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Existence and Uniqueness Theorems for a Variable-Order Fractional Differential Equation with Delay

Benoumran Telli [1], Mohammed Said Souid [2], Jehad Alzabut [3,4,*], Hasib Khan [3,5]

1 Department of Mathematics, University of Tiaret, Tiaret 14035, Algeria
2 Department of Economic Sciences, University of Tiaret, Tiaret 14035, Algeria
3 Department of Mathematics and Sciences, Prince Sultan University, Riyadh 11586, Saudi Arabia
4 Department of Industrial Engineering, OSTİM Technical University, 06374 Ankara, Türkiye
5 Department of Mathematics, Shaheed Benazir Bhutto University, Sheringal, Dir Upper, Khyber Pakhtunkhwa 18000, Pakistan
* Correspondence: jalzabut@psu.edu.sa or jehad.alzabut@ostimteknik.edu.tr

Abstract: This study establishes the existence and stability of solutions for a general class of Riemann–Liouville (RL) fractional differential equations (FDEs) with a variable order and finite delay. Our findings are confirmed by the fixed-point theorems (FPTs) from the available literature. We transform the RL FDE of variable order to alternate RL fractional integral structure, then with the use of classical FPTs, the existence results are studied and the Hyers–Ulam stability is established by the help of standard notions. The approach is more broad-based and the same methodology can be used for a number of additional issues.

Keywords: fractional differential equations of variable order; fixed-point theorems; existence of solutions; Hyers–Ulam stability

MSC: 26A33; 34A08; 35R11

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1. Introduction

Numerous studies in science and engineering have shown the importance of mathematical modelling and numerical simulations. Fractional-order modelling is one of the well-researched areas which has provided the scientists a useful technique for the generalization of classical results. Several fractional-order operators were recently used for capturing the dynamics of physical problems. These operators are based on the singular and nonsingular kernels as well as local and nonlocal kernels. Readers can peruse some related literature given in [1–3].

Several classes of FDEs were considered and analysed for scientific aims, including sequential FDEs, hybrid FDEs, mixed FDEs, and a large number of other classes that are yet understudied in this field. The use of perturbation methods substantially facilitates the understanding of system dynamics that are described by various mathematical methods in the area of nonlinear analysis. Even if a differential equation describing a specific dynamical system may occasionally be difficult to solve or assess, by perturbing the system in some way, we can gain some insight on the system [4–7].

In the recent years, fractional operators of variable orders have been investigated and mathematically formalized. Since it is possible to create evolutionary system of equations, these operators have been successfully used to represent complicated real-world issues in a variety of fields, including biology, mechanics, transport processes, and control theory. Fractional calculus is still an active topic of research. Due to the various properties of these fractional-order operators, several operators have been developed and are being used to solve research issues in the disciplines of science and engineering. The scientific community has been actively investigating VO-FC applications for the modelling of engineering and physical systems. The reader who is interested in learning more about this intriguing topic can study [8–12].

The basic idea of VO calculus is based on the generalization of the fractional operators from constant orders, say ζ^*, to some function $\zeta^*(.)$. In 2021 and 2022, some papers dealing with this topic were published, see e.g., ([8–15]).

In particular, Bai et al. [16] established results for the existence of a solution for the following nonlinear FDEs of nonvariable order

$$\begin{cases} D^{\xi}_{0^+}\vartheta(s) = \varphi(s,\vartheta_s), \ s \in \mathcal{M} := [0,M], \ \xi \in]0,1], \\ \vartheta(s) = \chi(s), \qquad\qquad\quad s \in [-\gamma,0], \end{cases}$$

where $D^{\xi}_{0^+}$ represents an RL fractional operator for the derivative, $\varphi : \mathcal{M} \times C([-\gamma,0],\mathbb{R}) \to \mathbb{R}$ is a given function and $\chi \in C([-\gamma,0],\mathbb{R})$ via $\chi(0) = 0$. For a well defined ϑ on $[-\gamma,M]$ and $s \in \mathcal{M}$, we assume ϑ_s be from $C([-\gamma,0],\mathbb{R})$, such that

$$\vartheta_s(\zeta^*) := \vartheta(s+\zeta^*),\ \zeta^* \in [-\gamma,0].$$

We have focused in this article on the following FDE with the variable order:

$$\begin{cases} D^{\mu(s)}_{0^+}\vartheta(s) = \varphi(s,\vartheta_s), \ s \in \mathcal{M} := [0,M], \\ \vartheta(s) = \chi(s), \qquad s \in [-\gamma,0], \gamma > 0 \end{cases} \tag{1}$$

where $0 < M < +\infty$, $0 < \mu(s) \le 1$, $\varphi : \mathcal{M} \times C([-\gamma,0],\mathbb{R}) \to \mathbb{R}$, $\chi \in C([-\gamma,0],\mathbb{R})$ via $\chi(0) = 0$, and $D^{\mu(s)}_{0^+}$ is the RL fractional differential operator with variable order $\mu(\cdot)$. Some related results can be studied in the work of Telli et al. [17].

The variable order differential equations are studied in several more related articles. One can see the analytical solutions of the VO-FDEs in [18], boundary value problems with the VO in [19], a detail description about the fractional calculus for the comparison in [20,21], the integral operation in [22] and VO operators with MIttag-Leffler in [23]. These results can be applied on the works in [24,25].

Here, we investigate an FDE for the variable order in the sense of an RL derivative and establish the requirements for the existence of solutions in the subintervals. In our research, the piecewise-constant functions have a vital role in transforming the variable-order RL fractional-order initial value problem to the normal RL fractional initial value problem.

Our purpose in this article was to analyse a variable-order mathematical problem. The desired FDE of variable order is given in Section 1, and relevant notions and results are presented in Section 2. With the help of an iterative sequential technique and a limit point procedure, the existence criterion and its uniqueness property are derived in Section 3. The HU stability of the solutions is demonstrated in Section 4. In the fifth section, an illustrative example is presented. The concluding remarks are summarised in Section 6. At the end, the paper provides basic references.

2. Preliminaries

This section contains the preliminary results which are related to our aims. Consider a Banach space $(E; \|\cdot\|)$ and $C(\mathcal{M},E)$ a space of E-valued functions which are continuous on $\mathcal{M}$ with norm

$$\|\vartheta\|_{\mathcal{M}} = sup\{\|\vartheta(s)\| : s \in \mathcal{M}\},$$

for any $\vartheta \in C(\mathcal{M},E)$.

Definition 1 ([26,27])**.** *For $-\infty < c < d < +\infty$, let us assume $u : [c, d] \to (0, +\infty)$. Then, the left RL fractional integral of order $\mu(\cdot)$ for $\vartheta(\cdot)$ is expressed as*

$$I_{c^+}^{\mu(s)}\vartheta(s) = \int_c^s \frac{(s-\zeta^*)^{\mu(\zeta^*)-1}}{\Gamma(\mu(\zeta^*))}\vartheta(\zeta^*)d\zeta^*, \ s > c, \tag{2}$$

where $\Gamma(.)$ represents the gamma function.

Definition 2 ([26,27])**.** *For $-\infty < c < d < +\infty$, we assume that $\mu : [c, d] \to (m-1, m)$, $m \in \mathbb{N}$. Then, the left RL fractional differential operator of order $\mu(\cdot)$ for ϑ is expressed as*

$$D_{c^+}^{\mu(s)}\vartheta(s) = \left(\frac{d}{ds}\right)^m I_{c^+}^{m-\mu(s)}\vartheta(s) = \left(\frac{d}{ds}\right)^m \int_c^s \frac{(s-\zeta^*)^{m-\mu(\zeta^*)-1}}{\Gamma(m-\mu(\zeta^*))}\vartheta(\zeta^*)d\zeta^*, \ s > c. \tag{3}$$

The following characteristics are added from the available literature:

Lemma 1 ([28])**.** *Assume $\xi > 0$, $c \geq 0$, $\vartheta \in L^1(c, d)$, $D_{c^+}^{\xi}\vartheta \in L^1(c, d)$. Then,*

$$D_{c^+}^{\xi}\vartheta = 0$$

gives us

$$\vartheta(s) = \varsigma_1(s-c)^{\xi-1} + \varsigma_2(s-c)^{\xi-2} + ... + \varsigma_\ell(s-c)^{\xi-\ell} + ... + \varsigma_m(s-c)^{\xi-m},$$

for $m = [\xi] + 1$, $\varsigma_\ell \in \mathbb{R}$, $\ell = 1, 2, \ldots, m$.

Lemma 2 ([28])**.** *For $\xi > 0$, $c \geq 0$, $\vartheta \in L^1(c, d)$, $D_{c^+}^{\xi}\vartheta \in L^1(c, d)$, then*

$$I_{c^+}^{\xi}D_{c^+}^{\xi}\vartheta(s) = \vartheta(s) + \varsigma_1(s-c)^{\xi-1} + \varsigma_2(s-c)^{\xi-2} + \ldots + \varsigma_\ell(s-c)^{\xi-\ell} + \ldots + \varsigma_m(s-c)^{\xi-m}, \tag{4}$$

where $m = [\xi] + 1$, $\varsigma_\ell \in \mathbb{R}$, $\ell = 1, 2, \ldots, m$.

Lemma 3 ([28])**.** *Let $\xi > 0$, $c \geq 0$, $\vartheta \in L^1(c, d)$, $D_{c^+}^{\xi}\vartheta \in L^1(c, d)$. Then,*

$$D_{c^+}^{\xi}I_{c^+}^{\xi}\vartheta(s) = \vartheta(s).$$

Lemma 4 ([28])**.** *Let ξ_1, $\xi_2 > 0$, $c \geq 0$, $\vartheta \in L^1(c, d)$. Then,*

$$I_{c^+}^{\xi_1}I_{c^+}^{\xi_2}\vartheta(s) = I_{c^+}^{\xi_2}I_{c^+}^{\xi_1}\vartheta(s) = I_{c^+}^{\xi_1+\xi_2}\vartheta(s).$$

Remark 1 ([29,30])**.** *Generally, for functions $\mu_1(s)$ and $\mu_2(s)$, the semigroup property does not hold, that is,*

$$I_{c^+}^{\mu_1(s)}I_{c^+}^{\mu_2(s)}\vartheta(s) \neq I_{c^+}^{\mu_1(s)+\mu_2(s)}\vartheta(s).$$

Definition 3 ([31])**.** *The considered Equation (1) is HU stable if for a real number $c_\varphi > 0$ such that for each $\epsilon > 0$ and for each solution $y \in C([-\gamma, M], \mathbb{R})$:*

$$\begin{cases} |D_{0^+}^{u(s)}y(s) - \varphi(s, y_s)| \leq \epsilon, & s \in \mathcal{M} := [0, M], \\ y(s) = \chi(s), & s \in [-\gamma, 0], \end{cases} \tag{5}$$

there is a solution $\vartheta \in C([-\gamma, M], \mathbb{R})$ of Equation (1), which implies

$$|y(s) - \vartheta(s)| \leq c_\varphi\epsilon, \ s \in [-\gamma, M].$$

Remark 2 ([31]). *A function $y \in C([0,M],\mathbb{R})$ is a solution of the inequality*

$$|D_{0^+}^{u(s)} y(s) - \varphi(s, y_s)| \leq \epsilon, \quad s \in \mathcal{M} = [0, M], \tag{6}$$

if and only if there exists a function $h \in C([0,M],\mathbb{R})$, which depends on a solution y, such that

(i) $|h(t)| \leq \epsilon$, *for all* $t \in [-\gamma, M]$.

(ii) $D_{0^+}^{\mu(t)} y(t) = \varphi(t, y_t) + h(t)$, *for all* $t \in J$.

Proof. Assume that $y \in C([0,M],\mathbb{R})$ is a solution of (6). Let $h(s) = D_{0^+}^{u(s)} y(s) - \varphi(s, y_s) + h(s)$, for all $s \in J$. Then, with the use of (6), we have $|h(s)| \leq \epsilon$ for all $s \in [0, M]$ and $D_{0^+}^{u(s)} y(s) = \varphi(s, y_s) + h(s)$ for all $s \in J$. Conversely, we have:

$$|D_{0^+}^{u(s)} y(s) - \varphi(s, y_s)| = |h(s)| \leq \epsilon, \quad s \in [0, M]. \tag{7}$$

□

3. Existence of Solutions

The assumptions below are important in the proof of our objective results.

(HY1) Let us consider $n \in \mathbb{N}$ a positive integer and $\{M_k\}_{k=0}^{n}$ a finite sequence such that $0 = M_0 < M_{k-1} < M_k < M_n = M$, $k = 2, \ldots, n-1$.
Denote $\mathcal{M}_k := (M_{k-1}, M_k]$, $k = 1, 2, \ldots, n$. Then, $\mathcal{P} = \{\mathcal{M}_k : k = 1, 2, \ldots, n\}$ is a partition of $\mathcal{M}$.

Consider a piecewise function $\mu : \mathcal{M} \to (0, 1]$, with respect to $\mathcal{P}$, given by:

$$\mu(s) = \sum_{k=1}^{n} \mu_k I_k(s) = \begin{cases} \mu_1, & if\ s \in \mathcal{M}_1, \\ \mu_2, & if\ s \in \mathcal{M}_2, \\ . \\ . \\ . \\ \mu_n, & if\ s \in \mathcal{M}_n, \end{cases}$$

where $1 < \mu_k \leq 1$ are positive constants and I_k are indicator of $\mathcal{M}_k$, for $k = 1, 2, \ldots, n$:

$$I_k(s) = \begin{cases} 1, & for\ s \in \mathcal{M}_k, \\ 0, & for\ elsewhere. \end{cases}$$

(HY2) Let $s^\sigma \varphi : \mathcal{M} \times C([-\gamma, 0], \mathbb{R}) \to \mathbb{R}$ be continuous, $(0 < \sigma = \min_{s \in \mathcal{M}} \mu(s) < 1)$, and there exists a $K > 0$, with $s^\sigma |\varphi(s, y) - \varphi(s, z)| \leq K \|y - z\|_{[-\gamma, 0]}$, for y, $z \in C([-\gamma, M], \mathbb{R})$ and $s \in \mathcal{M}$.

Then, for any $s \in \mathcal{M}_k$, $k = 1, 2, \ldots, n$, the left Riemann–Liouville fractional-order operator of variable order $\mu(.)$ for $\vartheta \in C(\mathcal{M}, \mathbb{R})$ could be sum of left Riemann–Liouville fractional-order derivatives of μ_k, $k = 1, 2, \ldots, n$, orders

$$D_{0^+}^{\mu(s)} \vartheta(s) = \frac{d}{ds} \Big(\int_0^{M_1} \frac{(s - \zeta^*)^{-\mu_1}}{\Gamma(1 - \mu_1)} \vartheta(\zeta^*) d\zeta^* + \ldots + \int_{M_{k-1}}^{s} \frac{(s - \zeta^*)^{-\mu_k}}{\Gamma(1 - \mu_k)} \vartheta(\zeta^*) d\zeta^* \Big) \tag{8}$$

Thus, according to (8), the equation of the R-fractional Equation (1) can be written for any $s \in \mathcal{M}_k$, $k = 1, 2, \ldots, n$, as

$$\frac{d}{ds} \Big(\int_0^{M_1} \frac{(s - \zeta^*)^{-\mu_1}}{\Gamma(1 - \mu_1)} \vartheta(\zeta^*) d\zeta^* + \ldots + \int_{M_{k-1}}^{s} \frac{(s - \zeta^*)^{-\mu_k}}{\Gamma(1 - \mu_k)} \vartheta(\zeta^*) d\zeta^* \Big) = \varphi(s, \vartheta_s), \ s \in \mathcal{M}_k. \tag{9}$$

Here, the solution of Equation (1) is stated, which is important in this paper.

Definition 4. *Subjected to $\vartheta_k, k = 1, 2, \ldots, n$, where $\vartheta_k \in C([-\gamma, M_k], \mathbb{R})$, (1) has a solution if (9) is satisfied for $s \in [0, M_k]$, $\vartheta(s) = \chi(s)$, and if $s \in [-\gamma, 0]$ with $\vartheta_k(M_{k-1}) = 0$.*

Consider $\vartheta \in C(\mathcal{M}, \mathbb{R})$ so that $\vartheta \equiv 0$ on $s \in [0, M_{k-1}]$, and it is the solution of (9). Then, (9) implies

$$D^{\mu_k}_{M^+_{k-1}} \vartheta(s) = \varphi(s, \vartheta_s), \quad s \in \mathcal{M}_k.$$

We shall deal with the following IVP:

$$\begin{cases} D^{\mu_k}_{M^+_{k-1}} \vartheta = \varphi(s, \vartheta_s), \quad s \in \mathcal{M}_k, \\ \vartheta(M_{k-1}) = 0, \\ \vartheta(s) = \chi_k, \; s \in [M_{k-1} - \gamma', M_{k-1}], \end{cases} \tag{10}$$

here $\gamma' = M_{k-1} + \gamma$ and

$$\chi_k(s) = \begin{cases} 0, \quad if \; s \in [0, M_{k-1}] \\ \chi(s), \quad if \; s \in [-\gamma, 0]. \end{cases}$$

For the existence of a solution of (10), the following lemma is important.

Lemma 5. *$\vartheta \in C([-\gamma, M_k], \mathbb{R})$ is a solution of (10) if and only if ϑ is such that*

$$\vartheta(s) = \begin{cases} \frac{1}{\Gamma(\mu_k)} \int_{M_{k-1}}^{s} (s - \zeta^*)^{\mu_k - 1} \varphi(\zeta^*, \vartheta_{\zeta^*}) d\zeta^*, \quad if \; s \in \mathcal{M}_k, \\ \chi_k(s), \quad if \; s \in [-\gamma, M_{k-1}]. \end{cases}, \tag{11}$$

where $k = 1, 2, \ldots, n$.

Proof. Let $\vartheta \in C([-\gamma, M_k], \mathbb{R})$ be a solution of Equation (10). From (4), we have

$$\vartheta(s) = \varsigma_1 (s - M_{k-1})^{\mu_k - 1} + I^{\mu_k}_{M^+_{k-1}} \varphi(s, \vartheta_s), \quad s \in \mathcal{M}_k, \; k \in \{1, 2, \ldots, n\}. \tag{12}$$

Using $\vartheta(M_{k-1}) = 0$, we find that $\varsigma_1 = 0$. By substituting the value of ς_1 on (12), we obtain

$$\vartheta(s) = I^{\mu_k}_{M^+_{k-1}} \varphi(s, \vartheta_s), \quad s \in \mathcal{M}_k.$$

Then, the solution of Equation (10) is given by

$$\vartheta(s) = \begin{cases} \frac{1}{\Gamma(\mu_k)} \int_{M_{k-1}}^{s} (s - \zeta^*)^{\mu_k - 1} \varphi(\zeta^*, \vartheta_{\zeta^*}) d\zeta^*, \quad if \; s \in \mathcal{M}_k, \\ \chi_k(s), \quad if \; s \in [-\gamma, M_{k-1}]. \end{cases},$$

For the converse proof, let $\vartheta \in C([-\gamma, M_k], \mathbb{R})$ be a solution (11); then, by Lemma 3, we have ϑ is the solution of (10). □

We prove the existence of a solution for Equation (10).

Theorem 1. *Assume the conditions (**HY1**) and (**HY2**), with*

$$\frac{K\Gamma(1 - \sigma)(M_k - M_{k-1})^{\mu_k - \sigma}}{\Gamma(\mu_k + 1 - \sigma)} < 1. \tag{13}$$

Then, (10) has a solution on $C([-\gamma, M_k], \mathbb{R})$.

Proof. We construct the operator
$\mathcal{N} : C([-\gamma, M_k], \mathbb{R}) \to C([-\gamma, M_k], \mathbb{R})$ as follows:

$$(\mathcal{N}\vartheta)(s) = \begin{cases} \chi_k(s),\ s \in [-\gamma, M_{k-1}] \\ \frac{1}{\Gamma(\mu_k)} \int_{M_{k-1}}^{s} (s-\zeta^*)^{\mu_k - 1} \varphi(\zeta^*, \vartheta_{\zeta^*}) d\zeta^*,\ \ if\ s \in \mathcal{M}_k \end{cases}$$

Let $v(.) : [-\gamma, M_k] \to \mathbb{R}$ given by:

$$v = \begin{cases} 0,\ \ if\ s \in \mathcal{M}_k \\ \chi_k(s),\ \ if\ s \in [-\gamma, M_{k-1}]. \end{cases}$$

For $z \in C([M_{k-1}, M_k], \mathbb{R})$, with $z(M_{k-1}) = 0$, we represent $\overline{z}$ as

$$\overline{z}(s) = \begin{cases} z(s),\ \ if\ s \in \mathcal{M}_k \\ 0,\ \ if\ s \in [-\gamma, M_{k-1}]. \end{cases}$$

If $\vartheta(.)$ satisfies

$$\vartheta(s) = \frac{1}{\Gamma(\mu_k)} \int_{M_{k-1}}^{s} (s-\zeta^*)^{\mu_k - 1} \varphi(\zeta^*, \vartheta_{\zeta^*}) d\zeta^*,\ \ if\ s \in \mathcal{M}_k,$$

we can divide in parts $\vartheta(.)$ as $\vartheta(s) = z(s) + v(s),\ M_{k-1} \le s \le M_k$, which yields $\vartheta_s = \overline{z}_s + v_s$, for $M_{k-1} \le s \le M_k$. Set

$$C_{M_{k-1}} = \{z \in C([M_{k-1}, M_k], \mathbb{R}) : z(M_{k-1}) = 0\},$$

and let $\|.\|_{M_k}$ be the norm in $C_{M_{k-1}}$ such that

$$\|z\|_{M_k} = \sup_{s \in \mathcal{M}_k} |z(s)|,\ z \in C_{M_{k-1}}.$$

Here, $C_{M_{k-1}}$ is a separable Banach space with norm $\|.\|_{M_k}$. Let us define an operator $\mathcal{E} : C_{M_{k-1}} \to C_{M_{k-1}}$ as

$$(\mathcal{E}z)(s) = \frac{1}{\Gamma(u_k)} \int_{M_{k-1}}^{s} (s-\zeta^*)^{\mu_k - 1} \varphi(\zeta^*, \overline{z}_{\zeta^*} + v_{\zeta^*}) d\zeta^*,\ s \in \mathcal{M}_k, \tag{14}$$

By the definition of $s^{\sigma}\varphi$, the operator $\mathcal{E} : C_{M_{k-1}} \to C_{M_{k-1}}$ is well-defined in (14).

Then, it is enough to show that the operator $\mathcal{E}$ has a fixed point z which guarantees that the operator $\mathcal{N}$ has a fixed point $\vartheta = \overline{z} + v$ and in consequence, this fixed point will correspond to a solution of Equation (10), indeed

$$\begin{aligned} \kappa(s) &= \overline{z}(s) + v(s) \\ &= \begin{cases} z(s),\ \ if\ s \in \mathcal{M}_k \\ \chi_k(s),\ \ if\ s \in [-\gamma, M_{k-1}]. \end{cases} \\ &= \begin{cases} \frac{1}{\Gamma(\mu_k)} \int_{M_{k-1}}^{s} (s-\zeta^*)^{\mu_k - 1} \varphi(\zeta^*, \overline{z}_{\zeta^*} + v_{\zeta^*}) d\zeta^*,\ \ if\ s \in \mathcal{M}_k \\ \chi_k(s),\ \ if\ s \in [-\gamma, M_{k-1}]. \end{cases} \\ &= \begin{cases} \frac{1}{\Gamma(\mu_k)} \int_{M_{k-1}}^{s} (s-\zeta^*)^{\mu_k - 1} \varphi(\zeta^*, \vartheta_{\zeta^*}) d\zeta^*,\ \ if\ s \in \mathcal{M}_k \\ \chi_i(s),\ \ if\ s \in [-\gamma, M_{k-1}]. \end{cases} \\ &= (\mathcal{N}\vartheta)(s). \end{aligned}$$

Let

$$R_k \ge \frac{\frac{\Gamma(1-\sigma)(M_k - M_{k-1})^{\mu_k - \sigma}}{\Gamma(\mu_k + 1 - \sigma)} \left(K\|\chi\|_{[-\gamma, 0]} + \varphi^{\star} \right)}{1 - \frac{K\Gamma(1-\sigma)(M_k - M_{k-1})^{\mu_i - \sigma}}{\Gamma(\mu_k + 1 - \sigma)}},$$

with $\varphi^\star = \sup_{s\in\mathcal{M}_k} s^\sigma|\varphi(s,0)|$. Let us consider:

$$B_{R_k} = \{z \in C_{M_{k-1}},\ \|z\|_{M_k} \le R_k\}.$$

Thus, B_{R_i} is nonempty, closed, convex, and a bounded set.

Let $z \in B_{R_k}$ and $s \in \mathcal{M}_k$; then,

$$\begin{aligned}
\|\overline{z}_s\|_{[-\gamma',0]} &= \sup_{-M_{k-1}-\gamma\le\theta\le 0} |\overline{z}_s(\theta)| \\
&= \sup_{-M_{k-1}-\gamma\le\theta\le 0} |\overline{z}(s+\theta)| \\
&\le \sup_{-\gamma\le\zeta^*\le M_k} |\overline{z}(\zeta^*)| \\
&= \sup_{\zeta^*\in\mathcal{M}_k} |z(\zeta^*)| = \|z\|_{M_k}
\end{aligned}$$

and

$$\begin{aligned}
\|v_s\|_{[-\gamma',0]} &= \sup_{-M_{k-1}-\gamma\le\theta\le 0} |v_s(\theta)| \\
&= \sup_{-M_{k-1}-\gamma\le\theta\le 0} |v(s+\theta)| \\
&\le \sup_{-\gamma\le\zeta^*\le M_k} |v(\zeta^*)| \\
&= \sup_{-\gamma\le\zeta^*\le 0} |v(\zeta^*)| \\
&= \sup_{-\gamma\le\zeta^*\le 0} |\chi(\zeta^*)| \\
&= \|\chi\|_{[-\gamma,0]}
\end{aligned}$$

Now, we show that $\mathcal{E}$ satisfies the requirements of Schauder's FPT. For this, we have:

Step 1: $\mathcal{E}(B_{R_k}) \subseteq (B_{R_k})$.

For $z \in B_{R_k}$, by **(HY2)**, we get

$$\begin{aligned}
|\mathcal{E}z(s)| &= \left|\frac{1}{\Gamma(\mu_k)}\int_{M_{k-1}}^{s}(s-\zeta^*)^{\mu_k-1}\varphi(\zeta^*,\overline{z}_{\zeta^*}+v_{\zeta^*})d\zeta^*\right| \\
&\le \frac{1}{\Gamma(\mu_k)}\int_{M_{k-1}}^{s}(s-\zeta^*)^{\mu_k-1}|\varphi(s,\overline{z}_{\zeta^*}+v_{\zeta^*})|d\zeta^* \\
&\le \frac{1}{\Gamma(\mu_k)}\int_{M_{k-1}}^{s}(s-\zeta^*)^{\mu_k-1}\zeta^{*-\sigma}\zeta^{*\sigma}\left|\varphi(\zeta^*,\overline{z}_{\zeta^*}+v_{\zeta^*})-\varphi(\zeta^*,0)\right|d\zeta^* \\
&+ \frac{1}{\Gamma(\mu_k)}\int_{M_{k-1}}^{s}(s-\zeta^*)^{\mu_k-1}\zeta^{*-\sigma}\zeta^{*\sigma}|\varphi(\zeta^*,0)|d\zeta^* \\
&\le \frac{1}{\Gamma(\mu_k)}\int_{M_{k-1}}^{s}(s-\zeta^*)^{\mu_k-1}\zeta^{*-\sigma}(K\|\overline{z}_{\zeta^*}+v_{\zeta^*}\|_{[-\gamma',0]})d\zeta^* \\
&+ \frac{\varphi^\star}{\Gamma(\mu_k)}\int_{M_{k-1}}^{s}(s-\zeta^*)^{\mu_k-1}\zeta^{*-\sigma}d\zeta^* \\
&\le \frac{K}{\Gamma(\mu_k)}\int_{M_{k-1}}^{s}(s-\zeta^*)^{\mu_k-1}(\|\overline{z}_{\zeta^*}\|_{[-\gamma',0]}+\|v_{\zeta^*}\|_{[-\gamma',0]})\zeta^{*-\sigma}d\zeta^* \\
&+ \frac{\varphi^\star}{\Gamma(\mu_k)}\int_{M_{k-1}}^{s}(s-\zeta^*)^{\mu_k-1}\zeta^{*-\sigma}d\zeta^* \\
&\le \frac{K}{\Gamma(\mu_k)}(\|z\|_{M_k}+\|\chi\|_{[-\gamma,0]})\int_{M_{k-1}}^{s}(s-\zeta^*)^{\mu_k-1}\zeta^{*-\sigma}d\zeta^* \\
&+ \frac{\varphi^\star}{\Gamma(u_i)}\int_{M_{i-1}}^{s}(s-\zeta^*)^{u_i-1}\zeta^{*-\sigma}d\zeta^*
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{K}{\Gamma(\mu_k)}(\|z\|_{M_k}+\|\chi\|_{[-\gamma,0]})\int_{M_{k-1}}^{s}(s-\zeta^*)^{\mu_k-1}(\zeta^*-M_{k-1})^{-\sigma}d\zeta^* \\
&+ \frac{\varphi^\star}{\Gamma(\mu_k)}\int_{M_{k-1}}^{s}(s-\zeta^*)^{\mu_k-1}(\zeta^*-M_{k-1})^{-\sigma}d\zeta^* \\
&\leq \frac{K\Gamma(1-\sigma)(M_k-M_{k-1})^{\mu_k-\sigma}}{\Gamma(\mu_k+1-\sigma)}(\|z\|_{M_k}+\|\chi\|_{[-\gamma,0]}) \\
&+ \frac{\varphi^\star\Gamma(1-\sigma)(M_k-M_{k-1})^{\mu_k-\sigma}}{\Gamma(\mu_k+1-\sigma)}
\end{aligned}$$

$$\begin{aligned}
|\mathcal{E}z(s)| &\leq \frac{K\Gamma(1-\sigma)(M_k-M_{k-1})^{\mu_k-\sigma}}{\Gamma(\mu_k+1-\sigma)}\|z\|_{M_k} \\
&+ \frac{K\Gamma(1-\sigma)(M_k-M_{k-1})^{\mu_k-\sigma}}{\Gamma(\mu_k+1-\sigma)}\|\chi\|_{[-\gamma,0]} \\
&+ \frac{\varphi^\star\Gamma(1-\sigma)(M_k-M_{k-1})^{\mu_k-\sigma}}{\Gamma(\mu_k+1-\sigma)} \\
&\leq \frac{K\Gamma(1-\sigma)(M_k-M_{k-1})^{\mu_k-\sigma}}{\Gamma(\mu_k+1-\sigma)}\|z\|_{M_k} \\
&+ \frac{\Gamma(1-\sigma)(M_k-M_{k-1})^{\mu_k-\sigma}}{\Gamma(\mu_k+1-\sigma)}\Big(K\|\chi\|_{[-\gamma,0]}+\varphi^\star\Big) \\
&\leq \frac{K\Gamma(1-\sigma)(M_k-M_{k-1})^{\mu_k-\sigma}}{\Gamma(\mu_k+1-\sigma)}R_k \\
&+ \frac{\Gamma(1-\sigma)(M_k-M_{k-1})^{\mu_k-\sigma}}{\Gamma(\mu_k+1-\sigma)}\Big(K\|\chi\|_{[-\gamma,0]}+\varphi^\star\Big) \\
&\leq R_k,
\end{aligned}$$

which means that $\mathcal{E}(B_{R_k}) \subseteq B_{R_k}$.

Step 2: $\mathcal{E}$ is continuous.

We assume that (z_n) converges to z in $C_{T_{k-1}}$ and $s \in \mathcal{M}_k$. Then,

$$\begin{aligned}
|(\mathcal{E}z_n)(s)-(\mathcal{E}z)(s)| &= \Big|\frac{1}{\Gamma(\mu_k)}\int_{M_{k-1}}^{s}(s-\zeta^*)^{\mu_k-1}\varphi(\zeta^*,\overline{z_n}_{\zeta^*}+v_{\zeta^*})d\zeta^* \\
&- \frac{1}{\Gamma(\mu_k)}\int_{M_{k-1}}^{s}(s-\zeta^*)^{\mu_k-1}\varphi(\zeta^*,\overline{z}_{\zeta^*}+v_{\zeta^*})d\zeta^*\Big| \\
&\leq \frac{1}{\Gamma(\mu_k)}\int_{M_{k-1}}^{s}(s-\zeta^*)^{\mu_k-1}\Big|\varphi(\zeta^*,\overline{z_n}_{\zeta^*}+v_{\zeta^*})-\varphi(\zeta^*,\overline{z}_{\zeta^*}+v_{\zeta^*})\Big|d\zeta^* \\
&\leq \frac{1}{\Gamma(\mu_k)}\int_{M_{k-1}}^{s}(s-\zeta^*)^{\mu_k-1}\zeta^{*-\sigma}\zeta^{*\sigma}\Big|\varphi(\zeta^*,\overline{z_n}_{\zeta^*}+v_{\zeta^*})-\varphi(\zeta^*,\overline{z}_{\zeta^*}+v_{\zeta^*})\Big|d\zeta^* \\
&\leq \frac{1}{\Gamma(\mu_k)}\int_{M_{k-1}}^{s}(s-\zeta^*)^{\mu_k-1}\zeta^{*-\sigma}K\|\overline{z_n}_{\zeta^*}-\overline{z}_{\zeta^*}\|_{[-\gamma',0]}d\zeta^* \\
&\leq (K\|z_n-z\|_{M_k})\frac{1}{\Gamma(\mu_k)}\int_{M_{k-1}}^{s}(s-\zeta^*)^{\mu_k-1}\zeta^{*-\sigma}d\zeta^* \\
&\leq (K\|z_n-z\|_{M_k})\frac{1}{\Gamma(\mu_k)}\int_{M_{k-1}}^{s}(s-\zeta^*)^{\mu_k-1}(\zeta^*-M_{k-1})^{-\sigma}d\zeta^* \\
&\leq (K\|z_n-z\|_{M_k})\frac{\Gamma(1-\sigma)(M_k-M_{k-1})^{\mu_k-\sigma}}{\Gamma(\mu_k+1-\sigma)} \\
&\leq \frac{K\Gamma(1-\sigma)(M_k-M_{k-1})^{\mu_k-\sigma}}{\Gamma(\mu_k+1-\sigma)}\|z_n-z\|_{M_k},
\end{aligned}$$

i.e., we get

$$\|(\mathcal{E}z_n)-(\mathcal{E}z)\|_{M_k} \to 0 \text{ as } n \to \infty.$$

Then, $\mathcal{E}$ is continuous on $C_{M_{k-1}}$.

Step 3: $\mathcal{E}$ maps bounded sets into bounded sets in $C_{M_{k-1}}$.

Let B_r be a bounded subset of B_{R_k}; as in Step 2, we have

$$\mathcal{E}(B_r) \subset \mathcal{E}(B_{R_k}) \subset B_{R_k}.$$

$\mathcal{E}$ maps bounded sets into bounded sets in $C_{M_{k-1}}$.

Step 4: $\mathcal{E}$ maps bounded sets into equicontinous sets in $C_{M_{k-1}}$.

We consider B_R a bounded set in $C_{M_{k-1}}$, for arbitrary $s_1, s_2 \in \mathcal{M}_k$, with $s_1 < s_2$, let $z \in B_R$, and we write

$$\begin{aligned}
|(\mathcal{E}z)(s_2) - (\mathcal{E}z)(s_1)| &= \Big|\frac{1}{\Gamma(\mu_k)}\int_{M_{k-1}}^{s_2}(s_2-\zeta^*)^{\mu_k-1}\varphi(\zeta^*,\bar{z}_{\zeta^*}+v_{\zeta^*})d\zeta^* \\
&- \frac{1}{\Gamma(\mu_k)}\int_{M_{k-1}}^{s_1}(s_1-\zeta^*)^{\mu_k-1}\varphi(\zeta^*,\bar{z}_{\zeta^*}+v_{\zeta^*})d\zeta^*\Big| \\
&\leq \frac{1}{\Gamma(\mu_k)}\int_{M_{k-1}}^{s_1}\Big|\Big((s_2-\zeta^*)^{\mu_k-1}-(s_1-\zeta^*)^{\mu_k-1}\Big)\varphi(\zeta^*,\bar{z}_{\zeta^*}+v_{\zeta^*})\Big|d\zeta^* \\
&+ \frac{1}{\Gamma(\mu_k)}\int_{s_1}^{s_2}\Big|(s_2-\zeta^*)^{\mu_k-1}\varphi(\zeta^*,\bar{z}_{\zeta^*}+v_{\zeta^*})\Big|d\zeta^* \\
&\leq \frac{1}{\Gamma(\mu_k)}\int_{M_{k-1}}^{s_1}\Big|\Big((s_2-\zeta^*)^{\mu_k-1}-(s_1-\zeta^*)^{\mu_k-1}\Big)\Big|\Big|\varphi(\zeta^*,\bar{z}_{\zeta^*}+v_{\zeta^*})\Big|d\zeta^* \\
&+ \frac{1}{\Gamma(\mu_k)}\int_{s_1}^{s_2}(s_2-\zeta^*)^{\mu_k-1}|\varphi(\zeta^*,\bar{z}_{\zeta^*}+v_{\zeta^*})\Big|d\zeta^* \\
&\leq \frac{1}{\Gamma(\mu_k)}\int_{M_{k-1}}^{s_1}\Big|\Big((s_2-\zeta^*)^{\mu_k-1}-(s_1-\zeta^*)^{\mu_k-1}\Big)\Big| \\
&\quad \zeta^{*-\sigma}\Big(\zeta^{*\sigma}\Big|\varphi(\zeta^*,\bar{z}_{\zeta^*}+v_{\zeta^*})-\varphi(\zeta^*,0)\Big|+\zeta^{*\sigma}|\varphi(\zeta^*,0)|\Big)d\zeta^* \\
&+ \frac{1}{\Gamma(\mu_k)}\int_{s_1}^{s_2}(s_2-\zeta^*)^{\mu_k-1}\zeta^{*-\sigma}\Big(\zeta^{*\sigma}\Big|\varphi(\zeta^*,\bar{z}_{\zeta^*}+v_{\zeta^*})-\varphi(\zeta^*,0)\Big|+\zeta^{*\sigma}|\varphi(\zeta^*,0)|\Big)d\zeta^* \\
&\leq \frac{1}{\Gamma(\mu_k)}\int_{M_{k-1}}^{s_1}\Big|\Big((s_2-\zeta^*)^{\mu_k-1}-(s_1-\zeta^*)^{\mu_k-1}\Big)\Big| \\
&\quad \Big[\zeta^{*-\sigma}(K\|\bar{z}_{\zeta^*}+v_{\zeta^*}\|_{[-\gamma',0]})+\zeta^{*-\sigma}\varphi^\star\Big]d\zeta^* \\
&+ \frac{1}{\Gamma(\mu_k)}\int_{s_1}^{s_2}(s_2-\zeta^*)^{\mu_k-1}\Big[\zeta^{*-\sigma}(K\|\bar{z}_{\zeta^*}+v_{\zeta^*}\|_{[-\gamma',0]})+\zeta^{*-\sigma}\varphi^\star\Big]d\zeta^*
\end{aligned}$$

$$\begin{aligned}
|(\mathcal{E}z)(s_2) - (\mathcal{E}z)(s_1)| &\leq \frac{1}{\Gamma(\mu_k)}\int_{M_{k-1}}^{s_1}\Big|\Big((s_2-\zeta^*)^{\mu_k-1}-(s_1-\zeta^*)^{\mu_k-1}\Big)\Big| \\
&\quad \Big[\zeta^{*-\sigma}K(\|\bar{z}_{\zeta^*}\|_{[-\gamma',0]}+\|v_{\zeta^*}\|_{[-\gamma',0]})+\zeta^{*-\sigma}\varphi^\star\Big]d\zeta^* \\
&+ \frac{1}{\Gamma(\mu_k)}\int_{s_1}^{s_2}(s_2-\zeta^*)^{\mu_k-1}\Big[\zeta^{*-\sigma}K(\|\bar{z}_{\zeta^*}\|_{[-\gamma',0]}+\|v_{\zeta^*}\|_{[-\gamma',0]})+\zeta^{*-\sigma}\varphi^\star\Big]d\zeta^* \\
&\leq \frac{1}{\Gamma(\mu_k)}\int_{M_{k-1}}^{s_1}\Big|\Big((s_2-\zeta^*)^{\mu_k-1}-(s_1-\zeta^*)^{\mu_k-1}\Big)\Big| \\
&\quad \Big[\zeta^{*-\sigma}K(\|z\|_{M_k}+\|\chi\|_{[-\gamma,0]})+\zeta^{*-\sigma}\varphi^\star\Big]d\zeta^* \\
&+ \frac{1}{\Gamma(\mu_k)}\int_{s_1}^{s_2}(s_2-s\zeta^*)^{\mu_k-1}\Big[\zeta^{*-\sigma}K(\|z\|_{M_k}+\|\chi\|_{[-\gamma,0]})+\zeta^{*-\sigma}\varphi^\star\Big]d\zeta^* \\
&\leq M_{k-1}^{-\sigma}\Big[K(\|z\|_{M_k}+\|\chi\|_{[-\gamma,0]})+\varphi^\star\Big]\times \\
&\quad \frac{1}{\Gamma(\mu_k)}\Big(\int_{M_{k-1}}^{s_1}\Big((s_1-\zeta^*)^{\mu_k-1}-(s_2-\zeta^*)^{\mu_k-1}\Big)d\zeta^*+\int_{s_1}^{s_2}(s_2-\zeta^*)^{\mu_k-1}d\zeta^*\Big) \\
&\leq \frac{M_{k-1}^{-\sigma}}{\Gamma(\mu_k+1)}\Big(K(\|z\|_{M_k}+\|\chi\|_{[-\gamma,0]})+\varphi^\star\Big)\times
\end{aligned}$$

$$
\begin{aligned}
&\Big((s_2-s_1)^{\mu_k}+(s_1-M_{k-1})^{\mu_k}-(s_2-M_{k-1})^{\mu_k}+(s_2-s_1)^{\mu_k}\Big),\\
\leq\;& \frac{2M_{k-1}^{-\sigma}}{\Gamma(\mu_k+1)}\Big(K(\|z\|_{M_k}+\|\chi\|_{[-\gamma,0]})+\varphi^{\star}\Big)(s_2-s_1)^{\mu_k}\\
\leq\;& \frac{2M_{k-1}^{-\sigma}}{\Gamma(\mu_k+1)}\Big(K(R+\|\chi\|_{[-\gamma,0]})+\varphi^{\star}\Big)(s_2-s_1)^{\mu_k}.
\end{aligned}
$$

Hence, $|(\mathcal{E}z)(s_2)-(\mathcal{E}z)(s_1)|\to 0$ as $|s_2-s_1|\to 0$. It implies that $\mathcal{E}(B_R)$ is equicontinuous.

As a consequence of steps one, two, three, and four together with the Ascoli–Arzela theorem, we deduce that $\mathcal{E}: C_{M_{k-1}}\to C_{M_{k-1}}$ is continuous and compact. Therefore, by Shauder's FP theorem, this implies $\mathcal{E}$ has an FP $z_k\in B_{R_k}$. Then, $\mathcal{N}$ has an FP; thus, the problem (10) has a solution $\vartheta_k=\overline{z_k}+v\in C([-\gamma,M_k],\mathbb{R})$. □

Theorem 2. *Assume that conditions* ***(HY1)***, ***(HY2)***, *and relation (13) are fulfilled* $\forall\, k\in\{1,2,\ldots,n\}$. *Then,* (1) *has a piecewise solution such that* $C([-\gamma,M],\mathbb{R})$.

Proof. By Theorem 1, Equation (10) has a piecewise solution such that $\vartheta_1\in C([-\gamma,M_1],\mathbb{R})$, $\vartheta_k\in PC([-\gamma,M_k],\mathbb{R})$, for any $k\in\{2,3,\ldots,n\}$, such that

$$
\vartheta_1(s)=\overline{z_1}(s)+v(s)=\begin{cases}\chi(s), & s\in[-\gamma,0]\\ z_1(s), & s\in\mathcal{M}_1.\end{cases}
$$

and for any $k\in\{2,\ldots,n\}$

$$
\vartheta_k(s)=\overline{z_k}_s+v_s=\begin{cases}\chi_s, & s\in[-\gamma,0],\\ 0, & s\in[0,M_{k-1}],\\ z_k(s), & s\in\mathcal{M}_k.\end{cases}
$$

Thus, $\vartheta_1\in C([-\gamma,M_1],\mathbb{R})$, $\vartheta_k\in PC([-\gamma,M_k],\mathbb{R})$, for any $k\in\{2,3,\ldots,n\}$, is a solution of (9) for $s\in\mathcal{M}_k$ with $\vartheta_k(M_{k-1})=z_k(M_{k-1})=0$ and $\vartheta_k(s)=\chi(s)$, for $s\in[-\gamma,0]$.

Then,

$$
\vartheta(s)=\begin{cases}
\vartheta_1(s)=\begin{cases}\chi(s), & s\in[-\gamma,0],\\ z_1(s), & s\in\mathcal{M}_1,\end{cases}\\
\vartheta_2(s)=\begin{cases}\chi(s), & s\in[-\gamma,0],\\ 0, & s\in\mathcal{M}_1,\\ z_2(s), & s\in\mathcal{M}_2,\end{cases}\\
\cdot\\
\cdot\\
\cdot\\
\vartheta_n(s)=\begin{cases}\chi_s, & s\in[-\gamma,0],\\ 0, & s\in[0,M_{n-1}],\\ z_n(s), & s\in\mathcal{M}_n,\end{cases}
\end{cases}.
$$

gives the solution for the Riemann–Liouville Equation (1). □

4. Ulam–Hyers Stability

Theorem 3. *Consider that* ***(HY1)***, ***(HY2)***, *and* (13) *hold true, then* (1) *is HU stable.*

Proof. Assume $\epsilon>0$ is an arbitrary number and $y\in C([-\gamma,M],\mathbb{R})$ satisfies:

$$
\begin{cases}|D_{0^+}^{\mu(s)}y-\varphi(s,y_s)|\leq\epsilon, & s\in\mathcal{M}:=[0,M],\\ y(s)=\chi(s), & s\in[-\gamma,0],\end{cases}\tag{15}
$$

We define the functions

$$
y_1=\begin{cases}y, & s\in[0,M_1],\\ \chi, & s\in[-\gamma,0].\end{cases}\tag{16}
$$

and for $k = 2, 3, \ldots, n$:

$$y_k = \begin{cases} \chi(s), & s \in [-\gamma, 0], \\ 0, & s \in [0, M_{k-1}], \\ y, & s \in \mathcal{M}_k. \end{cases} \tag{17}$$

For any $k \in \{1, 2, \ldots, n\}$, according to (9), for $s \in \mathcal{M}_k$, we have

$$D_{0^+}^{\mu(s)} y_k(s) = \frac{1}{\Gamma(1-\mu_k)} \frac{d}{ds} \int_{M_{k-1}}^{s} (s - \zeta^*)^{-\mu_k} y(\zeta^*) d\zeta^*.$$

Taking $I_{M_{k-1}^+}^{\mu_k}$ on both sides of (15) and from Remark (2), we obtain

$$\left| y(s) - \frac{1}{\Gamma(\mu_k)} \int_{M_{k-1}}^{s} (s - \zeta^*)^{\mu_k - 1} \varphi(\zeta^*, y_{\zeta^*}) d\zeta^* \right| \leq \frac{\epsilon}{\Gamma(\mu_k)} \int_{M_{k-1}}^{s} (s - \zeta^*)^{\mu_k - 1} d\zeta^*$$
$$\leq \epsilon \frac{\left(M_k - M_{k-1}\right)^{\mu_k}}{\Gamma(\mu_k + 1)}.$$

As per Theorem 2, the considered problem (1) has a solution $\vartheta \in C([-\gamma, M], \mathbb{R})$, where $\vartheta(s)$, for $s \in [0, M_k]$, $k = 1, 2, \ldots, n$, and

$$\vartheta_1(s) = \begin{cases} \chi(s), & s \in [-\gamma, 0] \\ z_1(s), & s \in \mathcal{M}_1 \end{cases} \tag{18}$$

and for any $k \in \{2, \ldots, n\}$

$$\vartheta_k(s) = \begin{cases} \chi(s), & s \in [-\gamma, 0], \\ 0, & s \in [0, M_{k-1}], \\ z_k(s), & s \in \mathcal{M}_k \end{cases} \tag{19}$$

and $\vartheta_k \in PC([-\gamma, M_k], \mathbb{R})$, for $k = 2, 3, \ldots$, is a solution of the RL Equation (10). According to Lemma 5, we have

$$\vartheta_k(s) = \frac{1}{\Gamma(\mu_k)} \int_{M_{k-1}}^{s} (s - \zeta^*)^{\mu_k - 1} \varphi(\zeta^*, (\vartheta_k)_{\zeta^*}) d\zeta^*, \tag{20}$$

Let $s \in \mathcal{M}_k$, where $k \in \{1, 2, \ldots, n\}$. Then, by Equations (17)–(20), we obtain

$$\begin{aligned}
|y(s) - \vartheta(s)| &= |y(s) - \vartheta_k(s)| = |y_k(s) - \vartheta_k(s)| \\
&= |y_k(s) - \frac{1}{\Gamma(\mu_k)} \int_{M_{k-1}}^{s} (s - \zeta^*)^{\mu_k - 1} \varphi(\zeta^*, (\vartheta_k)_{\zeta^*}) d\zeta^*| \\
&\leq \left| y_k(s) - \frac{1}{\Gamma(\mu_k)} \int_{M_{k-1}}^{s} (s - \zeta^*)^{\mu_k - 1} \varphi(\zeta^*, (y_k)_{\zeta^*}) d\zeta^* \right| \\
&+ \frac{1}{\Gamma(\mu_k)} \int_{M_{k-1}}^{s} (s - \zeta^*)^{\mu_k - 1} \left| \varphi(\zeta^*, (y_k)_{\zeta^*}) - \varphi(\zeta^*, (\vartheta_k)_{\zeta^*}) \right| d\zeta^* \\
&\leq \epsilon \frac{\left(M_k - M_{k-1}\right)^{\mu_k}}{\Gamma(\mu_k + 1)} \\
&+ \frac{K}{\Gamma(\mu_k)} \int_{M_{k-1}}^{s} (s - \zeta^*)^{\mu_k - 1} \zeta^{* - \sigma} \| (y_k)_{\zeta^*} - (\vartheta_k)_{\zeta^*} \|_{[-\gamma', 0]} d\zeta^* \\
&\leq \epsilon \frac{\left(M_k - M_{k-1}\right)^{\mu_k}}{\Gamma(\mu_k + 1)}
\end{aligned}$$

$$
\begin{aligned}
&+ \quad \frac{K}{\Gamma(\mu_k)} \int_{M_{k-1}}^{s} (s-\zeta^*)^{\mu_k-1} \zeta^{*-\sigma} \sup_{-M_{k-1}-\gamma \le \theta \le 0} |(y_k)_{\zeta^*}(\theta) - (\vartheta_k)_{\zeta^*}(\theta)| d\zeta^* \\
&\le \quad \epsilon \frac{\Big(M_k - M_{k-1}\Big)^{\mu_k}}{\Gamma(\mu_k+1)} \\
&+ \quad \frac{K}{\Gamma(\mu_k)} \int_{M_{k-1}}^{s} (s-\zeta^*)^{\mu_k-1} \zeta^{*-\sigma} \sup_{-M_{k-1}-\gamma \le \theta \le 0} |y_k(\zeta^*+\theta) - \vartheta_k(\zeta^*+\theta)| d\zeta^* \\
&\le \quad \epsilon \frac{\Big(M_k - M_{k-1}\Big)^{\mu_k}}{\Gamma(\mu_k+1)} + \frac{K}{\Gamma(\mu_k)} \int_{M_{k-1}}^{s} (s-\zeta^*)^{\mu_k-1} \zeta^{*-\sigma} \sup_{-\gamma \le t \le M_k} |y_k(t) - x_k(t)| d\zeta^* \\
&\le \quad \epsilon \frac{\Big(M_k - M_{k-1}\Big)^{\mu_k}}{\Gamma(\mu_k+1)} + \Bigg(\frac{K}{\Gamma(\mu_k)} \int_{M_{k-1}}^{s} (s-\zeta^*)^{\mu_k-1} \zeta^{*-\sigma} d\zeta^*\Bigg)) \|y_k - \vartheta_k\|_{[-\gamma, M_k]} \\
&\le \quad \epsilon \frac{\Big(M_k - M_{k-1}\Big)^{\mu_k}}{\Gamma(\mu_k+1)} + \Bigg(\frac{K}{\Gamma(\mu_k)} \int_{M_{k-1}}^{s} (s-\zeta^*)^{\mu_k-1} (\zeta^* - M_{k-1})^{-\sigma} d\zeta^*\Bigg)) \|y_k - \kappa_k\|_{[-\gamma, M_k]} \\
&\le \quad \epsilon \frac{\Big(M_k - M_{k-1}\Big)^{\mu_k}}{\Gamma(k_k+1)} + \frac{K\Gamma(1-\sigma)(M_k - M_{k-1})^{\mu_k-\sigma}}{\Gamma(\mu_k+1-\sigma)} \|y_k - \vartheta_k\|_{[-\gamma, M_k]} \\
&\le \quad \epsilon \frac{\Big(M_k - M_{k-1}\Big)^{\mu_k}}{\Gamma(\mu_k+1)} + \nu \|y_k - \vartheta_k\|_{[-\gamma, M_k]}
\end{aligned}
$$

where

$$\nu = \max_{k=1,2,\ldots,n} \frac{K\Gamma(1-\sigma)(M_k - M_{k-1})^{\mu_k-\sigma}}{\Gamma(\mu_k+1-\sigma)}.$$

Then,

$$\|y - \vartheta\|_{[-\gamma, M_k]}(1-\nu) \le \epsilon \frac{\Big(M_k - M_{k-1}\Big)^{\mu_k}}{\Gamma(\mu_k+1)},$$

and thus, by assuming $c_\varphi := \frac{\Big(M_k - M_{k-1}\Big)^{\mu_k}}{(1-\nu)\Gamma(\mu_k+1)}$,

$$\|y - \vartheta\|_{[-\gamma, M_k]} \le c_\varphi \epsilon,$$

i.e.,

$$|y(s) - \vartheta(s)| \le c_\varphi \epsilon \;\; s \in [-\gamma, M_k].$$

Then, with the help of Definition 3, the considered problem (1) is HU stable. □

5. Example

Consider:

$$\begin{cases} D_{0^+}^{\mu(s)} \vartheta(s) = \frac{7}{5\sqrt{\pi}} s^{\mu(s)} + \frac{s^{-\frac{1}{2}}}{s+4} \|\vartheta_s\|_{[-\gamma,0]}, & s \in \mathcal{M} :=]0,2], \\ \vartheta(s) = \chi(s), & s \in [-\gamma, 0], \end{cases} \tag{21}$$

where

$$\mu(s) = \begin{cases} \frac{5}{7}, & s \in \mathcal{M}_1 := [0,1], \\ \frac{1}{2}, & s \in \mathcal{M}_2 :=]1,2]. \end{cases} \tag{22}$$

Let

$$\varphi(s, y_s) = \frac{7}{5\sqrt{\pi}} s^{\mu(s)} + \frac{s^{-\frac{1}{2}}}{s+4} \|y_s\|_{[-\gamma,0]}, \; (s, y_s) \in [0,2] \times C([-\gamma, 0], \mathbb{R}).$$

Let $y, z \in C([-\gamma, 2], \mathbb{R})$ and $s \in \mathcal{M}$. Then, we have

$$\begin{aligned} s^{\frac{1}{2}}|\varphi(s,y_s)-\varphi(s,z_s)| &= \left|s^{\frac{1}{2}}\frac{7}{5\sqrt{\pi}}s^{\mu(s)}+\frac{1}{s+4}\|y\|_{[-\gamma,0]}-s^{\frac{1}{2}}\frac{7}{5\sqrt{\pi}}s^{\mu(s)}-\frac{1}{s+4}\|z_s\|_{[-\gamma,0]}\right| \\ &= \frac{1}{s+4}\Big|\|y_s\|_{[-\gamma,0]}-\|z_s\|_{[-\gamma,0]}\Big| \\ &\leq \frac{1}{s+4}(\|y-z\|_{[-\gamma,0]}) \\ &\leq \frac{1}{4}\|y-z\|_{[-\gamma,0]}. \end{aligned}$$

Thus, the condition **(HY2)** holds with $\sigma=\frac{1}{2}$ and $K=\frac{1}{4}$.
By (22), by (10), two related problems for RL FDE of constant order are considered:

$$\begin{cases} D_{0^+}^{\frac{5}{7}}\vartheta(s)=\frac{7}{5\sqrt{\pi}}s^{\frac{5}{7}}+\frac{s^{-\frac{1}{2}}}{s+4}\|\vartheta_s\|_{[-\gamma,0]}, & s\in\mathcal{M}_1, \\ \vartheta(0)=0, & , \\ \vartheta(s)=\chi_1(s), & s\in[-\gamma,0] \end{cases} \tag{23}$$

and

$$\begin{cases} D_{0^+}^{\frac{1}{2}}\vartheta(s)=\frac{7}{5\sqrt{\pi}}s^{\frac{1}{2}}+\frac{s^{-\frac{1}{2}}}{s+4}\|\vartheta_s\|_{[-\gamma,0]}, & s\in\mathcal{M}_2, \\ \vartheta(1)=0, & , \\ \vartheta(s)=\chi_2(s), & s\in[-\gamma,1] \end{cases} \tag{24}$$

where $\chi_1=\chi$ and

$$\chi_2(s)=\begin{cases} 0, \ , \ if\ s\in[0,1] \\ \chi(s), \ if\ s\in[-\gamma,0]. \end{cases}$$

Next, we prove that (13) is fulfilled for $k=1$. Indeed,

$$\begin{aligned} \frac{K\Gamma(1-\sigma)(M_1-M_0)^{\mu_1-\sigma}}{\Gamma(\mu_1+1-\sigma)} &= \frac{\frac{1}{4}\Gamma(1-\frac{1}{2})(1-0)^{\frac{5}{7}-\frac{1}{2}}}{\Gamma(\frac{5}{7}+1-\frac{1}{2})} \\ &\simeq 0.484540 \\ &< 1 \end{aligned}$$

Accordingly the condition (13) is achieved. With the help of Theorem 1, Equation (23) has a solution $\vartheta_1\in C([-\gamma,1],\mathbb{R})$, where

$$\vartheta_1(s) = \begin{cases} \chi(s), & s\in[-\gamma,0] \\ z_1(s), & s\in\mathcal{M}_1 \end{cases}$$

Indeed,

$$\begin{aligned} \frac{K\Gamma(1-\sigma)(M_2-M_1)^{\mu_2-\sigma}}{\Gamma(\mu_2+1-\sigma)} &= \frac{\frac{1}{4}\Gamma(1-\frac{1}{2})(2-1)^{\frac{1}{2}-\frac{1}{2}}}{\Gamma(\frac{1}{2}+1-\frac{1}{2})} \\ &\simeq 0.443113 \\ &< 1 \end{aligned}$$

Thus, condition (13) is satisfied.
With the help of Theorem 1, Equation (24) has a solution $\vartheta_2\in C([-\gamma,2],\mathbb{R})$, where

$$\vartheta_2(s)=\begin{cases} \chi(s), & s\in[-\gamma,0], \\ 0, & s\in[0,1], \\ z_2(s), & s\in\mathcal{M}_2 \end{cases}$$

Then, by Theorem 2, Equation (21) has a solution

$$\vartheta(s) = \begin{cases} \vartheta_1(s) = \begin{cases} \chi(s), & s \in [-\gamma, 0], \\ z_1(s), & s \in \mathcal{M}_1, \end{cases} \\ \vartheta_2(s) = \begin{cases} \chi(s), & s \in [-\gamma, 0], \\ 0, & s \in \mathcal{M}_1, \\ z_2(s), & s \in \mathcal{M}_2, \end{cases} \end{cases}$$

By Theorem 3, Equation (21) is HU stable.

6. Conclusions

We studied a general class of FDEs with a variable order for their existence results and stability criteria. We established that our problem was more complex and needed some essential conditions required for the solution existence and stability result. For the existence of a solution, the FPTs were utilized and an example was presented for the applicability of the work. This approach can be utilized for the extension of FDEs with multipoint boundary conditions for their theoretical and numerical aspects.

In this paper, we worked in a space of continuous functions. We introduced an abstract Riemann–Liouville (RL) fractional differential equations (FDEs) with a variable order and finite delay where the function stood for the variable order of the given system. Then, by defining a partition based on the generalized intervals, we introduced a piecewise-constant function μ(s) and converted the given variable–order Riemann–Liouville (RL) fractional differential equations (FDEs) with a variable order and finite delay (1) to equivalent standard Riemann–Liouville (RL) fractional differential equations (FDEs) (8) of a fractional constant order at each interval in this partition. This meant a solution of (1) $C([-\gamma, M_1], \mathbb{R})$, $PC([-\gamma, M_k], \mathbb{R})$, for $k = 2, 3, \ldots$ was an appellation which meant the existence of a finite sequence ϑ_k, $k = 2, ..., n$, so that $\vartheta_k \in PC([-\gamma, M_k], \mathbb{R})$ fulfilled Equation (9), for $s \in [0, M_k]$, $\vartheta(s) = \chi(s)$, for $s \in [-\gamma, 0]$ and $\vartheta_k(M_{k-1}) = 0$.

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