Y,\, T)\, f_*\, T$$
$$+\Big\{\Omega(X,Y) - \epsilon^2\, g^f_\Theta(f_* X,\, \tau^f_A\, f_*\, Y)\Big\}\,\Big\{T^f_A - \frac{1}{\epsilon^2 \mu_\epsilon}\, f_*\, T\Big\} = B_\epsilon(f)(X,Y)$$

or, by (163):

Proposition 7. *Let $f : M \to A$ be an isopseudohermitian immersion of (M, θ) into (A, Θ). The normal part (159) of the Gauss formula (141) for the isometric immersion $f : (M, g_\epsilon(f)) \to (A, g^A_\epsilon)$ along $H(M) \otimes H(M)$, is equivalent to*

$$\begin{aligned} &D^f_X f_* Y - \tfrac{1}{\mu_\epsilon}\,(f^* g_\Theta)(\nabla^\top_X Y,\, T)\, f_*\, T \\ &+\Big\{\Omega(X,Y) - \epsilon^2\, g^f_\Theta(f_* X,\, \tau^f_A\, f_*\, Y)\Big\}\,\Big\{T^f_A - \tfrac{1}{\epsilon^2 \mu_\epsilon}\, f_*\, T\Big\} \\ &= f_*\, \nabla_X Y + B_\epsilon(f)(X,Y) \end{aligned} \tag{168}$$

for any $X, Y \in H(M)$.

Another useful form of (168) is obtained by the substitution $f_* T = \mathfrak{X}_\Theta + T^f_A$; i.e.,

$$\begin{aligned} &D^f_X f_* Y - \tfrac{1}{\mu_\epsilon}\,(f^* g_\Theta)(\nabla^\top_X Y,\, T)\,(\mathfrak{X}_\Theta + T^f_A) \\ &+\Big\{\Omega(X,Y) - \epsilon^2\, g^f_\Theta(f_* X,\, \tau^f_A\, f_*\, Y)\Big\}\,\Big\{T^f_A - \tfrac{1}{\epsilon^2 \mu_\epsilon}\,(\mathfrak{X}_\Theta + T^f_A)\Big\} \\ &= f_*\, \nabla_X Y + B_\epsilon(f)(X,Y). \end{aligned} \tag{169}$$

Indeed, the left-hand side of (169) is already decomposed into a $f^{-1}H(A)$ component and an $\mathbb{R}T^f_A$ component. Consequently, we can compute the $\mathbb{R}T^f_A$ component of $B_\epsilon(f)(X,Y)$ by applying Θ^f to both sides of (169) and using the identities $\Theta^f(D^f_X f_* Y) = 0$, $\Theta^f(\mathfrak{X}_\Theta) = 0$, and $\Theta^f(f_*\, \nabla_X Y) = 0$], i.e.,

$$\begin{aligned} &\Theta^f(B_\epsilon(f)(X,Y)) = -\tfrac{1}{\mu_\epsilon}\,(f^* g_\Theta)(\nabla^\top_X Y,\, T) \\ &+\Big(1 - \tfrac{1}{\epsilon^2 \mu_\epsilon}\Big)\,\Big\{\Omega(X,Y) - \epsilon^2\, g^f_\Theta(f_* X,\, \tau^f_A\, f_*\, Y)\Big\}. \end{aligned} \tag{170}$$

To fully determine $B_\epsilon(f)(X,Y)$, one needs to compute its $f^{-1}H(A)$ component as well. To this end, we apply the projection $\Pi^f_{H(A)} : f^{-1}T(A) \to f^{-1}H(M)$ to both sides of (169) and use $\Pi_{H(A)} T_A = 0$, i.e.,

$$D^f_X f_* Y - \frac{1}{\mu_\epsilon}\,(f^* g_\Theta)(\nabla^\top_X Y,\, T)\,\mathfrak{X}_\Theta$$
$$-\frac{1}{\epsilon^2 \mu_\epsilon}\,\Big\{\Omega(X,Y) - \epsilon^2\, g^f_\Theta(f_* X,\, \tau^f_A\, f_*\, Y)\Big\}\,\mathfrak{X}_\Theta$$
$$= f_*\, \nabla_X Y + \Pi^f_{H(A)}\, B_\epsilon(f)(X,Y).$$

or

$$\begin{aligned} &\Pi^f_{H(A)}\, B_\epsilon(f)(X,Y) = D^f_X f_* Y - f_*\, \nabla_X Y \\ &+\tfrac{1}{\mu_\epsilon}\,\Big\{g^f_\Theta(f_* X,\, \tau^f_A\, f_*\, Y) - (f^* g_\Theta)(\nabla^\top_X Y,\, T) - \tfrac{1}{\epsilon^2}\,\Omega(X,Y)\Big\}\,\mathfrak{X}_\Theta \end{aligned} \tag{171}$$

for any X, $Y \in H(M)$.

A summary of the various decompositions and projections used so far is provided below. Given an isopseudohermitian immersion f of (M,θ) into (A,Θ), we adopted the following orthogonal decompositions.

Decomposition	Ambient inner product
$T_{f(x)}(A) = \left[(d_x f)T_x(M)\right] \oplus E_\epsilon(f)_x$	$g^A_{\epsilon, f(x)}$
$T_{f(x)}(A) = \left[(d_x f)T_x(M)\right] \oplus E(f)_x$	$g_{\Theta, f(x)}$
$H(A)_{f(x)} = \left[(d_x f)H(M)_x\right] \oplus E_H(f)_x$	$G_{\Theta, f(x)}$

The following projections correspond to the chosen decompositions:

$$\tan_\epsilon : f^{-1}T(A) \to T(M), \quad \mathrm{nor}_\epsilon : f^{-1}T(A) \to E_\epsilon(f),$$

$$\tan : f^{-1}T(A) \to T(M), \quad \mathrm{nor} : f^{-1}T(A) \to E(f),$$

$$\tan_H : f^{-1}H(A) \to H(M), \quad \mathrm{nor}_H : f^{-1}H(A) \to E_H(f).$$

The following Riemannian metrics appear in previous and further calculations:

Metrics on A		g_Θ	g^A_ϵ
Metrics on M	Intrinsic	g_θ	g^M_ϵ
	Extrinsic	$f^* g_\Theta$	$g_\epsilon(f)$

As a reminder, the adopted terminology g_θ and g_Θ are, respectively, the Webster metrics of (M,θ) and (A,Θ), g^M_ϵ and g^A_ϵ are respectively the ϵ-contractions of G_θ and G_Θ, $f^* g_\Theta$ and $g_\epsilon(f)$ are, respectively, the Riemannian metric on M induced by g_Θ *via* f, and the first fundamental form of $f : M \to (A,\, g^A_\epsilon)$.

The following connections (linear or in normal bundles) are of frequent use

D	Tanaka–Webster connection of	$(A,\, \Theta)$
D^ϵ	Levi–Civita connection of	$(A,\, g^A_\epsilon)$
∇	Tanaka–Webster connection of	$(M,\, \theta)$
∇^ϵ	Levi–Civita connection of	$(M,\, g^M_\epsilon)$
$\nabla^{f,\epsilon}$	Levi–Civita connection of	$(M,\, g_\epsilon(f))$
$\nabla^{\perp_\epsilon}$	normal connection of	$f : M \to (A,\, g^A_\epsilon)$
$\nabla^\top$	linear connection of M induced by	D *via* f
$\nabla^\perp$	normal Tanaka–Webster connection of	$f : M \to (A,\, \Theta)$

The list of Gauss–Weingarten and pseudohermitian Gauss–Weingarten formulas is

$$(D^\epsilon)^f_V f_* W = f_* \nabla^{f,\epsilon}_V W + B_\epsilon(f)(V,W),$$

$$(D^\epsilon)^f_V \xi = -f_* a^\epsilon_\xi V + \nabla^{\perp_\epsilon}_V \xi,$$

$$D^f_V f_* W = f_* \nabla^\top_V W + B(f)(V, W),$$

$$D^f_V \eta = -f_* a_\eta V + \nabla^\perp_V \eta\,,$$

$$V,\, W \in \mathfrak{X}(M), \quad \xi \in C^\infty\big(E_\epsilon(f)\big), \quad \eta \in C^\infty\big(E(f)\big).$$

Let $E_H(f) \to M$ be the Levi normal bundle. By Lemma 7, the Levi normal bundle is J_A-invariant. Let $J^\perp : E_H(f) \to E_h(f)$ be the restriction of J_A to $E_H(f)$. The complexification $E_H(f) \otimes \mathbb{C}$ decomposes as

$$E_H(f) \otimes \mathbb{C} = E_{1,0}(f) \oplus E_{0,1}(f),$$

$$E_{1,0}(f) = \text{Eigen}\big((J^\perp)^{\mathbb{C}}, i\big), \quad E_{0,1}(f) = \overline{E_{1,0}(f)}\,, \quad i = \sqrt{-1},$$

where $\big(J^\perp\big)^{\mathbb{C}}$ is the $\mathbb{C}$-linear extension of $J^\perp$ to $E_H(f) \otimes \mathbb{C}$. Let

$$\{X_a \,:\, 1 \le a \le 2n\} \equiv \{X_\alpha\,,\, JX_\alpha \,:\, 1 \le \alpha \le n\},$$

$$X_{n+\alpha} = JX_\alpha\,, \quad 1 \le \alpha \le n,$$

be a local G_θ-orthonormal, i.e., $G_\theta(X_a\,,\, X_b) = \delta_{ab}$, $1 \le a, b \le 2n$, frame of $H(M)$, adapted to the complex structure J, and defined on the open set $U \subset M$. Next, let us set

$$T_\alpha = \frac{1}{\sqrt{2}}\big(X_\alpha - i\, J\, X_\alpha\big), \quad 1 \le \alpha \le n,$$

so that $\{T_\alpha \,:\, 1 \le \alpha \le n\} \subset C^\infty\big(U,\, T_{1,0}(M)\big)$ is a local G_θ-orthonormal, i.e., $G_\theta(T_\alpha\,,\, T_{\overline{\beta}}) = \delta_{\alpha\beta}$ with $T_{\overline{\beta}} = \overline{T_\beta}$, frame of the CR structure $T_{1,0}(M)$. Moreover, let

$$\{\xi_p \,:\, 1 \le p \le 2k\} \subset C^\infty\big(U,\, E_H(f)\big)$$

be a local G^f_Θ-orthonormal, i.e., $G^f_\Theta(\xi_p\,,\, \xi_q) = \delta_{pq}$, frame, adapted to the complex structure $J^\perp$ i.e.,

$$J^\perp \xi_j = \xi_{k+j}\,, \quad 1 \le j \le k.$$

Let us set as customary

$$\zeta_j = \frac{1}{\sqrt{2}}\Big(\xi_j - i\, J^\perp\, \xi_j\Big) \in C^\infty\big(U,\, E_{1,0}(f)\big), \quad 1 \le j \le k,$$

so that $\{\zeta_j \,:\, 1 \le j \le k\}$ is a local G^f_Θ-orthonormal, i.e., $G^f_\Theta\big(\zeta_j\,,\, \zeta_{\overline{\ell}}\big) = \delta_{j\ell}$ with $\zeta_{\overline{\ell}} = \overline{\zeta_\ell}\,$, frame in $E_{1,0}(f)$. The conventions as to the range of the various indices are

$$a,\, b,\, c,\, \cdots \in \{1, \cdots, 2n\}, \quad \alpha,\, \beta,\, \gamma,\, \cdots \in \{1, \cdots, n\},$$

$$p,\, q,\, r,\, \cdots \in \{1, \cdots, 2k\}, \quad \sigma,\, \rho,\, \kappa,\, \cdots \in \{1, \cdots, k\}.$$

Lemma 16. *Let $f : M \to A$ be an isopseudohermitian immersion of (M, θ) into $(A,\, \Theta)$. The CR structure $T_{1,0}(A)$ of the ambient space decomposes as*

$$T_{1,0}(A)_{f(x)} = \big[(d_x f) T_{1,0}(M)_x\big] \oplus E_{1,0}(f)_x \tag{172}$$

for any $x \in M$. In particular,

$$\big\{\mathbf{T}_j \,:\, 1 \le j \le N\big\} \equiv \big\{f_* T_\alpha\,,\, \zeta_\sigma \,:\, 1 \le \alpha \le n,\ 1 \le \sigma \le k\big\}$$

is a local frame of $f^{-1} T_{1,0}(A)$ such that

$$G^f_\Theta\big(\mathbf{T}_j\,,\, \mathbf{T}_{\overline{\ell}}\big) = \delta_{j\ell}\,, \quad \mathbf{T}_{\overline{\ell}} \equiv \overline{\mathbf{T}_\ell}\,.$$

Proof. Let $w \in \left[(d_x f)T_{1,0}(M)_x\right] \oplus E_{1,0}(f)_x$ so that

$$w = (d_x f)Z = \xi - i\, J_x^{\perp}\, \xi$$

for some $Z \in T_{1,0}(M)_x$ and $\xi \in E_H(f)_x$. Of course, $Z = X - i\, J_x\, X$ for some $X \in H(M)_x$. Then,

$$G_{\Theta,\, f(x)}\big(w,\, \overline{w}\big) = G_{\Theta,\, f(x)}\big((d_x f)(X - i\, J_x\, X),\, \xi + i\, J_x^{\perp}\, \xi\big) = 0$$

as $(d_x f)X$, $(d_x f)J_x\, X \in (d_x f)H(M)_x \perp \xi$, $J_x^{\perp}\, \xi$, thus yielding $w = 0$; i.e., the sum $(d_x f)T_{1,0}(M)_x + E_{1,0}(f)_x$ is direct. $\square$

A comparison to the work by P. Ebenfelt et al. (see [12], p. 636) is at this point advisable. Let $M \subset \mathbb{C}^{n+1}$ and $A \subset \mathbb{C}^{N+1}$ be real hypersurfaces, such that the induced CR structures

$$T_{1,0}(M) = \big[T(M) \otimes \mathbb{C}\big] \cap T'\big(\mathbb{C}^{n+1}\big),$$

$$T_{1,0}(A) = \big[T(A) \otimes \mathbb{C}\big] \cap T'\big(\mathbb{C}^{N+1}\big),$$

are strictly pseudoconvex, where

$$T'\big(\mathbb{C}^{n+1}\big)_x = \text{Span}_{\mathbb{C}}\Big\{\Big(\frac{\partial}{\partial z^a}\Big)_x \;:\; 0 \le a \le n\Big\}, \quad x \in \mathbb{C}^{n+1}\,,$$

$$T'\big(\mathbb{C}^{N+1}\big)_y = \text{Span}_{\mathbb{C}}\Big\{\Big(\frac{\partial}{\partial Z^B}\Big)_y \;:\; 0 \le B \le N\Big\}, \quad y \in \mathbb{C}^{N+1}\,.$$

Let $f : M \to A$ be an isopseudohermitian immersion of (M, θ) into (A, Θ). Let $x \in M$ and let us set $\hat{x} = f(x) \in A$. Let $U \subset M$ and $V \subset A$ be, respectively, open neighborhoods of x and $\hat{x}$ in M and A, such that U is the domain of a (local) frame $\{T_\alpha \;:\; 1 \le \alpha \le n\}$ of $T_{1,0}(M)$, and V is described by a defining function $\hat{\rho} \in C^\infty\big(\mathbb{C}^{N+1}\big)$; i.e., $\hat{\rho}(y) = 0$ and $\hat{\rho}_Z(y) = (\partial\hat{\rho})(y) \equiv \big(\hat{\rho}_{Z_0}\,, \hat{\rho}_{Z_1}\,, \cdots, \hat{\rho}_{Z_N}\big) \neq 0$ for every $y \in V$. B. Lamel introduced (see [13,14]) the sequence of subspaces $E_\nu(x) \subset \mathbb{C}^{N+1}$, $\nu \in \mathbb{Z}_+$, where $E_\nu(x)$ is the span over $\mathbb{C}$ of

$$\Big\{T_1^{J_1} \cdots T_n^{J_n}\big(\hat{\rho}_Z \circ f\big)(x) \;:\; J = \big(J_1\,, \cdots, J_n\big) \in \mathbb{Z}_+^n\,, \;\; |J| \le \nu\Big\}.$$

Note that $E_\nu(x) \subset E_{\nu+1}(x)$ for any $\nu \in \mathbb{Z}_+$. Let us set

$$s(x) = \min_{\nu \in \mathbb{Z}_+}\Big[N + 1 - \dim_{\mathbb{C}}\, E_\nu(x)\Big].$$

Let $\nu_0 \in \mathbb{Z}_+$ and $s_0 \in \mathbb{Z}_+ \cap \big[1,\, N\big]$. According to B. Lamel (see [13,14]), $f : M \to A$ is $(\nu_0\,, s_0)$-*degenerate* at $x \in M$ if

$$s(x) = s_0 = \text{codim}_{\mathbb{C}}\, E_{\nu_0}(x).$$

Also, $\min_{x \in M} s(x)$ is the *degeneracy* of f at x. Let

$$\overline{E_1(x)} = \text{Span}_{\mathbb{C}}\big\{\big(\rho_{\overline{Z}} \circ f\big)(x),\;\; L_\alpha\big(\rho_{\overline{Z}} \circ f\big)(x) \;:\; 1 \le \alpha \le n\big\}$$

be the complex conjugate of $E_1(x)$. Let $f^{-1}T'\big(\mathbb{C}^{N+1}\big) \to M$ be the pullback of $T'\big(\mathbb{C}^{N+1}\big) \to \mathbb{C}^{N+1}$ *via* $f : M \to \mathbb{C}^{N+1}$. Let

$$\phi(y) : \mathbb{C}^{N+1} \to T'\big(\mathbb{C}^{N+1}\big)_y\,, \quad w = \big(w^0\,, w^1\,, \cdots, w^N\big) \longmapsto w^B\,\Big(\frac{\partial}{\partial Z^B}\Big)_y\,,$$

be the natural identification. For every subspace $S \subset \mathbb{C}^{N+1}$ and every $x \in M$, let

$$F(S)_y : \frac{\mathbb{C}^{N+1}}{S} \to \frac{T'\,\big(\mathbb{C}^{N+1}\big)_y}{\phi(y)\, S}\,, \quad y \in \mathbb{C}^{N+1}\,,$$

be the natural isomorphism. Let

$$\pi_x : \mathbb{C}^{N+1} \to \frac{\mathbb{C}^{N+1}}{E_1(x)}$$

be the projection, and let us consider the field of forms

$$\Pi_x : T_{1,0}(M)_x \times T_{1,0}(M)_x \to \frac{\left(f^{-1}T'\,\mathbb{C}^{N+1}\right)_x}{\phi(\hat{x})\,\overline{E_1(x)}},$$

$$\Pi_x(v,\,w) = F\Big[\overline{E_1(x)}\Big]_{\hat{x}}\,\overline{\pi_x\,\overline{V}\,\overline{W}\,(\rho_Z \circ f)(x)}\,, \quad v,\,w \in T_{1,0}(M)_x\,,$$

where $V,\,W \in C^\infty\big(T_{1,0}(M)\big)$ are smooth extensions of v and w to the whole of M; i.e., $V_x = v$ and $W_x = w$.

Theorem 4. *Let $M \subset \mathbb{C}^{n+1}$ and $A \subset \mathbb{C}^{N+1}$ be strictly pseudoconvex real hypersurfaces. Let $f : M \to A$ be an isopseudohermitan immersion of (M,θ) into (A,Θ), and let $x \in M$ and $\hat{x} = f(x) \in A$. The quotient space $\left(f^{-1}T'(\mathbb{C}^{N+1})\right)_x/\phi(\hat{x})\,\overline{E_1(x)}$ is isomorphic to $T_{1,0}(A)_{\hat{x}}/(d_x f)\,T_{1,0}(M)_x$. Hence, Π determines a field of $\mathbb{C}$-bilinear symmetric forms*

$$\Pi_x \;:\; T_{1,0}(M)_x \times T_{1,0}(M)_x \to E_{1,0}(f)_x\,.$$

Let $\left(Z^0\,,\,Z^1\,,\,\cdots\,,\,Z^N\right)$ be the Cartesian complex coordinates on $\mathbb{C}^{N+1}$, and let us set $f^B = Z^B \circ f$. As $f : M \to A$ is a CR map, its components f^B are CR functions, i.e.,

$$T_{\overline{\alpha}}\big(f^B\big) = 0, \quad 0 \le \alpha \le N, \tag{173}$$

for every local frame $T_\alpha \in C^\infty(U,\,T_{1,0}(M))$. Then,

$$\begin{gathered} T_{\overline{\alpha}}\big(\hat{\rho}_{Z_B} \circ f\big)(x) = \hat{\rho}_{Z_B\,\overline{Z}_C}(f(x))\,T_{\overline{\alpha}}\big(f^{\overline{C}}\big)_x\,, \\ f^{\overline{B}} = \overline{f^B}\,, \quad Z_B = Z^B\,, \quad u_{Z_B\overline{Z}_C} = \frac{\partial^2 u}{\partial Z_B\,\partial \overline{Z}_B}\,, \quad u \in C^2(\mathbb{C}^{N+1}). \end{gathered} \tag{174}$$

We shall need the local frame $\hat{T}_j \in C^\infty\big(V,\,T_{1,0}(A)\big)$

$$\begin{gathered} \hat{T}_j = \hat{\rho}_{Z_0}\,\frac{\partial}{\partial Z^j} - \hat{\rho}_{Z_j}\,\frac{\partial}{\partial Z^0}\,, \\ u_{Z_B} = \frac{\partial u}{\partial Z^B}\,, \quad u \in C^1\big(\mathbb{C}^{N+1}\big), \end{gathered} \tag{175}$$

defined on the open set $V = \{y \in A \;:\; \hat{\rho}_{Z_0}(y) \neq 0\}$. Then, by (173) and (175),

$$(d_x f) T_{\alpha\,,\,x} = T_{\alpha\,,\,x}\big(f^B\big)\left(\frac{\partial}{\partial Z^B}\right)_{\hat{x}}$$

$$= T_{\alpha\,,\,x}\big(f^j\big)\,\frac{1}{\rho_{Z_0}(\hat{x})}\left\{\hat{T}_{j,\,\hat{x}} + \hat{\rho}_{Z_j}(\hat{x})\left(\frac{\partial}{\partial Z_0}\right)_{\hat{x}}\right\} + T_{\alpha\,,\,x}\big(f^0\big)\left(\frac{\partial}{\partial Z^0}\right)_{\hat{x}}$$

or

$$f_*\,T_\alpha = \frac{1}{\hat{\rho}_{Z_0} \circ f}\;T_\alpha\big(f^j\big)\,\hat{T}_j^f + \left\{T_\alpha\big(f^0\big) + \frac{\hat{\rho}_{Z_j} \circ f}{\hat{\rho}_{Z_0} \circ f}\,T_\alpha\big(f^j\big)\right\}\left(\frac{\partial}{\partial Z_0}\right)^f.$$

On the other hand, $\hat{\rho} \circ f = 0$ on M yields, again by (173),

$$\begin{aligned} 0 = T_\alpha\big(\hat{\rho} \circ f\big) &= \big(\hat{\rho}_{Z_B} \circ f\big)\,T_\alpha\big(f^B\big) \\ \Longrightarrow T_\alpha\big(f^0\big) &+ \frac{\hat{\rho}_{Z_j} \circ f}{\hat{\rho}_{Z_0} \circ f}\,T_\alpha\big(f^j\big) = 0. \end{aligned}$$

We conclude that

$$f_* T_\alpha = \frac{1}{\hat{\rho}_{Z_0} \circ f} \, T_\alpha(f^j) \, \hat{T}_j^f \,. \tag{176}$$

It is an elementary matter that:

Lemma 17. *Let $\Theta = \frac{i}{2}(\overline{\partial} - \partial)\,\hat{\rho}$. The Levi form $G_{j\overline{k}} = G_\Theta(\hat{T}_j\,,\,\hat{T}_{\overline{k}})$ of $(A,\,\Theta)$ is given by*

$$\begin{aligned} \frac{2}{\left|\hat{\rho}_{Z_0}\right|^2} \, G_{j\overline{k}} &= \hat{\rho}_{Z_j \overline{Z}_k} - \frac{\hat{\rho}_{Z_j}}{\hat{\rho}_{Z_0}} \, \hat{\rho}_{Z_0 \overline{Z}_k} - \frac{\hat{\rho}_{\overline{Z}_k}}{\hat{\rho}_{\overline{Z}_0}} \left(\hat{\rho}_{Z_j \overline{Z}_0} - \frac{\hat{\rho}_{Z_j}}{\hat{\rho}_{Z_0}} \, \hat{\rho}_{Z_0 \overline{Z}_0} \right) \\ &= \frac{1}{\hat{\rho}_{Z_0}} \, \hat{T}_j(\hat{\rho}_{\overline{Z}_k}) - \frac{\hat{\rho}_{\overline{Z}_k}}{\left|\hat{\rho}_{Z_0}\right|^2} \, \hat{T}_j(\hat{\rho}_{\overline{Z}_0}) \end{aligned} \tag{177}$$

everywhere on V.

Lemma 18. *For every $x \in M$*

$$\dim_{\mathbb{C}} E_1(x) = n + 1. \tag{178}$$

Proof. Note that $E_1(x)$ is the span over $\mathbb{C}$ of

$$\Big\{ (\hat{\rho}_Z \circ f)(x), \;\; T_{\overline{\alpha}}(\hat{\rho}_Z \circ f)(x) \;:\; 1 \le \alpha \le n \Big\}.$$

Let $\lambda\,,\,\lambda^{\overline{\alpha}} \in \mathbb{C}$, $1 \le \alpha \le n$, such that

$$\lambda\,(\hat{\rho}_Z \circ f)(x) + \lambda^{\overline{\alpha}} \, T_{\overline{\alpha}}(\hat{\rho}_Z \circ f)(x) = 0$$

or

$$\lambda\,(\hat{\rho}_{Z_0} \circ f)(x) + \lambda^{\overline{\alpha}} \, T_{\overline{\alpha}}(\hat{\rho}_{Z_0} \circ f)(x) = 0, \tag{179}$$

$$\lambda\,(\hat{\rho}_{Z_j} \circ f)(x) + \lambda^{\overline{\alpha}} \, T_{\overline{\alpha}}(\hat{\rho}_{Z_j} \circ f)(x) = 0, \quad 1 \le j \le N. \tag{180}$$

Substitution from (179) into (180) gives

$$\lambda^{\overline{\alpha}} \left\{ T_{\overline{\alpha}\,,\,x}(\hat{\rho}_{Z_j} \circ f) - \frac{\hat{\rho}_{Z_j}(f(x))}{\hat{\rho}_{Z_0}(f(x))} \, T_{\overline{\alpha}\,,\,x}(\hat{\rho}_{Z_0} \circ f) \right\} = 0$$

or, by (174),

$$\overline{V}(f^{\overline{C}}) \left(\hat{\rho}_{Z_j \overline{Z}_C} - \frac{\hat{\rho}_{Z_j}}{\hat{\rho}_{Z_0}} \, \hat{\rho}_{Z_0 \overline{Z}_C} \right)_{\hat{x}} = 0 \tag{181}$$

where $\hat{x} = f(x)$ and

$$V := \lambda^\alpha \, T_{\alpha\,,\,x} \in T_{1,0}(M)_x\,, \quad \lambda^\alpha = \overline{\lambda^{\overline{\alpha}}}\,.$$

Let us substitute $\hat{\rho}_{Z_j \overline{Z}_k}$ from (177) into (181). We obtain

$$\begin{aligned} &\frac{2}{|\hat{\rho}_{Z_0}(\hat{x})|^2} \, \overline{V}(f^{\overline{k}}) \, G_{j\overline{k}}(\hat{x}) \\ &+ \left(\hat{\rho}_{Z_j \overline{Z}_0} - \frac{\hat{\rho}_{Z_j}}{\hat{\rho}_{Z_0}} \, \hat{\rho}_{Z_0 \overline{Z}_0} \right)_{\hat{x}} \left\{ \overline{V}(f^{\overline{0}}) + \frac{\hat{\rho}_{\overline{Z}_k}(\hat{x})}{\hat{\rho}_{\overline{Z}_0}(\hat{x})} \, \overline{V}(f^{\overline{k}}) \right\} = 0. \end{aligned} \tag{182}$$

On the other hand, $\hat{\rho} \circ f = 0$ everywhere on U, hence (as V is tangent to M and $\overline{V}(f^B) = 0$)

$$0 = \overline{V}(\hat{\rho} \circ f) = \hat{\rho}_{\overline{Z}_B}(\hat{x}) \, \overline{V}(f^{\overline{B}})$$

so that (182) simplifies to

$$\overline{V}(f^{\overline{k}}) \, G_{j\overline{k}}(\hat{x}) = 0$$

or, as $G_{\Theta,\hat{x}}$ is nondegenerate,

$$V(f^j) = 0, \quad 1 \le j \le N. \tag{183}$$

Substitution from (183) into (181) yields

$$\overline{V}(f^{\overline{0}})\,\hat{T}_j(\hat{\rho}_{\overline{Z}_0})_{\hat{x}} = 0$$

and hence

$$V(f^0) = 0. \tag{184}$$

Then, by (183) and (184) and $V(f^{\overline{B}}) = 0$,

$$(d_x f)V = V(f^B)\left(\frac{\partial}{\partial Z^B}\right)_{\hat{x}} = 0$$

hence, as $d_x f$ is a monomorphism, $V = 0$ and then $\lambda^\alpha = 0$, $1 \le \alpha \le n$. Finally, by (179), $\lambda = 0$. $\square$

Proof of Theorem 4. It suffices to show that

$$\Phi_x \;:\; E_{1,0}(f)_x \to \frac{\left(f^{-1}T'\mathbb{C}^{N+1}\right)_x}{\phi(\hat{x})\,\overline{E_1(x)}}\,,$$

$$\Phi_x(w) := w + \phi(\hat{x})\,\overline{E_1(x)}\,, \quad w \in E_{1,0}(f)_x\,,$$

is a monomorphism. We set

$$\zeta_\sigma = a^j_\sigma\,\hat{T}^f_j\,, \quad a^j_\sigma \in C^\infty(U, \mathbb{C}).$$

Let $w = \mu^\sigma\,\zeta_{\sigma,x} \in E_{1,0}(f)_x$ i.e.,

$$w = \mu^\sigma\,a^j_\sigma(x)\,\hat{T}_{j,\hat{x}} = \mu^\sigma\,a^j_\sigma(x)\left(\hat{\rho}_{Z_0}\,\frac{\partial}{\partial Z^j} - \hat{\rho}_{Z_j}\,\frac{\partial}{\partial Z_0}\right)_{\hat{x}}\,,$$

$$0 = G_{\Theta,\hat{x}}\big(w,\,(d_x f)T_{\overline{\alpha},x}\big) = \mu^\sigma\,a^j_\sigma(x)\,T_{\overline{\alpha}}(f^{\overline{k}})_x\,G_{j\overline{k}}(\hat{x}). \tag{185}$$

If $w \in \mathrm{Ker}(\Phi_x)$, then

$$w = \left\{\lambda\,\hat{\rho}_{\overline{Z}_B}(\hat{x}) + \lambda^\alpha\,T_{\alpha,x}\big(\hat{\rho}_{\overline{Z}_B}\circ f\big)\right\}\left(\frac{\partial}{\partial Z_B}\right)_{\hat{x}}$$

for some $\lambda\,,\,\lambda^\alpha \in \mathbb{C}$, yielding

$$-\mu^\sigma\,a^k_\sigma(x)\,\hat{\rho}_{Z_k}(\hat{x}) = \lambda\,\hat{\rho}_{\overline{Z}_0}(\hat{x}) + \lambda^\alpha\,T_{\alpha,x}\big(\hat{\rho}_{\overline{Z}_0}\circ f\big), \tag{186}$$

$$\mu^\sigma\,a^j_\sigma(x)\,\hat{\rho}_{Z_0}(\hat{x}) = \lambda\,\hat{\rho}_{\overline{Z}_j}(\hat{x}) + \lambda^\alpha\,T_{\alpha,x}\big(\hat{\rho}_{\overline{Z}_j}\circ f\big). \tag{187}$$

Let us substitute from (186) into (187). We obtain

$$\begin{aligned}&\mu^\sigma\left\{a^j_\sigma(x)\,\hat{\rho}_{Z_0}(\hat{x}) + \frac{\hat{\rho}_{\overline{Z}_j}(\hat{x})\hat{\rho}_{Z_k}(\hat{x})}{\hat{\rho}_{\overline{Z}_0}(\hat{x})}\,a^k_\sigma(x)\right\}\\ &= \lambda^\alpha\left\{T_{\alpha,x}\big(\hat{\rho}_{\overline{Z}_j}\circ f\big) - \frac{\hat{\rho}_{\overline{Z}_j}(\hat{x})}{\hat{\rho}_{\overline{Z}_0}(\hat{x})}\,T_{\alpha,x}\big(\hat{\rho}_{\overline{Z}_0}\circ f\big)\right\}.\end{aligned} \tag{188}$$

Let us substitute from (174) into (188). The right-hand side of (188) becomes

$$\left\{ \hat{\rho}_{\overline{Z}_j Z_B}(\hat{x}) - \frac{\hat{\rho}_{\overline{Z}_j}(\hat{x})}{\hat{\rho}_{\overline{Z}_0}(\hat{x})}\, \hat{\rho}_{\overline{Z}_0 Z_B}(\hat{x}) \right\} V(f^B)$$

where $V := \lambda^\alpha \, T_{\alpha\,,\,x} \in T_{1,0}(M)_x$

$$= \frac{1}{\hat{\rho}_{\overline{Z}_0}(\hat{x})}\, \hat{T}_{\bar{j}}(\hat{\rho}_{Z_B})_{\hat{x}}\, V(f^B)$$

by replacing $\hat{T}_{\bar{j}}(\hat{\rho}_{Z_k})$ in terms of the Levi form, from (177),

$$= \frac{2}{\left|\hat{\rho}_{Z_0}(\hat{x})\right|^2}\, G_{\bar{j}k}(\hat{x})\, V(f^k)$$

$$+ \frac{1}{\left|\hat{\rho}_{Z_0}(\hat{x})\right|^2} \left\{ \hat{\rho}_{Z_k}(\hat{x})\, V(f^k) + \hat{\rho}_{Z_0}(\hat{x})\, V(f^0) \right\} \hat{T}_{\bar{j}}(\hat{\rho}_{Z_0})_{\hat{x}}$$

and (188) is modified accordingly:

$$\begin{aligned} \mu^\sigma \left\{ a^j_\sigma(x)\, \hat{\rho}_{Z_0}(\hat{x}) + \frac{\hat{\rho}_{\overline{Z}_j}(\hat{x}) \hat{\rho}_{Z_k}(\hat{x})}{\hat{\rho}_{\overline{Z}_0}(\hat{x})}\, a^k_\sigma(x) \right\} \\ = \frac{2}{\left|\hat{\rho}_{Z_0}(\hat{x})\right|^2}\, G_{\bar{j}k}(\hat{x})\, V(f^k). \end{aligned} \tag{189}$$

Let us contract (189) with $\mu^\rho\, a^j_\rho(x)$ and observe, by (185), that the right-hand side of the resulting equation is zero, while the left-hand side is

$$\hat{\rho}_{Z_0}(\hat{x}) \left\{ \sum_{j=1}^{N} \left|\mu^\rho\, a^j_\rho(x)\right|^2 + \frac{1}{\left|\hat{\rho}_{Z_0}(\hat{x})\right|^2} \left|\mu^\rho\, a^j_\rho(x)\, \hat{\rho}_{Z_j}(\hat{x})\right|^2 \right\}.$$

Therefore, by (189), $\mu^\rho\, a^j_\rho(x) = 0$ for any $1 \le j \le N$, and hence $w = 0$. $\square$

The $\mathbb{C}$-bilinearity and symmetry of Π_x follows from

$$\overline{V}\,\overline{W}(\rho_Z \circ f)(x) \equiv v^{\bar{\alpha}}\, w^{\bar{\beta}}\, T_{\bar{\alpha}\,,\,x} \left\{ T_{\bar{\beta}}(\rho_Z \circ f) \right\}, \quad \bmod\ E_1(x),$$

$$v = v^\alpha\, T_{\alpha\,,\,x}\,, \quad w = w^\beta\, T_{\beta\,,\,x}\,, \quad v^{\bar{\alpha}} = \overline{v^\alpha}\,, \quad w^{\bar{\beta}} = \overline{w^\beta}\,,$$

together with the involutivity of $T_{0,1}(M)$.

Let $B_H(X,Y) := \mathrm{nor}_H \left\{ D^f_X f_* Y \right\}$ for any $X, Y \in C^\infty(H(M))$. We expect that

$$B_H(f)\big|_{T_{1,0}(M) \otimes T_{1,0}(M)} = \Pi.$$

The proof is relegated to further work.

9. Relating $R(\nabla^{f,\epsilon})$ to $R(\nabla)$

The scope of the present section is to relate the curvature tensor field $R(\nabla^{f,\epsilon})$ of the induced connection $\nabla^{f,\epsilon}$ to the curvature tensor field $R(\nabla)$ of the Tanaka–Webster connection ∇. To this end, we exploit the relationship (116)–(119) among $\nabla^{f,\epsilon}$ and ∇ i.e.,

$$\nabla^{f,\epsilon}_X Y = \nabla_X Y + \left[\Omega(X,Y) - \frac{1}{\mu_\epsilon}\, A(X,Y) \right] T, \tag{190}$$

$$\nabla^{f,\epsilon}_X T = \tau X + \mu_\epsilon\, J X + \frac{1}{2\mu_\epsilon}\, X(\lambda)\, T, \tag{191}$$

$$\nabla^{f,\epsilon}_T X = \nabla_T X + \mu_\epsilon \, JX + \frac{1}{2\,\mu_\epsilon} \, X(\lambda) \, T, \tag{192}$$

$$\nabla^{f,\epsilon}_T T = -\frac{1}{2} \, \nabla^H \lambda + \frac{1}{2\,\mu_\epsilon} \, T(\lambda) \, T, \tag{193}$$

for any $X, Y \in C^\infty(H(M))$. Then, for arbitrary $X, Y, Z \in C^\infty(H(M))$,

$$R(\nabla^{f,\epsilon})(X,Y)Z = [\nabla^{f,\epsilon}_X \,, \, \nabla^{f,\epsilon}_Y]Z - \nabla^{f,\epsilon}_{[X,Y]} Z$$

by (190) and the decomposition $[X,Y] = \Pi_H[X,Y] + \theta([X,Y])\,T$

$$= \nabla^{f,\epsilon}_X \left\{ \nabla_Y Z + \left[\Omega(Y,Z) - \frac{1}{\mu_\epsilon} \, A(Y,Z) \right] T \right\}$$

$$- \nabla^{f,\epsilon}_Y \left\{ \nabla_X Z + \left[\Omega(X,Z) - \frac{1}{\mu_\epsilon} \, A(X,Z) \right] T \right\}$$

$$- \nabla^{f,\epsilon}_{\Pi_H[X,Y]} Z - \theta([X,Y]) \, \nabla^{f,\epsilon}_T Z$$

by again applying (190), using the identity $\theta\big([X,Y]\big) = 2\,\Omega(X,Y)$, and taking covariant derivatives with respect to $\nabla^{f,\epsilon}$

$$= \nabla_X \nabla_Y Z + \big\{ \Omega\big(X, \, \nabla_Y Z\big) - \frac{1}{\mu_\epsilon} \, A\big(X, \, \nabla_Y Z\big) \big\} \, T$$

$$- \nabla_Y \nabla_X Z - \big\{ \Omega\big(Y, \, \nabla_X Z\big) - \frac{1}{\mu_\epsilon} \, A\big(Y, \, \nabla_X Z\big) \big\} \, T$$

$$- \nabla_{\Pi_H\,[X,Y]} Z - \big\{ \Omega\big(\Pi_H\,[X,Y], \, Z\big) - \frac{1}{\mu_\epsilon} \, A\big(\Pi_H[X,Y], \, Z\big) \big\} \, T -$$

$$-2\,\Omega(X,Y) \, \nabla^{f,\epsilon}_T Z$$

$$+ \big[X(\Omega(Y,Z) - \frac{1}{\mu_\epsilon} \, X(A(Y,Z)) \big] \, T + \big[\Omega(Y,Z) - \frac{1}{\mu_\epsilon} \, A(Y,Z) \big] \, \nabla^{f,\epsilon}_X T$$

$$- \big[Y(\Omega(X,Z) - \frac{1}{\mu_\epsilon} \, Y(A(X,Z)) \big] \, T - \big[\Omega(X,Z) - \frac{1}{\mu_\epsilon} \, A(X,Z) \big] \, \nabla^{f,\epsilon}_Y T.$$

Let us look at the term $\Omega\big(\Pi_H\,[X,Y], \, Z\big)$. Using again the decomposition of $[X,Y]$ into $H(M)$ and $\mathbb{R}T$ components, one has

$$\Omega\big(\Pi_H\,[X,Y], \, Z\big) = \Omega\big([X,Y], \, Z\big) - \theta([X,Y])\,\Omega(T,Z) =$$

as $T \rfloor \Omega = 0$ and $[X,Y] = \nabla_X Y - \nabla_Y X + 2\,\Omega(X,Y)\,T$

$$= \Omega\big(\nabla_X Y, \, Z\big) - \Omega\big(\nabla_Y X, \, Z\big).$$

The last identity leads (by $\nabla\Omega = 0$) to simplifications, i.e.,

$$\Omega\big(X, \, \nabla_Y Z\big) - \Omega\big(Y, \, \nabla_X Z\big) - \Omega\big(\Pi_H\,[X,Y], \, Z\big)$$

$$+ X(\Omega(Y,Z) - Y(\Omega(X,Z)$$

$$= \big(\nabla_X \Omega\big)(Y,Z) - \big(\nabla_Y \Omega\big)(X,Z) = 0.$$

The same arguments apply (as $T \rfloor A = 0$) to the pseudohermitian torsion A, hence leading one to recognize covariant derivatives of A with respect to the Tanaka–Webster connection,

$$-A\big(X, \, \nabla_Y Z\big) + A\big(Y, \, \nabla_X Z\big) + A\big(\Pi_H\,[X,Y], \, Z\big)$$

$$-X(A(Y,Z) + Y(A(X,Z)$$

$$= -(\nabla_X A)(Y,Z) + (\nabla_Y A)(X,Z).$$

Similarly,

$$\begin{aligned}\nabla_{\Pi_H [X,Y]} Z &= \nabla_{[X,Y]} Z - \theta([X,Y])\, \nabla_T Z \\ &= \nabla_{[X,Y]} Z - 2\,\Omega(X,Y)\, \nabla_T Z\end{aligned}$$

and one may recognize curvature i.e.,

$$\begin{aligned}&\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{\Pi_H [X,Y]} Z \\ &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z + 2\,\Omega(X,Y)\, \nabla_T Z \\ &= R^{\nabla}(X,Y)Z + 2\,\Omega(X,Y)\, \nabla_T Z.\end{aligned}$$

Then,

$$\begin{aligned}R(\nabla^{f,\epsilon})(X,Y)Z &= R^{\nabla}(X,Y)Z \\ &+ \frac{1}{\mu_\epsilon}\left[(\nabla_Y A)(X,Z) - (\nabla_X A)(Y,Z)\right] T + 2\,\Omega(X,Y)\, \nabla_T Z - 2\,\Omega(X,Y)\, \nabla^{f,\epsilon}_T Z \\ &+ \left[\Omega(Y,Z) - \frac{1}{\mu_\epsilon}\, A(Y,Z)\right] \nabla^{f,\epsilon}_X T - \left[\Omega(X,Z) - \frac{1}{\mu_\epsilon}\, A(X,Z)\right] \nabla^{f,\epsilon}_Y T\end{aligned}$$

by (191) and (192)

$$\begin{aligned}&= R^{\nabla}(X,Y)Z + \frac{1}{\mu_\epsilon}\left[(\nabla_Y A)(X,Z) - (\nabla_X A)(Y,Z)\right] T \\ &-2\,\Omega(X,Y)\left\{\mu_\epsilon\, JZ + \frac{1}{2\,\mu_\epsilon}\, Z(\lambda)\, T\right\} \\ &+\left[\Omega(Y,Z) - \frac{1}{\mu_\epsilon}\, A(Y,Z)\right]\left[\tau X + \mu_\epsilon\, JX + \frac{1}{2\mu_\epsilon}\, X(\lambda)\, T\right] \\ &-\left[\Omega(X,Z) - \frac{1}{\mu_\epsilon}\, A(X,Z)\right]\left[\tau Y + \mu_\epsilon\, JY + \frac{1}{2\mu_\epsilon}\, Y(\lambda)\, T\right],\end{aligned}$$

which leads to (194) in Proposition 8 below. Summing up the calculations above, we may state:

Proposition 8. *Let $f : M \to A$ be an isopseudohermitian immersion of (M,θ) into (A,Θ). Let $\nabla^{f,\epsilon}$ and ∇ be, respectively, the Levi–Civita connection of $(M,\, g_\epsilon(f))$ and the Tanaka–Webster connection of (M,θ). Then,*

$$\begin{aligned}R(\nabla^{f,\epsilon})(X,Y)Z &= R^{\nabla}(X,Y)Z \\ -2\,\Omega(X,Y)\Big\{\mu_\epsilon\, JZ + \tfrac{1}{2\,\mu_\epsilon}\, Z(\lambda)\, T\Big\} &+ \tfrac{1}{\mu_\epsilon}\{(\nabla_Y A)(X,Z) - (\nabla_X A)(Y,Z)\}\, T \\ +\Omega(Y,Z)\Big\{\tau X + \mu_\epsilon\, JX + \tfrac{1}{2\,\mu_\epsilon}\, X(\lambda)\, T\Big\} &- \Omega(X,Z)\Big\{\tau Y + \mu_\epsilon\, JY + \tfrac{1}{2\,\mu_\epsilon}\, Y(\lambda)\, T\Big\} \\ -\tfrac{1}{\mu_\epsilon}\, A(Y,Z)\Big\{\tau X + \mu_\epsilon\, JX - \tfrac{1}{2\,\mu_\epsilon}\, X(\lambda)T\Big\} &+ \tfrac{1}{\mu_\epsilon}\, A(X,Z)\Big\{\tau Y + \mu_\epsilon\, JY - \tfrac{1}{2\,\mu_\epsilon}\, Y(\lambda)T\Big\},\end{aligned} \tag{194}$$

$$\begin{aligned}R(\nabla^{f,\epsilon})(X,Y)T &= \Omega(X,Y)\, \nabla^H \lambda \\ &+ (\nabla_X \tau)Y - (\nabla_Y \tau)X \\ &+ \tfrac{1}{2}\{X(\lambda)\, JY - Y(\lambda)\, JX\} - \tfrac{1}{2\,\mu_\epsilon}\{X(\lambda)\, \tau Y - Y(\lambda)\, \tau X\},\end{aligned} \tag{195}$$

$$\begin{aligned} R\big(\nabla^{f,\epsilon}\big)(X,T)Y &= R^{\nabla}(X,T)Y \\ &+\tfrac{1}{\mu_\epsilon}\,(\nabla_T A)(X,Y)\,T + \tfrac{1}{2\,\mu_\epsilon}\,(\nabla_X d\lambda)Y \\ &+X(\lambda)\,JY + \tfrac{1}{2}\,Y(\lambda)\,JX \\ &-\mu_\epsilon\left\{g_\theta(X,Y) + \tfrac{1}{\mu_\epsilon}\,A(X,JY)\right\}T \\ &+\tfrac{1}{2\,\mu_\epsilon}\left\{Y(\lambda)\,\tau X - \tfrac{1}{2\,\mu_\epsilon}\,X(\lambda)\,Y(\lambda)\,T\right\} \\ &+\tfrac{1}{2}\,\Omega(X,Y)\left\{\nabla^H\lambda - \tfrac{1}{\mu_\epsilon}\,T(\lambda)\,T\right\} \\ &-\tfrac{1}{2\,\mu_\epsilon}\,A(X,Y)\left\{\nabla^H\lambda + \tfrac{1}{\mu_\epsilon}\,T(\lambda)\,T\right\} \\ &-\left\{\Omega(\tau X,\,Y) - \tfrac{1}{\mu_\epsilon}\,A(\tau X,\,Y)\right\}T. \end{aligned} \tag{196}$$

$$\begin{aligned} &R\big(\nabla^{f,\epsilon}\big)(X,T)T \\ &= -\tfrac{1}{2}\,\nabla_X\nabla^H\lambda - \tau^2 X + \mu_\epsilon^2 X + \tfrac{1}{2\,\mu_\epsilon}\,T(\lambda)\,\tau X - \big(\nabla_T\tau\big)X \\ &+\tfrac{1}{4\,\mu_\epsilon}\,X(\lambda)\,\nabla^H\lambda - \tfrac{1}{2}\,T(\lambda)\,JX - 2\,\mu_\epsilon\,J\tau X \\ &-\tfrac{1}{2}\left\{\Omega\big(X,\,\nabla^H\lambda\big) - \tfrac{1}{\mu_\epsilon}\,A\big(X,\,\nabla^H\lambda\big)\right\}T \\ &-\tfrac{1}{2\mu_\epsilon}\,(\tau X)(\lambda)\,T - \tfrac{1}{2}(JX)(\lambda)\,T \end{aligned} \tag{197}$$

for any $X, Y, Z \in H(M)$.

Proof. We are left with the proofs of (195)–(197). For all $X, Y \in H(M)$, one has

$$R\big(\nabla^{f,\epsilon}\big)(X,Y)T = \big[\nabla^{f,\epsilon}_X\,,\,\nabla^{f,\epsilon}_Y\big]T - \nabla^{f,\epsilon}_{[X,Y]}T$$

by (191)

$$\begin{aligned} &= \nabla^{f,\epsilon}_X\left\{\tau Y + \mu_\epsilon\,JY + \frac{1}{2\,\mu_\epsilon}\,Y(\lambda)\,T\right\} \\ &-\nabla^{f,\epsilon}_Y\left\{\tau X + \mu_\epsilon\,JX + \frac{1}{2\,\mu_\epsilon}\,X(\lambda)\,T\right\} \\ &-\nabla^{f,\epsilon}_{\Pi_H[X,Y]}T - \theta([X,Y])\,\nabla^{f,\epsilon}_T T \end{aligned}$$

by $\tau\,H(M) \subset H(M)$, $J\,H(M) \subset H(M)$, (190)–(191) and (193),

$$\begin{aligned} &= \nabla_X\tau Y + \left\{\Omega(X,\tau Y) - \frac{1}{\mu_\epsilon}\,A(X,\tau Y)\right\}T \\ &+X\big(\mu_\epsilon\big)\,JY + \mu_\epsilon\,\nabla^{f,\epsilon}_X JY - \frac{1}{2\,\mu_\epsilon^2}\,X\big(\mu_\epsilon\big)\,Y(\lambda)\,T \\ &+\frac{1}{2\,\mu_\epsilon}\left\{X\big(Y(\lambda)\big)\,T + Y(\lambda)\,\nabla^{f,\epsilon}_X T\right\} \\ &-\nabla_Y\tau X - \left\{\Omega(Y,\tau X) - \frac{1}{\mu_\epsilon}\,A(Y,\tau X)\right\}T \\ &-Y\big(\mu_\epsilon\big)\,JX - \mu_\epsilon\,\nabla^{f,\epsilon}_Y JX + \frac{1}{2\,\mu_\epsilon^2}\,Y\big(\mu_\epsilon\big)\,X(\lambda)\,T \\ &-\frac{1}{2\,\mu_\epsilon}\left\{Y\big(X(\lambda)\big)\,T + X(\lambda)\,\nabla^{f,\epsilon}_Y T\right\} \end{aligned}$$

$$-\tau\,\Pi_H[X,Y] - \mu_\epsilon\, J\,\Pi_H[X,Y] - \frac{1}{2\,\mu_\epsilon}\,(\Pi_H[X,Y])(\lambda)\,T$$

$$-2\,\Omega(X,Y)\{-\frac{1}{2}\,\nabla^H\lambda + \frac{1}{2\,\mu_\epsilon}\,T(\lambda)\,T\}.$$

As $\tau(T) = 0$, one has

$$\tau\,\Pi_H[X,Y] = \tau[X,Y]$$

$$= \tau\{\nabla_X Y - \nabla_Y X + 2\,\Omega(X,Y)\,T\} = \tau(\nabla_X Y) - \tau(\nabla_Y X),$$

and one may recognize the covariant derivatives of τ (with respect to ∇). The similar treatment of the term $J\,\Pi_H[X,Y]$ leads to covariant derivatives of J and $\nabla J = 0$, with the corresponding simplifications. Terms such as

$$\Omega(X,\tau Y) - \Omega(Y,\,\tau X)$$

and

$$-A(X,\tau Y) + A(Y,\,\tau X)$$

vanish because τ is symmetric [i.e., $g_\theta(X,\tau Y) = g_\theta(\tau X, Y)$] while J is skew symmetric, i.e., $g_\theta(X,JY) = -g_\theta(JX,Y)$, and because of the purity axiom $\tau\circ J + J\circ\tau = 0$. For instance,

$$\Omega(X,\tau Y) - \Omega(Y,\,\tau X) = g_\theta(X,\,J\tau Y) - g_\theta(Y,\,J\tau X)$$

$$= g_\theta(X,\,J\tau Y + \tau J Y) = 0.$$

In the end, one is left with (195). Q.E.D.

A bit of extra care should be put into the proof of (196). One has

$$R(\nabla^{f,\epsilon})(X,T)Y = \nabla^{f,\epsilon}_X\nabla^{f,\epsilon}_T Y - \nabla^{f,\epsilon}_T\nabla^{f,\epsilon}_X Y - \nabla^{f,\epsilon}_{[X,T]} Y$$

by (192) and (190), applied twice as $X,\ Y,\ [X,T] \in H(M)$,

$$= \nabla^{f,\epsilon}_X\Big\{\nabla_T Y + \mu_\epsilon\,JY + \frac{1}{2\,\mu_\epsilon}\,Y(\lambda)\,T\Big\}$$

$$-\nabla^{f,\epsilon}_T\Big\{\nabla_X Y + \Big[\Omega(X,Y) - \frac{1}{\mu_\epsilon}\,A(X,Y)\Big]\,T\Big\}$$

$$-\nabla_{[X,Y]}Y - \Big\{\Omega([X,T],\,Y) - \frac{1}{\mu_\epsilon}\,A([X,T],\,Y)\Big\}\,T$$

again by (190) and (192) and by taking covariant derivatives, together with $X(\mu_\epsilon) = X(\lambda)$,

$$= \nabla_X\nabla_T Y + \Big\{\Omega(X,\,\nabla_T Y) - \frac{1}{\mu_\epsilon}\,A(X,\,\nabla_T Y)\Big\}\,T$$

$$-\nabla_T\nabla_X Y - \mu_\epsilon\,J\nabla_X Y - \frac{1}{2\mu_\epsilon}\,(\nabla_X Y)(\lambda)\,T$$

$$+X(\lambda)\,JY + \mu_\epsilon\,\nabla^{f,\epsilon}_X JY - \frac{1}{2\,\mu_\epsilon^2}\,X(\lambda)\,Y(\lambda)\,T$$

$$+\frac{1}{2\,\mu_\epsilon}\Big\{X(Y(\lambda))\,T + Y(\lambda)\,\nabla^{f,\epsilon}_X T\Big\}$$

$$-\Big\{T(\Omega(X,Y)) - \frac{1}{\mu_\epsilon}\,T(A(X,Y)) + \frac{1}{\mu_\epsilon^2}\,T(\lambda)\,A(X,Y)\Big\}\,T$$

$$-\Big\{\Omega(X,Y) - \frac{1}{\mu_\epsilon}\,A(X,Y)\Big\}\,\nabla^{f,\epsilon}_T T - \nabla_{[X,T]}Y$$

as $[X,T] = \tau X - \nabla_T X$ by the very definition of pseudohermitian torsion together with $\nabla T = 0$

$$-\Big\{\Omega(\tau X,\, Y) - \Omega(\nabla_T X,\, Y) - \frac{1}{\mu_\epsilon}\, A(\tau X,\, Y) + \frac{1}{\mu_\epsilon}\, A(\nabla_T X,\, Y)\Big\}\, T$$

by recognizing the curvature of ∇ and the covariant derivatives of Ω and A

$$= R^\nabla(X,T)Y - (\nabla_T\Omega)(X,Y)\, T + \frac{1}{\mu_\epsilon}\, (\nabla_T A)(X,Y)\, T$$

$$-\mu_\epsilon\, J\nabla_X Y - \frac{1}{2\mu_\epsilon}(\nabla_X Y)(\lambda)\, T + X(\lambda)\, JY$$

by (190)

$$+\mu_\epsilon\, \Big\{\nabla_X JY + \Big[\Omega(X, JY) - \frac{1}{\mu_\epsilon}\, A(X, JY)\Big]\, T\Big\}$$

$$+\frac{1}{2\,\mu_\epsilon}\Big\{X(Y(\lambda)) - \frac{1}{\mu_\epsilon}\, X(\lambda)\, Y(\lambda)\Big\}\, T$$

by (191)

$$+\frac{1}{2\,\mu_\epsilon}\, Y(\lambda)\, \Big\{\tau X + \mu_\epsilon\, JX + \frac{1}{2\mu_\epsilon}\, X(\lambda)\, T\Big\} - \frac{1}{\mu_\epsilon^2}\, T(\lambda)\, A(X,Y)\, T$$

by (193)

$$+\Big\{\Omega(X,Y) - \frac{1}{\mu_\epsilon}\, A(X,Y)\Big\}\, \Big\{\frac{1}{2}\, \nabla^H\lambda - \frac{1}{2\,\mu_\epsilon}\, T(\lambda)\, T\Big\}$$

$$-\Big\{\Omega(\tau X,\, Y) - \frac{1}{\mu_\epsilon}\, A(\tau X,\, Y)\Big\}\, T$$

by recognizing the covariant derivatives of J and $d\lambda$, using $\nabla\Omega = 0$ and $\nabla J = 0$, and observing simplification of terms

$$= R^\nabla(X,T)Y + \frac{1}{\mu_\epsilon}\, (\nabla_T A)(X,Y)\, T + \frac{1}{2\,\mu_\epsilon}\, (\nabla_X d\lambda)Y$$

$$+X(\lambda)\, JY + \frac{1}{2}\, Y(\lambda)\, JX$$

$$-\mu_\epsilon\, \Big\{g_\theta(X,Y) + \frac{1}{\mu_\epsilon}\, A(X,JY)\Big\}\, T$$

$$+\frac{1}{2\,\mu_\epsilon}\, \Big\{Y(\lambda)\, \tau X - \frac{1}{2\,\mu_\epsilon}\, X(\lambda)\, Y(\lambda)\, T\Big\}$$

$$+\frac{1}{2}\, \Omega(X,Y)\, \Big\{\nabla^H\lambda - \frac{1}{\mu_\epsilon}\, T(\lambda)\, T\Big\}$$

$$-\frac{1}{2\,\mu_\epsilon}\, A(X,Y)\Big\{\nabla^H\lambda + \frac{1}{\mu_\epsilon}\, T(\lambda)\, T\Big\}$$

$$-\Big\{\Omega(\tau X,\, Y) - \frac{1}{\mu_\epsilon}\, A(\tau X,\, Y)\Big\}\, T,$$

which is (196). Q.E.D.

The proof of (197) is a similar, yet rather involved, calculation relying on (190)–(193). We give a few details for didactic purposes, as follows:

$$R(\nabla^{f,\epsilon})(X,T)T = \nabla^{f,\epsilon}_X \nabla^{f,\epsilon}_T T - \nabla^{f,\epsilon}_T \nabla^{f,\epsilon}_X T - \nabla^{f,\epsilon}_{[X,T]} T$$

by (193) and (191)

$$= \nabla^{f,\epsilon}_X \left\{ -\frac{1}{2}\nabla^H \lambda + \frac{1}{2\,\mu_\epsilon} T(\lambda)\, T \right\}$$

$$-\nabla^{f,\epsilon}_T \left\{ \tau X + \mu_\epsilon\, JX + \frac{1}{2\,\mu_\epsilon} X(\lambda)\, T \right\}$$

$$-\tau[X,T] - \mu_\epsilon\, J[X,T] - \frac{1}{2\,\mu_\epsilon}[X,T](\lambda)\, T$$

by (190) and (191) and computing covariant derivatives of products

$$= -\frac{1}{2}\left\{ \nabla_X \nabla^H \lambda + \left[\Omega(X,\, \nabla^H \lambda) - \frac{1}{\mu_\epsilon} A(X,\, \nabla^H \lambda) \right] T \right\}$$

$$-\frac{1}{2\,\mu_\epsilon^2} X(\mu_\epsilon)\, T(\lambda)\, T + \frac{1}{2\mu_\epsilon}\left[X(T(\lambda))\, T + T(\lambda)\, \nabla^{f,\epsilon}_X T \right]$$

by (192) and taking covariant derivatives

$$-\nabla_T \tau X - \mu_\epsilon\, J\tau X - \frac{1}{2\,\mu_\epsilon}(\tau X)(\lambda)\, T$$

$$-T(\mu_\epsilon)\, JX - \mu_\epsilon \nabla^{f,\epsilon}_T JX$$

$$+\frac{1}{2\,\mu_\epsilon^2} T(\mu_\epsilon)\, X(\lambda)\, T - \frac{1}{2\mu_\epsilon}\left\{ T(X(\lambda))\, T + X(\lambda)\, \nabla^{f,\epsilon}_T T \right\}$$

by the identity $[X,T] = \tau X - \nabla_T X$

$$-\tau\,\{\tau X - \nabla_T X\} - \mu_\epsilon\, J\,\{\tau X - \nabla_T X\} - \frac{1}{2\,\mu_\epsilon}[X,T](\lambda)\, T$$

by (192) and (193), and $\nabla J = 0$,

$$= -\frac{1}{2}\nabla_X \nabla^H \lambda - \tau^2 X + \mu_\epsilon^2 X + \frac{1}{2\,\mu_\epsilon} T(\lambda)\, \tau X - (\nabla_T \tau) X$$

$$+\frac{1}{4\,\mu_\epsilon} X(\lambda)\, \nabla^H \lambda - \frac{1}{2} T(\lambda)\, JX - 2\,\mu_\epsilon\, J\tau X$$

$$-\frac{1}{2}\left\{ \Omega(X,\, \nabla^H \lambda) - \frac{1}{\mu_\epsilon} A(X,\, \nabla^H \lambda) \right\} T$$

$$-\frac{1}{2\mu_\epsilon}(\tau X)(\lambda)\, T - \frac{1}{2}(JX)(\lambda)\, T,$$

which is (197). □

10. Gauss Equation for $f : (M,\, g_\epsilon(f)) \to (A,\, g^A_\epsilon)$

In this section,we start from the Gauss equation for the isometric immersion $f : (M,\, g_\epsilon(f)) \to (A,\, g^A_\epsilon)$, which relates the curvature $R(D^\epsilon)$ of the ambient space $(A,\, g^A_\epsilon)$ to the curvature $R(\nabla^{f,\epsilon})$ of the submanifold $(M,\, g_\epsilon(f))$, and the second fundamental form $B_\epsilon(f)$,

$$g^A_\epsilon\big(R(D^\epsilon)(X,Y)f_* Z,\, f_* W\big) = g_\epsilon(f)\big(R(\nabla^{f,\epsilon})(X,Y)Z,\, W\big) + \tag{198}$$
$$-g^A_\epsilon\big(B_\epsilon(f)(X,W),\, B_\epsilon(f)(Y,Z)\big) + g^A_\epsilon\big(B_\epsilon(f)(Y,W),\, B_\epsilon(f)(X,Z)\big)$$

for any $X, Y, Z, W \in \mathfrak{X}(M)$. According to our philosophy, through this work, we seek to relate the pseudohermitian geometry of the ambient space $(A,\, \Theta)$ to that of the submanifold $(M,\, \theta)$ and the pseudohermitian second fundamental form $B(f)$. The Gauss Equation (198) may be effectively used for the purpose because $R(D^\epsilon)$ and $R(D)$ are related by Lemma 5.

The identities in Lemma 5 are stated in terms of $R(\nabla^{\epsilon})$ and $R(\nabla)$ for an arbitrary strictly pseudoconvex CR manifold M endowed with the positively oriented contact form $\theta \in \mathcal{P}_{+}(M)$. These are easily transposed to (A, Θ). For instance, (28) prompts:

$$
\begin{gathered}
R(D^{\epsilon})(\mathbf{X}, \mathbf{Y})\mathbf{Z} = R^{D}(\mathbf{X}, \mathbf{Y})\mathbf{Z} \\
-\epsilon^{2}\{(D_{\mathbf{X}}A_{\Theta})(\mathbf{Y},\mathbf{Z}) - (D_{\mathbf{Y}}A_{\Theta})(\mathbf{X},\mathbf{Z})\}\, T_{A} \\
+\Omega_{A}(\mathbf{Y},\mathbf{Z})\,\tau_{A}\mathbf{X} - \Omega_{A}(\mathbf{X},\mathbf{Z})\,\tau_{A}\mathbf{Y} \\
-A_{\Theta}(\mathbf{Y},\mathbf{Z})\,J_{A}\mathbf{X} + A_{\Theta}(\mathbf{X},\mathbf{Z})\,J_{A}\mathbf{Y} \\
+\frac{1}{\epsilon^{2}}\{\Omega_{A}(\mathbf{Y},\mathbf{Z})\,J_{A}\mathbf{X} - \Omega_{A}(\mathbf{X},\mathbf{Z})\,J_{A}\mathbf{Y} - 2\,\Omega_{A}(\mathbf{X},\mathbf{Y})\,J_{A}\mathbf{Z}\} \\
+\epsilon^{2}\{A_{\Theta}(\mathbf{X},\mathbf{Z})\,\tau_{A}\mathbf{Y} - A_{\Theta}(\mathbf{Y},\mathbf{Z})\,\tau_{A}\mathbf{X}\}
\end{gathered}
\tag{199}
$$

for any $\mathbf{X}, \mathbf{Y}, \mathbf{Z} \in H(A)$. The fact that $R(\nabla^{f,\epsilon})$ is related to $R(\nabla)$ formed the topic of Section 9, while the relation between $B_{\epsilon}(f)$ and $B(f)$ was provided in Section 8.

Let $X, Y, Z, W \in H(M)$. Then, by (199),

$$
\begin{gathered}
(g_{\epsilon}^{A})^{f}(R((D^{\epsilon})^{f})(X,Y)\,f_{*}Z,\ f_{*}W) \\
= (g_{\epsilon}^{A})^{f}(R^{D^{f}}(X,Y)\,f_{*}Z,\ f_{*}W) \\
+\Omega_{A}^{f}(f_{*}Y,\ f_{*}Z)\,(g_{\epsilon}^{A})^{f}(\tau_{A}^{f}f_{*}X,\ f_{*}W) - \Omega_{A}^{f}(f_{*}X,\ f_{*}Z)\,(g_{\epsilon}^{A})^{f}(\tau_{A}^{f}f_{*}Y,\ f_{*}W) \\
+A_{\Theta}^{f}(f_{*}X,\ f_{*}Z)\,(g_{\epsilon}^{A})^{f}(J_{A}^{f}f_{*}Y,\ f_{*}W) - A_{\Theta}^{f}(f_{*}Y,\ f_{*}Z)\,(g_{\epsilon}^{A})^{f}(J_{A}^{f}f_{*}X,\ f_{*}W) \\
+\frac{1}{\epsilon^{2}}\Big\{\Omega_{A}^{f}(f_{*}Y,\ f_{*}Z)\,(g_{\epsilon}^{A})^{f}(J_{A}^{f}f_{*}X,\ f_{*}W) - \Omega_{A}^{f}(f_{*}X,\ f_{*}Z)\,(g_{\epsilon}^{A})^{f}(J_{A}^{f}f_{*}Y,\ f_{*}W) \\
-2\Omega_{A}^{f}(f_{*}X,\ f_{*}Y)\,(g_{\epsilon}^{A})^{f}(J_{A}^{f}f_{*}Z,\ f_{*}W)\Big\} \\
+\epsilon^{2}\Big\{A_{\Theta}^{f}(f_{*}X,\ f_{*}Z)\,(g_{\epsilon}^{A})^{f}(\tau_{A}^{f}f_{*}Y,\ f_{*}W) - A_{\Theta}^{f}(f_{*}Y,\ f_{*}Z)\,(g_{\epsilon}^{A})^{f}(\tau_{A}^{f}f_{*}X,\ f_{*}W)\Big\}.
\end{gathered}
\tag{200}
$$

Note that

$$\Omega_{A}^{f}(f_{*}X,\ f_{*}Y) = g_{\Theta}^{f}(f_{*}X,\ J_{A}^{f}f_{*}Y)$$

as $J_{A}^{f} \circ f_{*} = f_{*} \circ J$

$$= g_{\Theta}^{f}(f_{*}X,\ f_{*}JY)$$

as $f_{*}X,\ f_{*}JY \in C^{\infty}(f^{-1}H(M))$

$$= G_{\Theta}(f_{*}X,\ f_{*}JY)$$

as $f^{*}G_{\Theta} = G_{\theta}$

$$= G_{\theta}(X,Y) = \Omega(X,Y),$$

i.e.,

$$\Omega_{A}^{f}(f_{*}X,\ f_{*}Y) = \Omega(X,Y) \tag{201}$$

for any $X, Y \in H(M)$. In particular,

$$(g_{\epsilon}^{A})^{f}(J_{A}^{f}f_{*}X,\ f_{*}Y) = -\Omega(X,Y). \tag{202}$$

Substitution from (201) and (202) into (200) leads to

$$
\begin{aligned}
&(g^A_\epsilon)^f\big(R\big((D^\epsilon)^f\big)(X,Y)f_*Z,\, f_*W\big) \\
&= g^f_\Theta\big(R^{D^f}(X,Y)f_*Z,\, f_*W\big) \\
&+\Omega(Y,Z)\, g^f_\Theta\big(\tau^f_A f_*X,\, f_*W\big) - \Omega(X,Z)\, g^f_\Theta\big(\tau^f_A f_*Y,\, f_*W\big) \\
&-g^f_\Theta\big(\tau^f_A f_*X,\, f_*Z\big)\,\Omega(Y,W) + g^f_\Theta\big(\tau^f_A f_*Y,\, f_*Z\big)\,\Omega(X,W) \\
&+\epsilon^2\Big\{g^f_\Theta\big(\tau^f_A f_*X,\, f_*Z\big)\, g^f_\Theta\big(\tau^f_A f_*Y,\, f_*W\big) \\
&-g^f_\Theta\big(\tau^f_A f_*Y,\, f_*Z\big)\, g^f_\Theta\big(\tau^f_A f_*X,\, f_*W\big)\Big\} \\
&+\tfrac{1}{\epsilon^2}\{\Omega(X,Z)\,\Omega(Y,W) - \Omega(X,W)\,\Omega(Y,Z) + 2\,\Omega(X,Y)\,\Omega(Z,W)\}
\end{aligned}
\tag{203}
$$

for any $X,Y,Z,W \in H(M)$. Recalling (194) in Proposition 8, one has

$$
\begin{aligned}
&R(\nabla^{f,\epsilon})(X,Y)Z = R^\nabla(X,Y)Z \\
&2\,\Omega(X,Y)\Big\{\mu_\epsilon\, JZ + \tfrac{1}{2\,\mu_\epsilon}\, Z(\lambda)\, T\Big\} \\
&+\tfrac{1}{\mu_\epsilon}\{(\nabla_Y A)(X,Z) - (\nabla_X A)(Y,Z)\}\, T + \Omega(Y,Z)\Big\{\tau X + \mu_\epsilon\, JX + \tfrac{1}{2\,\mu_\epsilon}\, X(\lambda)\, T\Big\} \\
&-\Omega(X,Z)\Big\{\tau Y + \mu_\epsilon\, JY + \tfrac{1}{2\,\mu_\epsilon}\, Y(\lambda)\, T\Big\} - \tfrac{1}{\mu_\epsilon}\, A(Y,Z)\Big\{\tau X + \mu_\epsilon\, JX - \tfrac{1}{2\,\mu_\epsilon}\, X(\lambda) T\Big\} \\
&+\tfrac{1}{\mu_\epsilon}\, A(X,Z)\Big\{\tau Y + \mu_\epsilon\, JY - \tfrac{1}{2\,\mu_\epsilon}\, Y(\lambda) T\Big\}.
\end{aligned}
\tag{204}
$$

Let us take the inner product $g_\epsilon(f)$ of (204) with $W \in H(M)$. We obtain, because of $g_\epsilon(f) = G_\theta$ on $H(M) \otimes H(M)$ and $H(M) \perp T$ with respect to $g_\epsilon(f)$,

$$
\begin{aligned}
&g_\epsilon(f)\big(R(\nabla^{f,\epsilon})(X,Y)Z,\, W\big) = g_\epsilon(f)\big(R^\nabla(X,Y)Z,\, W\big) \\
&-2\,\mu_\epsilon\,\Omega(X,Y)\, G_\theta(JZ,W) \\
&+\Omega(Y,Z)\{G_\theta(\tau X,W) + \mu_\epsilon\, G_\theta(JX,W)\} \\
&-\Omega(X,Z)\{G_\theta(\tau Y,W) + \mu_\epsilon\, G_\theta(JX,W)\} \\
&-\tfrac{1}{\mu_\epsilon}\, A(Y,Z)\{G_\theta(\tau X,W) + \mu_\epsilon G_\theta(JX,W)\} \\
&+\tfrac{1}{\mu_\epsilon}\, A(X,Z)\{G_\theta(\tau Y,W) + \mu_\epsilon G_\theta(JY,W)\}.
\end{aligned}
\tag{205}
$$

At this point, we need to recall (168), i.e.,

$$
\begin{aligned}
&B_\epsilon(f)(X,W) = D_X f_*W - f_*\nabla_X W \\
&-\tfrac{1}{\mu_\epsilon}\,(f^*g_\Theta)\big(\nabla^\top_X W,\, T\big)\, f_*T \\
&+\Big\{\Omega(X,W) - \epsilon^2\, g_\Theta\big(f_*X,\, \tau^f_A\, f_*W\big)\Big\}\,\Big\{T^f_A - \tfrac{1}{\epsilon^2\,\mu_\epsilon}\, f_*T\Big\}.
\end{aligned}
\tag{206}
$$

Let us replace the pair (X,W) with (Y,Z) in (206). We obtain

$$
\begin{aligned}
&B_\epsilon(f)(Y,Z) = D_Y f_*Z - f_*\nabla_Y Z \\
&-\tfrac{1}{\mu_\epsilon}\,(f^*g_\Theta)\big(\nabla^\top_Y Z,\, T\big)\, f_*T \\
&+\Big\{\Omega(Y,Z) - \epsilon^2\, g_\Theta\big(f_*Y,\, \tau^f_A\, f_*Z\big)\Big\}\,\Big\{T^f_A - \tfrac{1}{\epsilon^2\,\mu_\epsilon}\, f_*T\Big\}.
\end{aligned}
\tag{207}
$$

Next, let us use (206) and (207) to compute the inner product of the normal vectors $B_\epsilon(f)(X,W)$ and $B_\epsilon(f)(Y,Z)$ with respect to g_ϵ^A. Specifically, we may conduct the following calculation

$$(g_\epsilon^A)^f\left(B_\epsilon(f)(X,W),\, B_\epsilon(f)(Y,Z)\right) = \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV} \tag{208}$$

where the terms I, II, III, and IV are given by

$$\begin{aligned}
\mathrm{I} = {} & (g_\epsilon^A)^f\left(D_X f_* W,\, D_Y f_* Z\right) - (g_\epsilon^A)^f\left(D_X f_* W,\, f_* \nabla_Y Z\right) \\
& - \frac{1}{\mu_\epsilon}\,(f^* g_\Theta)\left(\nabla_Y^\top Z,\, T\right)\,(g_\epsilon^A)^f\left(D_X f_* W,\, f_* T\right) \\
& + \left\{\Omega(Y,Z) - \epsilon^2\, g_\Theta\left(f_* Y,\, \tau_A^f\, f_* Z\right)\right\}\,(g_\epsilon^A)^f\left(D_X f_* W,\, T_A^f - \frac{1}{\epsilon^2\,\mu_\epsilon}\, f_* T\right), \\
\mathrm{II} = {} & -(g_\epsilon^A)^f\left(f_* \nabla_X W,\, D_Y f_* Z\right) + (g_\epsilon^A)^f\left(f_* \nabla_X W,\, f_* \nabla_Y Z\right) \\
& + \frac{1}{\mu_\epsilon}\,(f^* g_\Theta)\left(\nabla_Y^\top Z,\, T\right)\,(g_\epsilon^A)^f\left(f_* \nabla_X W,\, f_* T\right) \\
& - \left\{\Omega(Y,Z) - \epsilon^2\, g_\Theta\left(f_* Y,\, \tau_A^f\, f_* Z\right)\right\}\,(g_\epsilon^A)^f\left(f_* \nabla_X W,\, T_A^f - \frac{1}{\epsilon^2\,\mu_\epsilon}\, f_* T\right), \\
\mathrm{III} = {} & -\frac{1}{\mu_\epsilon}\,(f^* g_\Theta)\left(\nabla_X^\top W,\, T\right)\,(g_\epsilon^A)^f\left(f_* T,\, D_Y f_* Z\right) \\
& + \frac{1}{\mu_\epsilon}\,(f^* g_\Theta)\left(\nabla_X^\top W,\, T\right)\,(g_\epsilon^A)^f\left(f_* T,\, f_* \nabla_Y Z\right) \\
& + \frac{1}{\mu_\epsilon^2}\,(f^* g_\Theta)\left(\nabla_X^\top W,\, T\right)\,(f^* g_\Theta)\left(\nabla_Y^\top Z,\, T\right)\,(g_\epsilon^A)^f\left(f_* T,\, f_* T\right) \\
& - \frac{1}{\mu_\epsilon}\,(f^* g_\Theta)\left(\nabla_X^\top W,\, T\right)\left\{\Omega(Y,Z) - \epsilon^2\, g_\Theta\left(f_* Y,\, \tau_A^f\, f_* Z\right)\right\} \\
& \times (g_\epsilon^A)^f\left(f_* T,\, T_A^f - \frac{1}{\epsilon^2\,\mu_\epsilon}\, f_* T\right),
\end{aligned}$$

$$\begin{aligned}
\mathrm{IV} = {} & \left\{\Omega(X,W) - \epsilon^2\, g_\Theta\left(f_* X,\, \tau_A^f\, f_* W\right)\right\}\,(g_\epsilon^A)^f\left(T_A^f - \frac{1}{\epsilon^2\,\mu_\epsilon}\, f_* T,\, D_Y f_* Z\right) \\
& - \left\{\Omega(X,W) - \epsilon^2\, g_\Theta\left(f_* X,\, \tau_A^f\, f_* W\right)\right\}\,(g_\epsilon^A)^f\left(T_A^f - \frac{1}{\epsilon^2\,\mu_\epsilon}\, f_* T,\, f_* \nabla_Y Z\right) \\
& - \frac{1}{\mu_\epsilon}\left\{\Omega(X,W) - \epsilon^2\, g_\Theta\left(f_* X,\, \tau_A^f\, f_* W\right)\right\}\,(f^* g_\Theta)\left(\nabla_Y^\top Z,\, T\right) \\
& \times (g_\epsilon^A)^f\left(T_A^f - \frac{1}{\epsilon^2\,\mu_\epsilon}\, f_* T,\, f_* T\right) \\
& + \left\{\Omega(X,W) - \epsilon^2\, g_\Theta\left(f_* X,\, \tau_A^f\, f_* W\right)\right\}\left\{\Omega(Y,Z) - \epsilon^2\, g_\Theta\left(f_* Y,\, \tau_A^f\, f_* Z\right)\right\} \\
& \times (g_\epsilon^A)^f\left(T_A^f - \frac{1}{\epsilon^2\,\mu_\epsilon}\, f_* T,\, T_A^f - \frac{1}{\epsilon^2\,\mu_\epsilon}\, f_* T\right).
\end{aligned}$$

The calculation of the terms II, III, and IV requires the following:

Lemma 19.

(i) $\|f_* T\|_{g_\epsilon^A} = \sqrt{\mu_\epsilon}$.

(ii) $(g_\epsilon^A)^f\left(f_*T,\ T_A^f - \frac{1}{\epsilon^2\mu_\epsilon}\,f_*T\right) = 0.$

(iii) $\left\|T_A^f - \frac{1}{\epsilon^2\mu_\epsilon}\,f_*T\right\|_{g_\epsilon^A} = \sqrt{\frac{\lambda-1}{1+\epsilon^2(\lambda-1)}}.$

Here, we have set $\|V\|_{g_\epsilon^A} = \left(g_\epsilon^A\right)^f(V,V)^{1/2}$ for any $V \in C^\infty\left(f^{-1}T(A)\right)$.

Proof of Lemma 19. As g_Θ is a Riemannian metric, the function $\lambda = \left(f^* g_\Theta\right)(T,T) : M \to \mathbb{R}$ is everywhere positive, i.e., $\lambda(x) > 0$ for any $x \in M$. More is true, i.e.,

$$\lambda(x) \geq 1, \quad x \in M, \tag{209}$$

$$\lambda(x_0) = 1 \Longleftrightarrow (d_{x_0}f)T_{x_0} = T_{A,\,f(x_0)}\,. \tag{210}$$

Indeed

$$g_\Theta^f(f_*T,\ T_A^f) = g_\Theta^f(\mathfrak{X}_\Theta + T_A^f\,,\ T_A^f)$$

as $\mathfrak{X}_\Theta \in f^{-1}H(A) \perp T_A^f$ with respect to g_Θ^f

$$= g_\Theta^f(T_A^f\,,\ T_A^f) = g_\Theta(T_A\,,\ T_A)^f = 1,$$

i.e.,

$$g_\Theta^f(f_*T,\ T_A^f) = 1. \tag{211}$$

Hence,

$$0 \leq \|\mathfrak{X}_\Theta\|_{g_\Theta}^2 = g_\Theta^f(\mathfrak{X}_\Theta\,,\ \mathfrak{X}_\Theta) = \left(f^* g_\Theta\right)(T,T) - 2\,g_\Theta^f(f_*T,\ T_A^f) + 1$$

by (211)

$$= \lambda - 1,$$

yielding (209). □

To prove (210) let $x_0 \in M$ be a point such that $\lambda(x_0) = 1$. This is equivalent [by the preceding calculation] to $\mathfrak{X}_{\Theta,\,x_0} = 0$. Q.E.D.

In particular, by (209),

$$\mu_\epsilon = \frac{1}{\epsilon^2} + \lambda - 1 \geq 0$$

so that statement (i) in Lemma 19 is legitimate. As a byproduct of the calculations just done, one has

$$\|\mathfrak{X}_\Theta\|_{g_\Theta}^2 = \lambda - 1 \tag{212}$$

that we ought to keep for further applications.

The proofs of (i)–(iii) in Lemma 19 are straightforward calculations. We give a few details for pedagogical reasons.

(i) $\|f_*T\|_{g_\epsilon^A}^2 = \left\|\mathfrak{X}_\Theta + T_A^f\right\|_{g_\epsilon^A}^2$

as $\mathfrak{X}_\Theta \in f^{-1}H(A) \perp T_A^f$ with respect to g_Θ^f

$= \|\mathfrak{X}_\Theta\|_{g_\epsilon^A}^2 + \left\|T_A^f\right\|_{g_\epsilon^A}^2$

by (212) and the very definition of g_ϵ^A

$= \lambda - 1 + \epsilon^{-2} = \mu_\epsilon$. Q.E.D.

(ii) $(g_\epsilon^A)^f\left(f_*T,\ T_A^f - \frac{1}{\epsilon^2\mu_\epsilon}\,f_*T\right) =$

$$= (g_\epsilon^A)^f\left(\mathfrak{X}_\Theta + T_A^f\,,\, T_A^f - \frac{1}{\epsilon^2\mu_\epsilon}\,(\mathfrak{X}_\Theta + T_A)\right)$$

by $g_\epsilon^A = g_\Theta$ on $H(A)\otimes H(A)$ and $g_\epsilon^A(T_A\,,\,T_A) = \epsilon^{-2}$

$$= -\frac{1}{\epsilon^2\mu_\epsilon}\,g_\Theta^f(\mathfrak{X}_\Theta\,,\,\mathfrak{X}_\Theta) + \frac{1}{\epsilon^2} - \frac{1}{\epsilon^2\mu_\epsilon}\frac{1}{\epsilon^2}$$

by (212)

$$= \frac{1}{\epsilon^2}\left(1 - \frac{1}{\epsilon^2\mu_\epsilon}\right) - \frac{1}{\epsilon^2\mu_\epsilon}\,(\lambda - 1) = \frac{1}{\epsilon^2\mu_\epsilon}\left(\mu_\epsilon - \frac{1}{\epsilon^2} - \lambda + 1\right) = 0. \text{ Q.E.D.}$$

(iii) $$\left\|T_A^f - \frac{1}{\epsilon^2\mu_\epsilon}\,f_*T\right\|^2_{g_\epsilon^A} = \left\|T_A^f - \frac{1}{\epsilon^2\mu_\epsilon}\,(\mathfrak{X}_\Theta + T_A^f)\right\|^2_{g_\epsilon^A}$$

$$= \frac{1}{\epsilon^2}\left(1 - \frac{1}{\epsilon^2\mu_\epsilon}\right) + \frac{1}{\epsilon^4\mu_\epsilon^2}\,\|\mathfrak{X}_\Theta\|^2_{g_\Theta} = \frac{\epsilon^2\mu_\epsilon - 1}{\epsilon^4\mu_\epsilon} = \frac{\lambda - 1}{1 + \epsilon^2(\lambda - 1)}. \text{ Q.E.D.}$$

Let us go back to the calculation of the term I. One has, by the very definition of g_ϵ^A,

$$\begin{aligned} \mathrm{I} = {} & g_\Theta^f(D_X f_*W,\, D_Y f_*Z) - g_\Theta^f(D_X f_*W,\, f_*\nabla_Y Z) \\ & -\tfrac{1}{\mu_\epsilon}\,(f^*g_\Theta)(\nabla_Y^\top Z,\, T)\,g_\Theta^f(D_X f_*W,\, f_*T) \\ & -\tfrac{1}{\epsilon^2\mu_\epsilon}\left\{\Omega(Y,Z) - \epsilon^2\,g_\Theta^f(f_*Y,\, \tau_A^f\,f_*Z)\right\} g_\Theta^f(D_X f_*W,\, f_*T). \end{aligned} \tag{213}$$

Similarly, let us compute the term II. Again, by the relationship between g_ϵ^A and g_Θ, relative to the decomposition $T(A) = H(A) \oplus \mathbb{R}T_A$,

$$\begin{aligned} \mathrm{II} = {} & -g_\Theta^f(f_*\nabla_X W,\, D_Y f_*Z) + (f^*g_\Theta)(\nabla_X W,\, \nabla_Y Z) \\ & +\tfrac{1}{\mu_\epsilon}\,(f^*g_\Theta)(\nabla_Y^\top Z,\, T)\,(f^*g_\Theta)(\nabla_X W,\, T) \\ & \tfrac{1}{\epsilon^2\mu_\epsilon}\left\{\Omega(Y,Z) - \epsilon^2\,g_\Theta^f(f_*Y,\, \tau_A^f\,f_*Z)\right\}(f^*g_\Theta)(\nabla_X W,\, T). \end{aligned} \tag{214}$$

Similar to the above, the term III may be calculated as follows by (i)–(ii) in Lemma 19:

$$\begin{aligned} \mathrm{III} = {} & -\tfrac{1}{\mu_\epsilon}\,(f^*g_\Theta)(\nabla_X^\top W,\, T)\,g_\Theta^f(f_*T,\, D_Y f_*Z) \\ & +\tfrac{1}{\mu_\epsilon}\,(f^*g_\Theta)(\nabla_X^\top W,\, T)\,(f^*g_\Theta)(T,\, \nabla_Y Z) \\ & +\tfrac{1}{\mu_\epsilon}\,(f^*g_\Theta)(\nabla_X^\top W,\, T)\,(f^*g_\Theta)(\nabla_Y^\top Z,\, T). \end{aligned} \tag{215}$$

Finally, the term IV may be calculated as follows (by (ii)–(iii) in Lemma 19)

$$\begin{aligned} \mathrm{IV} = {} & -\tfrac{1}{\epsilon^2\mu_\epsilon}\left\{\Omega(X,W) - \epsilon^2\,g_\Theta^f(f_*X,\, \tau_A^f\,f_*W)\right\} \\ & \times g_\Theta^f(f_*T,\, D_Y f_*Z) \\ & +\tfrac{1}{\epsilon^2\mu_\epsilon}\left\{\Omega(X,W) - \epsilon^2\,g_\Theta^f(f_*X,\, \tau_A^f\,f_*W)\right\}(f^*g_\Theta)(T,\, \nabla_Y Z) \\ & +\left\{\Omega(X,W) - \epsilon^2\,g_\Theta^f(f_*X,\, \tau_A^f\,f_*W)\right\}\left\{\Omega(Y,Z) - \epsilon^2\,g_\Theta^f(f_*Y,\, \tau_A^f\,f_*Z)\right\}\times \\ & \times\,\tfrac{\lambda-1}{1+\epsilon^2(\lambda-1)}\,. \end{aligned} \tag{216}$$

We need:

Lemma 20.

$$\theta(T^f) = 1 - \|T^\perp\|^2_{g_\Theta}\,. \tag{217}$$

Proof. We start from $T_A^f = f_* T^\top + T^\perp$, an orthogonal decomposition with respect to g_Θ^f. Then,

$$
\begin{aligned}
1 = \Theta(T_A)^f = \Theta^f\big(T_A^f\big) &= \Theta^f\big(f_* T^\top\big) + \Theta^f\big(T^\perp\big) \\
&= (f^*\Theta)(T^\top) + g_\Theta^f\big(T_A^f,\, T^\perp\big) = \theta(T^\top) + \left\|T^\perp\right\|^2_{g_\Theta}.
\end{aligned}
$$

□

Let us go back to the Gauss Equation (198),

$$
\begin{aligned}
&(g_\epsilon^A)^f\big(R\big((D^\epsilon)^f\big)(X,Y)f_*Z,\, f_*W\big) \\
&= g_\epsilon(f)\big(R\big(\nabla^{f,\epsilon}\big)(X,Y)Z,\, W\big) \\
&-g_\epsilon^A\big(B_\epsilon(f)(X,W),\, B_\epsilon(f)(Y,Z)\big) + g_\epsilon^A\big(B_\epsilon(f)(Y,W),\, B_\epsilon(f)(X,Z)\big)
\end{aligned}
\tag{218}
$$

written for any $X, Y, Z, W \in H(M)$.

Collecting the calculations above, we shall substitute into (218) in three steps as follows.

(1) The curvature term

$$g_\epsilon^A\big(R\big(D^\epsilon\big)(X,Y)f_*Z,\, f_*W\big)$$

in the left-hand side of (218) is replaced by (203).

(2) The curvature term

$$g_\epsilon(f)\big(R\big(\nabla^{f,\epsilon}\big)(X,Y)Z,\, W\big)$$

in the right-hand side of (218) is replaced by (205).

(3) Finally, the terms

$$(g_\epsilon^A)^f\big(B_\epsilon(f)(X,W),\, B_\epsilon(f)(Y,Z)\big), \quad (g_\epsilon^A)^f\big(B_\epsilon(f)(Y,W),\, B_\epsilon(f)(X,Z)\big),$$

in the right-hand side of (218) are replaced by (208) together with the expressions (213)–(216) of the terms I, II, III, and IV.

The aim of Steps 1 and 2 is then to compute the term

$$(g_\epsilon^A)^f\big(R\big((D^\epsilon)^f\big)(X,Y)f_*Z,\, f_*W\big) - g_\epsilon(f)\big(R\big(\nabla^{f,\epsilon}\big)(X,Y)Z,\, W\big)$$

appearing in (198). Indeed one has

$$
\begin{aligned}
&(g_\epsilon^A)^f\big(R\big((D^\epsilon)^f\big)(X,Y)f_*Z,\, f_*W\big)\\
&-g_\epsilon(f)\big(R\big(\nabla^{f,\epsilon}\big)(X,Y)Z,\, W\big)\\
&= g_\Theta^f\big(R^{D^f}(X,Y)f_*Z,\, f_*W\big)\\
&+\Omega(Y,Z)\, g_\Theta^f\big(\tau_A^f f_*X,\, f_*W\big) - \Omega(X,Z)\, g_\Theta^f\big(\tau_A^f f_*Y,\, f_*W\big)\\
&-g_\Theta^f\big(\tau_A^f f_*X,\, f_*Z\big)\,\Omega(Y,W) + g_\Theta^f\big(\tau_A^f f_*Y,\, f_*Z\big)\,\Omega(X,W)\\
&+\epsilon^2\Big\{g_\Theta^f\big(\tau_A^f f_*X,\, f_*Z\big)\, g_\Theta^f\big(\tau_A^f f_*Y,\, f_*W\big) -\\
&-g_\Theta^f\big(\tau_A^f f_*Y,\, f_*Z\big)\, g_\Theta^f\big(\tau_A^f f_*X,\, f_*W\big)\Big\}\\
&+\tfrac{1}{\epsilon^2}\{\Omega(X,Z)\,\Omega(Y,W) - \Omega(X,W)\,\Omega(Y,Z) + 2\,\Omega(X,Y)\,\Omega(Z,W)\}\\
&-g_\epsilon(f)\big(R^\nabla(X,Y)Z,\, W\big)\\
&+2\,\mu_\epsilon\,\Omega(X,Y)\, G_\theta(JZ,W)\\
&-\Omega(Y,Z)\big\{G_\theta(\tau X,W) + \mu_\epsilon\, G_\theta(JX,W)\big\}\\
&+\Omega(X,Z)\big\{G_\theta(\tau Y,W) + \mu_\epsilon\, G_\theta(JY,W)\big\}\\
&+\tfrac{1}{\mu_\epsilon}\, A(Y,Z)\big\{G_\theta(\tau X,W) + \mu_\epsilon G_\theta(JX,W)\big\}\\
&-\tfrac{1}{\mu_\epsilon}\, A(X,Z)\big\{G_\theta(\tau Y,W) + \mu_\epsilon G_\theta(JY,W)\big\}.
\end{aligned} \tag{219}
$$

Let us substitute from (213)–(216) into (208). We obtain:

$$
\begin{aligned}
&(g_\epsilon^A)^f\big(B_\epsilon(f)(X,W),\, B_\epsilon(f)(Y,Z)\big)\\
&= g_\Theta^f\big(D_X f_*W,\, D_Y f_*Z\big) - g_\Theta^f\big(D_X f_*W,\, f_*\nabla_Y Z\big)\\
&-\frac{1}{\mu_\epsilon}\,(f^*g_\Theta)\big(\nabla_Y^\top Z,\, T\big)\, g_\Theta^f\big(D_X f_*W,\, f_*T\big)\\
&-\frac{1}{\epsilon^2\,\mu_\epsilon}\,\Big\{\Omega(Y,Z) - \epsilon^2\, g_\Theta^f\big(f_*Y,\, \tau_A^f\, f_*Z\big)\Big\}\, g_\Theta^f\big(D_X f_*W,\, f_*T\big)\\
&-g_\Theta^f\big(f_*\nabla_X W,\, D_Y f_*Z\big) + (f^*g_\Theta)\big(\nabla_X W,\, \nabla_Y Z\big)\\
&+\frac{1}{\mu_\epsilon}\,(f^*g_\Theta)\big(\nabla_Y^\top Z,\, T\big)\,(f^*g_\Theta)\big(\nabla_X W,\, T\big)\\
&+\frac{1}{\epsilon^2\,\mu_\epsilon}\,\Big\{\Omega(Y,Z) - \epsilon^2\, g_\Theta^f\big(f_*Y,\, \tau_A^f\, f_*Z\big)\Big\}\,(f^*g_\Theta)\big(\nabla_X W,\, T\big)\\
&-\frac{1}{\mu_\epsilon}\,(f^*g_\Theta)\big(\nabla_X^\top W,\, T\big)\, g_\Theta^f\big(f_*T,\, D_Y f_*Z\big)\\
&+\frac{1}{\mu_\epsilon}\,(f^*g_\Theta)\big(\nabla_X^\top W,\, T\big)\,(f^*g_\Theta)\big(T,\, \nabla_Y Z\big)\\
&+\frac{1}{\mu_\epsilon}\,(f^*g_\Theta)\big(\nabla_X^\top W,\, T\big)\,(f^*g_\Theta)\big(\nabla_Y^\top Z,\, T\big)\\
&-\frac{1}{\epsilon^2\,\mu_\epsilon}\,\Big\{\Omega(X,W) - \epsilon^2\, g_\Theta^f\big(f_*X,\, \tau_A^f\, f_*W\big)\Big\}\\
&\times g_\Theta^f\big(f_*T,\, D_Y f_*Z\big)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{\epsilon^2\,\mu_\epsilon}\left\{\Omega(X,W)-\epsilon^2\,g_\Theta^f(f_*X,\,\tau_A^f\,f_*W)\right\}(f^*g_\Theta)(T,\,\nabla_Y Z)\\
&+\left\{\Omega(X,W)-\epsilon^2\,g_\Theta^f(f_*X,\,\tau_A^f\,f_*W)\right\}\left\{\Omega(Y,Z)-\epsilon^2\,g_\Theta^f(f_*Y,\,\tau_A^f\,f_*Z)\right\}\\
&\times\frac{\lambda-1}{1+\epsilon^2(\lambda-1)}\,.
\end{aligned}
\tag{220}
$$

Let us interchange X and Y in (220). We obtain:

$$
\begin{aligned}
&big(g_\epsilon^A)^f\,(B_\epsilon(f)(Y,W),\,B_\epsilon(f)(X,Z))\\
&=g_\Theta^f(D_Yf_*W,\,D_Xf_*Z)-g_\Theta^f(D_Yf_*W,\,f_*\nabla_X Z)\\
&-\frac{1}{\mu_\epsilon}\,(f^*g_\Theta)(\nabla_X^\top Z,\,T)\,g_\Theta^f(D_Yf_*W,\,f_*T)\\
&-\frac{1}{\epsilon^2\,\mu_\epsilon}\left\{\Omega(X,Z)-\epsilon^2\,g_\Theta^f(f_*X,\,\tau_A^f\,f_*Z)\right\}g_\Theta^f(D_Yf_*W,\,f_*T)\\
&-g_\Theta^f(f_*\nabla_Y W,\,D_Xf_*Z)+(f^*g_\Theta)(\nabla_Y W,\,\nabla_X Z)\\
&+\frac{1}{\mu_\epsilon}\,(f^*g_\Theta)(\nabla_X^\top Z,\,T)\,(f^*g_\Theta)(\nabla_Y W,\,T)\\
&+\frac{1}{\epsilon^2\,\mu_\epsilon}\left\{\Omega(X,Z)-\epsilon^2\,g_\Theta^f(f_*X,\,\tau_A^f\,f_*Z)\right\}(f^*g_\Theta)(\nabla_Y W,\,T)\\
&-\frac{1}{\mu_\epsilon}\,(f^*g_\Theta)(\nabla_Y^\top W,\,T)\,g_\Theta^f(f_*T,\,D_Xf_*Z)\\
&+\frac{1}{\mu_\epsilon}\,(f^*g_\Theta)(\nabla_Y^\top W,\,T)\,(f^*g_\Theta)(T,\,\nabla_X Z)\\
&+\frac{1}{\mu_\epsilon}\,(f^*g_\Theta)(\nabla_Y^\top W,\,T)\,(f^*g_\Theta)(\nabla_X^\top Z,\,T)\\
&-\frac{1}{\epsilon^2\,\mu_\epsilon}\left\{\Omega(Y,W)-\epsilon^2\,g_\Theta^f(f_*Y,\,\tau_A^f\,f_*W)\right\}\\
&\times g_\Theta^f(f_*T,\,D_Xf_*Z)\\
&+\frac{1}{\epsilon^2\,\mu_\epsilon}\left\{\Omega(Y,W)-\epsilon^2\,g_\Theta^f(f_*Y,\,\tau_A^f\,f_*W)\right\}(f^*g_\Theta)(T,\,\nabla_X Z)\\
&+\left\{\Omega(Y,W)-\epsilon^2\,g_\Theta^f(f_*Y,\,\tau_A^f\,f_*W)\right\}\left\{\Omega(X,Z)-\epsilon^2\,g_\Theta^f(f_*X,\,\tau_A^f\,f_*Z)\right\}\\
&\times\frac{\lambda-1}{1+\epsilon^2(\lambda-1)}\,.
\end{aligned}
\tag{221}
$$

Consequently, by (220) and (221),

$$
\begin{aligned}
&-(g_\epsilon^A)^f\,(B_\epsilon(f)(X,W),\,B_\epsilon(f)(Y,Z))\\
&+(g_\epsilon^A)^f\,(B_\epsilon(f)(Y,W),\,B_\epsilon(f)(X,Z))\\
&=-g_\Theta^f(D_Xf_*W,\,D_Yf_*Z)+g_\Theta^f(D_Xf_*W,\,f_*\nabla_Y Z)\\
&+\frac{1}{\mu_\epsilon}\,(f^*g_\Theta)(\nabla_Y^\top Z,\,T)\,g_\Theta^f(D_Xf_*W,\,f_*T)\\
&+\frac{1}{\epsilon^2\,\mu_\epsilon}\left\{\Omega(Y,Z)-\epsilon^2\,g_\Theta^f(f_*Y,\,\tau_A^f\,f_*Z)\right\}g_\Theta^f(D_Xf_*W,\,f_*T)
\end{aligned}
$$

$$
\begin{aligned}
&+g_\Theta^f\big(f_*\nabla_X W,\, D_Y f_* Z\big) - \big(f^* g_\Theta\big)\big(\nabla_X W,\, \nabla_Y Z\big)\\
&-\frac{1}{\mu_\epsilon}\,\big(f^* g_\Theta\big)\big(\nabla_Y^\top Z,\, T\big)\,\big(f^* g_\Theta\big)\big(\nabla_X W,\, T\big)\\
&-\frac{1}{\epsilon^2\,\mu_\epsilon}\,\Big\{\Omega(Y,Z) - \epsilon^2\, g_\Theta^f\big(f_* Y,\, \tau_A^f\, f_* Z\big)\Big\}\,\big(f^* g_\Theta\big)\big(\nabla_X W,\ T\big)\\
&+\frac{1}{\mu_\epsilon}\,\big(f^* g_\Theta\big)\big(\nabla_X^\top W,\, T\big)\, g_\Theta^f\big(f_* T,\, D_Y f_* Z\big)\\
&-\frac{1}{\mu_\epsilon}\,\big(f^* g_\Theta\big)\big(\nabla_X^\top W,\, T\big)\,\big(f^* g_\Theta\big)\big(T,\, \nabla_Y Z\big)\\
&-\frac{1}{\mu_\epsilon}\,\big(f^* g_\Theta\big)\big(\nabla_X^\top W,\, T\big)\,\big(f^* g_\Theta\big)\big(\nabla_Y^\top Z,\, T\big)\\
&+\frac{1}{\epsilon^2\,\mu_\epsilon}\,\Big\{\Omega(X,W) - \epsilon^2\, g_\Theta^f\big(f_* X,\, \tau_A^f\, f_* W\big)\Big\}\\
&\times g_\Theta^f\big(f_* T,\, D_Y f_* Z\big)\\
&-\frac{1}{\epsilon^2\,\mu_\epsilon}\,\Big\{\Omega(X,W) - \epsilon^2\, g_\Theta^f\big(f_* X,\, \tau_A^f\, f_* W\big)\Big\}\,\big(f^* g_\Theta\big)\big(T,\, \nabla_Y Z\big)\\
&-\Big\{\Omega(X,W) - \epsilon^2\, g_\Theta^f\big(f_* X,\, \tau_A^f\, f_* W\big)\Big\}\,\Big\{\Omega(Y,Z) - \epsilon^2\, g_\Theta^f\big(f_* Y,\, \tau_A^f\, f_* Z\big)\Big\}\\
&\times\,\frac{\lambda-1}{1+\epsilon^2(\lambda-1)}\\
&+g_\Theta^f\big(D_Y f_* W,\, D_X f_* Z\big) - g_\Theta^f\big(D_Y f_* W,\, f_*\nabla_X Z\big)\\
&-\frac{1}{\mu_\epsilon}\,\big(f^* g_\Theta\big)\big(\nabla_X^\top Z,\, T\big)\, g_\Theta^f\big(D_Y f_* W,\, f_* T\big)\\
&-\frac{1}{\epsilon^2\,\mu_\epsilon}\,\Big\{\Omega(X,Z) - \epsilon^2\, g_\Theta^f\big(f_* X,\, \tau_A^f\, f_* Z\big)\Big\}\, g_\Theta^f\big(D_Y f_* W,\ f_* T\big)\\
&-g_\Theta^f\big(f_*\nabla_Y W,\, D_X f_* Z\big) + \big(f^* g_\Theta\big)\big(\nabla_Y W,\, \nabla_X Z\big)\\
&+\frac{1}{\mu_\epsilon}\,\big(f^* g_\Theta\big)\big(\nabla_X^\top Z,\, T\big)\,\big(f^* g_\Theta\big)\big(\nabla_Y W,\, T\big)\\
&+\frac{1}{\epsilon^2\,\mu_\epsilon}\,\Big\{\Omega(X,Z) - \epsilon^2\, g_\Theta^f\big(f_* X,\, \tau_A^f\, f_* Z\big)\Big\}\,\big(f^* g_\Theta\big)\big(\nabla_Y W,\ T\big)\\
&-\frac{1}{\mu_\epsilon}\,\big(f^* g_\Theta\big)\big(\nabla_Y^\top W,\, T\big)\, g_\Theta^f\big(f_* T,\, D_X f_* Z\big)\\
&+\frac{1}{\mu_\epsilon}\,\big(f^* g_\Theta\big)\big(\nabla_Y^\top W,\, T\big)\,\big(f^* g_\Theta\big)\big(T,\, \nabla_X Z\big)\\
&+\frac{1}{\mu_\epsilon}\,\big(f^* g_\Theta\big)\big(\nabla_Y^\top W,\, T\big)\,\big(f^* g_\Theta\big)\big(\nabla_X^\top Z,\, T\big)\\
&-\frac{1}{\epsilon^2\,\mu_\epsilon}\,\Big\{\Omega(Y,W) - \epsilon^2\, g_\Theta^f\big(f_* Y,\, \tau_A^f\, f_* W\big)\Big\}\\
&\times g_\Theta^f\big(f_* T,\, D_X f_* Z\big)\\
&+\frac{1}{\epsilon^2\,\mu_\epsilon}\,\Big\{\Omega(Y,W) - \epsilon^2\, g_\Theta^f\big(f_* Y,\, \tau_A^f\, f_* W\big)\Big\}\,\big(f^* g_\Theta\big)\big(T,\, \nabla_X Z\big)\\
&+\Big\{\Omega(Y,W) - \epsilon^2\, g_\Theta^f\big(f_* Y,\, \tau_A^f\, f_* W\big)\Big\}\,\Big\{\Omega(X,Z) - \epsilon^2\, g_\Theta^f\big(f_* X,\, \tau_A^f\, f_* Z\big)\Big\}
\end{aligned}
$$

$$\times \frac{\lambda - 1}{1 + \epsilon^2(\lambda - 1)}. \tag{222}$$

Finally, let us substitute from (219) and (222) into the Gauss Equation (218). We obtain the rather involved equation:

$$\begin{aligned}
& g^f_\Theta(R^D(X,Y)f_*Z,\, f_*W) \\
& +\Omega(Y,Z)\, g^f_\Theta(\tau^f_A f_*X,\, f_*W) - \Omega(X,Z)\, g^f_\Theta(\tau^f_A f_*Y,\, f_*W) \\
& -g^f_\Theta(\tau^f_A f_*X,\, f_*Z)\, \Omega(Y,W) + g^f_\Theta(\tau^f_A f_*Y,\, f_*Z)\, \Omega(X,W) \\
& +\epsilon^2 \Big\{ g^f_\Theta(\tau^f_A f_*X,\, f_*Z)\, g^f_\Theta(\tau^f_A f_*Y,\, f_*W) \\
& -g^f_\Theta(\tau^f_A f_*Y,\, f_*Z)\, g^f_\Theta(\tau^f_A f_*X,\, f_*W) \Big\} \\
& +\frac{1}{\epsilon^2}\{\Omega(X,Z)\,\Omega(Y,W) - \Omega(X,W)\,\Omega(Y,Z) + 2\,\Omega(X,Y)\,\Omega(Z,W)\} \\
& -g_\theta(R^\nabla(X,Y)Z,\, W) \\
& +2\,\mu_\epsilon\,\Omega(X,Y)\, G_\theta(JZ,W) \\
& -\Omega(Y,Z)\{G_\theta(\tau X,W) + \mu_\epsilon\, G_\theta(JX,W)\} \\
& +\Omega(X,Z)\{G_\theta(\tau Y,W) + \mu_\epsilon\, G_\theta(JY,W)\} \\
& +\frac{1}{\mu_\epsilon}\, A(Y,Z)\{G_\theta(\tau X,W) + \mu_\epsilon G_\theta(JX,W)\} \\
& -\frac{1}{\mu_\epsilon}\, A(X,Z)\{G_\theta(\tau Y,W) + \mu_\epsilon G_\theta(JY,W)\} \\
& = -g^f_\Theta(D_X f_*W,\, D_Y f_*Z) + g^f_\Theta(D_X f_*W,\, f_*\nabla_Y Z) \\
& +\frac{1}{\mu_\epsilon}\,(f^*g_\Theta)(\nabla^\top_Y Z,\, T)\, g^f_\Theta(D_X f_*W,\, f_*T) \\
& +\frac{1}{\epsilon^2\,\mu_\epsilon}\Big\{\Omega(Y,Z) - \epsilon^2\, g^f_\Theta(f_*Y,\, \tau^f_A\, f_*Z)\Big\}\, g^f_\Theta(D_X f_*W,\; f_*T) \\
& +g^f_\Theta(f_*\nabla_X W,\, D_Y f_*Z) - (f^*g_\Theta)(\nabla_X W,\, \nabla_Y Z) \\
& -\frac{1}{\mu_\epsilon}\,(f^*g_\Theta)(\nabla^\top_Y Z,\, T)\,(f^*g_\Theta)(\nabla_X W,\, T) \\
& -\frac{1}{\epsilon^2\,\mu_\epsilon}\Big\{\Omega(Y,Z) - \epsilon^2\, g^f_\Theta(f_*Y,\, \tau^f_A\, f_*Z)\Big\}\,(f^*g_\Theta)(\nabla_X W,\; T) \\
& +\frac{1}{\mu_\epsilon}\,(f^*g_\Theta)(\nabla^\top_X W,\, T)\, g^f_\Theta(f_*T,\, D_Y f_*Z) \\
& -\frac{1}{\mu_\epsilon}\,(f^*g_\Theta)(\nabla^\top_X W,\, T)\,(f^*g_\Theta)(T,\, \nabla_Y Z) \\
& -\frac{1}{\mu_\epsilon}\,(f^*g_\Theta)(\nabla^\top_X W,\, T)\,(f^*g_\Theta)(\nabla^\top_Y Z,\, T) \\
& +\frac{1}{\epsilon^2\,\mu_\epsilon}\Big\{\Omega(X,W) - \epsilon^2\, g^f_\Theta(f_*X,\, \tau^f_A\, f_*W)\Big\} \\
& \times g^f_\Theta(f_*T,\, D_Y f_*Z)
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{\epsilon^2\,\mu_\epsilon}\Big\{\Omega(X,W)-\epsilon^2\,g^f_\Theta\big(f_*X,\,\tau^f_A\,f_*W\big)\Big\}\,\big(f^*g_\Theta\big)\big(T,\,\nabla_Y Z\big)\\
&-\Big\{\Omega(X,W)-\epsilon^2\,g^f_\Theta\big(f_*X,\,\tau^f_A\,f_*W\big)\Big\}\Big\{\Omega(Y,Z)-\epsilon^2\,g^f_\Theta\big(f_*Y,\,\tau^f_A\,f_*Z\big)\Big\}\\
&\times\frac{\lambda-1}{1+\epsilon^2(\lambda-1)}\\
&+g^f_\Theta\big(D_Y f_*W,\,D_X f_*Z\big)-g^f_\Theta\big(D_Y f_*W,\,f_*\nabla_X Z\big)\\
&-\frac{1}{\mu_\epsilon}\,\big(f^*g_\Theta\big)\big(\nabla^\top_X Z,\,T\big)\,g^f_\Theta\big(D_Y f_*W,\,f_*T\big)\\
&-\frac{1}{\epsilon^2\,\mu_\epsilon}\Big\{\Omega(X,Z)-\epsilon^2\,g^f_\Theta\big(f_*X,\,\tau^f_A\,f_*Z\big)\Big\}\,g^f_\Theta\big(D_Y f_*W,\,f_*T\big)\\
&-g^f_\Theta\big(f_*\nabla_Y W,\,D_X f_*Z\big)+\big(f^*g_\Theta\big)\big(\nabla_Y W,\,\nabla_X Z\big)\\
&+\frac{1}{\mu_\epsilon}\,\big(f^*g_\Theta\big)\big(\nabla^\top_X Z,\,T\big)\,\big(f^*g_\Theta\big)\big(\nabla_Y W,\,T\big)\\
&+\frac{1}{\epsilon^2\,\mu_\epsilon}\Big\{\Omega(X,Z)-\epsilon^2\,g^f_\Theta\big(f_*X,\,\tau^f_A\,f_*Z\big)\Big\}\,\big(f^*g_\Theta\big)\big(\nabla_Y W,\,T\big)\\
&-\frac{1}{\mu_\epsilon}\,\big(f^*g_\Theta\big)\big(\nabla^\top_Y W,\,T\big)\,g^f_\Theta\big(f_*T,\,D_X f_*Z\big)\\
&+\frac{1}{\mu_\epsilon}\,\big(f^*g_\Theta\big)\big(\nabla^\top_Y W,\,T\big)\,\big(f^*g_\Theta\big)\big(T,\,\nabla_X Z\big)\\
&+\frac{1}{\mu_\epsilon}\,\big(f^*g_\Theta\big)\big(\nabla^\top_Y W,\,T\big)\,\big(f^*g_\Theta\big)\big(\nabla^\top_X Z,\,T\big)\\
&-\frac{1}{\epsilon^2\,\mu_\epsilon}\Big\{\Omega(Y,W)-\epsilon^2\,g^f_\Theta\big(f_*Y,\,\tau^f_A\,f_*W\big)\Big\}\\
&\times g^f_\Theta\big(f_*T,\,D_X f_*Z\big)\\
&+\frac{1}{\epsilon^2\,\mu_\epsilon}\Big\{\Omega(Y,W)-\epsilon^2\,g^f_\Theta\big(f_*Y,\,\tau^f_A\,f_*W\big)\Big\}\,\big(f^*g_\Theta\big)\big(T,\,\nabla_X Z\big)\\
&+\Big\{\Omega(Y,W)-\epsilon^2\,g^f_\Theta\big(f_*Y,\,\tau^f_A\,f_*W\big)\Big\}\Big\{\Omega(X,Z)-\epsilon^2\,g^f_\Theta\big(f_*X,\,\tau^f_A\,f_*Z\big)\Big\}\\
&\times\frac{\lambda-1}{1+\epsilon^2(\lambda-1)}\,. \qquad (223)
\end{aligned}
$$

This is the Gauss equation for the isometric immersion $f : (M,\, g_\epsilon(f)) \to (A, g^A_\epsilon)$, which we succeeded in fully writing in terms of pseudohermitian invariants of (M,θ) and (A,Θ) with coefficients that are rational functions of ϵ, i.e., coefficients of the form

$$
\epsilon^2\,,\quad \frac{1}{\epsilon^2}\,,\quad \mu_\epsilon\equiv\frac{1}{\epsilon^2}+\lambda-1,\quad \frac{1}{\mu_\epsilon}\,,\quad \frac{1}{\epsilon^2\mu_\epsilon}\,.
$$

To simplify (223) and examine its consequences as $\epsilon \to 0^+$ will be our job for the remainder of this section. To begin, note that simplifications occur in the term

$$
\begin{aligned}
&\frac{1}{\epsilon^2}\{\Omega(X,Y)\,\Omega(Y,W)-\Omega(X,W)\,\Omega(Y,Z)+2\,\Omega(X,Y)\,\Omega(Z,W)\}\\
&+2\mu_\epsilon\,\Omega(X,Y)\,G_\Theta(JZ,W)\\
&-\mu_\epsilon\,\Omega(Y,Z)\,G_\Theta(JX,W)+\mu_\epsilon\,\Omega(X,Z)\,G_\Theta(JY,W),
\end{aligned}
$$

which is, by $\mu_\epsilon = \epsilon^{-2} + \lambda - 1$, merely

$$(\lambda - 1)\big\{ - \Omega(X,Z)\,\Omega(Y,W) + \Omega(Y,Z)\,\Omega(X,W) - 2\,\Omega(X,Y)\,\Omega(Z,W)\big\}.$$

Therefore, the (unbounded) terms of order $O(\epsilon^{-2})$ simplify, and we may take $\epsilon \to 0^+$ in the resulting equation:

$$\begin{gathered}
g^f_\Theta\big(R^D(X,Y)f_*Z,\, f_*W\big) \\
+\Omega(Y,Z)\, g^f_\Theta\big(\tau^f_A f_*X,\, f_*W\big) - \Omega(X,Z)\, g^f_\Theta\big(\tau^f_A f_*Y,\, f_*W\big) \\
-g^f_\Theta\big(\tau^f_A f_*X,\, f_*Z\big)\,\Omega(Y,W) + g^f_\Theta\big(\tau^f_A f_*Y,\, f_*Z\big)\,\Omega(X,W) \\
+\epsilon^2\Big\{ g^f_\Theta\big(\tau^f_A f_*X,\, f_*Z\big)\, g^f_\Theta\big(\tau^f_A f_*Y,\, f_*W\big) \\
-g^f_\Theta\big(\tau^f_A f_*Y,\, f_*Z\big)\, g^f_\Theta\big(\tau^f_A f_*X,\, f_*W\big)\Big\} \\
-(\lambda-1)\{\Omega(X,Z)\,\Omega(Y,W) - \Omega(X,W)\,\Omega(Y,Z) + 2\,\Omega(X,Y)\,\Omega(Z,W)\} \\
-g_\theta\big(R^\nabla(X,Y)Z,\, W\big) \\
-\Omega(Y,Z)\,A(X,W) + \Omega(X,Z)\,A(Y,W) \\
+\frac{1}{\mu_\epsilon}\,A(Y,Z)\,A(X,W) - A(Y,Z)\,\Omega(X,W) \\
-\frac{1}{\mu_\epsilon}\,A(X,Z)\,A(Y,W) + A(X,Z)\,\Omega(Y,W) \\
= -g^f_\Theta\big(D_X f_*W,\, D_Y f_*Z\big) + g^f_\Theta\big(D_X f_*W,\, f_*\nabla_Y Z\big) \\
+\frac{1}{\mu_\epsilon}\,(f^*g_\Theta)\big(\nabla^\top_Y Z,\, T\big)\, g^f_\Theta\big(D_X f_*W,\, f_*T\big) \\
+\frac{1}{\epsilon^2\,\mu_\epsilon}\,\Big\{\Omega(Y,Z) - \epsilon^2\, g^f_\Theta\big(f_*Y,\, \tau^f_A\, f_*Z\big)\Big\}\, g^f_\Theta(D_X f_*W,\ f_*T) \\
+g^f_\Theta\big(f_*\nabla_X W,\, D_Y f_*Z\big) - (f^*g_\Theta)\big(\nabla_X W,\, \nabla_Y Z\big) \\
-\frac{1}{\mu_\epsilon}\,(f^*g_\Theta)\big(\nabla^\top_Y Z,\, T\big)\,(f^*g_\Theta)\big(\nabla_X W,\, T\big) \\
-\frac{1}{\epsilon^2\,\mu_\epsilon}\,\Big\{\Omega(Y,Z) - \epsilon^2\, g^f_\Theta\big(f_*Y,\, \tau^f_A\, f_*Z\big)\Big\}\,(f^*g_\Theta)(\nabla_X W,\ T) \\
+\frac{1}{\mu_\epsilon}\,(f^*g_\Theta)\big(\nabla^\top_X W,\, T\big)\, g^f_\Theta\big(f_*T,\, D_Y f_*Z\big) \\
-\frac{1}{\mu_\epsilon}\,(f^*g_\Theta)\big(\nabla^\top_X W,\, T\big)\,(f^*g_\Theta)\big(T,\, \nabla_Y Z\big) \\
-\frac{1}{\mu_\epsilon}\,(f^*g_\Theta)\big(\nabla^\top_X W,\, T\big)\,(f^*g_\Theta)\big(\nabla^\top_Y Z,\, T\big) \\
+\frac{1}{\epsilon^2\,\mu_\epsilon}\,\Big\{\Omega(X,W) - \epsilon^2\, g^f_\Theta\big(f_*X,\, \tau^f_A\, f_*W\big)\Big\} \\
\times g^f_\Theta\big(f_*T,\, D_Y f_*Z\big) \\
-\frac{1}{\epsilon^2\,\mu_\epsilon}\,\Big\{\Omega(X,W) - \epsilon^2\, g^f_\Theta\big(f_*X,\, \tau^f_A\, f_*W\big)\Big\}\,(f^*g_\Theta)\big(T,\, \nabla_Y Z\big) \\
-\Big\{\Omega(X,W) - \epsilon^2\, g^f_\Theta\big(f_*X,\, \tau^f_A\, f_*W\big)\Big\}\,\Big\{\Omega(Y,Z) - \epsilon^2\, g^f_\Theta\big(f_*Y,\, \tau^f_A\, f_*Z\big)\Big\}
\end{gathered}$$

$$
\begin{aligned}
&\times \frac{\lambda-1}{1+\epsilon^2(\lambda-1)} \\
&+g^f_\Theta\big(D_Y f_* W,\, D_X f_* Z\big) - g^f_\Theta\big(D_Y f_* W,\, f_* \nabla_X Z\big) \\
&-\frac{1}{\mu_\epsilon}\,(f^* g_\Theta)\big(\nabla^\top_X Z,\, T\big)\, g^f_\Theta\big(D_Y f_* W,\, f_* T\big) \\
&-\frac{1}{\epsilon^2\,\mu_\epsilon}\Big\{\Omega(X,Z) - \epsilon^2\, g^f_\Theta\big(f_* X,\, \tau^f_A\, f_* Z\big)\Big\}\, g^f_\Theta(D_Y f_* W,\, f_* T) \\
&-g^f_\Theta\big(f_* \nabla_Y W,\, D_X f_* Z\big) + (f^* g_\Theta)\big(\nabla_Y W,\, \nabla_X Z\big) \\
&+\frac{1}{\mu_\epsilon}\,(f^* g_\Theta)\big(\nabla^\top_X Z,\, T\big)\,(f^* g_\Theta)\big(\nabla_Y W,\, T\big) \\
&+\frac{1}{\epsilon^2\,\mu_\epsilon}\Big\{\Omega(X,Z) - \epsilon^2\, g^f_\Theta\big(f_* X,\, \tau^f_A\, f_* Z\big)\Big\}\,(f^* g_\Theta)(\nabla_Y W,\, T) \\
&-\frac{1}{\mu_\epsilon}\,(f^* g_\Theta)\big(\nabla^\top_Y W,\, T\big)\, g^f_\Theta\big(f_* T,\, D_X f_* Z\big) \\
&+\frac{1}{\mu_\epsilon}\,(f^* g_\Theta)\big(\nabla^\top_Y W,\, T\big)\,(f^* g_\Theta)\big(T,\, \nabla_X Z\big) \\
&+\frac{1}{\mu_\epsilon}\,(f^* g_\Theta)\big(\nabla^\top_Y W,\, T\big)\,(f^* g_\Theta)\big(\nabla^\top_X Z,\, T\big) \\
&-\frac{1}{\epsilon^2\,\mu_\epsilon}\Big\{\Omega(Y,W) - \epsilon^2\, g^f_\Theta\big(f_* Y,\, \tau^f_A\, f_* W\big)\Big\} \\
&\times g^f_\Theta\big(f_* T,\, D_X f_* Z\big) \\
&+\frac{1}{\epsilon^2\,\mu_\epsilon}\Big\{\Omega(Y,W) - \epsilon^2\, g^f_\Theta\big(f_* Y,\, \tau^f_A\, f_* W\big)\Big\}\,(f^* g_\Theta)\big(T,\, \nabla_X Z\big) \\
&+\Big\{\Omega(Y,W) - \epsilon^2\, g^f_\Theta\big(f_* Y,\, \tau^f_A\, f_* W\big)\Big\}\Big\{\Omega(X,Z) - \epsilon^2\, g^f_\Theta\big(f_* X,\, \tau^f_A\, f_* Z\big)\Big\} \\
&\times \frac{\lambda-1}{1+\epsilon^2(\lambda-1)}\,.
\end{aligned} \tag{224}
$$

To produce (224), one also has to recognize pseudohermitian torsion terms in (223); e.g., $A(X,W) = G_\theta(\tau X, W)$. Recall that $1/\mu_\epsilon \to 0$ and $1/(\epsilon^2\,\mu_\epsilon) \to 1$ as $\epsilon \to 0^+$. Therefore, (224) yields, as $\epsilon \to 0^+$,

$$
\begin{aligned}
& g^f_\Theta\big(R^{D^f}(X,Y) f_* Z,\, f_* W\big) \\
&+\Omega(Y,Z)\, g^f_\Theta\big(\tau^f_A f_* X,\, f_* W\big) - \Omega(X,Z)\, g^f_\Theta\big(\tau^f_A f_* Y,\, f_* W\big) \\
&-g^f_\Theta\big(\tau^f_A f_* X,\, f_* Z\big)\,\Omega(Y,W) + g^f_\Theta\big(\tau^f_A f_* Y,\, f_* Z\big)\,\Omega(X,W) \\
&-(\lambda-1)\{\Omega(X,Z)\,\Omega(Y,W) - \Omega(X,W)\,\Omega(Y,Z) + 2\,\Omega(X,Y)\,\Omega(Z,W)\} \\
&-g_\theta\big(R^\nabla(X,Y)Z,\, W\big) \\
&-\Omega(Y,Z)\,A(X,W) + \Omega(X,Z)\,A(Y,W) \\
&-A(Y,Z)\,\Omega(X,W) + A(X,Z)\,\Omega(Y,W) \\
&= -g^f_\Theta\big(D_X f_* W,\, D_Y f_* Z\big) + g^f_\Theta\big(D_X f_* W,\, f_* \nabla_Y Z\big) \\
&+\Omega(Y,Z)\, g^f_\Theta(D_X f_* W,\, f_* T) + \Omega(X,W)\, g^f_\Theta\big(f_* T,\, D_Y f_* Z\big) \\
&+g^f_\Theta\big(f_* \nabla_X W,\, D_Y f_* Z\big) - (f^* g_\Theta)\big(\nabla_X W,\, \nabla_Y Z\big) \\
&-\Omega(Y,Z)\,(f^* g_\Theta)\big(\nabla_X W,\, T\big) - \Omega(X,W)\,(f^* g_\Theta)\big(T,\, \nabla_Y Z\big)
\end{aligned}
$$

$$
\begin{aligned}
&+(\lambda-1)\,\{\Omega(Y,W)\,\Omega(X,Z)-\Omega(X,W)\,\Omega(Y,Z)\}+\\
&+g^f_\Theta(D_Y f_* W,\, D_X f_* Z)-g^f_\Theta(D_Y f_* W,\, f_*\nabla_X Z)\\
&-\Omega(X,Z)\,g^f_\Theta(D_Y f_* W,\, f_* T)-\Omega(Y,W)\,g^f_\Theta(f_* T,\, D_X f_* Z)\\
&-g^f_\Theta(f_*\nabla_Y W,\, D_X f_* Z)+(f^* g_\Theta)(\nabla_Y W,\, \nabla_X Z)\\
&+\Omega(X,Z)\,(f^* g_\Theta)(\nabla_Y W,\; T)+\Omega(Y,W)\,(f^* g_\Theta)(T,\, \nabla_X Z).
\end{aligned}
\tag{225}
$$

Equation (225) may be further simplified. First note that the term

$$
-(\lambda-1)\{\Omega(X,Z)\,\Omega(Y,W)-\Omega(X,W)\,\Omega(Y,Z)+2\,\Omega(X,Y)\,\Omega(Z,W)\}
$$

on the left-hand side of (225) and

$$
(\lambda-1)\,\{\Omega(Y,W)\,\Omega(X,Z)-\Omega(X,W)\,\Omega(Y,Z)\}
$$

on the right-hand side of (225) simplify to

$$
2(\lambda-1)\{\Omega(X,Z)\,\Omega(Y,W)-\Omega(X,W)\,\Omega(Y,Z)+\Omega(X,Y)\,\Omega(Z,W)\}
$$

to be written on the right-hand side of (225). Also, for further use, let us consider the $(1,2)$ tensor field

$$
U=U(f)=U(f,\,\Theta)\in\mathcal{T}^{1,2}(M)
$$

given by

$$
U(V,W)=\nabla^\top_V W-\nabla_V W,\quad V,W\in\mathfrak{X}(M),
\tag{226}
$$

expressing the difference between the induced connection $\nabla^\top$ and the (intrinsic) Tanaka–Webster connection ∇. A term of the form

$$
g^f_\Theta(D^f_X f_* W,\, f_* T)
$$

may be written

$$
g^f_\Theta(D^f_X f_* W,\, f_* T)
$$

by the pseudohermitian Gauss formula

$$
=g^f_\Theta(f_*\nabla^\top_X W,\, f_* T)=(f^* g_\Theta)(\nabla^\top_X W,\, T),
$$

hence pairs of terms such as

$$
\Omega(Y,Z)\,g^f_\Theta(D_X f_* W,\, f_* T)-\Omega(Y,Z)\,(f^* g_\Theta)(\nabla_X W,\, T)
$$

may be written

$$
\Omega(Y,Z)\,(f^* g_\Theta)(U(X,W),\, f_* T).
$$

Finally, terms of the form

$$
g^f_\Theta(D^f_Y f_* W,\, D_X f_* Z),\quad g^f_\Theta(D^f_Y f_* W,\, f_*\nabla_X Z),
$$

are computed by using the pseudohermitian Gauss formula. For instance,

$$
\begin{aligned}
&g^f_\Theta(D^f_Y f_* W,\, D^f_X f_* Z)\\
&=(f^* g_\Theta)(\nabla^\top_Y W,\, \nabla^\top_X Z)+g^f_\Theta(B(f)(Y,W),\, B(f)(X,Z)),\\
&g^f_\Theta(D_Y f_* W,\, f_*\nabla_X Z)=(f^* g_\Theta)(\nabla^\top_Y W,\, \nabla_X Z).
\end{aligned}
$$

Under the modifications above, Equation (225) simplifies down to:

$$
\begin{aligned}
& g^f_\Theta\big(R^D(X,Y)f_*Z,\ f_*W\big) - g_\theta\big(R^\nabla(X,Y)Z,\ W\big) \\
& +\Omega(Y,Z)\, g^f_\Theta\big(\tau^f_A f_*X,\ f_*W\big) - \Omega(X,Z)\, g^f_\Theta\big(\tau^f_A f_*Y,\ f_*W\big) \\
& -g^f_\Theta\big(\tau^f_A f_*X,\ f_*Z\big)\,\Omega(Y,W) + g^f_\Theta\big(\tau^f_A f_*Y,\ f_*Z\big)\,\Omega(X,W) \\
& -\Omega(Y,Z)\, A(X,W) + \Omega(X,Z)\, A(Y,W) \\
& -A(Y,Z)\,\Omega(X,W) + A(X,Z)\,\Omega(Y,W) \\
& = -(f^*g_\Theta)\big(\nabla^\top_X W,\ \nabla^\top_Y Z\big) - g^f_\Theta\big(B(f)(X,W),\ B(f)(Y,Z)\big) \\
& +(f^*g_\Theta)\big(\nabla^\top_X W,\ \nabla_Y Z\big) \\
& +\Omega(Y,Z)\ (f^*g_\Theta)\big(U(X,W),\ T\big) + \Omega(X,W)\ (f^*g_\Theta)\big(U(Y,Z),\ T\big) \\
& +(f^*g_\Theta)\big(\nabla_X W,\ \nabla^\top_Y Z\big) - (f^*g_\Theta)\big(\nabla_X W,\ \nabla_Y Z\big) \\
& +2(\lambda-1)\,\{\Omega(Y,W)\,\Omega(X,Z) - \Omega(X,W)\,\Omega(Y,Z) + \Omega(X,Y)\,\Omega(Z,W)\} \\
& +(f^*g_\Theta)\big(\nabla^\top_Y W,\ \nabla^\top_X Z\big) + g^f_\Theta\big(B(f)(Y,W),\ B(f)(X,Z)\big) \\
& -(f^*g_\Theta)\big(\nabla^\top_Y W,\ \nabla_X Z\big) \\
& -\Omega(X,Z)\ (f^*g_\Theta)\big(U(Y,W),\ T\big) - \Omega(Y,W)\ (f^*g_\Theta)\big(U(X,Z),\ T\big) \\
& -(f^*g_\Theta)\big(\nabla_Y W,\ \nabla^\top_X Z\big) + (f^*g_\Theta)\big(\nabla_Y W,\ \nabla_X Z\big).
\end{aligned}
\tag{227}
$$

Once again, using (226), we may write (227) as:

$$
\begin{aligned}
& g^f_\Theta\big(R^D(X,Y)f_*Z,\ f_*W\big) - g_\theta\big(R^\nabla(X,Y)Z,\ W\big) \\
& +\Omega(Y,Z)\, g^f_\Theta\big(\tau^f_A f_*X,\ f_*W\big) - \Omega(X,Z)\, g^f_\Theta\big(\tau^f_A f_*Y,\ f_*W\big) \\
& -g^f_\Theta\big(\tau^f_A f_*X,\ f_*Z\big)\,\Omega(Y,W) + g^f_\Theta\big(\tau^f_A f_*Y,\ f_*Z\big)\,\Omega(X,W) \\
& -\Omega(Y,Z)\, A(X,W) + \Omega(X,Z)\, A(Y,W) \\
& -A(Y,Z)\,\Omega(X,W) + A(X,Z)\,\Omega(Y,W) \\
& = -(f^*g_\Theta)\big(U(X,W),\ U(Y,Z)\big) - g^f_\Theta\big(B(f)(X,W),\ B(f)(Y,Z)\big) \\
& +\Omega(Y,Z)\ (f^*g_\Theta)\big(U(X,W),\ T\big) + \Omega(X,W)\ (f^*g_\Theta)\big(U(Y,Z),\ T\big) \\
& +2(\lambda-1)\,\{\Omega(Y,W)\,\Omega(X,Z) - \Omega(X,W)\,\Omega(Y,Z) + \Omega(X,Y)\,\Omega(Z,W)\} \\
& +(f^*g_\Theta)\big(U(Y,W),\ U(X,Z)\big) + g^f_\Theta\big(B(f)(Y,W),\ B(f)(X,Z)\big) \\
& -\Omega(X,Z)\ (f^*g_\Theta)\big(U(Y,W),\ T\big) - \Omega(Y,W)\ (f^*g_\Theta)\big(U(X,Z),\ T\big).
\end{aligned}
\tag{228}
$$

We organize terms in (228) so as to emphasize the similarity to the classical Gauss equation in the theory of isometric immersions between Riemannian manifolds:

$$
\begin{aligned}
& g^f_\Theta\big(R^D(X,Y)f_*Z,\ f_*W\big) = g_\theta\big(R^\nabla(X,Y)Z,\ W\big) \\
& -g^f_\Theta\big(B(f)(X,W),\ B(f)(Y,Z)\big) + g^f_\Theta\big(B(f)(Y,W),\ B(f)(X,Z)\big) \\
& -(f^*g_\Theta)\big(U(X,W),\ U(Y,Z)\big) + (f^*g_\Theta)\big(U(Y,W),\ U(X,Z)\big) \\
& +\Omega(Y,Z)\ (f^*g_\Theta)\big(U(X,W),\ T\big) \\
& +\Omega(X,W)\ (f^*g_\Theta)\big(U(Y,Z),\ T\big)
\end{aligned}
$$

$$
\begin{aligned}
&-\Omega(X,Z)\,(f^*g_\Theta)\,(U(Y,W),\,T) \\
&-\Omega(Y,W)\,(f^*g_\Theta)\,(U(X,Z),\,T) \\
&+2(\lambda-1)\big[\Omega(Y,W)\,\Omega(X,Z)-\Omega(X,W)\,\Omega(Y,Z)+\Omega(X,Y)\,\Omega(Z,W)\big] \\
&-\Omega(Y,Z)\left[g^f_\Theta(\tau^f_A f_*X,\,f_*W)-A(X,W)\right] \\
&-\Omega(X,W)\left[g^f_\Theta(\tau^f_A f_*Y,\,f_*Z)-A(Y,Z)\right] \\
&+\Omega(X,Z)\left[g^f_\Theta(\tau^f_A f_*Y,\,f_*W)-A(Y,W)\right] \\
&+\Omega(Y,W)\left[g^f_\Theta(\tau^f_A f_*X,\,f_*Z)-A(X,Z)\right]
\end{aligned} \tag{229}
$$

for any $X,Y,Z,W\in H(M)$.

Definition 39. We shall refer to (229) as the *pseudohermitian Gauss equation* of the isopseudohermitian immersion $f:(M,\theta)\to(A,\Theta)$.

The first two lines in (229) are entirely similar to the Gauss equation in Riemannian geometry except of course that the Levi–Civita connections of the ambient space and submanifold were replaced by the Tanaka–Webster connections, while the second fundamental form of the given immersion was replaced by the pseudohermitian second fundamental form. However, in the theory at hand, with respect to the theory of isometric immersions, where the induced connection and the (intrinsic) Levi–Civita connection of the induced metric coincide, there is a non-uniqueness of choice of connection on the submanifold and the $(1,2)$ tensor field U measuring the difference between the induced connection $\nabla^\top$, and the (intrinsic) Tanaka–Webster connection ∇ appears explicitly in (229). Also, unlike the Riemannian case, the canonical connections used (i.e., ∇, $\nabla^\top$ and D) are not symmetric, and their torsion appears explicitly in the embedding Equation (229); the last four terms in (229) depend upon the pseudohermitian torsion tensor fields τ_A and τ of the ambient space (A,Θ) and the submanifold (M,θ). The term in (229) containing the function $\lambda=\big(f^*g_\Theta\big)(T,T)$ is prompted by our more general treatment (with respect to the previous works [15,17,18]), including the case of pseudohermitian immersions where $\lambda=1$, but not confined to that case.

Keep in mind that (229) follows, as $\epsilon\to 0^+$, from the Gauss Equation (198) written for $X,Y,Z,W\in H(M)$. The same technique should then prompt other (pseudohermitian) Gauss-like equations corresponding to the cases where **not** all arguments X,Y,Z,W are horizontal. Deriving the remaining Gauss-like equations [springing from the various components of (198) with respect to the decomposition $T(M)=H(M)\oplus\mathbb{R}T$] is relegated to further work.

11. CR Immersions into Spheres

11.1. Mean Curvature

By a classic result of T. Takahashi (see [30]), an isometric immersion $\psi:M\to S^n(r)$ of an m-dimensional Riemannian manifold into the sphere $\mathbf{j}:S^n(r)\subset\mathbb{R}^{n+1}$ of radius $r>0$ is minimal if and only if $\Delta\Psi=-(m/r^2)\,\Psi$, where $\Psi=\mathbf{j}\circ\psi$ and Δ is the Laplace–Beltrami operator on M. Takahashi's theorem relies on the simple observation that given isometric immersions $f:M\to A$ and $\mathbf{j}:A\hookrightarrow\mathbb{R}^n$ where M and A are Riemannian manifolds, the mean curvature vector $H(f)$ is the tangential component in $f^{-1}T(A)$ of $(1/m)\,\Delta(\mathbf{j}\circ f)$. We wish to look at a similar configuration within CR geometry, starting with an isopseudohermitian immersion $f:(M,\theta)\to(A,\Theta)$ and, hence, for every $0<\epsilon<1$, with the isometric immersion $f:\big(M,\,g_\epsilon(f)\big)\to\big(A,\,g^A_\epsilon\big)$. By Nash's embedding theorem (see [31]), there is an integer $2n+1<\mathrm{K}\le(2N+1)(3N+7)$ and a C^∞ isometric embedding of $\big(A,\,g^A_\epsilon\big)$ into the Euclidean space $\mathbb{R}^{\mathrm{K}}$, yet both K and the immersion $A\to\mathbb{R}^{\mathrm{K}}$ depend on ϵ. [We conjecture that the nature of said dependence can be understood by inspecting the proof in [31]. See also R.E. Greene and H. Jacobowitz [32] for a simplified proof in the real analytic case.] To circumnavigate this difficulty, we confine ourselves to $A = S^{2N+1}$ and the pair of immersions $(M, g_\epsilon(f)) \xrightarrow{f} (S^{2N+1}, g^A_\epsilon) \overset{\mathbf{j}}{\hookrightarrow} \mathbb{C}^{N+1}$, where the ambient space $\mathbb{C}^{N+1}$ is thought of as carrying, in addition to the Euclidean metric g_0, the family of Riemannian metrics

$$g_{0,\epsilon} := g_0 + \Big(\frac{1}{\epsilon^2} - 1\Big)\, \omega_0 \otimes \omega_0\,, \quad 0 < \epsilon < 1, \tag{230}$$

built such that $\mathbf{j}^* g_{0,\epsilon} = g^A_\epsilon$. Then, we benefit from $\mathbf{j}^* g_0 = g_\Theta$ [a phenomenon known to hold only for the sphere — in general, for a strictly pseudoconvex real hypersurface $A \subset \mathbb{C}^{N+1}$, none of the Webster metrics $\{g_{e^u \Theta} : u \in C^\infty(A, \mathbb{R})\}$ is induced by the ambient Euclidean metric g_0 (see e.g., [3], p. 41–42)] to relate the mean curvature vectors $H_\epsilon(f)$ and $H_\epsilon(\mathbf{j} \circ f)$ of the isometric immersions $f : (M, g_\epsilon(f)) \to (S^{2N+1}, g^A_\epsilon)$ and $\mathbf{j} \circ f : (M, g_\epsilon(f)) \to (\mathbb{C}^{N+1}, g_{0,\epsilon})$.

Let $A = S^{2N+1} \subset \mathbb{C}^{N+1}$ be the standard sphere, equipped with the strictly pseudoconvex CR structure

$$T_{1,0}(S^{2N+1})_y = \big[T_y(S^{2N+1}) \otimes_{\mathbb{R}} \mathbb{C}\big] \oplus T'(\mathbb{C}^{N+1})_y\,, \quad y \in S^{2N+1}\,,$$

and the positively oriented contact form

$$\Theta \in \mathcal{P}_+(S^{2N+1}), \quad \Theta = \frac{i}{2}(\overline{\partial} - \partial)|Z|^2\,.$$

The Webster metric g_Θ is the first fundamental form of $\mathbf{j} : S^{2N+1} \to \mathbb{C}^{N+1}$, the inclusion of S^{2N+1} into $(\mathbb{C}^{N+1}, g_0)$; i.e., $\mathbf{j}^* g_0 = g_\Theta$. Let $\mathbf{X}_0 \in \mathfrak{X}(\mathbb{C}^{N+1})$ be the real tangent vector field defined by

$$\mathbf{X}_0 = \frac{i}{2}\Big(Z^B \frac{\partial}{\partial Z^B} - \overline{Z}_B \frac{\partial}{\partial \overline{Z}_B}\Big)\,.$$

The Reeb vector field T_A of (S^{2N+1}, Θ) is given by $\mathbf{j}_* T_A = 2\, X_0^{\mathbf{j}}$. Let $\omega_0 \in \Omega^1(\mathbb{C}^{N+1})$ be the differential 1-form

$$\omega_0(\mathbf{X}) = g_0(\mathbf{X}, \mathbf{X}_0), \quad \mathbf{X} \in \mathfrak{X}(\mathbb{C}^{N+1}).$$

Also,

$$\nu = \frac{1}{2}\Big(Z^B \frac{\partial}{\partial Z^B} + \overline{Z}_B \frac{\partial}{\partial \overline{Z}_B}\Big)^{\mathbf{j}}$$

is a unit normal vector field on S^{2N+1} in $(\mathbb{C}^{N+1}, g_0)$ and $J_0^{\mathbf{j}}\, \nu = X_0^{\mathbf{j}}$. We shall need the family of Riemannian metrics $\{g_{0,\epsilon}\}_{0<\epsilon<1}$ on $\mathbb{C}^{N+1}$

$$g_{0,\epsilon} = g_0 + \Big(\frac{1}{\epsilon^2} - 1\Big)\, \omega_0 \otimes \omega_0\,.$$

If $A = S^{2N+1}$, then $\mathbf{j}^* \omega_0 = \Theta$ and, hence, $\mathbf{j}^* g_{0,\epsilon} = g^A_\epsilon$. Let $f : M \to S^{2N+1}$ be an isopseudohermitian immersion, i.e., a CR immersion, such that $f^* \Theta = \theta$. Corresponding to the isometric immersions

$$(M, g_\epsilon(f)) \xrightarrow{f} (S^{2N+1}, g^A_\epsilon) \xrightarrow{\mathbf{j}} (\mathbb{C}^{N+1}, g_{0,\epsilon}) \tag{231}$$

one has the decompositions

$$T_{\hat{x}}(S^{2N+1}) = \big[(d_x f) T_x(M)\big] \oplus E_\epsilon(f)_x\,, \quad x \in M, \tag{232}$$

$$T_{\mathbf{j}(\hat{x})}(\mathbb{C}^{N+1}) = \big[(d_{\hat{x}} \mathbf{j}) T_{\hat{x}}(S^{2N+1})\big] \oplus E_\epsilon(\mathbf{j})_{\hat{x}}\,, \tag{233}$$

$$T_{(\mathbf{j}\circ f)(x)}\big(\mathbb{C}^{N+1}\big) = \big[d_x(\mathbf{j}\circ f)\, T_x(M)\big] \oplus E_\epsilon(\mathbf{j}\circ f)_x\,, \tag{234}$$

where

$$\begin{array}{ccc} E_\epsilon(f) & E_\epsilon(\mathbf{j}) & E_\epsilon(\mathbf{j}\circ f) \\ \downarrow & \downarrow & \downarrow \\ M & S^{2N+1} & M \end{array}$$

are the normal bundles of the given isometric immersions. The decompositions (232)–(234) imply.

Lemma 21. *For every $x \in M$*

$$E_\epsilon(\mathbf{j}\circ f)_x = \Big[(d_{\hat{x}}\,\mathbf{j})\; E_\epsilon(f)_x\Big] \oplus E_\epsilon(\mathbf{j})_{\hat{x}}\,. \tag{235}$$

The projections associated with the direct sum decompositions (232)–(234) are denoted by

$$T_x(M) \overset{\tan_{\epsilon,x}}{\longleftarrow} T_{\hat{x}}(S^{2N+1}) \overset{\mathrm{nor}_{\epsilon,x}}{\longrightarrow} E_\epsilon(f)_x\,,$$

$$T_{\hat{x}}\big(S^{2N+1}\big) \overset{\mathrm{Tan}_{\epsilon,\hat{x}}}{\longleftarrow} T_{\mathbf{j}(\hat{x})}\big(\mathbb{C}^{N+1}\big) \overset{\mathrm{Nor}_{\epsilon,\hat{x}}}{\longrightarrow} E_\epsilon(\mathbf{j})_{\hat{x}}\,,$$

$$T_x(M) \overset{\tan^0_{\epsilon,x}}{\longleftarrow} T_{(\mathbf{j}\circ f)(x)}\big(\mathbb{C}^{N+1}\big) \overset{\mathrm{nor}^0_{\epsilon,x}}{\longrightarrow} E_\epsilon(\mathbf{j}\circ f)_x\,.$$

Theorem 5. *Let M be a strictly pseudoconvex CR manifold of CR dimension n equipped with the contact form $\theta \in \mathcal{P}_+(M)$. Let $\Theta = (i/2)\,\mathbf{j}^*\big(\overline{\partial} - \partial\big)|Z|^2 \in \mathcal{P}_+\big(S^{2N+1}\big)$ be the canonical contact form. For any isopseudohermitian immersion $f : M \to S^{2N+1}$ of (M, θ) into (S^{2N+1}, Θ),*

$$\mathrm{nor}_{\epsilon,x}\circ \mathrm{Tan}_{\epsilon,\hat{x}} = \mathrm{Tan}_{\epsilon,\hat{x}}\circ \mathrm{nor}^0_{\epsilon,x} \tag{236}$$

for every $x \in M$ and $0 < \epsilon < 1$. Consequently, the mean curvature vectors of the isometric immersions $f : (M, g_\epsilon(f)) \to (S^{2N+1}, g^A_\epsilon)$ and $\mathbf{j}\circ f : (M, g_\epsilon(f)) \to (\mathbb{C}^{N+1}, g_{0,\epsilon})$ are related by

$$H_\epsilon(f)_x = \mathrm{Tan}_{\epsilon,\hat{x}}\, H_\epsilon(\mathbf{j}\circ f)_x\,, \quad x \in M. \tag{237}$$

Proof. The proof is organized in two steps, as follows. First, we show that $\mathrm{Tan}_{\epsilon,\hat{x}}$ maps $E_\epsilon(\mathbf{j}\circ f)_x$ into $E_\epsilon(f)_x$. Indeed, for any $X \in T_x(M)$ and any $V \in E_\epsilon(\mathbf{j}\circ f)_x$

$$\big(g^A_\epsilon\big)_{\hat{x}}\big((d_x f)X,\, \mathrm{Tan}_{\epsilon,\hat{x}}\, V\big)$$

as $g^A_\epsilon = \mathbf{j}^*\, g_{0,\epsilon}$

$$= \big(g_{0,\epsilon}\big)_{\mathbf{j}(\hat{x})}\big((d_{\hat{x}}\,\mathbf{j})(d_x f)X,\, (d_{\hat{x}}\,\mathbf{j})\mathrm{Tan}_{\epsilon,\hat{x}}\, V\big)$$

as $V = (d_{\hat{x}}\,\mathbf{j})\mathrm{Tan}_{\epsilon,\hat{x}}\, V + \mathrm{Nor}_{\epsilon,\hat{x}}\, V$ and

$$\mathrm{Nor}_{\epsilon,\hat{x}}\, V \in E_\epsilon(\mathbf{j})_{\hat{x}} \perp (d_{\hat{x}}\,\mathbf{j})\, T_{\hat{x}}(S^{2N+1}) \ni (d_{\hat{x}}\,\mathbf{j})(d_x f)X$$

$$= \big(g_{0,\epsilon}\big)_{\mathbf{j}(\hat{x})}\big(d_x(\mathbf{j}\circ f)X,\, V\big) = 0,$$

hence

$$\mathrm{Tan}_{\epsilon,\hat{x}}\, E_\epsilon(\mathbf{j}\circ f)_x \subset E_\epsilon(f)_x\,.$$

□

At this point, (236) is equivalent to the commutativity of the diagram

$$\begin{array}{ccc} T_{\hat{x}}\big(\mathbb{C}^{N+1}\big) & \stackrel{\mathrm{Tan}_{\epsilon,\hat{x}}}{\longrightarrow} & T_{\hat{x}}\big(S^{2N+1}\big) \\ \mathrm{nor}^0_{\epsilon,x}\downarrow & & \downarrow \mathrm{nor}_{\epsilon,x} \\ E_\epsilon(\mathbf{j}\circ f)_x & \stackrel{\mathrm{Tan}_{\epsilon,\hat{x}}}{\longrightarrow} & E_\epsilon(f)_x \end{array}$$

Let $V \in T_{\hat{x}}\big(\mathbb{C}^{N+1}\big)$. Then, by (232) and (233),

$$\mathrm{Tan}_{\epsilon,\hat{x}}\, V = (d_x f)\,\tan_{\epsilon,x}\big(\mathrm{Tan}_{\epsilon,\hat{x}}\, V\big) + \mathrm{nor}_{\epsilon,x}\big(\mathrm{Tan}_{\epsilon,\hat{x}}\, V\big), \tag{238}$$

$$\mathrm{nor}^0_{\epsilon,x}\, V = (d_{\hat{x}}\,\mathbf{j})\,\mathrm{Tan}_{\epsilon,\hat{x}}\big(\mathrm{nor}^0_{\epsilon,x}\, V\big) + \mathrm{Nor}_{\epsilon,\hat{x}}\big(\mathrm{nor}^0_{\epsilon,x}\, V\big). \tag{239}$$

Then, by (233),

$$V = (d_{\hat{x}}\,\mathbf{j})\,\mathrm{Tan}_{\epsilon,\hat{x}}\, V + \mathrm{Nor}_{\epsilon,\hat{x}}\, V =$$

by substitution from (238)

$$= \underbrace{d_x(\mathbf{j}\circ f)\,\tan_{\epsilon,x}\big(\mathrm{Tan}_{\epsilon,\hat{x}}\, V\big)}_{\in d_x(\mathbf{j}\circ f)\,T_x(M)} + \underbrace{(d_{\hat{x}}\,\mathbf{j})\,\mathrm{nor}_{\epsilon,x}\big(\mathrm{Tan}_{\epsilon,\hat{x}}\, V\big)}_{\in (d_{\hat{x}}\,\mathbf{j})\,E_\epsilon(f)_x} + \underbrace{\mathrm{Nor}_{\epsilon,\hat{x}}\, V}_{\in E_\epsilon(\mathbf{j})_{\hat{x}}}$$

by (235) together with comparison to $V = d_x(\mathbf{j}\circ f)\,\tan^0_{\epsilon,x}\, V + \mathrm{nor}^0_{\epsilon,x}\, V$

$$\tan_{\epsilon,x}\big(\mathrm{Tan}_{\epsilon,\hat{x}}\, V\big) = \tan^0_{\epsilon,x}\, V,$$

$$(d_{\hat{x}}\,\mathbf{j})\,\mathrm{nor}_{\epsilon,x}\big(\mathrm{Tan}_{\epsilon,\hat{x}}\, V\big) + \mathrm{Nor}_{\epsilon,\hat{x}}\, V = \mathrm{nor}^0_{\epsilon,x}\, V. \tag{240}$$

Finally, (239) and (240) yield (236). Q.E.D.

Let $\{E_a \,:\, 1 \le a \le 2n\}$ be a local G_θ-orthonormal frame of $H(M)$, and let us set $T_\epsilon = \big(1/\sqrt{\mu_\epsilon}\big)\,T$, so that

$$\{E_p \,:\, 0 \le p \le 2n\} \equiv \{E_a\,,\; T_\epsilon \;:\; 1 \le a \le 2n\}, \quad E_0 = T_\epsilon\,,$$

is a local $g_\epsilon(f)$-orthonormal frame of $T(M)$. The mean curvature vector of the isometric immersion f of $(M,\, g_\epsilon(f))$ into $(S^{2N+1},\, g^A_\epsilon)$ is given by

$$H_\epsilon(f) = \frac{1}{2n+1}\,\mathrm{Trace}_{g_\epsilon(f)}\, B_\epsilon(f)$$

(locally)

$$= \frac{1}{2n+1}\sum_{p=0}^{2n} B_\epsilon(f)\big(E_p\,,\, E_p\big).$$

To prove (237), we recall the Gauss formulas

$$(D^\epsilon)^f_X f_* Y = f_* \nabla^{f,\epsilon}_X Y + B_\epsilon(f)(X,Y),$$

$$\big(D^{g_{0,\epsilon}}\big)^{\mathbf{j}}_{\mathbf{X}}\,\mathbf{j}_*\,\mathbf{Y} = \mathbf{j}_*\, D^\epsilon_{\mathbf{X}}\mathbf{Y} + B_\epsilon(\mathbf{j})(\mathbf{X},\,\mathbf{Y}),$$

$$X,\, Y \in \mathfrak{X}(M), \quad \mathbf{X},\, \mathbf{Y} \in \mathfrak{X}(S^{2N+1}),$$

for the isometric immersions (231). Here, $D^{g_{0,\epsilon}}$ denotes the Levi–Civita connection of $(\mathbb{C}^{n+1},\, g_{0,\epsilon})$. Also, $\nabla^{f,\epsilon} \equiv \nabla^{g_\epsilon(f)}$ and $D^\epsilon \equiv D^{g^A_\epsilon}$ are, respectively, the Levi–Civita connections of $(M,\, g_\epsilon(f))$ and $(S^{2N+1},\, g^A_\epsilon)$, as considered earlier in this paper. For every tangent vector field $X \in \mathfrak{X}(M)$, let $\tilde{X} \in \mathfrak{X}(S^{2N+1})$ denote a tangent vector field such that $\tilde{X}_{f(x)} = (d_x f)X_x$, for every $x \in M$. Then,

$$B_\epsilon(f)(E_p\,,\,E_p)_x = \mathrm{nor}_{\epsilon,x}\left\{(D^\epsilon)^f_{E_p} f_*\,E_p\right\}_x$$

$$= \mathrm{nor}_{\epsilon,x}\left\{(D^\epsilon_{\tilde{E}_p}\,\tilde{E}_p)_{f(x)}\right\} = \mathrm{nor}_{\epsilon,x}\,\mathrm{Tan}_{\epsilon,\hat{x}}\left\{(D^{g_{0,\epsilon}})^{\mathbf{j}}_{\tilde{E}_p}\,\mathbf{j}_*\,\tilde{E}_p\right\}_{\hat{x}}$$

(by Lemma 5)

$$= \mathrm{Tan}_{\epsilon,\hat{x}}\,\mathrm{nor}^0_{\epsilon,x}\left\{(D^{g_{0,\epsilon}})^{\mathbf{j}}_{\tilde{E}_p}\,\mathbf{j}_*\,\tilde{E}_p\right\}_{\hat{x}}.$$

Let $\tilde{\tilde{X}} \in \mathfrak{X}(\mathbb{C}^{N+1})$ be a tangent vector field such that $\tilde{\tilde{X}}_{\mathbf{j}(y)} = (d_y\,\mathbf{j})\tilde{X}_y$ for any $y \in S^{2N+1}$. Then,

$$\tilde{\tilde{X}}_{(\mathbf{j}\circ f)(x)} = d_x(\mathbf{j}\circ f)\,X_x,$$

hence

$$B_\epsilon(f)(E_p\,,\,E_p)_x = \mathrm{Tan}_{\epsilon,\hat{x}}\,\mathrm{nor}^0_{\epsilon,x}\left(D^{g_{0,\epsilon}}_{\tilde{\tilde{E}}_p}\tilde{\tilde{E}}_p\right)_{\mathbf{j}(\hat{x})}$$

$$= \mathrm{Tan}_{\epsilon,\hat{x}}\,\mathrm{nor}^0_{\epsilon,x}\left\{(D^{g_{0,\epsilon}})^{\mathbf{j}\circ f}_{E_p}(\mathbf{j}\circ f)E_p\right\}_{\hat{x}} = \mathrm{Tan}_{\epsilon,\hat{x}}\,B_\epsilon(\mathbf{j}\circ f)(E_p\,,\,E_p)_x\,.$$

Q.E.D.

It is an open problem to relate the mean curvature vector $H(\mathbf{j}\circ f)$ to $\Delta_{g_\epsilon(f)}(\mathbf{j}\circ f)$, where $\Delta_{g_\epsilon(f)}$ is the Laplace–Beltrami operator of the Riemannian manifold $(M,\,g_\epsilon(f))$. This would be the first step towards a pseudohermitian analog to Takahashi's theorem (see [30]). We expect that the solution to the problem may be obtained along the following lines. Start with the Gauss formula

$$(D^{g_{0,\epsilon}})^{\mathbf{j}\circ f}_X(\mathbf{j}\circ f)_*\,Y - (\mathbf{j}\circ f)_*\nabla^{f,\epsilon}_X Y = B_\epsilon(\mathbf{j}\circ f)(X,\,Y) \tag{241}$$

for the isometric immersion $\mathbf{j}\circ f : (M,\,g_\epsilon(f)) \to (\mathbb{C}^{N+1},\,g_{0,\epsilon})$, and take the trace with respect to $g_\epsilon(f)$ of both members. Exploit the relationship between $g_{0,\epsilon}$ and the Euclidean metric g_0 (see (230)) to relate the Levi–Civita connections $D^{g_{0,\epsilon}}$ and $D^0 \equiv D^{g_0}$.

11.2. On a Theorem by S-S. Chern

Given the result of S-S. Chern (see [21]), for every minimal isometric immersion $f : M \to A$ of an m-dimensional Riemannian manifold M into a Riemannian manifold A of constant sectional curvature κ, the scalar curvature R of M obeys to $R \le m(m-1)\kappa$ with equality if and only if f is totally geodesic. The proof is to take traces twice in the Gauss equation for f. To search for a pseudohermitian analog to S-S. Chern's result (see op. cit.), let $f : M \to S^{2N+1}$ be an isopseudohermitian immersion, and let $\mathbf{X},\,\mathbf{Y},\,\mathbf{Z} \in H(S^{2N+1})$. Then (see [3], p. 60),

$$R(D)(\mathbf{X},\,\mathbf{Y})\mathbf{Z} = g_\Theta(\mathbf{Y},\,\mathbf{Z})\,\mathbf{X} - g_\Theta(\mathbf{X},\,\mathbf{Z})\mathbf{Y}$$
$$+g_\Theta(J_A\mathbf{Y},\,\mathbf{Z})\,J_A\mathbf{X} - g_\Theta(J_A\mathbf{X},\,\mathbf{Z})\,J_A\mathbf{Y} - 2\,g_\Theta(J_A\mathbf{X},\,\mathbf{Y})\,J_A\mathbf{Z}\,. \tag{242}$$

Let $f : M \to S^{2N+1}$ be an isopseudohermitian immersion of $(M,\,\theta)$ into $(S^{2N+1},\,\Theta)$. Then, by (242) and $f_*\circ J = J_A\circ f_*$,

$$R^{D^f}(X,\,Y)\,f_*\,Z = (f^*g_\Theta)(Y,\,Z)\,f_*X - (f^*g_\Theta)(X,\,Z)\,f_*\,Y$$
$$+(f^*g_\Theta)(JY,\,Z)\,f_*\,J\,X - (f^*g_\Theta)(JX,\,Z)\,f_*\,JY$$
$$-2\,(f^*g_\Theta)(J\,X,\,Y)\,f_*\,J\,Z \tag{243}$$

for any $X,\,Y,\,Z \in H(M)$. Let $W \in H(M)$, and let us take the inner product of (243) with $f_*\,W$; i.e.,

$$g^f_\Theta(R^{D^f}(X,\,Y)\,f_*\,Z\,,\,f_*\,W)$$

$$= g_\theta(Y, Z)\, g_\theta(X, W) - g_\theta(X, Z)\, g_\theta(Y, W)$$
$$+\Omega(Y, Z)\, \Omega(X, W) - \Omega(X, Z)\, \Omega(Y, W) - 2\,\Omega(X, Y)\, \Omega(Z, W). \quad (244)$$

The pseudohermitian torsion of the sphere $A = S^{2N+1}$ vanishes; i.e., $\tau_A = 0$. Then, by substitution from (244) into (229),

$$g_\theta\big(R^\nabla(X,Y)Z,\, W\big)$$
$$= g_\theta(Y, Z)\, g_\theta(X, W) - g_\theta(X, Z)\, g_\theta(Y, W)$$
$$+g^f_\Theta\big(B(f)(X,W),\, B(f)(Y,Z)\big) - g^f_\Theta\big(B(f)(Y,W),\, B(f)(X,Z)\big)$$
$$+(2\lambda - 1)\{\Omega(X,W)\,\Omega(Y,Z) - \Omega(Y,W)\,\Omega(X,Z)\}$$
$$-2\lambda\,\Omega(X,Y)\,\Omega(Z,W)$$
$$-\Omega(Y,Z)\, A(X,W) - \Omega(X,W)\, A(Y,Z)$$
$$+\Omega(X,Z)\, A(Y,W) + \Omega(Y,W)\, A(X,Z)$$
$$+\big(f^*g_\Theta\big)\big(U(X,W),\, U(Y,Z)\big) - \big(f^*g_\Theta\big)\big(U(Y,W),\, U(X,Z)\big)$$
$$-\Omega(Y,Z)\, \big(f^*g_\Theta\big)\big(U(X,W),\, T\big) - \Omega(X,W)\, \big(f^*g_\Theta\big)\big(U(Y,Z),\, T\big)$$
$$+\Omega(X,Z)\, \big(f^*g_\Theta\big)\big(U(Y,W),\, T\big) + \Omega(Y,W)\, \big(f^*g_\Theta\big)\big(U(X,Z),\, T\big). \quad (245)$$

The Ricci curvature of the Tanaka–Webster connection ∇ of (M, θ) is

$$\mathrm{Ric}_\nabla(V,W) = \mathrm{trace}\{U \longmapsto R^\nabla(U,\, W)V\}$$

for any $V, W \in \mathfrak{X}(M)$. Let $\{E_a\, : 1 \le a \le 2n\}$ be a local G_θ-orthonormal frame of $H(M)$, defined on an open subset $\Omega \subset M$, so that $\{E_a\,,\, T\, :\, 1 \le a \le 2n\}$ is a local g_θ-orthonormal frame of $T(M)$. Then,

$$\mathrm{Ric}_\nabla(V,W) = \sum_{a=1}^{2n} g_\theta\big(R^\nabla(E_a\,,\, W)V,\, E_a\big) + \theta\big(R^\nabla(T,\, W)V\big).$$

As $H(M)$ is parallel with respect to ∇, the curvature transformation $R^\nabla(V,W)$ maps $H(M)$ into itself. Let us substitute $X = W = E_a$ in (245) and take the sum over $1 \le a \le 2n$. We obtain, as Ω is skew-symmetric, $\mathrm{trace}(\tau) = 0$, and $\tau \circ J + J \circ \tau = 0$,

$$\mathrm{Ric}_\nabla(Z,\, Y) = 2(2\lambda + n - 1)\, g_\theta(Y,\, Z)$$
$$+g^f_\Theta\big(H(f),\, B(f)(Y,Z)\big)$$
$$-\sum_{a=1}^{2n} g^f_\Theta\big(B(f)(Y,\, E_a),\, B(f)(E_a\,,\, Z)\big)$$
$$+\big(f^*g_\Theta\big)\big(\mathrm{trace}_{G_\theta}\, \Pi_{H(M)}\, U,\, U(Y,Z)\big)$$
$$-\sum_{a=1}^{2n} \big(f^*g_\Theta\big)\big(U(Y,\, E_a),\, U(E_a\,,\, Z)\big)$$
$$-\Omega(Y,Z)\, \big(f^*g_\Theta\big)\big(\mathrm{trace}_{G_\theta}\, \Pi_{H(M)}\, U,\, T\big)$$
$$+\big(f^*g_\Theta\big)\big(U(Y,\, JZ),\, T\big) - \big(f^*g_\Theta\big)\big(U(JY\,,\, Z),\, T\big) \quad (246)$$

where

$$H(f) := \mathrm{trace}_{G_\theta}\, \Pi_{H(M)}\, B(f) \in C^\infty\big(E(f)\big).$$

Let us set $Z = T_\alpha$ and $Y = T_{\overline{\beta}}$ in (246) and contract with $g^{\alpha\overline{\beta}}$. We obtain

$$R = 2n(2\lambda + n - 1)$$
$$+ g^f_\Theta\big(H(f),\, g^{\alpha\overline{\beta}}\, B(f)(T_{\overline{\beta}}\,,\, T_\alpha)\big)$$
$$- g^{\alpha\overline{\beta}} \sum_{b=1}^{2n} g^f_\Theta\big(B(f)(T_{\overline{\beta}}\,,\, E_b),\, B(f)(E_b\,,\, T_\alpha)\big)$$
$$+ (f^* g_\Theta)\big(\mathrm{trace}_{G_\theta}\, \Pi_{H(M)}\, U,\, g^{\alpha\overline{\beta}}\, U(T_{\overline{\beta}}\,,\, T_\alpha)\big)$$
$$- g^{\alpha\overline{\beta}} \sum_{b=1}^{2n} (f^* g_\Theta)\big(U(T_{\overline{\beta}}\,,\, E_b),\, U(E_b\,,\, T_\alpha)\big)$$
$$- i\, n\, (f^* g_\Theta)\big(\mathrm{trace}_{G_\theta}\, \Pi_{H(M)}\, U,\, T\big) + 2\, i\, (f^* g_\Theta)\big(g^{\alpha\overline{\beta}}\, U(T_{\overline{\beta}}\,,\, T_\alpha),\, T\big), \tag{247}$$

where $R = g^{\alpha\overline{\beta}}\, R_{\alpha\overline{\beta}}$ is the pseudohermitian scalar curvature of $(M,\, \theta)$ (see e.g., [3], p. 50). Note that for every bilinear form B on $T(M)$

$$\mathrm{trace}_{G_\theta}\, \Pi_{H(M)}\, B = g^{\alpha\overline{\beta}}\, B(T_\alpha\,,\, T_{\overline{\beta}}) + \text{complex conjugate}.$$

The identity (147) with $\tau_A = 0$ is

$$B(f)(V, W) = B(f)(W, V) - 2\, (f^* g_\Theta)(V,\, JW)\, T_A^\perp$$

for any $V, W \in \mathfrak{X}(M)$; hence

$$B(f)(T_{\overline{\beta}}\,,\, T_\alpha) = B(T_\alpha\,,\, T_{\overline{\beta}}) - 2\, i\, (f^* g_\Theta)(T_\alpha\,,\, T_{\overline{\beta}})\, T_A^\perp,$$

yielding

$$H(f) = 2\, g^{\overline{\beta}\alpha}\, B(f)\big(T_{\overline{\beta}}\,,\, T_\alpha\big) + 2\, i\, g^{\alpha\overline{\beta}}\, (f^* g_\Theta)(T_\alpha\,,\, T_{\overline{\beta}})\, T_A^\perp\,. \tag{248}$$

Note that for any $X,\, Y \in H(M)$, as f is a CR map,

$$(f^* g_\Theta)(X, Y) = g^f_\Theta(f_* X,\, f_* Y) = G^f_\Theta(f_*\,,\, f_* Y) = (f^* G_\Theta)(X, Y),$$

that is,

$$(f^* g_\Theta)(X, Y) = G_\theta(X, Y). \tag{249}$$

Hence, as $f^* g_\Theta$ is symmetric, and by (249),

$$2\, g^{\alpha\overline{\beta}}\, (f^* g_\Theta)(T_\alpha\,,\, T_{\overline{\beta}}) = \mathrm{trace}_{G_\theta}\, \Pi_{H(M)}\, f^* g_\Theta$$
$$= \sum_{a=1}^{2n} (f^* g_\Theta)(E_a\,,\, E_a) = \sum_{a=1}^{2n} G_\theta(E_a\,,\, E_a),$$

that is,

$$g^{\alpha\overline{\beta}}\, (f^* g_\Theta)(T_\alpha\,,\, T_{\overline{\beta}}) = n. \tag{250}$$

Consequently, (248) simplifies to

$$g^{\overline{\beta}\alpha}\, B(f)\big(T_{\overline{\beta}}\,,\, T_\alpha\big) = \frac{1}{2}\, H(f) - 2\, i\, n\, T_A^\perp\,. \tag{251}$$

Next, by (146) with $\tau_A = 0$,

$$U(V, W) - U(W, V) = \mathrm{Tor}_{\nabla^\top}(V, W) - \mathrm{Tor}_\nabla(V, W)$$
$$= -2\, (f^* g_\Theta)(V,\, JW)\, T_A^\top - 2\{\theta \wedge \tau - \Omega \otimes T\}(V, W), \tag{252}$$

hence

$$U(T_\alpha\,,\,T_{\overline{\beta}}) = U(T_{\overline{\beta}}\,,\,T_\alpha) + 2\,i\,(f^*g_\Theta)(T_\alpha\,,\,T_{\overline{\beta}})\,T_A^\top - 2\,i\,g_{\alpha\overline{\beta}}\,T,$$

yielding

$$\mathrm{trace}_{G_\theta}\,\Pi_{H(M)}\,U = 2\,g^{\overline{\beta}\alpha}\,U(T_{\overline{\beta}}\,,\,T_\alpha)$$
$$+2\,i\,g^{\alpha\overline{\beta}}\,(f^*g_\Theta)(T_\alpha\,,\,T_{\overline{\beta}})\,T_A^\top - 2\,i\,n\,T$$

or, by (250),

$$g^{\overline{\beta}\alpha}\,U(T_{\overline{\beta}}\,,\,T_\alpha) = \frac{1}{2}\,\mathrm{trace}_{G_\theta}\,\Pi_{H(M)}\,U - i\,n\,T_A^\top + i\,n\,T. \tag{253}$$

Substitution from (251)–(253) into (247) leads to

$$R = 2n(\lambda + n) + \frac{1}{2}\,\|H(f)\|^2_{g_\Theta} - i\,n\,\Theta^f\,(H(f))$$
$$-g^{\alpha\overline{\beta}}\sum_{b=1}^{2n} g^f_\Theta\big(B(f)(T_{\overline{\beta}}\,,\,E_b),\,B(f)(E_b\,,\,T_\alpha)\big)$$
$$+\frac{1}{2}\,\big\|\mathrm{trace}_{G_\theta}\,\Pi_{H(M)}\,U\big\|^2_{f^*g_\Theta}$$
$$-\,i\,n\,(f^*g_\Theta)\big(\mathrm{trace}_{G_\theta}\,\Pi_{H(M)}\,U,\ T_A^\top\big)$$
$$-g^{\alpha\overline{\beta}}\sum_{b=1}^{2n}\,(f^*g_\Theta)\,\big(U(T_{\overline{\beta}}\,,\,E_b),\,U(E_b\,,\,T_\alpha)\big)$$
$$+i\,(f^*g_\Theta)(\mathrm{trace}_{G_\theta}\,\Pi_{H(M)}\,U,\,T) \tag{254}$$

because, as $H(f) \in C^\infty(E(f))$,

$$g^f_\Theta\big(H(f),\,T_A^\perp\big) = g^f_\Theta\big(H(f),\,T_A\big) = \Theta^f\,(H(f)),$$
$$\big(f^*g_\Theta\big)\,\big(T_A^\top\,,\,T\big) = g^f_\Theta\big(f_*\,T_A^\top\,,\,f_*\,T\big) = g^f_\Theta\big(T_A\,,\,f_*\,T\big)$$
$$= \Theta^f\,\big(f_*\,T\big) = (f^*\Theta)(T) = \theta(T) = 1.$$

Let $\{\xi_\sigma\,:\,1 \le \sigma \le 2k\}$ be a local g^f_Θ-orthonormal frame of $E(f)$, so that $B(f) = B^\sigma(f) \otimes \xi_\sigma$ for some field $B^\sigma(f)$ of scalar $C^\infty(M)$-bilinear forms on $T(M)$. Then,

$$B^\sigma(f)(V,W) = (f^*g_\Theta)(a_{\xi_\sigma}V,\,W).$$

In particular, for any $V,\,W,\,Y,\,Z \in \mathfrak{X}(M)$

$$g^f_\Theta\big(B(f)(V,W),\,B(f)(Y,Z)\big) = \sum_{\sigma=1}^{2k}(f^*g_\Theta)(a_\sigma V,\,W)\,(f^*g_\Theta)(a_\sigma Y,\,Z)$$

where $a_\sigma \equiv a_{\xi_\sigma}$. Consequently,

$$g^{\alpha\overline{\beta}}\sum_{b=1}^{2n} g^f_\Theta\big(B(f)(T_{\overline{\beta}}\,,\,E_b),\,B(f)(E_b\,,\,T_\alpha)\big)$$
$$= \sum_{b=1}^{2n}\sum_{\sigma=1}^{2k} g^{\overline{\beta}\alpha}\,(f^*g_\Theta)(a_\sigma T_{\overline{\beta}}\,,\,E_b)\,(f^*g_\Theta)(a_\sigma E_b\,,\,T_\alpha). \tag{255}$$

Next, for any $V,W \in \mathfrak{X}(M)$ and $\xi \in C^\infty(E(f))$

$$(f^*g_\Theta)(a_\xi V,\,W) = g^f_\Theta\big(B(f)(V,W),\,\xi\big)$$

$$= g^f_\Theta(B(f)(W,V),\,\xi) - 2\,(f^*g_\Theta)(V,JW)\,g^f_\Theta(T_A^\perp\,,\,\xi)$$

or

$$(f^*g_\Theta)(a_\xi V,\, W) = (f^*g_\Theta)(a_\xi W,\, V) - 2\,(f^*g_\Theta)(V,\, JW)\,\Theta^f(\xi). \tag{256}$$

Therefore, the failure (as compared to the theory of isometric immersions) of the pseudohermitian Weingarten operator a_ξ to be self-adjoint (with respect to f^*g_Θ) is compensated by (256). Consequently,

$$(f^*g_\Theta)(a_\sigma T_{\overline{\beta}}\,,\,E_a) = (f^*g_\Theta)(a_\sigma E_a\,,\,T_{\overline{\beta}}) - 2(f^*g_\Theta)(T_{\overline{\beta}}\,,\,JE_a)\,\Theta^f(\xi_\sigma),$$

and (255) becomes

$$\begin{aligned}
&g^{\alpha\overline{\beta}}\sum_{a=1}^{2n} g^f_\Theta\big(B(f)(T_{\overline{\beta}}\,,\,E_a)\,B(f)(E_a\,,\,T_\alpha)\big)\\
&= \sum_{a,\sigma} g^{\alpha\overline{\beta}}\,(f^*g_\Theta)(a_\sigma E_a\,,\,T_\alpha)\,(f^*g_\Theta)(a_\sigma E_a\,,\,T_{\overline{\beta}})\\
&-2\sum_{a,\sigma}\Theta^f(\xi_\sigma)\,g^{\alpha\overline{\beta}}\,(f^*g_\Theta)(a_\sigma E_a\,,\,T_\alpha)\,(f^*g_\Theta)(T_{\overline{\beta}}\,,\,JE_a),
\end{aligned}$$

and, as $\{E_a\,:\,1 \le a \le 2n\}$ is G_θ-orthonormal,

$$\begin{aligned}
&-2\sum_{a,\sigma} g^{\alpha\overline{\beta}}\,(f^*g_\Theta)(a_\sigma E_a\,,\,T_\alpha)\,(f^*g_\Theta)(T_{\overline{\beta}}\,,\,JE_a)\\
&= -2\sum_{a,\sigma} g^{\alpha\overline{\beta}}\,(f^*g_\Theta)(a_\sigma E_a\,,\,T_\alpha)\,G_\theta(T_{\overline{\beta}}\,,\,JE_a)\\
&= -2i\sum_{a,\sigma} g^{\alpha\overline{\beta}}\,(f^*g_\Theta)(a_\sigma E_a\,,\,T_\alpha)\,G_\theta(T_{\overline{\beta}}\,,\,E_a)
\end{aligned}$$

so that

$$\begin{aligned}
&g^{\alpha\overline{\beta}}\sum_{a=1}^{2n} g^f_\Theta\big(B(f)(T_{\overline{\beta}}\,,\,E_a)\,B(f)(E_a\,,\,T_\alpha)\big)\\
&= \frac{1}{2}\sum_{a,\sigma}\mathrm{trace}_{G_\theta}\,\Pi_{H(M)}\Big[(f^*g_\Theta)(a_\sigma E_a\,,\,\cdot\,)\otimes(f^*g_\Theta)(a_\sigma E_a\,,\,\cdot\,)\Big]\\
&\quad -2i\sum_{a,\sigma} g^{\alpha\overline{\beta}}\,(f^*g_\Theta)(a_\sigma T_{\overline{\beta}}\,,\,T_\alpha).
\end{aligned} \tag{257}$$

Starting again from

$$(f^*g_\Theta)(a_\xi X,\, Y) = (f^*g_\Theta)(a_\xi Y,\, X) - 2\,\Omega(X,Y)\,\Theta^f(\xi)$$

for $X = T_\alpha$, $Y = T_{\overline{\beta}}$, and $\xi = \xi_\sigma$, one has

$$(f^*g_\Theta)(a_\sigma T_\alpha\,,\,T_{\overline{\beta}}) = (f^*g_\Theta)(a_\sigma T_{\overline{\beta}}\,,\,T_\alpha) + 2\,i\,g_{\alpha\overline{\beta}}\,\Theta^f(\xi_\sigma)$$

or, by contraction with $g^{\alpha\overline{\beta}}$,

$$g^{\alpha\overline{\beta}}\,(f^*g_\Theta)(a_\sigma T_\alpha\,,\,T_{\overline{\beta}}) = g^{\overline{\beta}\alpha}\,(f^*g_\Theta)(a_\sigma T_{\overline{\beta}}\,,\,T_\alpha) + 2\,i\,n\,\Theta^f(\xi_\sigma),$$

yielding

$$\begin{aligned}
&\mathrm{trace}_{G_\theta}\,\Pi_{H(M)}\,(f^*g_\Theta)(a_\sigma\,\cdot\,,\,\cdot\,)\\
&= 2\,g^{\overline{\beta}\alpha}\,(f^*g_\Theta)(a_\sigma T_{\overline{\beta}}\,,\,T_\alpha) + 2\,i\,n\,\Theta^f(\xi_\sigma).
\end{aligned} \tag{258}$$

Finally, the identity

$$(f^*g_\Theta)(a_\xi X,\, Y) = g^f_\Theta(B(f)(X,\, Y),\, \xi), \quad X,\, Y \in H(M)$$

furnishes

$$\mathrm{trace}_{G_\theta}\, \Pi_{H(M)}\, (f^* g_\Theta)(a_\xi \cdot\,,\,\cdot\,) = g^f_\Theta(H(f),\, \xi). \tag{259}$$

Therefore, the identities (258) and (259) yield

$$2\, g^{\overline{\beta}\alpha}\, (f^* g_\Theta)(a_\sigma\, T_{\overline{\beta}}\,,\, T_\alpha) = g^f_\Theta(H(f),\, \xi_\sigma) - 2\, i\, n\, \Theta^f(\xi_\sigma). \tag{260}$$

Substitution from (260) into (257) gives

$$g^{\overline{\beta}\alpha} \sum_a g^f_\Theta\big(B(f)(T_{\overline{\beta}}\,,\, E_a),\, B(E_a\,,\, T_\alpha)\big)$$

$$= \frac{1}{2} \sum_{a,b,\sigma} \Big[(f^* g_\Theta)\,(a_\sigma\, E_a\,,\, E_b)\Big]^2 - 2\, n \sum_\sigma \big[\Theta^f(\xi_\sigma)\big]^2 - i\, \Theta^f\big(H(f)\big). \tag{261}$$

Note that

$$\sum_{\sigma=1}^{2k} \big[\Theta^f(\xi_\sigma)\big]^2 = \sum_\sigma g^f_\Theta(T^f_A\,,\, \xi_\sigma)^2 = \|T^\perp_A\|^2_{g^f_\Theta}\,.$$

Again, by (252) for $V = E_b$ and $W = T_\alpha$,

$$U(E_b\,,\, T_\alpha) = U(T_\alpha\,,\, E_b) - 2\, i\, G_\theta(E_b\,,\, T_\alpha)\, T^\perp_A$$

so that

$$g^{\alpha\overline{\beta}} \sum_a (f^* g_\Theta)\big(U(T_{\overline{\beta}}\,,\, E_a),\, U(E_a\,,\, T_\alpha)\big)$$

$$= g^{\alpha\overline{\beta}} \sum_a (f^* g_\Theta)\big(U(T_{\overline{\beta}}\,,\, E_a),\, U(T_\alpha\,,\, E_a)\big)$$

$$-2\, i\, g^{\alpha\overline{\beta}} \sum_a (f^* g_\Theta)\big(U(T_{\overline{\beta}}\,,\, E_a),\, T^\perp_A\big)\, G_\theta(E_a\,,\, T_\alpha)$$

$$= \frac{1}{2} \sum_a \mathrm{trace}_{G_\theta}\, \Pi_{H(M)}\, (f^* g_\Theta)\big(U(\,\cdot\,,\, E_a),\, (\,\cdot\,,\, E_a)\big)$$

$$-2\, i\, g^{\alpha\overline{\beta}} \sum_a \theta\big(U(T_{\overline{\beta}}\,,\, E_a)\big)\, G_\theta(E_a\,,\, T_\alpha)$$

or

$$g^{\alpha\overline{\beta}} \sum_a (f^* g_\Theta)\big(U(T_{\overline{\beta}}\,,\, E_a),\, U(E_a\,,\, T_\alpha)\big)$$

$$= \frac{1}{2} \sum_{a,b} \|U(E_b\,,\, E_a)\|^2_{f^* g_\Theta} - 2\, i\, g^{\alpha\overline{\beta}}\, \theta\big(U(T_{\overline{\beta}}\,,\, T_\alpha)\big). \tag{262}$$

Substitution from (253) into (262) yields, by $\theta(T) = 1$ and

$$\theta(T^\top_A) = 1 - \|T^\perp_A\|^2_{g^f_\Theta}\,, \quad (f^* g_\Theta)(V,\, T^\top_A) = \theta(V),$$

$$g^{\overline{\beta}\alpha} \textstyle\sum_a g^f_\Theta\big(B(f)(T_{\overline{\beta}}\,,\, E_a),\, B(E_a\,,\, T_\alpha)\big)$$

$$= \tfrac{1}{2} \textstyle\sum_{a,b,\sigma} \Big[(f^* g_\Theta)\,(a_\sigma\, E_a\,,\, E_b)\Big]^2 - i\, \theta\big(\mathrm{trace}_{G_\theta}\, \Pi_{H(M)}\, U\big) + 2\, n\, \|T^\perp_A\|^2_{g^f_\Theta}\,. \tag{263}$$

Finally, let us substitute from (261) and (263) into (254) so as to obtain

$$R = 2n(\lambda + n + 1) + \frac{1}{2}\Big(\|H(f)\|^2_{g^f_\Theta} + \|\mathrm{trace}_{G_\theta}\, \Pi_{H(M)}\, U\|^2_{f^* g_\Theta}\Big)$$

$$-\frac{1}{2} \sum_{a,b} \Big\{ \sum_\sigma \big[(f^* g_\Theta)(a_\sigma\, E_a\,,\, E_b)\big]^2 + \|U(E_a\,,\, E_b)\|^2_{f^* g_\Theta} \Big\}$$

$$-i(n-1)\left\{\Theta^f\left(H(f)\right)+\theta\left(\mathrm{trace}_{G_\theta}\,\Pi_{H(M)}\,U\right)\right\},$$

and the last term vanishes because of

$$\Theta^f\left(B(f)(X,Y)\right)+\theta\left(U(X,Y)\right)$$

$$=g^f_\Theta\left(B(f)(X,Y),\,T_A^\perp\right)+\theta(\nabla^\top_X Y)$$

$$=g^f_\Theta\left(B(f)(X,Y),\,T_A^\perp\right)+(f^*g_\Theta)(\nabla^\top_X Y,\,T_A^\top)$$

by the pseudohermitian Gauss equation

$$=g^f_\Theta\left(D^f_X f_*Y,\,T_A^\perp\right)+g^f_\Theta\left(f_*\nabla^\top_X Y,\,f_*T_A^\top\right)$$

$$=g^f_\Theta\left(D^f_X f_*Y,\,T_A^\perp\right)+g^f_\Theta\left(D^f_X f_*Y,\,f_*T_A^\top\right)$$

by $T_A^\perp+f_*T_A^\top=T_A^f$

$$=g^f_\Theta\left(D^f_X f_*Y,\,T_A^f\right)=\Theta^f\left(D^f_X f_*Y\right)=0$$

for any $X,Y\in H(M)$, because $H(A)$ is parallel with respect to D. We may conclude that

$$\begin{aligned}R&=2n(\lambda+n+1)+\tfrac{1}{2}\left(\|H(f)\|^2_{g^f_\Theta}+\|\mathrm{trace}_{G_\theta}\,\Pi_{H(M)}\,U\|^2_{f^*g_\Theta}\right)\\&-\tfrac{1}{2}\textstyle\sum_{a,b}\left\{\sum_\sigma\left[(f^*g_\Theta)(a_\sigma\,E_a\,,\,E_b)\right]^2+\|U(E_a\,,\,E_b)\|^2_{f^*g_\Theta}\right\},\end{aligned}\tag{264}$$

yielding the inequality

$$R\leq 2n(\lambda+n+1)+\frac{1}{2}\left(\|H(f)\|^2_{g^f_\Theta}+\|\mathrm{trace}_{G_\theta}\,\Pi_{H(M)}\,U\|^2_{f^*g_\Theta}\right)$$

with equality if and only if $B(f)(E_a\,,\,E_b)=0$ and $U(E_a\,,\,E_b)=0$. Theorem 1 is proved.

12. Final Comments and Open Problems

Proper holomorphic maps of balls $\Phi:\mathbb{B}_N\to\mathbb{B}_{N+1}$, $N\geq 2$, and their boundary values $f:S^{2N-1}\to S^{2N+1}$ are fairly well understood from the point of view of complex analysis of functions of several complex variables. Here, $\mathbb{B}_{N+1}=\{Z\in\mathbb{C}^{N+1}\,:\,|Z|<1\}$. For instance, by a classic result of S.M. Webster (see [11]), if $N\geq 3$ and Φ is C^3 up to the boundary, then Φ is linear fractional. While Webster's theorem does not apply to the case $n=2$ (Alexander's map $\Phi(z,w)=\left(z^2\,,\,\sqrt{2}\,z\,w\,,\,w^2\right)$ is indeed a counterexample; see [33]), proper holomorphic maps $\Phi:\mathbb{B}_2\to\mathbb{B}_3$ that are C^3 up to the boundary were fully classified by J.J. Faran up to spherical equivalence (see [34]). [Two maps Φ, $\Psi:\mathbb{B}_2\to\mathbb{B}_3$ are *spherically equivalent* if $\Psi=\zeta\circ\Phi\circ\xi^{-1}$ for some $\xi\in\mathrm{Hol}(\mathbb{B}_2)$ and $\zeta\in\mathrm{Hol}(\mathbb{B}_3)$.] Let $O(2,3)$ be the set of all proper holomorphic maps from $\mathbb{B}_2$ into $\mathbb{B}_3$. Let $P(2,3)$ consist of all $\Phi\in O(2,3)$ such that Φ extends holomorphically past the boundary of $\mathbb{B}_2$, and let $P^*(2,3)$ be the corresponding quotient space, modulo spherical equivalence. Faran's result is that

$$P^*(2,3)=\left\{\mathbb{F},\,\mathbb{A}_0\,,\,\mathbb{A}_1\,,\,\mathbb{I}\right\},$$

$$\Phi_{\mathbb{F}}\in\mathbb{F},\quad\Phi_{\mathbb{A}_t}\in\mathbb{A}_t\,,\quad\Phi_{\mathbb{I}}\in\mathbb{I},\quad t\in\{0,\,1\},$$

$$\Phi_{\mathbb{F}}(z,w)=\left(z^3\,,\,\sqrt{3}\,z\,w,\,w^3\right),\quad\Phi_{\mathbb{A}_0}(z,w)=\left(z^2\,,\,\sqrt{2}\,z\,w,\,w^2\right),$$

$$\Phi_{\mathbb{A}_1}(z,w)=\left(z,\,zw,\,w^2\right),\quad\Phi_{\mathbb{I}}(z,w)=\left(0,\,z,\,w\right),$$

for any $(z,w)\in\mathbb{B}_2$. Let $\mathcal{C}_b$ be the boundary values of the class $\mathcal{C}\in P^*(2,3)$; i.e., $\mathcal{C}_b$ consists of all maps $f:S^3\to S^5$ such that $\mathbf{j}_2\circ f=\Phi\circ\mathbf{j}_1$ as Φ ranges over $\mathcal{C}$. Here, $\mathbf{j}_N:S^{2N+1}\to\mathbb{C}^{N+1}$ denotes the inclusion. When it comes to pseudohermitian geometry, however, the properties of the maps $f\in\mathcal{C}_b$ are not well understood so far. That is, aside from the natural CR structures, when one endows S^3 and S^5 with the standard Riemannian metrics (coinciding with the Webster metrics) associated with the canonical choice of contact forms $\theta = \frac{i}{2}(\overline{\partial} - \partial)|z|^2$ and $\Theta = \frac{i}{2}(\overline{\partial} - \partial)|Z|^2$, the study of the geometry of the second fundamental form of maps $f \in \mathcal{C}_b$ is an open problem. A pioneering paper in this direction is [35], aiming to find subelliptic harmonic representatives of $\mathcal{C}_b$ for each of the four classes $\mathcal{C} \in P^*(2,3)$ (see Corollary 1 in [35], p. 1470).

A classification of CR maps $f : S^3 \to \Lambda_+ \times i\,\mathbb{R}^3$, where $\Lambda_+ = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 - x_3^2 = 0, \;\; x_3 > 0\}$, was undertaken by M. Reiter and D-N. Son (see [36]), whose list, similar to Faran's list, consists of four algebraic maps and classifies the proper holomorphic maps $\Phi : \mathbb{B}_2 \to \mathcal{D}$ (with $\mathcal{D} \subset \mathbb{C}^3$ biholomorphically equivalent to the bounded symmetric domain $\mathcal{D}_3 = \{Z \in \mathbb{C}^3 \, : \, |Z| < 1, \;\; 1 - 2|Z|^2 + \left|Z^\tau Z\right|^2 > 0\}$; see, e.g., [37]) that extend past the boundary of $\mathbb{B}_2$. It is an open problem to investigate the geometry of the second fundamental form of the CR maps in Reiter and Son's list. Two of the maps in the list admit higher dimensional analogs $\Phi : \mathbb{B}_n \to \bullet$ and are rigid when $n \geq 4$, in the sense of M. Xiao and Y. Yuan [38]. A parallel between the Xiao–Yuan rigidity theory (op. cit.) and the pseudohermitian analog to classical Riemannian rigidity (see, e.g., Theorem 6.2 in [10], Volume II, p. 43) for CR immersions $f : M \to S^{2N+1}$ will require a pseudohermitian version of the Codazzi equation to be derived, by an elementary asymptotic analysis, as $\epsilon \to 0^+$, from the Codazzi Equation (95) for the isometric immersion $f : (M, g_\epsilon(f)) \to (S^{2N+1}, g_\epsilon^A)$. The Authors are grateful to the anonymous referee for suggesting a connection between the present paper and the work by Y. Li et al. [39].

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