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Local Existence and Blow-Up of Solutions for Wave Equation Involving the Fractional Laplacian with Nonlinear Source Term

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Abstract: The aim of this paper is to investigate the local weak existence and vacuum isolating of solutions, asymptotic behavior, and blow-up of the solutions for a wave equation involving the fractional Laplacian with nonlinear source. By means of the Galerkin approximations, we prove the local weak existence and finite time blow-up of the solutions and we give the upper and lower bounds for blow-up time.

Keywords: fractional derivatives; existence; fractional Laplacian; local solution; blow-up

MSC: 35K59; 35K55; 35B40

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1. Introduction

In the study of thermodynamic processes, the need to generalize the concept of differentiation of an integer order from a function to a fractional order differential arises often. For example, it is known that differential equations of order $1/2$ describe some physical processes, in particular, temperature changes in climate control objects, diffusion processes during super capacitor charging (see [1–8]).

The difficulty of generalization lies in the lack of physical interpretation of the concept of fractional derivatives and fractional integrals. For example, we all know that the first order derivatives with respect to time are the "average speed" of a point object over some finite time interval. Accordingly, the second-order derivatives set the "average acceleration" over a certain time interval. Fractional derivatives do not give similar interpretations that can be attributed to the point properties of an object or a small interval—rather, they are properties of the entire path-trajectory as a whole. Therefore, the semantic interpretations of fractional derivatives (more precisely, differential equations with fractional derivatives) are reflected in the descriptions of diffusion processes, in which the modeling object is fundamentally considered to be distributed, and all possible density distributions are considered (see [9–17]).

Here, we will discuss some of the system-related works that have helped pave the road for understanding and posing our primary problem. Pan, Pucci and Zhang [6] studied the initial-boundary value problem of degenerate Kirchhoff-type

$$\partial_t^2 w + [w]_r^{2(\theta-1)}(-\Delta_x)^r w = |w|^{p-1}w, \quad \text{in } \Omega_1 \times \mathbb{R}_+, \tag{1}$$

where $\Omega_1 \subset \mathbb{R}^n, 1 \leq n, p, r \geq 2, t > 0$ is a bounded domain with Lipshcitz boundary, $\theta \in [1, 2^\star],\ 2^* = \frac{2n}{n-2}$ and $[A]_r$ is the Gagliardo seminorm of A defined by

$$[A]_r = \Big(\int_\Omega \int_\Omega \frac{A(x) - A(z)|^2}{|x-z|^{n+2r}} dxdz \Big)^{\frac{1}{2}}.$$

The global existence, vacuum isolating and blow-up of solutions for (1) are obtained owing to the Galerkin method combined with the potential wells theory under some appropriate conditions (see [18–20]). Moreover, in [6], the authors also investigated the global existence of solutions under critical initial conditions. Furthermore, Pan et al. in [7] considered the following degenerate Kirchhoff equation with the nonlinear damping term,

$$\partial_t^2 w + [w]_r^{2(\theta-1)}(-\Delta_x)^r w + |\partial_t w|^{\alpha-1}\partial_t w + w = |w|^{p-2}w, \quad \text{in } \Omega \times \mathbb{R}_+, \tag{2}$$

where $2 < \alpha < 2\theta < p < 2^\star < r$. The global existence, vacuum isolating, asymptotic behavior and blow-up of solutions for (2) are obtained by combining the Galerkin method with potential wells theory under some natural assumptions (see [11,21–28]). In [29], Lin et al. studied the initial-boundary value problem of Kirchhoff wave equation

$$\partial_t^2 w + [w]_r^{2(\theta-1)}(-\Delta_x)^r w = |w|^{p-2}w, \quad \text{in } \Omega \times \mathbb{R}_+, \tag{3}$$

Here, motivated by the previous works, IVP involving the fractional Laplacian with nonlinearity is considered, let $\Omega_1 \subset \mathbb{R}^n, 1 \leq n$ with Lipschitz boundary $\partial\Omega_1$ and $\Omega_2 = \mathbb{R}^n \setminus \Omega_1$,

$$\begin{cases} \partial_t^2 w + (-\Delta_x)^r w + (-\Delta_x)^r \partial_t w = w|w|^{p-2} & x \in \Omega_1,\ t > 0 \\ w = 0, & x \in \Omega_2,\ t > 0 \\ w(x,0) = w_0(x), \partial_t w(x,0) = w_1(x) & x \in \Omega_1, \end{cases} \tag{4}$$

where $r \in (0,1)$. The exponent p satisfies

$$2 < p < \frac{2n}{n-2r} = 2_r^\star,\ n > 2r. \tag{5}$$

Our article is structured as follows. In Section 2, we describe our system and review several pertinent features and definitions pertaining to fractional Sobolev spaces. In Section 3, we discuss the local and global existence of solutions for the problem (4). As we will see, Section 4 will concentrate on the vacuum isolating of solutions. In addition, Section 4 examines decay estimates for solutions to the issue. In Section 6, we prove the solution to this issue, when $\mathcal{E}(0) > 0$ is defined in (13), and give the upper and lower bounds for blow-up time.

2. Preliminaries

When solving many differential equations containing fractional derivatives, the Laplace transform of a function is often used. Here, we try to collect and state some tools, properties and definitions of the fractional Sobolev spaces, for further details see [21].

The fractional Laplacian $(-\Delta_x)^r w$ of the function w is given by

$$(-\Delta_x)^r w(x) = const \int_{\mathbb{R}^n} \frac{w(x) - w(z)}{|x-z|^{n+2r}} dz,\ \forall x \in \mathbb{R}^n, \tag{6}$$

We define the fractional Sobolev space by

$$H^r(\Omega_1) = \Big\{ w \in L^2(\Omega_1) : \int_{\Omega_1} \int_{\Omega_1} \frac{|w(x) - w(z)|^2}{|x-z|^{n+2r}} dxdz < \infty \Big\}, \tag{7}$$

equipped with the norm,

$$\|w\|_{H^r(\Omega_1)} = \Big(\int_{\Omega_1} |w|^2 dx + \int_{\Omega_1} \int_{\Omega_1} \frac{|w(x) - w(z)|^2}{|x - z|^{n+2r}} dxdz \Big)^{\frac{1}{2}}, \tag{8}$$

Let

$$H_0^r(\Omega_1) = \{w \in H^r(\Omega_1) : w = 0 \; a.e. \; in \; \Omega_2\}, \tag{9}$$

is a closed linear subspace of $H^r(\Omega_1)$, and its norm

$$\|w\|_{H_0^r(\Omega_1)} = \Big(\int_{\Omega_1} \int_{\Omega_1} \frac{|w(x) - w(z)|^2}{|x - z|^{n+2r}} dxdz \Big)^{\frac{1}{2}}. \tag{10}$$

The space $H_0^r(\Omega_1)$ is a Hilbert space with inner product

$$\langle w, u \rangle_{H_0^r(\Omega_1)} = \int_{\Omega_1} \int_{\Omega_1} \frac{(w(x) - w(z))(u(x) - u(z))}{|x - z|^{n+2r}} dxdz, \tag{11}$$

Lemma 1. *For $s \in [1, 2_r^\star]$, there exists a constant $C_0 = C_0(n, s, r)$ positive, such that, for any $w \in H_0^r(\Omega_1)$,*

$$\|w\|_{L^s(\Omega_1)} \le C_0 \int_{\Omega_1} \int_{\Omega_1} \frac{|w(x) - w(z)|^2}{|x - y|^{n+2r}} dxdy. \tag{12}$$

In fact, if we replace w in what is to come by $w(t)$ for any $t \in [0, T)$, all the facts are still valid. The energy $\mathcal{E}$ of solution at time t to (4) is given by,

$$\mathcal{E}(t) = \frac{1}{2} \|\partial_t w(t)\|_2^2 + \mathcal{J}(w), \tag{13}$$

with the functional $\mathcal{J} : H_0^r(\Omega_1) \longrightarrow \mathbb{R}$ associated with model (4) is defined by

$$\mathcal{J}(w) = \frac{1}{2} \|w\|^2_{H_0^r(\Omega_1)} - \frac{1}{p} \|w\|_p^p, \tag{14}$$

and

$$\mathcal{I}(w) = \|w\|^2_{H_0^r(\Omega_1)} - \|w\|_p^p, \tag{15}$$

Now, we are ready to introduce a stable set as follows

$$\mathcal{W} = \{w \in H_0^r(\Omega_1) : \mathcal{I}(w) > 0, \; \mathcal{J}(w) < d\} \cup \{0\}, \tag{16}$$

where the mountain pass level d is defined by

$$d = \inf_{w \in H_0^r(\Omega_1) \setminus \{0\}} \{\sup_{\mu \ge 0} \mathcal{J}(\mu w)\}, \tag{17}$$

and the so-called *"Nehari manifold"* is introduced as

$$\mathcal{N} = \{w \in H_0^r(\Omega_1) \setminus \{0\} : \mathcal{I}(w) = 0\}, \tag{18}$$

where d is given by

$$d = \inf_{w \in \mathcal{N}} \mathcal{J}(w), \tag{19}$$

Lemma 2.

1. *The constant d is positive.*
2. *The functional $\mathcal{J}(\mu w)$ attains maximum, with respect to μ, at*

$$\mu^\star = \left(\frac{\|w\|^2_{H^r_0(\Omega_1)}}{\|w(t)\|^p_p}\right)^{1/(p-2)}. \tag{20}$$

Lemma 3. *$\mathcal{W}$ is a bounded neighbourhood of 0 in $H^r_0(\Omega_1)$.*

For problem (4) and $\delta \in (0,1)$ we define

$$\mathcal{J}_\delta(w) \quad = \quad \frac{\delta}{2}\|w\|^2_{H^r_0(\Omega_1)} - \frac{1}{p}\|w\|^p_p. \tag{21}$$

$$d(\delta) = \frac{1-\delta}{2}\left(\frac{p}{2K^p}\delta\right)^{\frac{2}{p-2}}, \tag{22}$$

$$r(\delta) = \left(\frac{p}{2K^p}\delta\right)^{\frac{1}{p-2}}, \tag{23}$$

where K is the best embedding constant of $H^r_0(\Omega_1)$ into $L^p(\Omega_1)$.

We have

$$\mathcal{J}(w) = \mathcal{J}_\delta(w) + \frac{1-\delta}{2}\|w\|^2_{H^r_0(\Omega_1)}, \quad \forall\delta \in [0,1], \tag{24}$$

Lemma 4. *Let $w \in H^r_0(\Omega_1)$ and $\mathcal{J}(w) \le d(\delta)$, we have the following property,*

$$\mathcal{J}_\delta(w) > 0(<0, =0) \text{ if and only if } \|w\|^2_{H^r_0(\Omega_1)} < r(\delta)(> r(\delta), = r(\delta)).$$

Lemma 5. *The next properties regarding the function $d : [0,1] \to \mathbb{R}$ satisfied:*

- $d(0) = d(1) = 0$.
- *The function d takes the maximum value at $\delta_0 = \frac{2}{p}$ and $d(\delta_0) = e$.*
- *The function d is strictly increasing in $[0, \delta_0]$, is strictly decreasing in $[\delta_0, 1]$.*
- *For any $e \in [0, d(\delta_0)]$, the equation $d(\delta) = e$ has exactly two roots $\delta_1 \in [0, \delta_0]$ and $\delta_2 \in [\delta_0, 1]$.*

Lemma 6.

1. *for all $\delta \in (0,1)$, $d(\delta) = \inf\Big\{\mathcal{J}(w),\ w \in H^r_0(\Omega_1),\ \|w\|_{H^r_0(\Omega_1)} \neq 0,\ \mathcal{J}_\delta(w) = 0\Big\}$*
2. $d = d(\delta_0) = \inf\Big\{\mathcal{J}(w),\ w \in H^r_0(\Omega_1),\ \|w\|_{H^r_0(\Omega_1)} \neq 0,\ \mathcal{J}_{\delta_0}(w) = 0\Big\}$

Lemma 7. *Let $\delta \in (0,1)$ and $w \in H^r_0(\Omega_1)$, we have*

1. *if $\mathcal{J}(w) \le d(\delta)$ and $\mathcal{J}_\delta(w) > 0$, then $\|w\|^2_{H^r_0(\Omega_1)} < 2\frac{d(\delta)}{1-\delta}$.*
   In particular, if $\mathcal{J}_\delta(w) \le d$ and $\mathcal{I}(w) > 0$, then $0 \le \|w\|^2_{H^r_0(\Omega_1)} < \frac{2p}{p-1}d$.
2. *if $\mathcal{J}(w) \le d(\delta)$ and $\|w\|^2_{H^r_0(\Omega_1)} > 2\frac{d(\delta)}{1-\delta}$, then $\mathcal{J}_\delta(w) < 0$.*
   Furthermore, if $\mathcal{J}(w) \le d$ and $\|w\|^2_{H^r_0(\Omega_1)} > \frac{2p}{p-1}d$, then $\mathcal{I}(w) < 0$.

Let us define the following family of groups for all $\delta \in (0,1)$, we put,

$$F \quad = \quad \{w \in H^r_0(\Omega_1) :\ \mathcal{J}(w) < d\},$$

$$\begin{aligned}
\mathcal{W}_\delta &= \{w \in H_0^r(\Omega_1): \mathcal{J}_\delta(w) > 0, \mathcal{J}(w) < d(\delta)\} \cup \{0\}, \ \overline{\mathcal{W}}_\delta = \mathcal{W}_\delta \cup \partial \mathcal{W}_\delta, \\
F_\delta &= \{w \in H_0^r(\Omega_1): \mathcal{J}_\delta(w) < 0, \mathcal{J}(w) < d(\delta)\}, \ \overline{F}_\delta = F_\delta \cup \partial F_\delta, \\
G_\delta &= \left\{w \in H_0^r(\Omega_1): \|w\|^2_{H_0^r(\Omega_1)} < r(\delta)\right\}, \ \overline{G}_\delta = G_\delta \cup \partial G_\delta,
\end{aligned}$$

3. Local Existence of Weak Solution

Definition 1. *A function $w = w(x,t)$ is said to be a Local weak solution of problem (4), if*

$$\begin{aligned}
&w \in L^\infty(0,T,H_0^r(\Omega_1)), && w_t \in L^\infty(0,T,L^2(\Omega_1)) \\
&w_0 \in L^\infty(0,T,H_0^r(\Omega_1)), && w_1 \in L^\infty(0,T,L^2(\Omega_1))
\end{aligned}$$

and for any $\phi \in L^\infty(0,T,H_0^r(\Omega_1))$, $t \in [0,T]$,

$$\begin{aligned}
(\partial_t w, \phi(.,t)) + \tfrac{1}{2}\int_0^t (w(.,\tau), \phi(.,\tau))_{H_0^r(\Omega_1)} d\tau + \int_0^t (w_t(.,\tau), \phi(.,\tau))_{H_0^r(\Omega_1)} d\tau \\
= (w_1, \phi(.,0)) + \int_0^t (w(.,\tau)|w(.,\tau)|^{p-2}, \phi(.,\tau)) d\tau,
\end{aligned}$$

If a local solution w belongs to $C(0,T,H_0^r(\Omega_1))$, we say that w a strong local solution of problem (4).

Theorem 1. *Let $(w_0, w_1) \in H_0^r(\Omega_1) \times L^2(\Omega_1)$, suppose that $\mathcal{E}(0) < d$. Then there exists a positive constant T such that problem (4) has solution $w(t)$ on $\Omega_1 \times (0,T)$.*

Proof. In order to prove the existence of weak solutions for problem (4) by applying the Galerkin method, The proof is divided into the following steps:
Step 1: Let $\{V_n\}_{n\in\mathbb{N}}$ be a Galerkin space of the separable Banach space $H_0^r(\Omega_1)$, i.e.,

$$V_n = Span\{\varphi_1, \varphi_2, \cdots, \varphi_n\} \text{ and } \overline{\bigcup_{n\in\mathbb{N}} V_n} = H_0^r(\Omega_1),$$

with $\{\varphi_j\}_{j=1}^n$ is an orthonormal basis in $L^2(\Omega_1)$.
Let $w_0 \in H_0^r(\Omega_1)$, then we can find $w_{0n} \in V_n$ such that,

$$w_n(0) = w_{0n} \to w_0 \text{ strongly in } H_0^r(\Omega_1), \text{ as } n \to \infty, \tag{25}$$

we approximate solutions

$$w_n(x,t) = \sum_{j=1}^n e_j^n(t) u_j(x), \qquad n = 1,2,3,\ldots$$

by Galerkin's approximation, we have

$$\begin{cases}
(\partial_t^2 w_n(.,t), u_j) + (w_n(.,t), u_j)_{H_0^r(\Omega_1)} + (\partial_t w_n(.,t), u_j)_{H_0^r(\Omega_1)} = (w_n(.,t)|w_n(.,t)|^{p-2}, u_j), \\
w_n(.,0) = \sum\limits_{j=1}^n A_j u_j \to w_0, n \to \infty \text{ in } H_0^r(\Omega_1), j = \overline{1,n}, \\
w_{nt}(.,0) = \sum\limits_{j=1}^n B_j u_j \to w_1, n \to \infty \text{ in } L^2(\Omega_1).
\end{cases}$$

Substituting w_n into (4), to get,

$$\begin{cases}
e_j^{n''} + \mu_j e_j^n + \mu_j e_j^{n'} = \sum\limits_{l=1}^n e_j^l |e_j^l|^{p-2} \int_{\Omega_1} u_l |u_l|^{p-2} u_j dx, \\
e_j^n(0) = a_j, \quad j = 1, \ldots\ldots, n, \\
e_j^{n'}(0) = b_j, \quad j = 1, \ldots\ldots, n,
\end{cases}$$

According to standard ordinary differential equations theory, the problem admits a solution in $C^1([0,T])$ for all n.

Step 2: Multiplying the problem (4) by $e_j^{n'}$, summing for j and integrating with respect to τ, we get for all $t \in [0,T]$

$$\frac{1}{2}\int_0^t \frac{d}{dt}\left(\int_{\Omega_1} |\partial_t w_n(x,\tau)|^2 dx\right) d\tau + \frac{1}{2}\int_0^t \frac{d}{dt}\left(\int_{\Omega_1}\int_{\Omega_1} \frac{(w_n(x,\tau) - w_n(z,\tau))^2}{|x-z|^{n+2r}} dxdz\right) d\tau$$
$$+ \int_0^t \|\partial_t w_n(z,\tau)\|^2_{H_0^r(\Omega_1)} d\tau = \frac{1}{p}\int_0^t \frac{d}{dt}\left(\int_{\Omega_1} |w_n(x,\tau)|^p dx\right) d\tau,$$

for all $t \in [0,T]$, we obtain

$$\frac{1}{2}\|\partial_t w_n(.,t)\|_2^2 + \frac{1}{2}\|w_n(.,t)\|^2_{H_0^r(\Omega_1)} + \int_0^t \|\partial_t w_n(z,\tau)\|^2_{H_0^r(\Omega_1)} d\tau - \frac{1}{p}\|w_n(.,t)\|_p^p$$
$$= \frac{1}{2}\|\partial_t w_n(.,0)\|_2^2 + \frac{1}{2}\|w_n(.,0)\|^2_{H_0^r(\Omega_1)} - \frac{1}{p}\|w_n(.,0)\|_p^p = \mathcal{E}_n(0).$$

Hence, we obtain

$$\mathcal{E}_n(0) = \underbrace{\frac{1}{2}\|\partial_t w_n(.,t)\|_2^2 + \mathcal{J}(w_n(.,t))}_{\mathcal{E}_n(t)} + \int_0^t \|\partial_t w_n(z,\tau)\|^2_{H_0^r(\Omega_1)} d\tau,$$

where,

$$\mathcal{E}_n(t) = \frac{1}{2}\|w_{nt}\|_2^2 + \mathcal{J}(w_n).$$

For sufficiently large n, we can get $\mathcal{E}_n(0) < d$ and $\mathcal{E}(0) < d$.

Then for sufficiently large n, it follows from (29) that,

$$\mathcal{E}_n(t) = \frac{1}{2}\|w_{nt}\|_2^2 + \mathcal{J}(w_n) < d. \tag{26}$$

We can infer that $w_{nt} \in \mathcal{W}$ for sufficiently large n. Next, for sufficiently large m and any $t \in [0,T]$, we will show $w_n(t) \in \mathcal{W}$. Indeed, if not, then there is a sufficiently large n and a $t_\star \in (0,T]$ such that $w_n(t_\star) = 0$ and $\mathcal{I}(w_n(t_\star)) = 0$, which implies $w(t_\star) \in \mathcal{N}$. Then $\mathcal{J}(w_n(t_\star)) \geq d$, which conflicts with (26). Hence, by (26) and for sufficiently large n and any $t \in [0,T]$, we get $w_n(t) \in \mathcal{W}$. We have,

$$\frac{1}{2}\|\partial_t w_n(.,t)\|_2^2 + \mathcal{J}(w_n(.,t)) + \int_0^t \|\partial_t w_n(z,\tau)\|^2_{H_0^r(\Omega_1)} d\tau = \mathcal{E}_n(0) \leq d, \tag{27}$$

by (27), for all $t \in [0,T]$,

$$\begin{cases} \|w_n(.,t)\|^2_{H_0^r(\Omega_1)} < C_1 \\ \|\partial_t w_n(.,t)\|_2^2 \leq C_2, \\ \|w_n(.,t)\|_p^p \leq C_3, \\ \|\partial_t w_n(.,t)\|^2_{H_0^r(\Omega_1)} \leq C_4, \end{cases}$$

such that C_1, C_2, C_3, C_4 constants.

Step 3: We see that there must be a function $w \in L^\infty(0,T,H_0^r(\Omega_1))$ with $\partial_t w \in L^\infty(0,T,L^2(\Omega_1))$, $\xi \in L^2(0,T,L^{p'}(\Omega_1))$ and a subsequence of $\{w_i\}_{i=1}^n$, as $n \to \infty$, such that,

$$w_n \rightharpoonup^\star w \text{ in } L^\infty(0,T,H_0^r(\Omega_1)) \text{ and } w_n \to w \text{ in } \Omega_1 \times [0,T],$$

$$\partial_t w_n \rightharpoonup^\star \partial_t w \text{ in } L^\infty(0,T,H_0^r(\Omega_1)) \text{ and } \partial_t w_n \to \partial_t w \text{ in } \Omega_1 \times [0,T],$$
$$w_n|w_n|^{p-2} \to \xi \text{ in } L^\infty(0,T,L^{p'}(\Omega_1)) \text{ and } w_n|w_n|^{p-2} \to \xi \text{ in } \Omega_1 \times [0,T],$$
$$\partial_t w_n \rightharpoonup^\star \partial_t w \text{ in } L^\infty(0,T,L^2(\Omega_1)) \text{ and } \partial_t w_n \to \partial_t w \text{ in } L^\infty(0,T,L^2(\Omega_1)),$$

as $n \to \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$.

Multiplying the problem (4) by u_j, summing for j and Integrating with respect to τ from 0 to t, we have,

$$(\partial_t w, u_j) + \tfrac{1}{2}\int_0^t (w(.,\tau),u_j)_{H_0^r(\Omega_1)} d\tau + \int_0^t (\partial_t w(.,\tau),u_j)_{H_0^r(\Omega_1)} d\tau$$
$$= (w_1,u_j) + \int_0^t (\xi,u_j) d\tau.$$

Therefore, since $C_0^\infty(\Omega_1)$ is dense in $H_0^r(\Omega_1)$. The fact that $(u_j)_j \subset C_0^\infty(\Omega_1)$ is an orthonormal basis of $L^2(\Omega_1)$, we obtain for all $v \in H_0^r(\Omega_1)$

$$(\partial_t w, v(x)) + \tfrac{1}{2}\int_0^t (w(.,\tau),v(x))_{H_0^r(\Omega_1)} d\tau + \int_0^t (\partial_t w(.,\tau),v(x))_{H_0^r(\Omega_1)} d\tau$$
$$= (w_1,v(x)) + \int_0^t (\xi,v(x)) d\tau,$$

for any $\phi \in L^1(0,T,H_0^{r,2}(\Omega_1))$, putting $v(x) = \phi(x,t)$, with t fixed, and integration with respect to t, we conclude that w is a Local weak solution of the problem.

□

With regard to proving the global solutions to problem (4), it is shown in [30] by using Gallerkin's method. This theorem is given as follows,

Theorem 2. *Let $(w_0,w_1) \in H_0^{r,2}(\Omega_1) \times L^2(\Omega_1)$, suppose that $0 < \mathcal{E}(0) < d$, δ_1 and δ_2, with $\delta_2 > \delta_1 > 0$, are the two solutions of the equation $d(\delta) = \mathcal{E}(0)$, that either $0 < \mathcal{J}(w)$ or $\|w_0\|_{H_0^r(\Omega_1))} = 0$. Then (4) has a global solution w in $L^\infty(0,\infty,H_0^{r,2}(\Omega_1))$, with $\partial_t w \in L^\infty(0,\infty,L^2(\Omega_1))$.*

If a global solution w belongs to $C(0,\infty,H_0^r(\Omega_1))$, we say that w a strong global solution of problem (4).

4. Vacuum Isolating of Solution

Lemma 8. *Let w be the weak solution of problem (4). If $(w_0,w_1) \in \mathcal{W} \times L^2(\Omega_1)$, then $\mathcal{E}(t) \le \mathcal{E}(0)$.*

Theorem 3. *Let $w \in H_0^r(\Omega_1)$. Fixing $e \in (0,d)$ and let $\delta_2 > \delta_1 > 0$ be the two solutions for $d(\delta) = e$. Then, for all strong solutions w of (4), with $\mathcal{E}(0) = e$, we have*

1. *$w(.,t)$ belongs to $\mathcal{W}_\delta$ for all $\delta \in (\delta_1,\delta_2)$ and $t \in [0,T)$, provided that $\mathcal{I}(w_0) > 0$.*
2. *$w(.,t)$ belongs to F_δ for all $\delta \in (\delta_1,\delta_2)$ and $t \in [0,T)$, provided that $\mathcal{I}(w_0) < 0$.*

Proof.

1. Let w be a solution of (4), with $\mathcal{E}(0) = e$. Assume that $\mathcal{I}(w_0) > 0$, then,

$$\frac{1}{2}\|w_1\|_2^2 + \mathcal{J}(w_0) = \mathcal{E}(0) = d(\delta_1) = d(\delta_2) < d(\delta),$$

yield that $\mathcal{J}_\delta(w_0) > 0$ and $\mathcal{J}(w_0) < d(\delta)$, i.e, $w_0 \in \mathcal{W}_\delta$ for all $\delta \in (\delta_1, \delta_2)$.
We claim that $w \in \mathcal{W}_\delta$ for all $\delta \in (\delta_1, \delta_2)$ and $t \in [0, T)$.
Assume by contradiction that there exists $t_0 \in [0, T)$,
such that $w(., t_0) \in \partial\mathcal{W}_\delta$ for all $\delta \in (\delta_1, \delta_2)$. That is $\mathcal{J}_\delta(w(., t_0)) = 0$, and either $\|w(., t_0)\|_{H_0^r(\Omega_1)} \neq 0$ or $\mathcal{J}(w(., t_0)) = d(\delta)$,we have,

$$\frac{1}{2}\|\partial_t w\|_2^2 + \mathcal{J}(w) \leq \mathcal{E}(0) < d(\delta), \quad \forall\delta \in (\delta_1, \delta_2), \tag{28}$$

the case $\mathcal{J}(w(., t_0)) = d(\delta)$ is impossible.
if $\mathcal{J}_\delta(w(., t_0)) = 0$ and $\|w(., t_0)\|_{H_0^r(\Omega_1)} \neq 0$, the $\mathcal{J}(w(., t_0)) > d(\delta)$, this contradicts (28), which completes the proof.

2. Let w be a solution of (4), with $\mathcal{E}(0) = e$.
Assume that either $\mathcal{I}(w_0) > 0$, since $\delta \to \mathcal{J}_\delta(w)$ does not change sign in (δ_1, δ_2), then $\mathcal{J}_\delta(w_0) < 0$ for all $\delta \in (\delta_1, \delta_2)$. This fact and $\mathcal{J}_\delta(w_0) < 0$ for all $\delta \in (\delta_1, \delta_2)$ give $w_0 \in F_\delta$ for $\delta \in (\delta_1, \delta_2)$.
we claim that $w \in F_\delta$ for all $t \in (0, T)$ and all $\delta \in (\delta_1, \delta_2)$.
Otherwise, let $t_0 \in [0, T)$ be the first time such that $w \in F_\delta$ for all $t \in [0, t_0)$ and $w(., t_0) \in \partial F_\delta$ for some $\delta \in (\delta_1, \delta_2)$, i.e., either $\mathcal{J}_\delta(w(., t_0)) = 0$ or $\mathcal{J}(w(., t_0)) = d(\delta)$, the case $\mathcal{J}(w(., t_0)) = d(\delta)$ impossible.
If $\mathcal{J}_\delta(w(., t_0)) = 0$, then $\mathcal{J}_\delta(w(., t_0)) < 0$ for all $t \in [0, t_0)$,
yields $\|w\|_{H_0^r(\Omega_1)} > r(\delta)$, for $t \in [0, t_0)$ and $\|w\|_{H_0^r(\Omega_1)} = r(\delta)$. Hence, $\mathcal{J}(w(., t_0)) \geq d(\delta)$, this contradicts (28), which completes the proof.

□

Theorem 4. *Let $w \in H_0^r(\Omega_1)$. Fix $e \in (0, d)$ and let $\delta_2 > \delta_1 > 0$ be the two solutions of $d(\delta) = e$. Then, for all strong solutions w of (4), with initial energy $\mathcal{E}(0)$ satisfying $\mathcal{E}(0) \in (0, e]$,*

1. *$w(., t)$ belongs to $\mathcal{W}_\delta$ for all $\delta \in (\delta_1, \delta_2)$ and $t \in [0, T)$, provided that $\mathcal{I}(w_0) > 0$.*
2. *$w(., t)$ belongs to F_δ for all $\delta \in (\delta_1, \delta_2)$ and $t \in [0, T)$, provided that $\mathcal{I}(w_0) < 0$.*

Theorem 5. *Let $w \in H_0^r(\Omega_1)$. Fix $e \in (0, d)$ and let $\delta_2 > \delta_1 > 0$ be two solutions of $d(\delta) = e$. Then, for all strong solutions w of (4), with initial energy $\mathcal{E}(0)$ satisfying $\mathcal{E}(0) \in (0, e]$,*

(i) *$w(., t)$ belongs to $\overline{\mathcal{W}_{\delta_1}}$ for all $\delta \in (\delta_1, \delta_2)$ and $t \in [0, T)$, provided that $\mathcal{I}(w_0) > 0$.*
(ii) *$w(., t)$ belongs to $\overline{F_{\delta_2}}$ for all $\delta \in (\delta_1, \delta_2)$ and $t \in [0, T)$, provided that $\mathcal{I}(w_0) < 0$.*

Proof. we have $\mathcal{J}(w) \leq d(\delta_2)$ (or $d(\delta_1)$) for all $t \in [0, T)$. Fix $t \in [0, T)$. Letting $\delta \to \delta_1$ (or $\delta \to \delta_2$) in $\mathcal{J}_\delta(w) > 0$ (or $\mathcal{J}_\delta(w) < 0$) for the case (i) (or case (ii)), we have $\mathcal{J}_{\delta_1}(w) \geq 0$ (or $\mathcal{J}_{\delta_2}(w) \leq 0$) for all $t \in [0, T)$. This completes the proof. □

Theorem 6. *Let $w \in H_0^r(\Omega_1)$. Fix $e \in (0, d)$ and let $\delta_2 > \delta_1 > 0$ be two solutions of $d(\delta) = e$. Then, for all strong solutions w of (4), with initial energy $\mathcal{E}(0)$ satisfying $\mathcal{E}(0) \in (0, e]$,*

(i) *$w(., t)$ lies inside the ball $\overline{G_{\delta_1}}$ for all $\delta \in (\delta_1, \delta_2)$ and $t \in [0, T)$, provided that $w_0 \in G_{\delta_0}$.*
(ii) *$w(., t)$ lies outside the ball G_{δ_2} for all $\delta \in (\delta_1, \delta_2)$ and $t \in [0, T)$, provided that $w_0 \in \overline{G_{\delta_0}}^c$.*

Proof. Let $w \in H_0^r(\Omega_1)$. Fix $e \in (0, d)$ and let $\delta_2 > \delta_1 > 0$ be two solutions of $d(\delta) = e$. Then, for all strong solutions w of (4), with initial energy $\mathcal{E}(0)$ satisfying $\mathcal{E}(0) \in (0, e]$,

(i) By Theorem 5, we have that $w \in \overline{\mathcal{W}_{\delta_1}}$, so that $\mathcal{J}_{\delta_1}(w) \geq 0$ and $\mathcal{J}(w) \leq d(\delta)$, using Lemma 4, then there are $\|w\|_{H_0^r(\Omega_1)} \leq r(\delta_1)$, and from him $w \in \overline{G_{\delta_1}}$.

(ii) By Theorem 5, we have that $w \in F_{\delta_1}$, so that $\mathcal{J}_{\delta_2}(w) \leq 0$ and $\mathcal{J}(w) \leq d(\delta)$, using Lemma 4, consequently there are $\|w\|_{H_0^r(\Omega_1)} \geq r(\delta_2)$, Then $w \in \overline{G_{\delta_2}}^c$.

□

From the results in Theorem 6, one can show that for any given $e \in (0, d)$, there exists a corresponding vacuum region

$$V_e = \Big\{ w \in H_0^r(\Omega_1), \quad r(\delta_1) < \|w\|_{H_0^r(\Omega_1)} < r(\delta_2) \Big\},$$

for the set of strong solutions of the problem (4), with initial energy $\mathcal{E}(0)$ satisfying $0 < \mathcal{E}(0) \leq e$, i.e., there are no strong solutions w such that $w \in V_e$ for all $t \in [0, T)$.

The vacuum region V_e becomes bigger when e decreases to 0. As limiting case, we obtain,

$$V_1 = \Big\{ w \in H_0^r(\Omega_1), \quad \|w\|_{H_0^r(\Omega_1)} < r(1) \Big\}.$$

Theorem 7. *Let $w \in H_0^r(\Omega_1)$. Fix $e \in (0, d)$ and let $\delta_2 > \delta_1 > 0$ be two solutions of $d(\delta) = e$. Then, any nontrivial strong solution w of (4), with initial energy $\mathcal{E}(0)$ satisfying $\mathcal{E}(0) \in (0, e]$, is such that w lies outside the ball $\overline{G_1}^c$ for all $t \in [0, T)$.*

Proof. Let w be any solution of (4), with $\mathcal{E}(0) = 0$. We have,

$$\frac{1}{2}\|\partial_t w\|_2^2 + \mathcal{J}(w) \leq \mathcal{E}(0) = 0, \tag{29}$$

we get $\mathcal{J}(w) \leq 0$ for all $t \in [0, T)$. Furthermore, for all $t \in [0, T)$,

$$p\|w\|_{H_0^r(\Omega_1)}^2 \leq \|w\|_p^p \leq k^p \|w\|_{H_0^r(\Omega_1)}^{p-1} \|w\|_{H_0^r(\Omega_1)},$$

We claim that either $\|w\|_{H_0^r(\Omega_1)}^2 = 0$ for all $t \in [0, T)$, or $\|w\|_{H_0^r(\Omega_1)}^2 > r(1)$ for all $t \in [0, T)$.

Otherwise, there are $t \in [0, T)$ and $\tau \in [0, T)$ such that $\|w\|_{H_0^r(\Omega_1)}^2 = 0$ and $\|w_0(., t)\|_{H_0^r(\Omega_1)}^2 > r(1)$, this contradicts the mean value theorem since a strong solution of (4) is of class $C(0, T, H_0^r(\Omega_1))$. □

Theorem 8. *Let $w \in H_0^r(\Omega_1)$, $e \in (0, d)$ and $\delta_2 > \delta_1 > 0$ be two solutions of $d(\delta) = e$. Then, for any nontrivial strong solution w of the problem (4), with initial energy $\mathcal{E}(0) < 0$,satisfy $\|w\|_{H_0^r(\Omega_1)} > r(1)$ for all $t \in [0, T)$.*

Furthermore,

$$\|w\|_{H_0^r(\Omega_1)} \geq \left(\frac{p}{k^p} \sqrt{-2\mathcal{E}(0)} \right)^{\dfrac{1}{p-1}},$$

Proof. Theorem 7 give $\|w\|_{H_0^r(\Omega_1)}^2 > r(1)$ for all $t \in [0, T)$. We have,

$$\frac{1}{2}\|\partial_t w\|_2^2 + \mathcal{J}(w) \leq \mathcal{E}(0) < 0, \tag{30}$$

for all $t \in [0, T)$, so that

$$\mathcal{E}(0) - \mathcal{J}(w) \geq 0,$$

so,

$$\mathcal{E}(0) - \frac{1}{2}\|w\|_{H_0^r(\Omega_1)}^2 + \frac{1}{p}\|w\|_p^p \geq 0,$$

Therefore,

$$p\|w\|_p^p \geq \frac{1}{2}\|w\|^2_{H_0^r(\Omega_1)} - \mathcal{E}(0) \geq \sqrt{2}\|w\|_{H_0^r(\Omega_1)}\sqrt{-\mathcal{E}(0)},$$

since,

$$p\|w\|^2_{H_0^r(\Omega_1)} \leq \|w\|_p^p \leq k^p||w\|^{p-1}_{H_0^r(\Omega_1)}\|w\|_{H_0^r(\Omega_1)},$$

which completes the proof. □

5. Decay Estimate of Solution

In this section, we can pass to the qualitative properties and prove the rate of decay for solutions to model (4).

Lemma 9 ([22])**.** *Let φ be a bounded positive (nonnegative nonincreasing) function on $\mathbb{R}_+$ satisfying, for some canstants k, if $\alpha > 0$,*

$$k\varphi(t)^{\alpha+1} \leq (\varphi(t) - \varphi(t+1)), \quad t \geq 0,$$

Then we have

$$\varphi(t) \leq (\alpha k(t-1) + M^{-\alpha})^{-\frac{1}{\alpha}}, \quad \forall t \geq 1, \tag{31}$$

where,

$$M = \max_{t\in[0,1]} \varphi(t).$$

Theorem 9. *Let $(w_0, w_1) \in \mathcal{W} \times L^2(\Omega_1)$. we suppose $\mathcal{E}(0) < d$, and w a global solution of problem (4), he has the following decay estimate,*

$$\mathcal{E}(t) \leq \frac{1}{\left(\frac{1}{k'^2}(t-1) + \frac{1}{M}\right)}, \quad \forall t \geq 1,$$

where $M = \max_{t\in[0,1]} \mathcal{E}(t)$, $k' = const(k, d)$.

k: the constant of the embedding $H_0^r(\Omega_1)$ into $L^2(\Omega_1)$.

Proof. Next, multiplying the equation of (4) by $\partial_t w$ and integrating in $[t, t+1]$, we get

$$\int_t^{t+1} \frac{d\mathcal{E}(\tau)}{d\tau} d\tau = \int_t^{t+1} \|\partial_t w(.,\tau)\|^2_{H_0^r(\Omega_1)} d\tau = \mathcal{E}(t) - \mathcal{E}(t+1) \geq 0. \tag{32}$$

Thus, $\mathcal{E}$ is nonincreasing in $\mathbb{R}_+^\star$.

It follows from (32) and the embedding $H_0^r(\Omega_1)$ into $L^p(\Omega_1)$ that

$$\begin{aligned}\int_t^{t+1} \|\partial_t w\|_2^2 dt \leq k\int_t^{t+1} \|\partial_t w\|^2_{H_0^r(\Omega_1)} dt &\leq k[\mathcal{E}(t) - \mathcal{E}(t+1)] \\ &\leq kd^{\frac{1}{2}}[\mathcal{E}(t) - \mathcal{E}(t+1)]^{\frac{1}{2}},\end{aligned} \tag{33}$$

Owing to the mean value theorem for (33), there exist $t_1 \in \left[t, t+\frac{1}{4}\right]$ and $t_2 \in \left[t+\frac{3}{4}, t+1\right]$ such that

$$\|w_t(.,t_i)\|_2 \le k^{\frac{1}{2}}[\mathcal{E}(t)-\mathcal{E}(t+1)]^{\frac{1}{2}}, \quad (i=1,2), \tag{34}$$

Multiplying the equation of (4) by w and integrating in $\Omega_1 \times [t_1, t_2]$, we can see that

$$\int_{t_1}^{t_2}\int_{\Omega_1} \partial_{tt} w w dx dt + \int_{t_1}^{t_2} \|w\|^2_{H_0^r(\Omega_1)} dt + \int_{t_1}^{t_2} (\partial_t w, w)_{H_0^r(\Omega_1)} dt = \int_{t_1}^{t_2} \|w\|_p^p dt, \tag{35}$$

by (35), we have,

$$\int_{t_1}^{t_2} \|w\|^2_{H_0^r(\Omega_1)} dt - \int_{t_1}^{t_2} \|w\|_p^p dt = -\int_{t_1}^{t_2}\int_{\Omega_1} \partial_{tt} w w dx dt - \int_{t_1}^{t_2} (\partial_t w, w)_{H_0^r(\Omega_1)} dt, \tag{36}$$

so that,

$$\begin{aligned}
&\int_{t_1}^{t_2} \|w\|^2_{H_0^r(\Omega_1)} dt - \int_{t_1}^{t_2} \|w\|_p^p dt = -\int_{t_1}^{t_2}\int_{\Omega_1} \partial_{tt} w w dx dt - \int_{t_1}^{t_2} (\partial_t w, w)_{H_0^r(\Omega_1)} dt \\
&\le \left[\int_{\Omega_1} -\partial_{tt} w w dx\right]_{t_1}^{t_2} + \int_{t_1}^{t_2} \|w_t(.,t)\|_2^2 dt + \int_{t_1}^{t_2} \|-\partial_t w\|_{H_0^r(\Omega_1)} \|w\|_{H_0^r(\Omega_1)} dt \\
&\le \sum_{i=1}^{2} \|\partial_t w(.,t_i)\|_2 \|w(.,t_i)\|_2 + \int_{t_1}^{t_2} \|\partial_t w\|_2^2 dt + \int_{t_1}^{t_2} \|\partial_t w\|_{H_0^r(\Omega_1)} \|w\|_{H_0^r(\Omega_1)} dt.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\|\partial_t w(.,t_i)\|_2 \|w(.,t_i)\|_2 &\le k^{\frac{1}{2}}[\mathcal{E}(t)-\mathcal{E}(t+1)]^{\frac{1}{2}} \sup_{[t,t+1]} \mathcal{E}(s)^{\frac{1}{2}} \\
&\le k^{\frac{1}{2}} d^{\frac{1}{2}} [\mathcal{E}(t)-\mathcal{E}(t+1)]^{\frac{1}{2}},
\end{aligned} \tag{37}$$

and,

$$\begin{aligned}
\int_{t_1}^{t_2} \|\partial_t w\|_{H_0^r(\Omega_1)} \|w\|_{H_0^r(\Omega_1)} dt &\le \left(\int_{t_1}^{t_2} \|\partial_t w\|_{H_0^r(\Omega_1)} dt\right)^{\frac{1}{2}} \left(\int_{t_1}^{t_2} \|w\|_{H_0^r(\Omega_1)} dt\right)^{\frac{1}{2}} \\
&\le k^{\frac{1}{2}}[\mathcal{E}(t)-\mathcal{E}(t+1)]^{\frac{1}{2}} \left(\int_{t_1}^{t_2} \mathcal{E}(t) dt\right)^{\frac{1}{2}} \\
&\le k^{\frac{1}{2}} (d)^{\frac{1}{2}} [\mathcal{E}(t)-\mathcal{E}(t+1)]^{\frac{1}{2}},
\end{aligned} \tag{38}$$

In view of (33), (37) and (38), we get

$$\mathcal{E}(t) \le k'[\mathcal{E}(t)-\mathcal{E}(t+1)]^{\frac{1}{2}} \left(k' = const(k,d) \text{ is a constant, it is exist}\right) \tag{39}$$

Then

$$\mathcal{E}(t) \le \frac{1}{\left(\frac{1}{k'^2}(t-1) + \frac{1}{M}\right)}, \quad \forall t \ge 1,$$

□

6. Blow-Up Time of Solution

In this section, we prove blow-up results of the solution to problem (4), when $\mathcal{E}(0) > 0$. we give the proofs of the upper and lower bounds for blow-up time.

In [23], we demonstrated the blow-up of solutions with many conditions. However, in our current article, we replaced these conditions with a few conditions.

We begin by presenting the most important theorems that help us prove the blow-up of this solution.

Definition 2. *We say that the solution w blow-up in finite time if there exists $T^\star \in (0, \infty)$ such that,*

$$\lim_{t \to T^\star} \|w\|_2 = +\infty, \tag{40}$$

Lemma 10. *Let w be a local solution of (4) on its maximal existence interval $[0, T)$. If there exists $t_0 \in [0, T)$ such that $w(., t_0) \in \overline{V}$ and $E(t_0) < d$, then*

$$w \in \overline{V} \text{ for all } t \in [t_0, T),$$

Proof. Let w be a local solution of (4) as in the statement.

Assume, by contradiction, that there exists $t^\star \in [t_0, T)$ such that

$$w \in \overline{V} \text{ for all } t \in [t_0, t^\star) \text{ and } u(., t^\star) \notin \overline{V}.$$

The continuity of $\mathcal{J}$ and $\mathcal{I}$ yields that either $\mathcal{J}(w(., t^\star)) = d$ or $\mathcal{I}(w(., t^\star)) = 0$.

On the other hand, $\mathcal{J}(w(., t^\star)) \le \mathcal{E}(t) < d$, so that the case $\mathcal{J}(w(., t)) = d$ cannot.

It remains to consider the case when $\mathcal{I}(w(., t^\star) = 0$. Then

$$\|w(., t^\star)\|^2_{H^r_0(\Omega_1)} = \|w(., t^\star)\|^p_p,$$

and so,

$$\frac{d}{d\mu}\mathcal{J}(\mu w(., t^\star) = \mu^2(1 - \mu^{p-2})\|w(., t^\star)\|^p_p,$$

and,

$$\sup_{\mu \in \mathbb{R}} \mathcal{J}(\mu w(., t^\star)) < d,$$

which contradicts the definition of d. □

Lemma 11. *Let w be a local solution of (4) on its maximal existence interval $[0, T)$. If there exists $t_0 \in [0, T)$ such that $w(., t_0) \in \overline{V}$ and $\mathcal{E}(t_0) < d$, then*

$$\|w(., t^\star)\|^2_{H^r_0(\Omega_1)} > \frac{2p}{p-2}d,$$

for all $t \in [0, T)$.

Proof. by too under Lemmas, we have $\mathcal{I}(w < 0$ .i.e, $\|w\|^2_{H^r_0(\Omega_1)} < \|w\|^p_p$ for all $t \in [0, T)$. Then,

$$d < \mathcal{J}(\mu^\star w) = \sup_{\mu \in \mathbb{R}} \mathcal{J}(\mu w)$$

$$= \frac{p-2}{2p}\frac{\left(\|w\|^2_{H^r_0(\Omega_1)}\right)^{\frac{p}{p-2}}}{\left(\|w\|^p_p\right)^{\frac{2}{p-2}}} < \frac{p-2}{2p}\|w\|^2_{H^r_0(\Omega_1)},$$

□

Theorem 10. *Let w be a local solution of (4) on $[0, T)$. If there exists $t_0 \in [0, T)$ such that $w(., t_0) \in \overline{V}$ and $\mathcal{E}(t_0) < d$, then $T < \infty$, i.e., the solution w blows up in a finite time.*

Proof. Let w be a local solution of (4). suppose, by contradiction, that w is a global solution, then $0 \le \mathcal{E}(t)$ for all $0 \le t$, we have,

$$-\int_{t_0}^{t}\frac{d\mathcal{E}(\tau)}{d\tau}d\tau = \int_{t_0}^{t}||\partial_t w(., \tau)||^2_{H^r_0(\Omega_1)}d\tau = \mathcal{E}(t_0) - \mathcal{E}(t) \le d, \tag{41}$$

and,

$$(\partial^2_{tt}w(.t), w(.t)) = \|w\|^p_p - \|w\|^2_{H^r_0(\Omega_1)} - (\partial_t w(.t), w(.t))_{H^r_0(\Omega_1)},$$

We put $M(t) = \|w\|^2_2$, we obtain,

$$\begin{aligned}
&\frac{1}{2}M''(t) + C(\varepsilon)\|\partial_t w\|_{H^r(\Omega_1)} = (\partial_t w, \partial_t w) + (w, \partial^2_{tt}w) + C(\varepsilon)\|\partial_t w\|_{H^r(\Omega_1)}\\
&= \|\partial_t w\|^2_2 + \|w\|^p_p - \|w\|_{H^r_0(\Omega_1)} - (w, \partial_t w)_{H^r_0(\Omega_1)} + C(\varepsilon)\|\partial_t w\|_{H^r(\Omega_1)}\\
&\ge \|\partial_t w\|^2_2 + \|w\|^p_p - \|w\|_{H^r_0(\Omega_1)} - \|w\|_{H^r_0(\Omega_1)}\|\partial_t w\|_{H^r_0(\Omega_1)} + C(\varepsilon)\|\partial_t w\|_{H^r(\Omega_1)}\\
&\ge \|\partial_t w\|^2_2 + \|w\|^p_p - \|w\|_{H^r_0(\Omega_1)} - \varepsilon\|w\|^2_{H^r_0(\Omega_1)} - C(\varepsilon)\|\partial_t w\|^2_{H^r_0(\Omega_1)} + C(\varepsilon)\|\partial_t w\|_{H^r(\Omega_1)}\\
&\ge \left(1 + \frac{\delta}{2}\right)\|\partial_t w\|^2_2 + \left(1 - \frac{\delta}{p}\right)\|w\|^p_p + \left(-1 + \frac{\delta}{2} - \varepsilon\right)\|w\|_{H^r_0(\Omega_1)} - \delta\mathcal{E}(t_0)
\end{aligned}$$

where $\varepsilon > 0$ and $\delta > 2$, satisfy,

$$h = -1 + \frac{\delta}{2} - \varepsilon, \text{ and } \mathcal{E}(t_0) \le \frac{p(\delta - 1)}{\delta(p-2)},$$

So and take ($\delta < p(1 - \delta_0)$), we obtain,

$$\frac{1}{2}M''(t) + C(\varepsilon)\|\partial_t w\|_{H^r(\Omega_1)} \ge \left(1 - \frac{\delta}{p}\right)\|w\|^p_p \ge \delta_0\|w\|^p_p \ge C\delta_0\|w\|^p_{H^r_0(\Omega_1)} > C_1,$$

Integrating twice on over $[t_0, t]$, we get,

$$M(t) \ge C_1 t^2 + C_2 t + C_3,$$

such that,

$$\begin{cases} C_2 = M'(t_0) - 2C(\varepsilon)d), \\ C_3 = 2M(t_0), \end{cases}$$

Thus, there exist $t_0 > 0$ and $V > 0$ such that,

$$M(t) > Vt^2, \quad \text{for all} t \ge t_1, \tag{42}$$

On the other hand, owing to the Hölder's inequality, we get

$$
\begin{aligned}
\|w\|_2 &\le \|w(.,t_0)\|_2 + \int\limits_{t_0}^{t} \|\partial_t w(.,\tau)\|_2 d\tau \le \|w(.,t_0)\|_2 + |\Omega_1|^{\frac{1}{l}} \int\limits_{t_0}^{t} \|\partial_t w(.,\tau)\|_p d\tau, \left(\frac{1}{l}+\frac{1}{p}=\frac{1}{2}\right) \\
&\le \|w(.,t_0)\|_2 + |\int\limits_{t_0}^{t} (M+\tau)^{\frac{n}{l}} \|\partial_t w(.,\tau)\|_p d\tau \\
&\le \|w(.,t_0)\|_2 + C_\star \int\limits_{t_0}^{t} (M+\tau)^{\frac{n}{l}} \|\partial_t w(.,\tau)\|_{H_0^r(\Omega)} d\tau, \left(\|\partial_t w(.,\tau)\|_p \le C_\star \|\partial_t w(.,\tau)\|_{H_0^r(\Omega_1)}\right) \\
&\le \|w(.,t_0)\|_2 + C_\star \left(\int\limits_{t_0}^{t} (M+\tau)^{\frac{np}{l(p-1)}}\right)^{\frac{p-1}{p}} \left(\int\limits_{t_0}^{t} \|\partial_t w(.,\tau)\|_{H_0^r(\Omega)}^{p} d\tau\right)^{\frac{1}{p}} \\
&\le \|w(.,t_0)\|_2 + (M+t)^{\left[1+\frac{np}{(p-1)l}\right]\frac{p-1}{p}} d^{\frac{1}{p}},
\end{aligned}
$$

where M is a positive constant to be chosen large enough.

Note that $p \in [2, 2^\star]$, then we have,

$$\left[1+\frac{np}{(p-1)l}\right]\frac{p-1}{p} < 2.$$

which contradicts (42) as $t \to \infty$. The proof is now completed. □

The following results (Lemma 12 and Theorem 11) are present with their proof in [23].

Lemma 12. *We put $\mathcal{M} = \{w \in H_0^r(\Omega_1),\ \mathcal{I}(w) < 0\}$. Let w be the weak solution of the problem (4). If $(w_0, w_1) \in \mathcal{M} \times L^2(\Omega_1)$, satisfies that,*

$$\|w_0\|^2 \ge \frac{2p}{p-2} K\mathcal{E}(0), \tag{43}$$

$$(\partial_t w, w)_{H_0^r(\Omega_1)} < 0, \tag{44}$$

$$\int_{\Omega_1} w_0 w_1 dx > 0, \tag{45}$$

Then w belongs to $\mathcal{M}$.

K is the constant of the embedding $H_0^r(\Omega_1)$ into $L^2(\Omega_1)$.

Theorem 11. *Let $w \in H_0^r(\Omega_1)$ and $w_1 \in L^2(\Omega_1)$. Assume that $w_0 \in \mathcal{M}$, $\mathcal{E}(0) > 0$ and $\int\limits_{\Omega_1} w_0 w_1 dx > 0$, then the solution to problem (4) blow-up in finite time.*

Lemma 13. *Assume that φ in $C^2([0,T])$ satisfying,*

$$\varphi(t)\varphi_{tt}(t) - a\varphi_t^2(t) + b\varphi(t) + c\varphi(t)\varphi_t(t) \ge 0, \ \ a > 1,\ b, c \ge 0,$$

and,

$$\begin{cases} \varphi_t(0) > \dfrac{c}{a-1}\varphi(0), \\ \left(\varphi_t(0) - \dfrac{c}{a-1}\varphi(0)\right)^2 > \dfrac{2b}{2a-1}\varphi(0), \end{cases}$$

then,

$$\lim_{t \to T^\star} \sup \varphi(t) = +\infty, \ \ T^\star \le \frac{\varphi(0)^{1-a}}{B},$$

where,

$$B=\sqrt{(a-1)^2\varphi(0)^{-2a}\left[\left(\varphi_t(0)-\frac{c}{a-1}\varphi(0)\right)^2-\frac{2b}{2a-1}\varphi(0)\right]},$$

Moreover, φ satisfies,

$$\varphi(t)\geq\frac{e^{\frac{ct}{a-1}}}{(\varphi(0)^{1-a}-Bt)^{\frac{1}{a-1}}},$$

Theorem 12. *Assume that $\mathcal{E}(0)>0$. Let $w_0\in H_0^r(\Omega_1)$ and $w_1\in L^2(\Omega_1)$ such that $\int\limits_{\Omega_1} w_0w_1dx>0$, $(w,\partial_t w)_{H_0^r(\Omega_1)}<0$ and $\left(\int\limits_{\Omega_1} w_0w_1dx\right)^2>8\mathcal{E}(0)\|w_0\|^2$, then there exists,*

$$T^\star\leq\frac{4\|w_0\|^2}{\sqrt{(p-2)^2\left[\left(\int\limits_{\Omega_1} w_0w_1dx\right)^2-8\mathcal{E}(0)\|w_0\|^2\right]}},$$

such that,

$$\lim_{t\to T^\star}\|w\|=+\infty,$$

Moreover,

$$\|w\|_2^2\geq\frac{\left(4\|w_0\|_2^{\frac{p+6}{4}}-\sqrt{(p-2)^2\left[\left(\int\limits_{\Omega_1} w_0w_1dx\right)^2-8\mathcal{E}(0)\|w_0\|^2\right]}\,t\right)^{\frac{4}{p-2}}}{4^{\frac{4}{p-2}}\|w_0\|^{\frac{2p+4}{p-2}}},$$

Proof. Multiplying $(4)_1$ by w and by integration over Ω_1, we have

$$\frac{1d^2}{2dt^2}M(t)-H(t)+\|w\|^2_{H_0^r(\Omega_1)}+(w,\partial_t w)_{H_0^r(\Omega_1)}=\|w\|_p^p, \tag{46}$$

such that,

$$\begin{cases} M(t)=\|w\|^2,\\ H(t)=\int\limits_{\Omega_1}\|\partial_t w\|^2dx,\end{cases}$$

In a similar way, multiplying $(4)_1$ by $\partial_t w$ and by integration over Ω_1, we get,

$$\frac{d}{dt}\left(\frac{1}{2}H(t)+\frac{1}{2}\|w\|^2_{H_0^r(\Omega_1)}\right)+\|\partial_t w\|^2_{H_0^r(\Omega_1)}=\frac{d}{dt}\left(\frac{1}{p}\|w\|_p^p\right), \tag{47}$$

integrating on $[0,t]$, we find,

$$\frac{1}{2}H(t)+\frac{1}{2}\|w\|^2_{H_0^r(\Omega_1)}-\mathcal{E}(0)+\int\limits_0^t\|\partial_t w\|^2_{H_0^r(\Omega_1)}dt=\frac{1}{p}\|w\|_p^p, \tag{48}$$

by subtracting $p\times$ (48) from (46), we obtained,

$$\frac{1d^2}{2dt^2}M(t)+p\mathcal{E}(0)+(w,\partial_t w)_{H_0^r(\Omega_1)}-p\int_0^t\|\partial_t w\|^2_{H_0^r(\Omega_1)}dt=\frac{p-2}{2}\|w\|_{H_0^r(\Omega_1)}+\frac{p+2}{2}H(t), \tag{49}$$

Since $p>2$, we have,

$$\frac{1d^2}{2dt^2}M(t)+p\mathcal{E}(0)>\frac{p+2}{2}H(t), \tag{50}$$

multiply (51) by M, and using Cauchy–Schwartz inequality to get

$$\frac{1}{2}M''(t)M(t)+p\mathcal{E}(0)M(t)>\frac{p+2}{2}H(t)M(t)>\frac{p+2}{2}\left(\frac{1}{4}M'(t)^2\right)>\frac{p+2}{8}M'(t)^2, \tag{51}$$

then,

$$\begin{cases} a=\dfrac{p+2}{4}, \\ b=2p\mathcal{E}(0), \\ c=0, \end{cases}$$

and therefore we find,

$$\begin{cases} \dfrac{c}{a-1}=0, \\ \dfrac{2b}{2a-1}=\dfrac{8p\mathcal{E}(0)}{p}=8\mathcal{E}(0), \end{cases}$$

by Lemma 13, we have,

$$\begin{aligned} B&=\sqrt{\left(\frac{p-2}{4}\right)^2M(0)^{-\frac{p+2}{2}}\left[(M'(0))^2-8\mathcal{E}(0)M(0)\right]} \\ &=\sqrt{\left(\frac{p-2}{4}\right)^2\|w_0\|^{-(p+2)}\left[\left(\int_{\Omega_1}w_0w_1dx\right)^2-8\mathcal{E}(0)\|w_0\|^2\right]} \\ &=\frac{\sqrt{(p-2)^2\left[\left(\int_{\Omega_1}w_0w_1dx\right)^2-8\mathcal{E}(0)\|w_0\|^2\right]}}{4\|w_0\|^{\frac{p+2}{2}}}, \end{aligned}$$

then,

$$T^\star\le\frac{\|w_0\|^{\frac{2-p}{2}}}{B}=\frac{4\|w_0\|^{\frac{p+2}{2}}\|w_0\|^{\frac{2-p}{2}}}{\sqrt{(p-2)^2\left[\left(\int_{\Omega_1}w_0w_1dx\right)^2-8\mathcal{E}(0)\|w_0\|^2\right]}},$$

Furthermore, therefore, we find,

$$T^\star\le\frac{4\|w_0\|^2}{\sqrt{(p-2)^2\left[\left(\int_{\Omega_1}w_0w_1dx\right)^2-8\mathcal{E}(0)\|w_0\|^2\right]}},$$

Moreover, we get,

$$\|w\|_2^2 \geq \left(\|w_0\|_2^{\frac{2-p}{4}} - \frac{\sqrt{(p-2)^2\left[\left(\int\limits_{\Omega_1} w_0 w_1 dx\right)^2 - 8\mathcal{E}(0)\|w_0\|^2\right]}}{4\|w_0\|^{\frac{p+2}{2}}} t \right)^{\frac{4}{p-2}},$$

By completing the calculation, the proof is completed. □

Theorem 13. *Assume that* $p \in \left(2, \frac{2_r^\star}{2}\right]$ *and* $0 < \mathcal{E}(0)$. *Let* w *be a solution of model* (4)*, which blows up at a finite time* T. *Then,*

$$T \geq \int\limits_{\|w_0\|_p^p}^{+\infty} \frac{1}{p\mathcal{E}(0) + x + C_0^p 2^{2p-2}(p\mathcal{E}(0)^p + p^{1-p}x^p)} dx \tag{52}$$

C_0 *give by* (1).

Proof. We have $L(t) = \int\limits_{\Omega_1} |w|^p dx$, by Chauchy's inequality and inequality of arithmetic and geometric means, we have,

$$L^{'}(t) \leq p\left(\int\limits_{\Omega_1} |\partial_t w|^2 dx\right)^{\frac{1}{2}} \left(\int\limits_{\Omega_1} |w|^{2p} dx\right)^{\frac{1}{2}} \leq \frac{p}{2}\left(\int\limits_{\Omega_1} |\partial_t w|^2 dx + \int\limits_{\Omega_1} |w|^{2p} dx\right), \tag{53}$$

we using Lemma 1, we obtain,

$$\begin{aligned} L^{'}(t) &\leq \frac{p}{2}\left(\|\partial_t w\|_2^2 + C_0^p \|w\|_{H_0^r(\Omega_1)}^{2p}\right) \\ &\leq \frac{p}{2}\left(2\mathcal{E}(t) + \frac{2}{p}\|w\|_p^p + C_0^p\left[2\mathcal{E}(t) + \frac{2}{p}\|w\|_p^p\right]^p\right) \\ &\leq p\mathcal{E}(t) + L(t) + pC_0^p 2^{2p-2}\left((2\mathcal{E}(t))^p + \left(\frac{2}{p}L(t)\right)^p\right), \end{aligned}$$

We can obtain the next result,

$$\int\limits_0^T \frac{L^{'}(t)}{p\mathcal{E}(0) + L(t) + C_0^p 2^{2p-2}(p\mathcal{E}(0)^p + p^{1-p}L(t)^p)} dt \leq \int\limits_0^T dt \tag{54}$$

We put $x = L(t)$, then,

$$\begin{cases} dx = L^{'}(t)dt, \\ \text{Si } t = 0, \text{ then } x = L(0) = \|w_0\|_p^p, \\ \text{Si } t = T, \text{ then } x = +\infty \end{cases}$$

Finally, we obtain my result,

$$\int_{\|w_0\|_p^p}^{\infty} \frac{1}{p\mathcal{E}(0) + x + C_0^p 2^{2p-2}(p\mathcal{E}(0)^p + p^{1-p}x^p)} dx \leq T \tag{55}$$

The proof is therefore completed. □

7. Conclusions

To study such systems as a complete study, it is necessary to study them quantitatively and qualitatively. In a previous article [30], the global existence and behavior of the solution were studied during the evolution of time at infinity, where very good results were found. Supplementing these results, in this research, the same problem is studied from another angle and under other conditions contrary to what they were in the previous article [30], where we found a solution in a local time and then the solution blows up to announce the global non-existence. In this study, we used the latest methods and developed some computational techniques to make it easier for the reader to follow and understand the article. We created a new interaction between the various parameters r and p in order to find simple and clear results.

The article is devoted to a study of the question of the local (in time) existence of weak solutions and to the derivation of qualitative properties of such solutions for the wave equation with the fractional Laplace operator and a minor nonlinear damping term. At the beginning of the article, in Sections 1 and 2, we make a brief overview of the literature on the topic and outline some preparatory information from the theory of fractional derivatives and equations involving such derivatives. In Section 3, by means of the basic properties of fractional derivatives and the application of the standard Galerkin method, the existence of a weak solution is established on some time segment $(0, T)$. After this, in Sections 4–6, the qualitative properties are established for the solutions, namely, the isolation of solutions from zero (i.e., from the vacuum), the conditional result on the decay of solutions, and the conditional result on the blow-up of solutions.

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