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A New Reliability Class-Test Statistic for Life Distributions under Convolution, Mixture and Homogeneous Shock Model: Characterizations and Applications in Engineering and Medical Fields

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Abstract: Over the past few decades, a new area of reliability known as classes of life distributions has developed as a result of the creation of metrics for evaluating the success or failure of reliability. This paper proposes a new reliability class-test statistic for life distributions. In some reliability processes, such as convolution, mixture, and homogeneous shock models, the closure characteristics of the proposed class-test statistic are investigated. To compare the proposed class-test against some competitive tests, the Weibull, linear failure rate (LFR), and Makeham distributions are evaluated. In addition, the relationship between sample size, level of confidence, and critical values is considered to assess the efficacy of the proposed class-test. Furthermore, a Monte Carlo null distribution critical points simulation and some applications of the censored and uncensored data are performed to demonstrate the validity of the proposed class-test in reliability analysis.

Keywords: aging; convolution; Poisson shock model; simulation; goodness-of-fit approach; COVID-19; statistics; numerical data

MSC: 62N01; 62N03; 62N05; 62F05; 62F40

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1. Introduction

Several reliability analysts and statisticians have shown a strong interest in presenting survival data using rankings of life distributions based on different aging concepts that explain how the number of units or systems improves or deteriorates with age. There are several important types of life distributions used in applications that can be seen in reliability, including bio-metrics, engineering, medical and biological research. It was discovered that the basic distribution of the statistical reliability theory is the exponential distribution; see Mahmoud et al. [1]. Various types of life distributions have been put forward over the past few decades to model different aspects of aging. The most famous of these categories are: increasing failure rate (IFA), increasing failure rate average (IFRA), *new better than used* (NBU), *new better than used in expectation* (NBUE), *harmonic new better* *than used in expectation* (HNBUE), decreasing mean residual life (DMRL), and *new better than renewal used* (NBRU). Bryson and Siddiqui [2] and Barlow and Proschan [3] proposed some properties for these aging concepts and their duals, including decreasing failure rate (DFR), decreasing failure rate average (DFRA), *new worse than used* (NWU), *new worse than used in expectation* (NWUE). El-Arishy et al. [4] investigated the characterizations and testing hypotheses for decreasing the Laplace transform of the time to failure (DLTTF) class. Abouammoh et al. [5] studied some properties for the NBRU class. Klefsjo [6] introduced some properties for HNBUE and *harmonic new worse than used in expectation* (HNWUE) classes, and EL-Sagheer et al. [7] introduced characterizations and testing hypotheses for the NBRUL-$t_\circ$ class.

Many authors provided tests for the exponentiality of specific types of life distributions based on the Laplace transform approach technique. As an example, Gadallah et al. [8] tested *new better than used in the increasing concave order* (NBU(2)), Mansour [9] tested the NBU class, and Bakr et al. [10] tested better than aged in the moment generating function order (UBA_{mgf}). For testing exponentiality versus the *new better than renewal used in Laplace transform order* (NBRUL) class, see Mahmoud et al. [11,12], EL-Sagheer et al. [13], and Kumazawa [14]) for the NBU class. The random variable $X \in$ NBRU, if

$$\overline{W}_F(x+t) \leq \overline{F}(x)\overline{W}_F(t), \quad x, t \geq 0, \tag{1}$$

whereas the random variable $X \in$ NBRUL, if

$$\int_0^\infty e^{-sx}\overline{W}_F(x+t)dx \leq \overline{W}_F(t)\int_0^\infty e^{-sx}\overline{F}(x)dx, \quad x, t, s \geq 0,$$

or

$$\int_0^\infty \int_{x+t}^\infty e^{-sx}\overline{F}(u)dudx \leq \int_0^\infty \int_t^\infty e^{-sx}\overline{F}(x)\overline{F}(u)dudx,$$

where $\overline{W}_F(x+t) = \frac{1}{\mu}\int_{x+t}^\infty \overline{F}(u)du$ and $\overline{F}(u)$ represent the survival function. It is obvious that NBRU $\subset$ NBRUL $\subset$ NBRUE. Based on the goodness-of-fit approach, many authors offered tests for exponentiality against some classes of life distributions. For instance, Kayid et al. [15] tested the *new better than used in the increasing concave order* "NBU(2)" class, Abu-Youssef and El-Toony [16] tested used better than aged in increasing concave (UBAC(2)L), Mahmoud and Abdul Alim [17] tested *new better than used renewal failure rate* "NBURFR" and *new better than used average renewal failure rate* "NBARFR" classes, Bakr et al. [18] tested *used better than aged in Laplace* "UBAL" transform order, Abu-Youssef and Gerges [19] tested *new better than used convex order at the moment generating function* "$NBUC_{mgf}$", Mahmoud et al. [20] tested *renewal new better than used in Laplace transform order* "RNBUL", and Abu-Youssef et al. [21] tested *used better than aged in moment generating function* "UBA_{mgf}".

The goal of creating a systematic method for the study of any event and process occurring in the world was forcefully pushed forward by the essential requirements of modern science and technology. It follows that the need for such an approach in the investigation of the issue of the technological product and system reliability is quite natural. There are instances in real life where the system's components gradually degrade over time t, the amount of time covered by the manufacturer's warranty, and then there is a need for it to be renewed through the replacement of spare parts. In this case, renewal is intended to enhance the system's functionality but cannot return it to a superior state than it had at age t. For instance, after several hours of flight, the aviation administration wished to replace a portion of an airplane engine. The airlines contend that this replacement is, at best, unneeded and may potentially be detrimental to the aircraft. Airlines examine if an aviation engine after hours of renewal is as good as a new engine using operational data to support their claim.

We found that there is a lack of test efficiency and a weak test power in the nonparametric tests of life distributions. As a result, in this paper, we have established a brand-new class of life distribution that takes into account the effectiveness and power of the test. This paper can be organized as follows: some definitions for the NBRU and NBRUL classes of life distributions are listed in Section 2. In Section 3, we discuss preservation for the NBRUL class of life distribution under convolution, mixture and homogeneous Poisson shock models. A goodness-of-fit based on a test of exponentiality is discussed against the NBRUL class in Section 4. Section 5 provides the Pitman asymptotic for some life distributions. The power estimates and critical points of the Monte Carlo null distribution are simulated in Section 6. Section 7 deals with data that have been right-censored and tabulates a few critical values. Further, several real data applications are discussed based on the statistical test suggested. Finally, concluding remarks are listed in Section 8.

2. Closure Properties

In this section, the closure characteristics of the NBRUL class under some reliability operations are given as follows.

1. Property of convolution: The NBRUL class is preserved under convolution, where

$$\int_0^\infty \int_t^\infty e^{-sx}\overline{F}(x+y)dydx \le \int_0^\infty \int_t^\infty e^{-sx}\overline{F}(x)\overline{F}(y)dydx. \tag{2}$$

The following example is presented to show that the NWRUL class is not preserved under convolution.

Example 1. *The convolution of the exponential distribution $F(x) = 1 - e^{-x}$ with itself yields the gamma distribution of order 2: $G(x) = 1 - (1+x)e^{-x}$, with strictly increasing failure rate. Thus, $G(x)$ is not NWRUL.*

2. Property of mixture: The NWRUL class is preserved under mixture, where

$$\int_0^\infty \int_t^\infty e^{-sx}\overline{F}(x+y)dydx \ge \int_0^\infty \int_t^\infty e^{-sx}\overline{F}(x)\overline{F}(y)dydx. \tag{3}$$

The following example shows that the NBRUL class is not preserved under mixtures.

Example 2. *Let $\overline{F}_\alpha(u) = e^{-\alpha u}, \alpha > 0$ "scale parameter" and $\overline{G}(u) = \int_0^\infty \overline{F}_\alpha(u)e^{-\alpha}d\alpha = (u+1)^{-1}$. Then the failure rate function is $r_g(u) = (u+1)^{-1}$, which is strictly decreasing; thus, $\overline{G}(u)$ is not NBRUL.*

3. The shock model under a homogeneous Poisson process: Suppose the device is subjected to a series of shocks that occur at random time intervals using the Poisson process with intensity λ. Further suppose that the device has a probability $\overline{P}_k$. From surviving the first shock k, where $1 = \overline{P}_0 \ge \overline{P}_1 \ge ....$ and $P_j = \overline{P}_{j-1} - \overline{P}_j,\ j \ge 1$. Then, the survival function of the device is given by

$$\overline{H}(t) = \sum_{k=0}^{\infty} \overline{P}_k \frac{(\lambda t)^k}{k!} e^{-\lambda t}, \qquad t \ge 0. \tag{4}$$

If $\overline{P}_k$ is discrete NBRUL, then $\overline{H}(t)$ is given by (4) is NBRUL, where

$$\int_0^\infty \int_t^\infty e^{-sx}\overline{H}(x+y)dydx \le \int_0^\infty \int_t^\infty e^{-sx}\overline{H}(x)\overline{H}(y)dydx. \tag{5}$$

For more details about the proofs of closure properties, see EL-Sagheer et al. [13]).

3. NBRUL Comparative Testing Alternatives

In this section, a test statistic based on the goodness-of-fit approach is presented for testing the null hypothesis $H_\circ$: the distribution function F is exponential (does not belong to the NBRUL class), against the alternative hypothesis H_1 : the distribution function F is not exponential (belongs to the NBRUL class).

Lemma 1. *Let X be an NBRUL random variable with distribution function F. Then,*

$$(s^2 - s)\zeta(s)\zeta(1) + (s^2 - s)\mu\zeta(s) + (s - s^2)\zeta(s) + s\zeta(1) - s \geq \zeta(s) - 1, \quad s \geq 0, s \neq 1, \tag{6}$$

where

$$\zeta(s) = Ee^{-sX} = -\int_0^\infty e^{-sx} d\overline{F}(x).$$

Proof. Since F is NBRUL, then

$$\int_0^\infty e^{-sx}\overline{W}_F(x+t)dx \leq \overline{W}_F(t)\int_0^\infty e^{-sx}\overline{F}(x)dx, \;\; x, t \geq 0. \tag{7}$$

Consider the following integral

$$\int_0^\infty \int_0^\infty e^{-t}e^{-sx}\overline{W}_F(x+t)dxdt \leq \int_0^\infty e^{-t}\overline{W}_F(t)\int_0^\infty e^{-sx}\overline{F}(x)dxdt. \tag{8}$$

Setting,

$$I_1 = \int_0^\infty \int_0^\infty e^{-t}e^{-sx}\overline{W}_F(x+t)dxdt.$$

Hence,

$$\begin{aligned} I_1 &= \int_0^\infty \int_v^\infty e^{-v}e^{-s(u-v)}\overline{W}_F(u)dudv \\ &= \int_0^\infty \int_0^v e^{-u}e^{-s(v-u)}\overline{W}_F(v)dudv \\ &= \frac{1}{1-s}[\int_0^\infty e^{-sv}\overline{W}_F(v)dv - \int_0^\infty e^{-v}\overline{W}_F(v)dv], s \neq 1. \end{aligned}$$

Note ,

$$\begin{aligned} \int_0^\infty e^{-sv}\overline{W}_F(v)dv &= \mu_F^{-1}\int_0^\infty e^{-sv}\int_v^\infty \overline{F}(u)dudv \\ &= \mu_F^{-1}\int_0^\infty \overline{F}(v)\int_0^v e^{-su}dudv \\ &= \frac{\mu_F^{-1}}{s}[\mu - \frac{1}{s}(1-\zeta(s))]. \end{aligned}$$

Therefore,

$$I_1 = \frac{1}{1-s}\{\frac{\mu_F^{-1}}{s}[\mu - \frac{1}{s}(1-\zeta(s))] - \mu_F^{-1}[\mu - (1-\zeta(1))]\}. \tag{9}$$

Setting

$$I_2 = \int_0^\infty e^{-t}\overline{W}_F(t)\int_0^\infty e^{-sx}\overline{F}(x)dxdt,$$

gives

$$\begin{aligned} I_2 &= E\int_0^\infty e^{-t}\overline{W}_F(t)\int_0^\infty e^{-sx}I(X > x)dxdt \\ &= E\int_0^\infty e^{-t}\overline{W}_F(t)\int_0^X e^{-sx}dxdt \\ &= \frac{1}{s}(1-\zeta(s))\int_0^\infty e^{-t}\overline{W}_F(t)dt. \end{aligned}$$

Therefore,

$$I_2 = \frac{\mu_F^{-1}}{s}[\mu(1-\zeta(s)) - (1-\zeta(s))(1-\zeta(1))]. \tag{10}$$

Substituting (9) and (10) into (8), we obtain

$$(s^2-s)\zeta(1)\zeta(s) + (s^2-s)\mu\zeta(s) + (s-s^2)\zeta(s) + s\zeta(1) - s \geq \zeta(s) - 1. \tag{11}$$

This completes the proof. □

Let $X_1, X_2, ..., X_n$ be a random sample from a population with distribution F. Based on Lemma 1, $\delta(s)$ "a measure of departure from exponentiality" can be reported as

$$\delta(s) = (s^2-s)\zeta(1)\zeta(s) + (s^2-s)\mu\zeta(s) + (s-s^2-1)\zeta(s) + s\zeta(1) - s + 1. \tag{12}$$

Under $H_\circ$, $\delta(s) = 0$, whereas under H_1, $\delta(s) > 0$. The empirical estimate $\hat{\delta}(s)$ of $\delta(s)$ can be obtained as

$$\hat{\delta}(s) = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}[(s^2-s)e^{-X_i}e^{-sX_j} + (s^2-s)X_i e^{-sX_j} + (s-s^2-1)e^{-sX_i} + se^{-X_i} - s + 1].$$

To make the test invariant, let $\Delta(s) = \frac{\delta(s)}{\mu},$ which is estimated by $\hat{\Delta}(s) = \frac{\hat{\delta}(s)}{\overline{X}}$, where $\overline{X}$ is the sample mean. Then,

$$\hat{\Delta}(s) = \frac{1}{n^2\overline{X}}\sum_{i=1}^{n}\sum_{j=1}^{n}[(s^2-s)e^{-X_i}e^{-sX_j} + (s^2-s)X_i e^{-sX_j} + (s-s^2-1)e^{-sX_i} + se^{-X_i} - s + 1]. \tag{13}$$

One can note that $\hat{\Delta}(s)$ is an unbiased estimator of $\delta(s)$. Now, set

$$\phi(X_i, X_j) = (s^2-s)e^{-X_i}e^{-sX_j} + (s^2-s)X_i e^{-sX_j} + (s-s^2-1)e^{-sX_i} + se^{-X_i} - s + 1, \tag{14}$$

and define the symmetric kernel

$$\psi(X_i, X_j) = \frac{1}{2!}\sum \phi(X_i, X_j),$$

where $\hat{\Delta}(s)$ in (13) is equivalent to the U_n statistic supplied by the summation over all configurations of X_i, X_j.

$$U_n = \frac{1}{\binom{n}{2}}\sum_{i<j}^{n}\psi(X_i, X_j). \tag{15}$$

The following theorem provides a concise statement of the asymptotic normality of $\hat{\Delta}(s)$.

Theorem 1. ***(i)*** *As $n \to \infty$, $\sqrt{n}(\hat{\Delta}(s) - \Delta(s))$ is asymptotically normal with zero mean and variance $\sigma^2(s)$, where*

$$\begin{aligned}\sigma^2(s) \quad = \quad & Var\{(s^2-s)e^{-X}\zeta(s) + (s^2-s)X\zeta(s) + (s-s^2-1)e^{-sX} \\ & +se^{-X} + (s^2-s)e^{-sX}\zeta(1) + (s^2-s)\mu e^{-sX} \\ & +(s-s^2-1)\zeta(s) + s\zeta(1) - 2s + 2\}. \end{aligned} \tag{16}$$

(ii) *Under $H_\circ$, the variance $\sigma_\circ^2(s)$ can be expressed as*

$$\sigma_\circ^2(s) = \frac{(-1+s)^2 s^4(14+3s)}{12(1+s)^2(2+s)(1+2s)}. \tag{17}$$

Proof. Using standard U-statistics theory (see Lee, [22]) yields

$$\sigma^2 = V\{E[\phi(X_1, X_2) \mid X_1] + E[\phi(X_1, X_2) \mid X_2]\}. \tag{18}$$

Utilizing (14), $E[\phi(X_1, X_2) \mid X_1]$ and $E[\phi(X_1, X_2) \mid X_2]$ can be formulated as

$$\begin{aligned} E(\phi(X_1, X_2) \quad \mid \quad X_1) &= (s^2 - s)e^{-X}\int_0^\infty e^{-sx}dF(x) + (s^2 - s)X\int_0^\infty e^{-sx}dF(x) \\ &+ (s - s^2 - 1)e^{-sX} + se^{-X} - s + 1, \end{aligned}$$

and

$$\begin{aligned} E(\phi(X_1, X_2) \quad \mid \quad X_2) &= (s^2 - s)e^{-sX}\int_0^\infty e^{-x}dF(x) + (s^2 - s)e^{-sX}\int_0^\infty xdF(x) \\ &+ (s - s^2 - 1)\int_0^\infty e^{-sx}dF(x) + s\int_0^\infty e^{-x}dF(x) - s + 1, \end{aligned}$$

as long as,

$$\begin{aligned} \sigma^2(s) \quad = \quad & Var\{(s^2 - s)e^{-X}\zeta(s) + (s^2 - s)X\zeta(s) + (s - s^2 - 1)e^{-sX} \\ & + se^{-X} + (s^2 - s)e^{-sX}\zeta(1) + (s^2 - s)\mu e^{-sX} \\ & + (s - s^2 - 1)\zeta(s) + s\zeta(1) - 2s + 2\}. \end{aligned}$$

Under $H_\circ$, $\sigma_\circ^2(s)$, it can be proposed as

$$\sigma_\circ^2(s) = \frac{(-1 + s)^2 s^4 (14 + 3s)}{12(1 + s)^2(2 + s)(1 + 2s)}. \tag{19}$$

□

4. The Pitman Asymptotic Efficiency (PAE) of $\hat{\Delta}(s)$

In this section, the PAE technique's effectiveness for the Weibull, near-failure rate (LFR), and Makeham distributions is evaluated using the following probability distributions: $\bar{F}_1(x) = e^{-x^\theta}$, $x \geq 0$, $\theta > 0$ (Weibull); $\bar{F}_2(x) = e^{-x - \frac{\theta}{2}x^2}$, $x \geq 0$, $\theta \geq 0$ (LFR), and $\bar{F}_3(x) = e^{-x - \theta(x + e^{-x} - 1)}$, $x \geq 0$, $\theta \geq 0$ (Makeham). The exponential distribution is produced from these distributions by setting $\theta = 1$ for $\bar{F}_1(x)$ and $\theta = 0$ for $\bar{F}_2(x)$ and $\bar{F}_3(x)$. The PAE of $\hat{\Delta}(s)$ is defined by

$$PAE(\Delta(s)) = \frac{1}{\sigma_\circ(s)}\left|\frac{d}{d\theta}\Delta(s)\right|_{\theta \to \theta_\circ}. \tag{20}$$

$$\delta_\theta(s) = (s^2 - s)\zeta_\theta(1)\zeta_\theta(s) + (s^2 - s)\mu_\theta\zeta_\theta(s) + (s - s^2 - 1)\zeta_\theta(s) + s\zeta_\theta(1) - s + 1,$$

where

$$\mu_\theta = \int_0^\infty \overline{F}_\theta(x)dx, \zeta_\theta(s) = \int_0^\infty e^{-sx}dF_\theta(x).$$

Hence,

$$\begin{aligned} \frac{d}{d\theta}\delta_\theta(s) \quad = \quad & (s^2 - s)[\zeta_\theta(1)\zeta_\theta^{\grave{}}(s) + \zeta_\theta^{\grave{}}(1)\zeta_\theta(s)] + (s^2 - s)[\mu_\theta\zeta_\theta^{\grave{}}(s) + \mu_\theta^{\grave{}}\zeta_\theta(s)] \\ & + (s - s^2 - 1)\zeta_\theta^{\grave{}}(s) + s\zeta_\theta^{\grave{}}(1), \end{aligned}$$

where

$$\mu_\theta^{\grave{}} = \int_0^\infty \overline{F}_\theta^{\grave{}}(x)dx, \zeta_\theta^{\grave{}}(s) = -\int_0^\infty e^{-sx}d\overline{F}_\theta^{\grave{}}(x). \tag{21}$$

Making use of the definition of (20), we have

$$PAE(\delta(s)) = \frac{1}{\sigma_\circ} +(s - s^2 - 1)\zeta^{\backslash}_{\theta_\circ}(s) + s\zeta^{\backslash}_{\theta_\circ}(1)_{\theta\to\theta_\circ}, \quad (22)$$

evaluating (22) at $s = 0.99$ and $\sigma_\circ(0.99) = 0.00196211$,gives

$$PAE[\Delta(0.99), \text{ Weibull}] = 1.11564, \quad (23)$$

$$PAE[\Delta(0.99), LFR] = 0.946023 \quad (24)$$

and

$$PAE[\Delta(0.99), \text{ Makeham}] = 0.279834. \quad (25)$$

Table 1 indicates a comparison of the proposed PAE test with various other tests based on some probability models.

Table 1. Comparison between PAE test and some competitive tests.

	Models		
Test	**Makeham**	**LFR**	**Weibull**
Mugdadi and Ahmad [23]	0.039	0.408	0.170
Kango [24]	0.144	0.433	0.132
Abdel-Aziz [25]	0.184	0.535	0.223
Etman et al. [26]	0.233	0.932	1.046
EL-Sagheer et al. [13]	0.287	0.901	1.158
Proposed test $\hat{\Delta}(0.99)$	**0.280**	**0.946**	**1.116**

$\hat{\Delta}(0.99)$ is superior to the alternative tests based on the PAEs. However, EL-Sagheer et al. [12] used the Laplace transform technique with the s and β parameters, whereas in our paper, we used the goodness of fit technique with the s parameter only, and the results were as follows: the PAE used by EL-Sagheer et al. [12] is better in the case of Weibull and Makeham distributions, whereas our paper is better in the case of LFR distribution.

5. Critical Points for Monte Carlo Distribution

Using 10,000 size-generated samples with $n = 5(5)50, 29, 43, 59$, this section simulates the critical points of the Monte Carlo null distribution. We used the *Mathematica* 12 program for the common exponential distribution.

For various levels of confidence, 90%, 95%, and 99%, Table 2 provides the upper percentile values of statistic $\hat{\Delta}(0.99)$. Figure 1 shows our empirical results where the critical values increase with increasing confidence levels and approximately decrease with increasing sample size.

Table 2. Critical values of the statistic $\hat{\Delta}(0.99)$.

Sample Size	**Confidence Levels**		
n	**90%**	**95%**	**99%**
5	0.001238	0.001473	0.001939
10	0.000818	0.000955	0.001259
15	0.000657	0.000791	0.001005
20	0.000576	0.000689	0.000884
25	0.000512	0.000612	0.000793
29	0.000476	0.000565	0.000736
30	0.000470	0.000564	0.000734
35	0.000434	0.000518	0.000675
40	0.000409	0.000488	0.000633
43	0.000393	0.000474	0.000619
45	0.000392	0.000478	0.000605
50	0.000361	0.000432	0.000566
59	0.000337	0.000408	0.000530

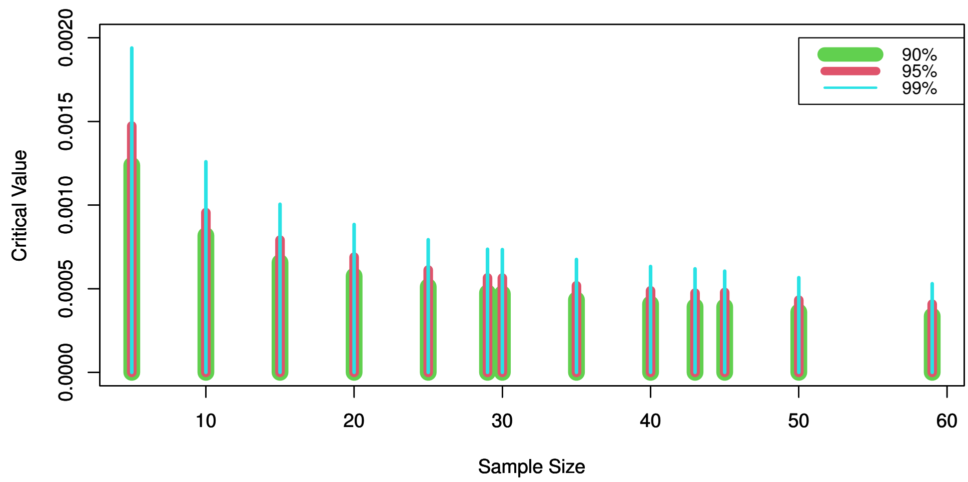

Figure 1. Relationship between the sample size, the level of confidence, and the critical values.

Estimations of Test Power

For some commonly-used distributions, such as the Weibull and gamma distributions, based on 10,000 samples, the power of the proposed test will be estimated in this section at a $(1-\alpha)\%$ confidence level, $\alpha = 0.05$ and appropriate parameter values of θ at $n = 10, 20$ and 30. The results are summarized in Table 3. It is noted that the test's power estimates $\hat{\Delta}(0.99)$ are good for all substitutions and rise when increasing the parameter value and sample size.

Table 3. The power estimates of $\hat{\Delta}(0.99)$.

n	θ	Weibull	Gamma
10	2	0.9345	0.6718
	3	0.9997	0.9371
	4	1.0000	0.9902
20	2	0.9959	0.8372
	3	1.0000	0.9962
	4	1.0000	0.9999
30	2	0.9998	0.9235
	3	1.0000	0.9996
	4	1.0000	1.0000

6. Censoring Data Testing

In this section, a test statistic for testing $H_\circ$ versus H_1 with randomly right-controlled data is proposed. Such censored data are usually the only information available in a life-test form or in a clinical study where patients may be missed (censored) before the completion of the study. This demo/experimental situation can be formally modeled as follows: suppose n objects are tested, where $X_1, X_2, ..., X_n$ denote their true lifetime. The lifetimes are independent and identically distributed (i.i.d.) according to a continuous life distribution F. Let $Y_1, Y_2, ..., Y_n$ be i.i.d. according to a continuous life distribution G. Further, assume X's and Y's are independent. In the randomly right-censored model, we observe the pairs (Z_j, δ_j), $j = 1, ..., n$, where $Z_j = \min(Xj, Y_j)$ and

$$\delta_j = \begin{cases} 1, & \text{if} \quad Z_j = X_j \ (j\text{-th observation is uncensored}), \\ 0, & \text{if} \quad Z_j = Y_j \ (j\text{-th observation is censored}). \end{cases}$$

Let $Z_{(0)} = 0 < Z_{(1)} < Z_{(2)} < .... < Z_{(n)}$ denote the ordered Z's, and $\delta_{(j)}$ is δ_j corresponding to $Z_{(j)}$. Using the censored data (Z_j, δ_j), $j = 1, ..., n$. Kaplan and Meier [27] proposed the product limit estimator, where

$$\bar{F}_n(X) =_{[j:Z(j) \leq X]} \{(n-j)/(n-j+1)\}^{\delta_{(j)}}, X \in [0, Z_{(n)}. \tag{26}$$

Now, for testing $H_\circ : \hat{\phi}_c = 0$ against $H_1 : \hat{\phi}_c > 0$, we propose the following test statistic

$$\hat{\phi}_c = (s^2 - s)\zeta(1)\zeta(s) + (s^2 - s)\mu\zeta(s) + (s - s^2 - 1)\zeta(s) + s\zeta(1) - s + 1, \tag{27}$$

where $\zeta(s) = \int_0^\infty e^{-sx} dF_n(x)$. For computational purposes, $\hat{\phi}_c$ may be rewritten as

$$\hat{\phi}_c = (s^2 - s)\tau\eta + (s^2 - s)\Omega\eta + (s - s^2 - 1)\eta + s\tau - s + 1, \tag{28}$$

where

$$\Omega = \sum_{k=1}^{n} [_{m=1}^{k-1} C_m^{\delta(m)} \left(Z_{(k)} - Z_{(k-1)} \right)], \tag{29}$$

$$\eta = \sum_{j=1}^{n} e^{-sZ_{(j)}} [_{p=1}^{j-2} C_p^{\delta(p)} -_{p=1}^{j-1} C_p^{\delta(p)}], \tag{30}$$

$$\tau = \sum_{j=1}^{n} e^{-Z_{(j)}} [_{p=1}^{j-2} C_p^{\delta(p)} -_{p=1}^{j-1} C_p^{\delta(p)}], \tag{31}$$

and

$$dF_n(Z_j) = \bar{F}_n(Z_{j-1}) - \bar{F}_n(Z_j), c_k = [n-k][n-k+1]^{-1}. \quad (32)$$

To make the test invariant, let

$$\hat{\Delta}_c = \frac{\hat{\phi}_c}{\bar{Z}}, \text{ where } \bar{Z} = \sum_{i=1}^{n} \frac{Z_{(i)}}{n}. \quad (33)$$

By utilizing the *Mathematica* 8 software, the common exponential distribution is used to simulate the Monte Carlo null distribution critical values of $\hat{\Delta}_c$ at $s = 0.99$ for sample sizes of $n = 10(10)80$, 51, and 81 with 10,000 replications. Table 4 displays the critical value percentile points for the statistic $\hat{\Delta}_c$. The critical values rise as the confidence level rises, and they fall as the sample size rises, respectively, as shown in Table 4 and Figure 2.

Table 4. The upper percentile of $\hat{\Delta}_c$ at $s = 0.99$.

Sample Size	Confidence Intervals		
n	90%	95%	99%
10	0.011499	0.014484	0.022093
20	0.007095	0.008642	0.012312
30	0.005581	0.006751	0.009710
40	0.004488	0.005425	0.007276
50	0.004075	0.004885	0.006603
51	0.004014	0.004857	0.006554
60	0.003660	0.004448	0.005918
70	0.003254	0.003928	0.005155
80	0.003016	0.003589	0.004904
81	0.002998	0.003615	0.004873

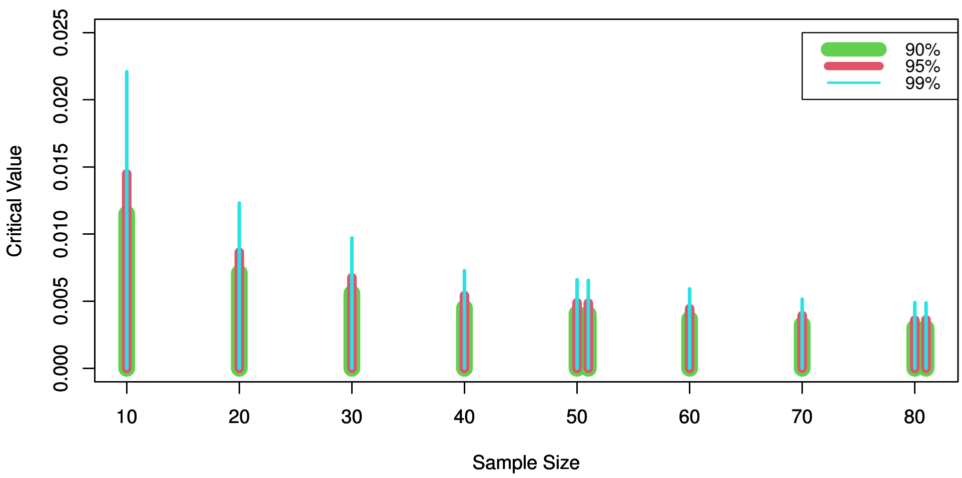

Figure 2. Relationship between the sample size, the level of confidence, and the critical values.

$\hat{\Delta}_c(s)$ Test Power Estimates

This section will assess the test's power at $(1-\alpha)\%$ confidence level, $\alpha = 0.05$, with appropriate parameter values of θ at $n = 10, 20$ and 30, concerning three alternative distributions based on 10,000 samples: Weibull, LFR, and gamma distributions. The results are listed in Table 5. It is clear that the test's power estimates $\hat{\Delta}_c(0.99)$ are good for all substitutions and rise when increasing the parameter value.

Table 5. Power estimates of $\hat{\Delta}_c(0.99)$.

n	θ	Weibull	LFR	Gamma
10	2	1.0000	1.0000	1.0000
	3	1.0000	1.0000	1.0000
	4	1.0000	1.0000	1.0000
20	2	1.0000	1.0000	1.0000
	3	1.0000	1.0000	1.0000
	4	1.0000	1.0000	1.0000
30	2	1.0000	1.0000	1.0000
	3	1.0000	1.0000	1.0000
	4	1.0000	1.0000	1.0000

7. Applications: Uncensored and Censored Observations

In this section, utilizing both censored and uncensored data at a 95% confidence level, the proposed test is applied to a number of applications in the engineering and medical sciences.

7.1. Uncensored Data

7.1.1. Data Set I: COVID-19-Italy

The COVID-19 death rate in Italy from 27 February 2020, to 27 April 2020 is represented in this data set (see Almongy et al. [28]). The data set size is 622.

4.571	7.201	3.606	8.479	11.410	8.961	10.919	10.908	6.503
18.474	11.010	17.337	16.561	13.226	15.137	8.697	15.787	13.333
11.822	14.242	11.273	14.330	16.046	11.950	10.282	11.775	10.138
9.037	12.396	10.644	8.646	8.905	8.906	7.407	7.445	7.214
6.194	4.640	5.452	5.073	4.416	4.859	4.408	4.639	3.148
4.040	4.253	4.011	3.564	3.827	3.134	2.780	2.881	3.341
2.686	2.814	2.508	2.450	1.518				

Non-parametric plots are required to discuss the shape of the data set (see Figure 3). The data display an asymmetric dimorphic shape with no extreme observations. In this example, the value shown in Table 2 is less than $\hat{\Delta}(0.99) = 0.001003$, indicating that the data set has the property NBRUL.

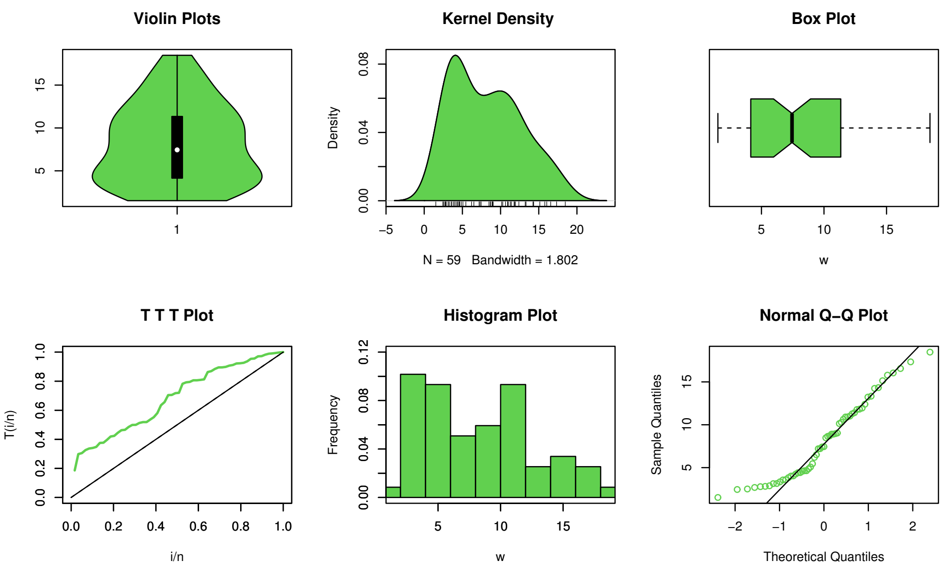

Figure 3. Non-parametric plots for data set I.

7.1.2. Data Set II: COVID-19-Netherlands

This data set represents a COVID-19 mortality rate in the Netherlands from 31 March to 30 April 2020 (see, EL-Sagheer et al. [29]). The data set is as follows

14.918	10.656	12.274	10.289	10.832	7.099	5.928	13.211
7.968	7.584	5.555	6.027	4.097	3.611	4.960	7.498
6.940	5.307	5.048	2.857	2.254	5.431	4.462	3.883
3.461	3.647	1.974	1.273	1.416	4.235		

It is clear from the non-parametric plots in Figure 4 that the data has an asymmetric dimorphic with an extreme observation. In addition, the data set has the property NBRUL since $\hat{\Delta}(0.99) = 0.001167$ is higher than the critical value displayed in Table 2 at a 95% confidence level.

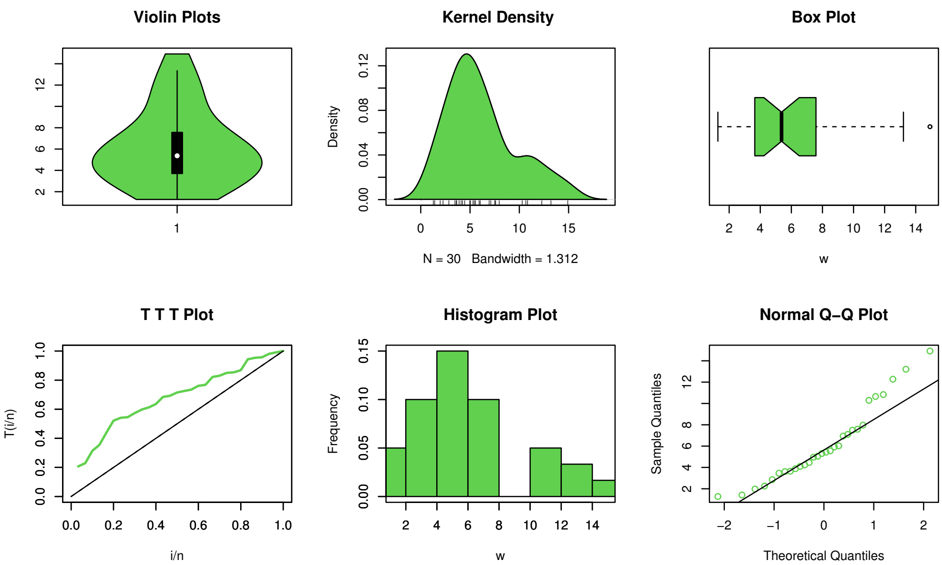

Figure 4. Non-parametric plots for data set II.

7.1.3. Data Set III: Aircraft Air Conditioning

Consider the classical real data in the study by Keating et al. [30], which were specified at times on operating days between successive malfunctions of the aircraft's air conditioning equipment. This data set is recorded as

3.750	0.417	2.500	7.750	2.542	2.042	0.583
1.000	2.333	0.833	3.292	3.500	1.833	2.458
1.208	4.917	1.042	6.500	12.917	3.167	1.083
1.833	0.958	2.583	5.417	8.667	2.917	4.208
8.667						

According to the non-parametric plots (see Figure 5), it is noted that the data have an asymmetric multimodal with an extreme observation. In this case, $\hat{\Delta}(0.99) = 0.000716$ is more than the critical value indicated in Table 2 at a 95% confidence level. Then, we disregard $H_\circ$, which asserts that the dataset exhibits an exponential property.

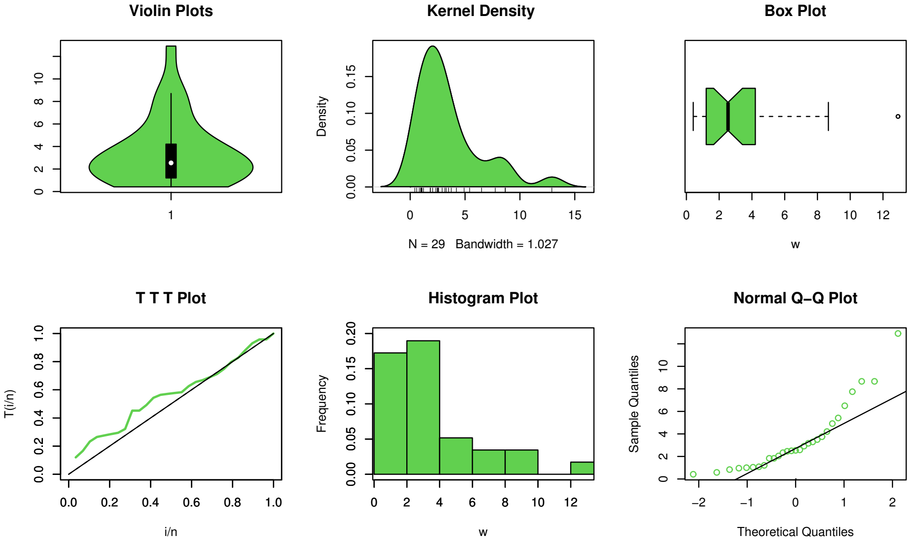

Figure 5. Non-parametric plots for data set III.

7.1.4. Data Set IV: Leukemia

We take into account the data set in the study by Kotz and Johnson [31], which shows the post-diagnosis survival times (in years) of 43 individuals with a particular type of leukemia.

0.019	0.129	0.159	0.203	0.485	0.636	0.748
0.781	0.869	1.175	1.206	1.219	1.219	1.282
1.356	1.362	1.458	1.564	1.586	1.592	1.781
1.923	1.959	2.134	2.413	2.466	2.548	2.652
2.951	3.038	3.6	3.655	3.745	4.203	4.690
4.888	5.143	5.167	5.603	5.633	6.192	6.655
6.874						

The data set contains a binary form in an asymmetric shape with no extreme values, as shown by the non-parametric plots (see Figure 6). At a 95% level of confidence, it is evident that $\hat{\Delta}(0.99) = 0.000603$ is higher than the equivalent critical value in Table 2. Therefore, the NBRUL property applies to this data set.

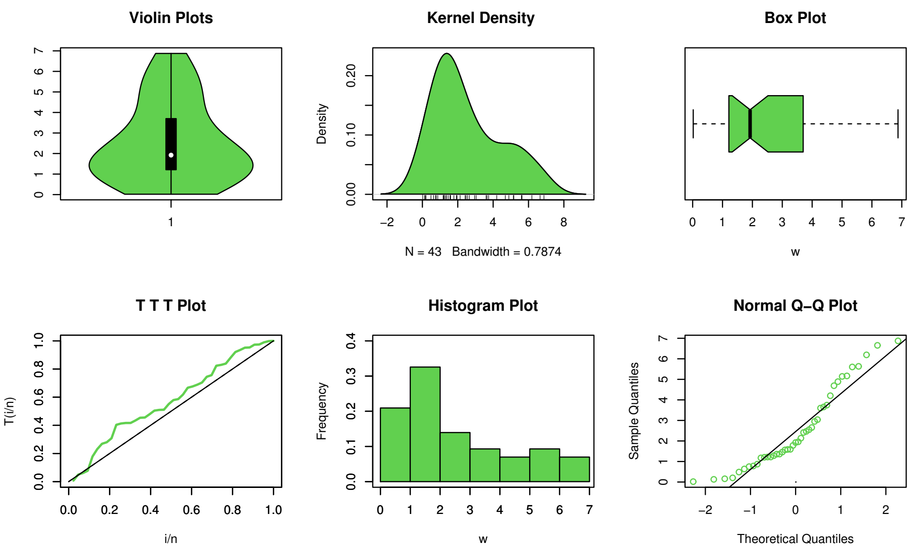

Figure 6. Non-parametric plots for data set IV.

7.2. Censored Data

7.2.1. Data Set V: Melanoma Patients

Consider the data set in the study by Susarla and Van Ryzin [32], which displays the survival rates of 46 melanoma patients, 35 of which correspond to entire lifetimes (non-censored data). The order of the non-censored observations is given by

13	14	19	19	20	21	23	23	25	26	26	27
27	31	32	34	34	37	38	38	40	46	50	53
54	57	58	59	60	65	65	66	70	85	90	98
102	103	110	118	124	130	136	138	141	234		

whereas the censored observations are ordered as follows

16	21	44	50	55	67	73	76	80	81	86	93
100	108	114	120	124	125	129	130	132	134	140	147
148	151	152	152	158	181	190	193	194	213	215	

In this example, $\hat{\Delta}_c(0.99) = -1.24863 \times 10^{88}$ is obtained by considering the entire set of survival data, both censored and uncensored. As shown in Table 4, this result is below the critical value, indicating that the data has exponential properties.

7.2.2. Data Set VI: Blood Cancer

The International Bone Marrow Transplant Registry received 101 reports from patients with advanced acute myelogenous blood malignancy (see Ghitany and Al-Awadhi [33] for further information). In order to restore their immune systems, 50 of these patients experienced an allogeneic bone marrow transplant using the marrow of histocompatibility leukocyte antigen (HLA) matched sibling. After receiving high doses of chemotherapy, 51 individuals experienced a tautologous bone marrow transplant in which their marrow was re-infused to restore their immune systems. The 50 allogeneic transplant patients' leukemia-free survival times (in months) that represent censored observations are as follows

0.030	0.493	0.855	1.184	1.283	1.480	1.776	2.138
2.500	2.763	2.993	3.224	3.421	4.178	4.441+	5.691
5.855+	6.941+	6.941	7.993+	8.882	8.882	9.145+	11.480
11.513	12.105+	12.796	12.993+	13.849+	16.612+	17.138+	20.066
20.329+	22.368+	26.776+	28.717+	28.717+	32.928+	33.783+	34.221+
34.770+	39.539+	41.118+	45.033+	46.053+	46.941+	48.289+	57.401+
58.322+	60.625+						

In the case of the complete set of survival data, which includes both censored and uncensored data, $\hat{\Delta}_c(0.99) = -9.531 \times 10^{38}$ is smaller than the critical value mentioned in Table 4 at the confidence level of 95%. Therefore, $H_\circ$ is accepted, which claims that the data set has exponential properties. For the 51 autologous transplant recipients, the leukemia-free survival periods were (in months)

0.658	0.822	1.414	2.500	3.322	3.816	4.737	4.836+
4.934	5.033	5.757	5.855	5.987	6.151	6.217	6.447+
8.651	8.717	9.441+	10.329	11.480	12.007	12.007+	12.237
12.401+	13.059+	14.474+	15.000+	15.461	15.757	16.480	16.711
17.204+	17.237	17.303+	17.664+	18.092	18.092+	18.750+	20.625+
23.158	27.730+	31.184+	32.434+	35.921+	42.237+	44.638+	46.480+
47.467+	48.322+	56.086					

Taking into account the complete set of survival data (both censored and uncensored) yielded $\hat{\Delta}_c(0.99) = -7.581 \times 10^{37}$. The data set meets the NBRUL property requirements since the $\hat{\Delta}_c(0.99)$ value is higher than the critical value in Table 4.

8. Conclusions

This paper introduces a new reliability class-test statistic. The closure characteristics of the new class-test statistic have been discussed using some reliability processes, such as convolution, mixture, and homogeneous shock models. The Weibull, near-failure rate (LFR), and Makeham distributions were evaluated to compare the new class-test to some competitive tests, and a new hypothesis test based on the goodness-of-fit approach was suggested. In order to assess the effectiveness of the new class-test, a Monte Carlo null distribution critical points simulation was performed, and the relationship between sample size, level of confidence, and critical values was considered. For different levels of confidence, 90%, 95%, and 99%, the upper percentile values of the statistic $\hat{\Delta}(0.99)$ increase with increasing levels of confidence and nearly drop with growing levels of sample size.

In order to demonstrate the validity of the proposed class-test, some applications in medical and engineering fields are discussed. In both censored and uncensored scenarios, if the $\hat{\Delta}_c(s)$ value is lower than the critical value at a 95% level of confidence, the data have exponential properties; however, if the $\hat{\Delta}_c(s)$ value is greater than the critical value at a 95% level of confidence, the data have NBRUL properties. For all substitutions, it is clear that test power estimates $\hat{\Delta}_c(0.99)$ are significant and increase when increasing the parameter value.

In future work, we aim to study the problem of testing exponentiality against the NBRUL class based on the Kernel method, defining another class of life distributions, such as NBRUL-$t_\circ$ (new better than renewal used in Laplace transform order after a specific time $t_\circ$). We also aim to study the problem of testing exponentiality against these classes from different points of view.

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