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Graphs with Strong Proper Connection Numbers and Large Cliques

Yingbin Ma *, Xiaoxue Zhang and Yanfeng Xue

College of Mathematics and Information Science, Henan Normal University, Xinxiang 453007, China
* Correspondence: mayingbincw@htu.cn

Abstract: In this paper, we mainly investigate graphs with a small (strong) proper connection number and a large clique number. First, we discuss the (strong) proper connection number of a graph G of order n and $\omega(G) = n - i$ for $1 \leqslant i \leqslant 3$. Next, we investigate the rainbow connection number of a graph G of order n, $diam(G) \geq 3$ and $\omega(G) = n - i$ for $2 \leqslant i \leqslant 3$.

Keywords: edge-coloring; proper connection number; strong proper connection number; rainbow connection number; clique number

MSC: 05C15; 05C40

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1. Introduction

We only consider graphs that are undirected, simple, finite, and connected in this paper. For terminology and notation that are not defined here, we refer to [1].

In 2008, Chartrand et al. [2] introduced the concept of rainbow connection. For an edge-colored graph G, if each pair of vertices is connected by a rainbow path, where its edges are assigned different colors, then G is said to be *rainbow-connected*. An edge-coloring that makes G rainbow-connected is said to be a *rainbow coloring* of G. The *rainbow connection number* of G, denoted by $rc(G)$, is the smallest number of colors that are needed to make G rainbow-connected. Obviously, $rc(G) = 1$ if and only if G is complete, and $rc(G) \geq diam(G)$. As a natural generalization of the rainbow connection number, the concept of the vertex rainbow connection number was presented by Krivelevich et al. [3], and the concept of the total rainbow connection number was introduced by Liu et al. [4]. There are abundant research results on this topic. In [5], Schiermeyer proved that a connected graph G with n vertices has $rc(G) < \frac{4n}{\delta(G)+1} + 4$. Huang et al. [6] provided upper bounds of the rainbow connection number of outerplanar graphs with small diameters. In [7], Li et al. studied the vertex rainbow connection numbers of some graph operations. Ma et al. [8] investigated the total rainbow connection numbers of some special graphs. The reader should also consult [9] for a survey and [10] for a monograph.

Inspired by the concept of rainbow connection, Borozan et al. [11] proposed the concept of proper connection, and Andrews et al. [12] presented the concept of strong proper connection. A path is called a *proper path* in an edge-colored graph if its adjacent edges are assigned distinct colors. An edge-colored graph G is said to be *properly connected* if any two vertices are connected by a proper path, and G is said to be *strongly properly connected* if every pair of vertices is connected by a proper geodesic. An edge-coloring θ of graph G is called a *proper-path coloring* if it makes G properly connected, and θ is called a *strong proper coloring* if it makes G strongly properly connected. The *proper connection number* of G, denoted by $pc(G)$, is the smallest number of colors that are needed to make G properly connected. The *strong proper connection number* of G, denoted by $spc(G)$, is the smallest number of colors that are needed to make G strongly properly connected. From these definitions, it is easy to establish that $pc(G) = spc(G) = 1$ if and only if G is complete. In [13,14], Huang et al. presented an upper bound for the proper connection number of a graph in terms of the bridge-block tree of the graph and investigated the proper connection number of the complement of a graph. Li et al. [15] used dominating sets to study the proper connection number of a graph. Ma and Zhang [16] characterized all connected graphs of size m with (strong) proper connection number $m-4$. For more details, we refer the reader to a survey [17].

Some results regarding the (vertex) rainbow connection numbers of graphs with a large clique number are available; see [18,19]. These results motivated us to consider the (strong) proper connection numbers of graphs with a large clique number. In this paper, we mainly discuss the (strong) proper connection number of a graph G of order n and $\omega(G) = n - i$ for $1 \leqslant i \leqslant 3$. Moreover, we also investigate the rainbow connection number of a graph G of order n, $diam(G) \geq 3$ and $\omega(G) = n - i$ for $2 \leqslant i \leqslant 3$.

2. (Strong) Proper Connection and Clique Number

In this section, we investigate graphs with a small (strong) proper connection number and a large clique number. We first introduce some definitions that will be used later.

A Hamiltonian path in a graph G is a path containing every vertex of G. A graph with a Hamiltonian path is called a traceable graph. Recall that a clique of a graph is a set of mutually adjacent vertices, and that the maximum size of a clique of graph G, i.e., the clique number of G, is denoted $\omega(G)$. For a connected graph G, we say Q is a subgraph of G which induces a maximum clique and $V(F) = V(G) \setminus V(Q)$. We say $N_Q(u)$ is the set of neighbors of u in Q and $d_Q(u) = |N_Q(u)|$. Additionally, we say $E[V(F), V(Q)]$ is the set of edges of G between vertices of $V(F)$ and vertices of $V(Q)$. Next, we present the following three useful propositions.

Proposition 1 ([12])**.** *Let G be a non-complete graph. If G is traceable, then $pc(G) = 2$.*

Proposition 2 ([12])**.** *For a non-trivial connected graph G that contains a bridge, if b is the maximum number of bridges incident with a vertex in G, then $spc(G) \geq pc(G) \geq b$.*

Proposition 3 ([18])**.** *Let G be a connected graph of order n and size m. If $\binom{n-1}{2} + 1 \leq m \leq \binom{n}{2} - 1$, then $rc(G) = 2$.*

As an immediate consequence of Proposition 3, we have the following Lemma.

Lemma 1. *Let G be a connected graph of order n and size m. If $\binom{n-1}{2} + 1 \leq m \leq \binom{n}{2} - 1$, then $pc(G) = spc(G) = 2$.*

Theorem 1. *Let G be a connected graph of order n. If $\omega(G) = n + 1 - i$ for $i \in \{1, 2\}$, then $pc(G) = spc(G) = i$.*

Proof. If $i = 1$, then $\omega(G) = n$, which implies that G is a complete graph. Thus, $pc(G) = spc(G) = 1$. If $i = 2$, then $\omega(G) = n - 1$. Since G is connected, we obtain $|E(G)| \geq \binom{n-1}{2} + 1$, and so $\binom{n-1}{2} + 1 \leq |E(G)| \leq \binom{n}{2} - 1$. Hence, $pc(G) = spc(G) = 2$ by Lemma 1. □

Theorem 2. *Let G be a connected graph of order $n \geq 4$ and $\omega(G) = n - 2$. Let Q be a maximum clique of G and $V(G) \setminus V(Q) = \{u_1, u_2\}$. Then, either $pc(G) = spc(G) = 2$ or one of the following holds:*

(i) $4 \leq n \leq 5$, $G[V(G) \setminus V(Q)] \cong 2K_1$ *and* $N_Q(u_1) = N_Q(u_2) = \{v\}$.
(ii) $n \geq 6$, $G[V(G) \setminus V(Q)] \cong 2K_1$ *and* $N_Q(u_1) = N_Q(u_2) = \{v\}$.

Moreover, we have $pc(G) = spc(G) = 3$ for (i), *$pc(G) = 2$, and $spc(G) = 3$ for (ii).*

Proof. Let $F = G[V(G) \setminus V(Q)]$ and let θ be an edge-coloring of G. We prove this theorem by analyzing the structure of F.

Case 1. $F \cong K_2$. Since G is connected, it follows that $max\{d_Q(u_1), d_Q(u_2)\} \geq 1$. Note that G is traceable, and we have $pc(G) = 2$ by Proposition 1. The following edge-coloring θ with two colors makes G strongly properly connected: color u_1u_2 and all edges of $E(Q)$ with 1, and color all edges of $E[V(F), V(Q)]$ with 2. Thus, $spc(G) = 2$.

Case 2. $F \cong 2K_1$. Since G is connected, it follows that $min\{d_Q(u_1), d_Q(u_2)\} \geq 1$. Assume that $N_Q(u_1) \cap N_Q(u_2) = \emptyset$. Observe that G is traceable, and we have $pc(G) = 2$ by Proposition 1. Assign an edge-coloring θ with two colors to G as follows: color all edges of $E(Q)$ with 1 and all edges of $E[V(F), V(Q)]$ with 2. It is clear that G is strongly properly connected with the above edge-coloring. Hence, $spc(G) = 2$.

Assume that $N_Q(u_1) \cap N_Q(u_2) \neq \emptyset$ and $d_Q(u_1) = d_Q(u_2) = 1$. Without a loss of generality, let $v \in N_Q(u_1) \cap N_Q(u_2)$. If $n = 4$, then $G \cong K_{1,3}$. Hence, $pc(G) = spc(G) = 3$. If $n = 5$, then $G \cong G_1$, where G_1 is obtained by adding two pendant edges to a vertex of K_3. Thus, $pc(G) = spc(G) = 3$. Now we consider $n \geq 6$. Let $V(Q) = \{v, w_1, w_2, \ldots, w_{n-3}\}$. Define an edge-coloring θ of G with two colors as follows: $\theta(u_1v) = \theta(w_1w_{n-3}) = 1$; $\theta(u_2v) = \theta(vw_{n-4}) = 2$; color the sequence $vw_1w_2 \cdots w_{n-3}v$ alternately with 1 and 2 starting with $\theta(vw_1) = 1$; and color the remaining edges arbitrarily with 1 and 2. We can check that G is properly connected with the above edge-coloring, and so $pc(G) = 2$. If θ is a strong proper coloring of G, then $\theta(u_1v) \neq \theta(u_2v) \neq \theta(vw_1)$, and thus $spc(G) \geq 3$. On the other hand, we define a strong proper coloring θ' of G with three colors as follows: $\theta'(u_1v) = 1$, $\theta'(u_2v) = 2$, and color all edges of $E(Q)$ with 3. Thus, $spc(G) = 3$.

Assume that $N_Q(u_1) \cap N_Q(u_2) \neq \emptyset$ and $max\{d_Q(u_1), d_Q(u_2)\} \geq 2$. Without a loss of generality, let $v \in N_Q(u_1) \cap N_Q(u_2)$ and $d_Q(u_1) \geq 2$. Observe that G is traceable, and we obtain $pc(G) = 2$ by Proposition 1. Assign an edge-coloring θ with two colors to G as follows: $\theta(u_1v) = 1$; $\theta(u_2v) = \theta(u_1w) = 2$ for any $w \in N_Q(u_1) \setminus \{v\}$; and color the remaining edges with 1. It is clear that θ is a strong proper coloring of G. Hence, $spc(G) = 2$. □

Theorem 3. *Let G be a connected graph of order $n \geq 5$, $diam(G) = 2$, and $\omega(G) = n - 3$. Let Q be a maximum clique of G and $V(G) \setminus V(Q) = \{u_1, u_2, u_3\}$. Then, either $pc(G) = spc(G) = 2$ or one of the following holds:*

(i) $G[V(G) \setminus V(Q)] \cong P_3$, *where* $u_1u_2, u_2u_3 \in E(G)$, $d_Q(u_2) = 0$, $min\{d_Q(u_1), d_Q(u_3)\} = 1$, $N_Q(u_1) \cup N_Q(u_3) = V(Q)$ *and* $N_Q(u_1) \cap N_Q(u_3) = \emptyset$.

(ii) $n = 6$, $G[V(G) \setminus V(Q)] \cong K_2 + K_1$, *where* $u_1u_2 \in E(G)$, *and* $N_Q(u_1) = N_Q(u_2) = N_Q(u_3) = \{v\}$.

(iii) $n \geq 7$, $G[V(G) \setminus V(Q)] \cong K_2 + K_1$, *where* $u_1u_2 \in E(G)$, *and* $N_Q(u_1) = N_Q(u_2) = N_Q(u_3) = \{v\}$.

(iv) $G[V(G) \setminus V(Q)] \cong K_2 + K_1$, *where* $u_1u_2 \in E(G)$, $N_Q(u_1) \cap N_Q(u_2) \cap N_Q(u_3) = \{v\}$, $min\{d_Q(u_1), d_Q(u_2)\} = d_Q(u_3) = 1$ *and* $d_Q(u_1) + d_Q(u_2) \geq 3$.

(v) $G[V(G) \setminus V(Q)] \cong K_2 + K_1$, *where* $u_1u_2 \in E(G)$, $N_Q(u_1) \cap N_Q(u_3) \neq \emptyset$, $N_Q(u_2) \cap N_Q(u_3) \neq \emptyset$, $d_Q(u_1) = d_Q(u_2) = 1$, $d_Q(u_3) = 2$ *and* $N_Q(u_1) \cap N_Q(u_2) = \emptyset$.

(vi) $n = 5$, $G[V(G) \setminus V(Q)] \cong 3K_1$ *and* $N_Q(u_1) = N_Q(u_2) = N_Q(u_3) = \{v\}$.

(vii) $n \geq 6$, $G[V(G) \setminus V(Q)] \cong 3K_1$ *and* $N_Q(u_1) = N_Q(u_2) = N_Q(u_3) = \{v\}$.

(viii) $G[V(G) \setminus V(Q)] \cong 3K_1$, $|(N_Q(u_1) \cap N_Q(u_2)) \cup (N_Q(u_1) \cap N_Q(u_3)) \cup (N_Q(u_2) \cap N_Q(u_3))| = 1$, $N_Q(u_1) \cap N_Q(u_2) \cap N_Q(u_3) \neq \emptyset$ *and* $d_Q(u_1) + d_Q(u_2) + d_Q(u_3) \geq 4$.

(ix) $G[V(G) \setminus V(Q)] \cong 3K_1$, $|(N_Q(u_1) \cap N_Q(u_2)) \cup (N_Q(u_1) \cap N_Q(u_3)) \cup (N_Q(u_2) \cap N_Q(u_3))| = 2$, $N_Q(u_1) \cap N_Q(u_2) \cap N_Q(u_3) \neq \emptyset$ *and* $d_Q(u_1) + d_Q(u_2) + d_Q(u_3) = 5$.

Moreover, we have $pc(G) = 2$ *and* $spc(G) = 3$ *for (i), (iii), (iv), (v), (viii), and (ix);* $pc(G) = spc(G) = 3$ *for (ii);* $pc(G) = spc(G) = 4$ *for (vi); and* $pc(G) = 3$ *and* $spc(G) = 4$ *for (vii).*

Proof. Let $F = G[V(G) \setminus V(Q)]$ and let θ be an edge-coloring of G. We prove this theorem by analyzing the structure of F.

Case 1. $F \cong K_3$. Observe that G is traceable, and so $pc(G) = 2$ by Proposition 1. The following edge-coloring θ with two colors induces a strong proper coloring of G: color all edges of $E(F)$ and $E(Q)$ with 1, and color all edges of $E[V(F), V(Q)]$ with 2. Thus, $spc(G) = 2$.

Case 2. $F \cong P_3$, where $u_1u_2, u_2u_3 \in E(G)$. Assume that $min\{d_Q(u_1), d_Q(u_2), d_Q(u_3)\} \geq 1$. Note that G is traceable, and we have $pc(G) = 2$ by Proposition 1. Assign a strong proper coloring θ with two colors to G as follows: $\theta(u_1u_2) = 1$; $\theta(u_2u_3) = 2$; and color all edges of $E(Q)$ with 1 and all edges of $E[V(F), V(Q)]$ with 2. Hence, $spc(G) = 2$.

Assume that $min\{d_Q(u_1), d_Q(u_2), d_Q(u_3)\} = 0$. Since $diam(G) = 2$, it follows that $d_Q(u_1) \geq 1$, $d_Q(u_2) = 0$, $d_Q(u_3) \geq 1$, and $N_Q(u_1) \cup N_Q(u_3) = V(Q)$. Observe that G is traceable, and we obtain $pc(G) = 2$ by Proposition 1. Next, we only consider the strong proper connection number of graph G under this assumption.

Suppose $N_Q(u_1) \cap N_Q(u_3) = \emptyset$ and $min\{d_Q(u_1), d_Q(u_3)\} = 1$. Without a loss of generality, let $d_Q(u_1) = 1$ and $N_Q(u_1) = \{v\}$. If there exists a strong proper coloring θ of G with two colors, then $\theta(u_1u_2) \neq \theta(u_2u_3)$. Without a loss of generality, let $\theta(u_1u_2) = 1$ and $\theta(u_2u_3) = 2$. Since u_2u_1v is the unique $u_2 - v$ geodesic and u_2u_3w is the unique $u_2 - w$ geodesic for any $w \in N_Q(u_3)$, it follows that $\theta(u_1v) = 2$ and $\theta(u_3w) = 1$. Note that u_1vw is the unique $u_1 - w$ geodesic for any $w \in N_Q(u_3)$, and so $\theta(vw) = 1$. There is no proper geodesic between u_3 and v, which is a contradiction. Thus, $spc(G) \geq 3$. Assign an edge-coloring θ' with three colors to G as follows: $\theta'(u_1u_2) = \theta'(u_3w) = 1$ for any $w \in N_Q(u_3)$, $\theta'(u_2u_3) = \theta'(u_1v) = 2$, and color all edges of $E(Q)$ with 3. Obviously, θ' is a strong proper coloring of G, and so $spc(G) = 3$.

Suppose $N_Q(u_1) \cap N_Q(u_3) = \emptyset$ and $min\{d_Q(u_1), d_Q(u_3)\} \geq 2$. Let $N_Q(u_1) = \{v_1, v_2, \ldots, v_t\}$ and $N_Q(u_3) = \{w_1, w_2, \ldots, w_k\}$, where $t + k = n - 3$. Assign an edge-coloring θ with two colors to G such that G is strongly properly connected: $\theta(u_1u_2) = \theta(v_1w_1) = \theta(u_3w_1) = \theta(u_3w_i) = \theta(v_2w_i) = 1$ for $2 \leq i \leq k$, $\theta(u_2u_3) = \theta(v_1w_k) = \theta(u_1v_1) = \theta(u_1v_j) = \theta(w_1v_j) = 2$ for $2 \leq j \leq t$, and color the remaining edges arbitrarily with 1 and 2. Hence, $spc(G) = 2$.

Suppose $N_Q(u_1) \cap N_Q(u_3) \neq \emptyset$, and say $v \in N_Q(u_1) \cap N_Q(u_3)$. This implies that $min\{d_Q(u_1), d_Q(u_3)\} \geq 2$. Color u_1u_2, u_2u_3, u_1v and all edges of $E(Q)$ with 1, and color the remaining edges with 2. Clearly, G is strongly properly connected with the above edge-coloring, and so $spc(G) = 2$.

Case 3. $F \cong K_2 + K_1$, where $u_1u_2 \in E(G)$. Since G is connected, we obtain $d_Q(u_3) \geq 1$. We distinguish the following three subcases.

Subcase 3.1. $d_Q(u_3) = 1$. Let $N_Q(u_3) = \{v\}$. Since $diam(G) = 2$, we have $d_Q(u_1) \geq 1$, $d_Q(u_2) \geq 1$ and $v \in N_Q(u_1) \cap N_Q(u_2)$. Assume that $d_Q(u_1) = d_Q(u_2) = 1$. This implies $n \geq 6$. If $n = 6$, then $G \cong G_2$, where G_2 is displayed in Figure 1. Thus, $pc(G) = spc(G) = 3$. Now we consider $n \geq 7$. Let $V(Q) = \{w_1, w_2, \ldots, w_{n-4}, v\}$. Assign an edge-coloring θ with two colors to G as follows: $\theta(u_1u_2) = \theta(u_1v) = \theta(u_2v) = \theta(w_iv) = \theta(w_jw_{n-4}) = 1$ for $1 \leq i \leq n-5$ and $2 \leq j \leq n-5$, $\theta(u_3v) = \theta(w_{n-4}v) = \theta(w_1w_{n-4}) = \theta(w_1w_2) = 2$, and color the remaining edges arbitrarily with 1 and 2. It is easy to verify that θ is a proper-path coloring of G. Thus, $pc(G) = 2$. If G is strongly properly connected with an edge-coloring θ, then $\theta(u_1v) \neq \theta(u_3v) \neq \theta(w_1v)$, and so $spc(G) \geq 3$. Assign an edge-coloring θ' with three colors to G as follows: $\theta'(u_1u_2) = \theta'(u_1v) = \theta'(u_2v) = 1$, $\theta'(u_3v) = 2$, and color all edges of $E(Q)$ with 3. We can check that G is strongly properly connected with the above edge-coloring. Hence, $spc(G) = 3$.

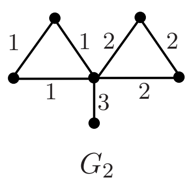

Figure 1. The graph G_2 with a strong proper coloring.

Assume that $min\{d_Q(u_1), d_Q(u_2)\} = 1$ and $d_Q(u_1) + d_Q(u_2) \geq 3$. Without a loss of generality, let $d_Q(u_1) \geq 2$ and $d_Q(u_2) = 1$. Observe that G is traceable, and we have

$pc(G) = 2$ by Proposition 1. If G is strongly properly connected with an edge-coloring θ, then $\theta(u_2v) \neq \theta(u_3v) \neq \theta(wv)$, where $w \in V(Q) \setminus N_Q(u_1)$. Hence, $spc(G) \geq 3$. Define an edge-coloring θ' of G with three colors such that G is strongly properly connected: $\theta'(u_1u_2) = \theta'(u_1v) = \theta'(u_2v) = 1$, and color all edges of $E(Q)$ with 3 and the remaining edges with 2. Thus, $spc(G) = 3$.

Assume that $min\{d_Q(u_1), d_Q(u_2)\} \geq 2$. Note that G is traceable, and so $pc(G) = 2$ by Proposition 1. Assign a strong proper coloring θ with two colors to G as follows: $\theta(u_1u_2) = \theta(u_1v) = \theta(u_2v) = 1$, and color all edges of $E(Q)$ with 1 and the remaining edges with 2. Hence, $spc(G) = 2$.

Subcase 3.2. $d_Q(u_3) = 2$. Let $N_Q(u_3) = \{u, v\}$. Since $diam(G) = 2$, we obtain $N_Q(u_1) \cap N_Q(u_3) \neq \emptyset$ and $N_Q(u_2) \cap N_Q(u_3) \neq \emptyset$. Observe that G is traceable, and we have $pc(G) = 2$ by Proposition 1.

Assume that $d_Q(u_1) = d_Q(u_2) = 1$ and $N_Q(u_1) \cap N_Q(u_2) \neq \emptyset$. Let $u \in N_Q(u_1) \cap N_Q(u_2)$. There exists a strong proper coloring θ of G with two colors as follows: $\theta(u_1u_2) = \theta(u_1u) = \theta(u_2u) = \theta(u_3v) = 1$, $\theta(u_3u) = 2$, and color all edges of $E(Q)$ with 2. Thus, $spc(G) = 2$.

Assume that $d_Q(u_1) = d_Q(u_2) = 1$ and $N_Q(u_1) \cap N_Q(u_2) = \emptyset$. Let $N_Q(u_1) = \{u\}$, $N_Q(u_2) = \{v\}$ and $V(Q) = \{w_1, w_2, \ldots, w_{n-5}, u, v\}$. If there exists a strong proper coloring θ of G with two colors, then $\theta(u_1u) \neq \theta(uu_3)$. Without a loss of generality, let $\theta(u_1u) = 1$ and $\theta(uu_3) = 2$. Since u_1uw_1 is the unique $u_1 - w_1$ geodesic, it follows that $\theta(uw_1) = 2$. Note that u_2vu_3 is the unique $u_2 - u_3$ geodesic, and so $\theta(u_2v) \neq \theta(vu_3)$. We first consider $\theta(u_2v) = 1$ and $\theta(vu_3) = 2$. Since u_2vw_1 is the unique $u_2 - w_1$ geodesic, we have $\theta(vw_1) = 2$. There is no proper geodesic between u_3 and w_1, which is a contradiction. Next, we consider $\theta(u_2v) = 2$ and $\theta(vu_3) = 1$. Note that u_2vw_1 is the unique $u_2 - w_1$ geodesic, so we obtain $\theta(vw_1) = 1$. There is no proper geodesic between u_3 and w_1, which is a contradiction. Hence, $spc(G) \geq 3$. Allocate a strong proper coloring θ' with three colors to G as follows: $\theta'(u_1u_2) = \theta'(u_1u) = \theta'(u_2v) = 1$, $\theta'(uu_3) = \theta'(vu_3) = 2$, and color all edges of $E(Q)$ with 3. Thus, $spc(G) = 3$.

Assume that $d_Q(u_1) + d_Q(u_2) \geq 3$ and $N_Q(u_1) \cap N_Q(u_2) \neq \emptyset$. Without a loss of generality, let $d_Q(u_1) \geq 2$ and $w \in N_Q(u_1) \cap N_Q(u_2)$. Consider $u \in N_Q(u_1) \cap N_Q(u_2)$ or $v \in N_Q(u_1) \cap N_Q(u_2)$. Without a loss of generality, let $u \in N_Q(u_1) \cap N_Q(u_2)$. The following edge-coloring θ with two colors makes G strongly properly connected: $\theta(u_1u) = \theta(u_2u) = \theta(u_1u_2) = 1$, $\theta(u_3u) = 2$, and color all edges of $E(Q)$ with 2 and the remaining edges with 1. Hence, $spc(G) = 2$. Consider $u \notin N_Q(u_1) \cap N_Q(u_2)$ and $v \notin N_Q(u_1) \cap N_Q(u_2)$. Then, $min\{d_Q(u_1), d_Q(u_2)\} \geq 2$. Without a loss of generality, let $u \in N_Q(u_1)$ and $v \in N_Q(u_2)$. Assign an edge-coloring θ with two colors to G as follows: $\theta(u_1w) = \theta(u_2w) = \theta(u_1u_2) = \theta(u_1u) = \theta(u_3v) = 1$, $\theta(u_3u) = \theta(u_2v) = 2$, and color all edges of $E(Q)$ with 2 and the remaining edges with 1. It is not difficult to verify that θ is a strong proper coloring of G, and so $spc(G) = 2$.

Assume that $d_Q(u_1) + d_Q(u_2) \geq 3$ and $N_Q(u_1) \cap N_Q(u_2) = \emptyset$. Without a loss of generality, let $d_Q(u_1) \geq 2$, $u \in N_Q(u_1)$, and $v \in N_Q(u_2)$. There exists an edge-coloring θ with two colors such that G is strongly properly connected, as follows: $\theta(u_1u_2) = \theta(u_2v) = \theta(u_3u) = 1$, $\theta(u_1u) = \theta(u_3v) = 2$, and color all edges of $E(Q)$ with 2 and the remaining edges with 1. Hence, $spc(G) = 2$.

Subcase 3.3. $d_Q(u_3) \geq 3$. Note that G is traceable, and we obtain $pc(G) = 2$ by Proposition 1. Assume that $d_Q(u_1) = d_Q(u_2) = 1$ and $N_Q(u_1) \cap N_Q(u_2) = \emptyset$. Let $N_Q(u_1) \cap N_Q(u_3) = \{u\}$ and $N_Q(u_2) \cap N_Q(u_3) = \{v\}$. Assign a strong proper coloring θ with two colors to G as follows: $\theta(u_1u_2) = \theta(u_1u) = \theta(u_2v) = 1$, $\theta(u_3u) = \theta(u_3v) = 2$, and color all edges of $E(Q)$ with 2 and the remaining edges with 1. Thus, $spc(G) = 2$.

Assume that either $d_Q(u_1) = d_Q(u_2) = 1$ and $N_Q(u_1) \cap N_Q(u_2) \neq \emptyset$, or $d_Q(u_1) + d_Q(u_2) \geq 3$. An analogous edge-coloring to that presented in Subcase 3.2 induces a strong proper coloring of G with $spc(G) = 2$.

Case 4. $F \cong 3K_1$. Since $diam(G) = 2$, it follows that $N_Q(u_1) \cap N_Q(u_2) \neq \emptyset$, $N_Q(u_1) \cap N_Q(u_3) \neq \emptyset$ and $N_Q(u_2) \cap N_Q(u_3) \neq \emptyset$. This case is demonstrated by the following three subcases.

Subcase 4.1. $|(N_Q(u_1) \cap N_Q(u_2)) \cup (N_Q(u_1) \cap N_Q(u_3)) \cup (N_Q(u_2) \cap N_Q(u_3))| = 1$. This implies that $|N_Q(u_1) \cap N_Q(u_2) \cap N_Q(u_3)| = 1$. Let $N_Q(u_1) \cap N_Q(u_2) \cap N_Q(u_3) = \{v\}$. Assume that $d_Q(u_1) = d_Q(u_2) = d_Q(u_3) = 1$. Then, $spc(G) \geq pc(G) \geq 3$ by Proposition 2. If $n = 5$, then $G \cong K_{1,4}$. Hence, $pc(G) = spc(G) = 4$. Now we consider $n \geq 6$. Let $V(Q) = \{v, w_1, w_2, \ldots, w_{n-4}\}$. Assign an edge-coloring θ with three colors to G as follows: $\theta(u_1v) = 1$; $\theta(u_2v) = 2$; $\theta(u_3v) = 3$; $\theta(w_1w_{n-4}) = 3$ if n is even, $\theta(w_1w_{n-4}) = 2$ if n is odd; color the sequence $w_1vw_2w_3 \cdots w_{n-4}$ alternately with 1 and 2 starting with $\theta(w_1v) = 1$; and color the remaining edges arbitrarily with 1 and 2. It is not difficult to check that θ is a proper-path coloring of G. Thus, $pc(G) = 3$. Suppose G has a strong proper coloring θ, we have $\theta(u_1v) \neq \theta(u_2v) \neq \theta(u_3v) \neq \theta(w_1v)$, and so $spc(G) \geq 4$. On the other hand, there exists a strong proper coloring θ' of G with four colors, as follows: $\theta'(u_1v) = 1$, $\theta'(u_2v) = 2$, $\theta'(u_3v) = 3$, and color all edges of $E(Q)$ with 4. Therefore, we have $spc(G) = 4$.

Assume that $d_Q(u_1) + d_Q(u_2) + d_Q(u_3) \geq 4$. Without a loss of generality, let $d_Q(u_1) \geq 2$, and say $u \in N_Q(u_1) \setminus \{v\}$. Let $V(Q) = \{u, v, w_1, w_2, \ldots, w_{n-5}\}$ with $n \geq 6$. The following edge-coloring θ with two colors makes G properly connected: $\theta(u_1v) = \theta(u_2v) = \theta(uv) = 1$, $\theta(u_3v) = 2$, color the sequence $vw_1w_2 \cdots w_{n-5}uu_1$ alternately with 2 and 1 starting with $\theta(vw_1) = 2$, and color the remaining edges arbitrarily with 1 and 2. Thus, $pc(G) = 2$. Suppose G has a strong proper coloring θ, we have $\theta(u_1v) \neq \theta(u_2v) \neq \theta(u_3v)$, and so $spc(G) \geq 3$. On the other hand, there exists a strong proper coloring θ' of G with three colors, as follows: $\theta'(u_1u) = \theta'(u_2v) = 1$, $\theta'(u_3v) = 2$, $\theta'(u_1v) = 3$, and color all edges of $E(Q)$ with 3 and the remaining edges with 1. Hence, $spc(G) = 3$.

Subcase 4.2. $|(N_Q(u_1) \cap N_Q(u_2)) \cup (N_Q(u_1) \cap N_Q(u_3)) \cup (N_Q(u_2) \cap N_Q(u_3))| = 2$. Since $diam(G) = 2$, we obtain $N_Q(u_1) \cap N_Q(u_2) \cap N_Q(u_3) \neq \emptyset$, and say $v \in N_Q(u_1) \cap N_Q(u_2) \cap N_Q(u_3)$. Without a loss of generality, we consider $|N_Q(u_1) \cap N_Q(u_2)| = 2$, and say $u \in (N_Q(u_1) \cap N_Q(u_2)) \setminus \{v\}$. Assign an analogous edge-coloring to that presented in Subcase 4.1 to G that satisfies $d_Q(u_1) + d_Q(u_2) + d_Q(u_3) \geq 4$. Obviously, G is properly connected, and so $pc(G) = 2$.

Assume that $d_Q(u_1) + d_Q(u_2) + d_Q(u_3) = 5$. Suppose that there exists a strong proper coloring θ of G with two colors. Note that u_1vu_3 is the unique $u_1 - u_3$ geodesic, and u_2vu_3 is the unique $u_2 - u_3$ geodesic. Without a loss of generality, let $\theta(u_1v) = \theta(u_2v) = 1$ and $\theta(u_3v) = 2$. Since u_3vw is the unique $u_3 - w$ geodesic, where $w \in V(Q) \setminus \{u, v\}$, it follows that $\theta(vw) = 1$. In order to have a proper geodesic connecting u_2 and w, we have $\theta(u_2u) \neq \theta(uw)$. Similarly, for the sake of having a proper geodesic between u_1 and u_2, we obtain $\theta(u_1u) \neq \theta(u_2u)$. Then, $\theta(uw) = \theta(u_1u)$, and so there is no proper geodesic connecting u_1 and w, which is a contradiction. Thus, $spc(G) \geq 3$. Now we assign a strong proper coloring θ' with three colors to G as follows: $\theta'(u_1u) = \theta'(u_1v) = \theta'(u_2v) = 1$, $\theta'(u_2u) = \theta'(u_3v) = 2$, and color all edges of $E(Q)$ with 3. Hence, $spc(G) = 3$.

Assume that $d_Q(u_1) + d_Q(u_2) + d_Q(u_3) \geq 6$. Suppose $d_Q(u_3) = 1$. This implies that $max\{d_Q(u_1), d_Q(u_2)\} \geq 3$. Without a loss of generality, we consider $d_Q(u_1) \geq 3$, and say $w \in N_Q(u_1) \setminus \{u, v\}$. The following edge-coloring θ with two colors makes G strongly properly connected: $\theta(u_1u) = \theta(u_1v) = \theta(u_2v) = 1$, $\theta(u_2u) = \theta(u_3v) = \theta(u_1w) = 2$, and color all edges of $E(Q)$ with 1 and the remaining edges with 2. Thus, $spc(G) = 2$. Suppose $d_Q(u_3) \geq 2$. Let $z \in N_Q(u_3) \setminus \{v\}$, where $u = z$ is possible. Assign an edge-coloring θ with two colors to G as follows: $\theta(u_1u) = \theta(u_1v) = \theta(u_2v) = \theta(u_3z) = 1$, $\theta(u_2u) = \theta(u_3v) = 2$, and color all edges of $E(Q)$ with 2 and the remaining edges with 1. Obviously, θ is a strong proper coloring of G, and so $spc(G) = 2$.

Subcase 4.3. $|(N_Q(u_1) \cap N_Q(u_2)) \cup (N_Q(u_1) \cap N_Q(u_3)) \cup (N_Q(u_2) \cap N_Q(u_3))| \geq 3$, and let $\{w_1, w_2, w_3\} \subseteq (N_Q(u_1) \cap N_Q(u_2)) \cup (N_Q(u_1) \cap N_Q(u_3)) \cup (N_Q(u_2) \cap N_Q(u_3))$. Up to isomorphism, we only need to consider the following two cases.

Let $\{u_1w_1, u_1w_2, u_2w_1, u_2w_3, u_3w_2, u_3w_3\} \subseteq E[V(F), V(Q)]$. Assign an edge-coloring θ with two colors to G such that G is strongly properly connected: $\theta(u_1w_1) = \theta(u_2w_3) =$

$\theta(u_3w_2) = 1$, $\theta(u_1w_2) = \theta(u_2w_1) = \theta(u_3w_3) = 2$, and color all edges of $E(Q)$ with 2 and the remaining edges with 1. Hence, $pc(G) = spc(G) = 2$.

Let $\{u_1w_1, u_1w_2, u_1w_3, u_2w_1, u_2w_2, u_2w_3, u_3w_1\} \subseteq E[V(F), V(Q)]$. The following edge-coloring θ with two colors makes G strongly properly connected: $\theta(u_1w_2) = \theta(u_2w_3) = \theta(u_3w_1) = 1$, $\theta(u_1w_1) = \theta(u_2w_1) = \theta(u_2w_2) = 2$, and color all edges of $E(Q)$ with 2 and the remaining edges with 1. Thus, $pc(G) = spc(G) = 2$. □

Theorem 4. *Let G be a connected graph of order $n \geq 5$, $diam(G) \geq 3$, and $\omega(G) = n - 3$. Let Q be a maximum clique of G and $V(G) \setminus V(Q) = \{u_1, u_2, u_3\}$. Then, either $pc(G) = spc(G) = 2$ or one of the following holds:*

(i) $G[V(G) \setminus V(Q)] \cong P_3$, *where* $u_1u_2, u_2u_3 \in E(G)$, $d_Q(u_1) = d_Q(u_3) = 0$, *and* $d_Q(u_2) = 1$.
(ii) $G[V(G) \setminus V(Q)] \cong P_3$, *where* $u_1u_2, u_2u_3 \in E(G)$, $d_Q(u_1) = d_Q(u_3) = 0$, *and* $d_Q(u_2) \geq 2$.
(iii) $G[V(G) \setminus V(Q)] \cong P_3$, *where* $u_1u_2, u_2u_3 \in E(G)$, $min\{d_Q(u_1), d_Q(u_3)\} = 1$, $d_Q(u_2) = 0$, $N_Q(u_1) \cup N_Q(u_3) \neq V(Q)$, *and* $N_Q(u_1) \cap N_Q(u_3) = \emptyset$.
(iv) $5 \leq n \leq 6$, $G[V(G) \setminus V(Q)] \cong K_2 + K_1$, *where* $u_1u_2 \in E(G)$, $N_Q(u_1) = N_Q(u_3) = \{v\}$, *and* $d_Q(u_2) = 0$.
(v) $n \geq 7$, $G[V(G) \setminus V(Q)] \cong K_2 + K_1$, *where* $u_1u_2 \in E(G)$, $N_Q(u_1) = N_Q(u_3) = \{v\}$, *and* $d_Q(u_2) = 0$.
(vi) $G[V(G) \setminus V(Q)] \cong K_2 + K_1$, *where* $u_1u_2 \in E(G)$, $d_Q(u_2) \geq 1$, $N_Q(u_1) = N_Q(u_3) = \{v\}$, $N_Q(u_2) \cap N_Q(u_3) = \emptyset$, *and* $N_Q(u_1) \cup N_Q(u_2) \neq V(Q)$.
(vii) $5 \leq n \leq 6$, $G[V(G) \setminus V(Q)] \cong 3K_1$, $N_Q(u_1) \cap N_Q(u_2) \cap N_Q(u_3) = \emptyset$, $|N_Q(u_1) \cap N_Q(u_2)| = 1$, *and* $d_Q(u_1) = d_Q(u_2) = d_Q(u_3) = 1$.
(viii) $n = 6$, $G[V(G) \setminus V(Q)] \cong 3K_1$, $N_Q(u_1) \cap N_Q(u_2) \cap N_Q(u_3) = \emptyset$, $|N_Q(u_1) \cap N_Q(u_2)| = 1$, $d_Q(u_1) = d_Q(u_2) = 1$, *and* $d_Q(u_3) = 2$.
(ix) $n \geq 7$, $G[V(G) \setminus V(Q)] \cong 3K_1$, $N_Q(u_1) \cap N_Q(u_2) \cap N_Q(u_3) = \emptyset$, $|N_Q(u_1) \cap N_Q(u_2)| = 1$, $d_Q(u_1) = d_Q(u_2) = 1$, *and* $d_Q(u_3) \geq 1$.
(x) $G[V(G) \setminus V(Q)] \cong 3K_1$, $N_Q(u_1) \cap N_Q(u_2) \cap N_Q(u_3) = \emptyset$, $|N_Q(u_1) \cap N_Q(u_2)| = 1$, $N_Q(u_1) \cap N_Q(u_3) \neq \emptyset$, $d_Q(u_1) = 2$, *and* $d_Q(u_2) = d_Q(u_3) = 1$.

Moreover, we have $pc(G) = 2$ and $spc(G) = 3$ for (ii), (iii), (v), (vi), (viii), (ix), and (x) and $pc(G) = spc(G) = 3$ for (i), (iv), and (vii).

Proof. Let $F = G[V(G) \setminus V(Q)]$, and let θ be an edge-coloring of G. We prove this theorem by the following two cases.

Case 1. $diam(G) = 3$. We distinguish the following four subcases by analyzing the structure of F.

Subcase 1.1. $F \cong K_3$. Note that G is traceable, and we have $pc(G) = 2$ by Proposition 1. Assign an edge-coloring θ with two colors to G as follows: color all edges of $E(F)$ and $E(Q)$ with 1, and color all edges of $E[V(F), V(Q)]$ with 2. It is obvious that θ is a strong proper coloring of G, and so $spc(G) = 2$.

Subcase 1.2. $F \cong P_3$, where $u_1u_2, u_2u_3 \in E(G)$. Assume that $d_Q(u_1) = d_Q(u_3) = 0$. Suppose $d_Q(u_2) = 1$, and let $N_Q(u_2) = \{v\}$. Then, $spc(G) \geq pc(G) \geq 3$ by Proposition 2. Now we define a strong proper coloring θ of G with three colors as follows: $\theta(u_1u_2) = 1$, $\theta(u_2u_3) = 2$, $\theta(u_2v) = 3$, and color all edges of $E(Q)$ with 1. Thus, $pc(G) = spc(G) = 3$. Suppose $d_Q(u_2) \geq 2$, and let $u, v \in N_Q(u_2)$. Assign an edge-coloring θ with two colors to G as follows: $\theta(u_1u_2) = \theta(u_2u) = \theta(vw) = 1$ for any $w \in V(Q) \setminus \{u, v\}$, $\theta(u_2u_3) = \theta(u_2v) = \theta(uv) = \theta(uw) = 2$ for any $w \in V(Q) \setminus \{u, v\}$, and color the remaining edges arbitrarily with 1 and 2. We can check that G is properly connected with the above edge-coloring, and so $pc(G) = 2$. If G is strongly properly connected with an edge-coloring θ, then $\theta(u_1u_2) \neq \theta(u_2u_3) \neq \theta(u_2u)$. Thus, $spc(G) \geq 3$. Assign a strong proper coloring θ' with three colors to G as follows: $\theta'(u_1u_2) = 1$, $\theta'(u_2u_3) = 2$, and color all edges of $E[V(F), V(Q)]$ with 3 and all edges of $E(Q)$ with 1. Thus, $spc(G) = 3$.

Assume that $min\{d_Q(u_1), d_Q(u_3)\} = 0$ and $max\{d_Q(u_1), d_Q(u_3)\} \geq 1$. Without a loss of generality, let $d_Q(u_3) = 0$ and $d_Q(u_1) \geq 1$. Since $diam(G) = 3$, it follows that $d_Q(u_2) \geq 1$. Note that G is traceable, and we have $pc(G) = 2$ by Proposition 1. The following edge-coloring θ with two colors makes G strongly properly connected: $\theta(u_1u_2) = 1$, $\theta(u_2u_3) = 2$, and color all edges of $E(Q)$ with 2 and all edges of $E[V(F), V(Q)]$ with 1. Hence, $spc(G) = 2$.

Assume that $d_Q(u_1) \geq 1$ and $d_Q(u_3) \geq 1$. Since $diam(G) = 3$, it follows that $d_Q(u_2) = 0$ and $N_Q(u_1) \cup N_Q(u_3) \neq V(Q)$. Observe that G is traceable, and we have $pc(G) = 2$ by Proposition 1. Now, we only consider the strong proper connection number of graph G under this assumption.

Suppose $N_Q(u_1) \cap N_Q(u_3) = \emptyset$ and $min\{d_Q(u_1), d_Q(u_3)\} = 1$. Without a loss of generality, we consider $d_Q(u_1) = 1$, and say $N_Q(u_1) = \{u\}$. If there exists a strong proper coloring θ of G with two colors, then $\theta(u_1u_2) \neq \theta(u_2u_3)$. Without a loss of generality, let $\theta(u_1u_2) = 1$ and $\theta(u_2u_3) = 2$. Note that u_2u_1u is the unique $u_2 - u$ geodesic, and u_2u_3v is the unique $u_2 - v$ geodesic for any $v \in N_Q(u_3)$; then, $\theta(u_1u) = 2$ and $\theta(u_3v) = 1$. Since u_1uv is the unique $u_1 - v$ geodesic for any $v \in N_Q(u_3)$, we have $\theta(uv) = 1$. There is no proper geodesic between u_3 and u, which is a contradiction. Thus, $spc(G) \geq 3$. On the other hand, we assign a strong proper coloring θ' with three colors to G as follows: $\theta'(u_1u_2) = \theta'(u_3v) = 1$ for any $v \in N_Q(u_3)$, $\theta'(u_2u_3) = \theta'(u_1u) = 2$, and color all edges of $E(Q)$ with 3. Hence, $spc(G) = 3$.

Suppose $N_Q(u_1) \cap N_Q(u_3) = \emptyset$ and $min\{d_Q(u_1), d_Q(u_3)\} \geq 2$. Let $N_Q(u_1) = \{w_1, w_2, \cdots, w_t\}$ and $N_Q(u_3) = \{v_1, v_2, \cdots, v_k\}$, where $t + k < n - 3$. Assign an edge-coloring θ with two colors to G as follows: $\theta(u_1u_2) = \theta(w_1v_1) = \theta(u_3v_1) = \theta(u_3v_i) = \theta(w_2v_i) = 1$ for $2 \leq i \leq k$, $\theta(u_2u_3) = \theta(w_1v_k) = \theta(u_1w_1) = \theta(u_1w_j) = \theta(v_1w_j) = 2$ for $2 \leq j \leq t$, $\theta(v_1w) = 2$ and $\theta(w_1w) = 1$ for any $w \in V(Q) \setminus \{N_Q(u_1) \cup N_Q(u_3)\}$, and color the remaining edges arbitrarily with 1 and 2. It is clear that θ is a strong proper coloring of G, and so $spc(G) = 2$.

Suppose $N_Q(u_1) \cap N_Q(u_3) \neq \emptyset$, and let $v \in N_Q(u_1) \cap N_Q(u_3)$. Consider $d_Q(u_1) = d_Q(u_3) = 1$. Color u_1u_2 and all edges of $E(Q)$ with 1, and color u_2u_3, u_1v and u_3v with 2. Obviously, the above edge-coloring makes G strongly properly connected. Thus, $spc(G) = 2$. Consider $min\{d_Q(u_1), d_Q(u_3)\} = 1$ and $max\{d_Q(u_1), d_Q(u_3)\} \geq 2$. Without a loss of generality, let $d_Q(u_1) = 1$ and $d_Q(u_3) \geq 2$. Assign a strong proper coloring θ with two colors to G as follows: $\theta(u_1u_2) = \theta(u_2u_3) = \theta(u_3v) = 1$, $\theta(u_1v) = \theta(u_3w) = 2$ for any $w \in N_Q(u_3) \setminus \{v\}$, and color all edges of $E(Q)$ with 1. Hence, $spc(G) = 2$. Consider $min\{d_Q(u_1), d_Q(u_3)\} \geq 2$. Allocate a strong proper coloring θ with two colors to G as follows: $\theta(u_1u_2) = \theta(u_2u_3) = \theta(u_3v) = 1$, and color all edges of $E(Q)$ with 1 and the remaining edges with 2. Thus, $spc(G) = 2$.

Subcase 1.3. $F \cong K_2 + K_1$, where $u_1u_2 \in E(G)$. Since G is connected, we have $d_Q(u_3) \geq 1$ and $max\{d_Q(u_1), d_Q(u_2)\} \geq 1$. Without a loss of generality, let $d_Q(u_1) \geq 1$. Assume that $d_Q(u_2) = 0$. Since $diam(G) = 3$, it follows that $N_Q(u_1) \cap N_Q(u_3) \neq \emptyset$, and let $v \in N_Q(u_1) \cap N_Q(u_3)$.

Suppose $d_Q(u_1) = d_Q(u_3) = 1$. If $n = 5$, then $G \cong G_3$, where G_3 is displayed in Figure 2. Hence, $pc(G) = spc(G) = 3$. If $n = 6$, then $G \cong G_4$, where G_4 is shown in Figure 2. Thus, $pc(G) = spc(G) = 3$. Now, we consider $n \geq 7$. Let $V(Q) = \{w_1, w_2, \ldots, w_{n-4}, v\}$. Assign an edge-coloring θ with two colors to G as follows: $\theta(u_1u_2) = \theta(u_3v) = \theta(w_{n-4}w_1) = 1$, $\theta(u_1v) = \theta(w_2v) = 2$, color the sequence $vw_1w_2 \cdots w_{n-4}v$ alternately with 1 and 2 starting with $\theta(vw_1) = 1$, and color the remaining edges arbitrarily with 1 and 2. We can verify that θ is a proper-path coloring of G. Thus, $pc(G) = 2$. If G has a strong proper coloring θ, then $\theta(u_1v) \neq \theta(u_3v) \neq \theta(vw_1)$, and so $spc(G) \geq 3$. On the other hand, there exists a strong proper coloring θ' of G with three colors: assign 1 to u_1u_2 and u_3v, assign 2 to u_1v, and assign 3 to all edges of $E(Q)$. Therefore, $spc(G) = 3$.

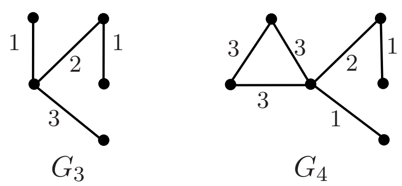

Figure 2. The graphs G_3 and G_4 with a strong proper coloring.

Suppose $d_Q(u_1) = 1$ and $d_Q(u_3) \geq 2$. Note that G is traceable, and we have $pc(G) = 2$ by Proposition 1. Allocate an edge-coloring θ with two colors to G as follows: $\theta(u_1v) = \theta(u_3w) = 1$ for any $w \in N_Q(u_3) \setminus \{v\}$, $\theta(u_1u_2) = \theta(u_3v) = 2$, and color all edges of $E(Q)$ with 2. Obviously, θ is a strong proper coloring of G, and so $spc(G) = 2$.

Suppose $d_Q(u_1) \geq 2$ and $d_Q(u_3) = 1$. Observe that G is traceable, and we obtain $pc(G) = 2$ by Proposition 1. The following edge-coloring θ with two colors makes G strongly properly connected: $\theta(u_1v) = \theta(u_1w_1) = 1$ for any $w_1 \in N_Q(u_1) \setminus \{v\}$, $\theta(u_1u_2) = \theta(u_3v) = 2$, and color all edges incident with v in $E(Q)$ with 1 and the remaining edges with 2. Hence, $spc(G) = 2$.

Suppose $d_Q(u_1) \geq 2$ and $d_Q(u_3) \geq 2$. Note that G is traceable, and we have $pc(G) = 2$ by Proposition 1. Define a strong proper coloring θ of G with two colors as follows: $\theta(u_1v) = 1$, $\theta(u_1u_2) = \theta(u_3v) = 2$, and color all edges of $E(Q)$ with 2 and the remaining edges with 1. Thus, $spc(G) = 2$.

Assume that $d_Q(u_2) \geq 1$. Since $diam(G) = 3$, it follows that $min\{|N_Q(u_1) \cap N_Q(u_3)|, |N_Q(u_2) \cap N_Q(u_3)|\} = 0$. Suppose $max\{|N_Q(u_1) \cap N_Q(u_3)|, |N_Q(u_2) \cap N_Q(u_3)|\} \geq 1$. Without a loss of generality, we consider $|N_Q(u_2) \cap N_Q(u_3)|\} = 0$ and $|N_Q(u_1) \cap N_Q(u_3)| \geq 1$, and say $v \in N_Q(u_1) \cap N_Q(u_3)$. Observe that G is traceable, and we have $pc(G) = 2$ by Proposition 1. Now, we only consider the strong proper connection number of graph G under this supposition.

We first consider $|N_Q(u_1) \cap N_Q(u_3)| \geq 2$. The following edge-coloring θ with two colors makes G strongly properly connected: color u_1u_2, u_3v and all edges of $E(Q)$ with 2, and color the remaining edges with 1. Hence, $spc(G) = 2$.

Next, we consider $|N_Q(u_1) \cap N_Q(u_3)| = 1$. Let $d_Q(u_1) \geq 2$. Assign a strong proper coloring θ with two colors to G: color u_1v and all edges of $E(Q)$ with 1, and color the remaining edges with 2. Hence, $spc(G) = 2$. Let $d_Q(u_3) \geq 2$. Define a strong proper coloring θ of G with two colors as follows: color u_1u_2, u_3v and all edges of $E(Q)$ with 2, and color the remaining edges with 1. Thus, $spc(G) = 2$. Let $d_Q(u_1) = d_Q(u_3) = 1$ and $N_Q(u_1) \cup N_Q(u_2) = V(Q)$. Allocate an edge-coloring θ with two colors to G: color u_1u_2, u_1v and all edges of $E(Q)$ with 1, and color the remaining edges with 2. We can check that G is strongly properly connected with the above edge-coloring, and so $spc(G) = 2$. Let $d_Q(u_1) = d_Q(u_3) = 1$ and $N_Q(u_1) \cup N_Q(u_2) \neq V(Q)$. If θ is a strong proper coloring of G, then $\theta(u_1v) \neq \theta(u_3v) \neq \theta(vw)$, where $w \in V(Q) \setminus \{N_Q(u_1) \cup N_Q(u_2)\}$. Thus, $spc(G) \geq 3$. On the other hand, there exists an edge-coloring θ' with three colors such that G is strongly properly connected: color u_1v with 1 and all edges of $E(Q)$ with 3, and color the remaining edges with 2. Hence, $spc(G) = 3$.

Suppose $max\{|N_Q(u_1) \cap N_Q(u_3)|, |N_Q(u_2) \cap N_Q(u_3)|\} = 0$. Observe that G is traceable, and we have $pc(G) = 2$ by Proposition 1. Assign an edge-coloring θ with two colors to G as follows: color u_1u_2 and all edges of $E(Q)$ with 2, and color all edges of $E[V(F), V(Q)]$ with 1. It is clear that θ is a strong proper coloring of G, and so $spc(G) = 2$.

Subcase 1.4. $F \cong 3K_1$. Since $diam(G) = 3$, it follows that $min\{|N_Q(u_1) \cap N_Q(u_2)|, |N_Q(u_1) \cap N_Q(u_3)|, |N_Q(u_2) \cap N_Q(u_3)|\} = 0$. Assume that $max\{|N_Q(u_1) \cap N_Q(u_2)|, |N_Q(u_1) \cap N_Q(u_3)|, |N_Q(u_2) \cap N_Q(u_3)|\} = 0$. The following edge-coloring θ with two colors makes G strongly properly connected: color all edges of $E(Q)$ with 2, and color all edges of $E[V(F), V(Q)]$ with 1. Thus, $pc(G) = spc(G) = 2$. Assume that $max\{|N_Q(u_1) \cap N_Q(u_2)|, |N_Q(u_1) \cap N_Q(u_3)|, |N_Q(u_2) \cap N_Q(u_3)|\} \geq 1$. Without a loss of generality, we consider $|N_Q(u_1) \cap N_Q(u_2)| \geq 1$, and say $u \in N_Q(u_1) \cap N_Q(u_2)$.

Suppose $d_Q(u_1) = d_Q(u_2) = 1$. If θ is a strong proper coloring of G, then $\theta(u_1u) \neq \theta(u_2u) \neq \theta(uw)$, where $w \in V(Q) \setminus \{u\}$. Hence, $spc(G) \geq 3$. On the other hand, there exists a strong proper coloring θ' of G with three colors, as follows: $\theta'(u_1u) = 1$, $\theta'(u_2u) = 2$, and color all edges of $E(Q)$ with 3 and the remaining edges with 1. Thus, $spc(G) = 3$. Next, we discuss the proper connection number of G. If $n = 5$, then $G \cong G_3$, where G_3 is displayed in Figure 2. Hence, $pc(G) = 3$. We consider $n = 6$. If $d_Q(u_3) = 1$, then $G \cong G_5$. Thus, $pc(G) = 3$. If $d_Q(u_3) = 2$, then $G \cong G_6$. Hence, $pc(G) = 2$. The graphs G_5 and G_6 are shown in Figure 3. Now, we consider $n \geq 7$. Let $V(Q) = \{u, v, w_1, w_2, \ldots, w_{n-5}\}$

and $v \in N_Q(u_3)$. Assign an edge-coloring θ with two colors to G as follows: $\theta(u_1u) = \theta(u_3v) = \theta(uw_{n-6}) = 1$; $\theta(u_2u) = \theta(w_{n-5}v) = 2$; color $w_{n-6}v$ with 1 for $n = 7$ and $w_{n-6}v$ with 2 for $n \geq 8$; color the sequence $uvw_1w_2 \cdots w_{n-5}u$ alternately with 2 and 1 starting with $\theta(uv) = 2$; and color the remaining edges arbitrarily with 1 and 2. We can check that G is properly connected with the above edge-coloring, and so $pc(G) = 2$.

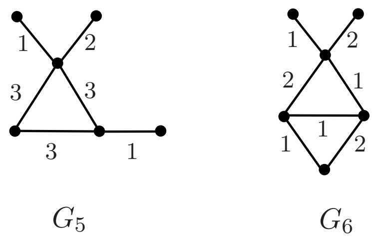

Figure 3. The graphs G_5 and G_6 with a proper-path coloring.

Suppose $max\{d_Q(u_1), d_Q(u_2)\} \geq 2$. Without a loss of generality, let $d_Q(u_1) \geq 2$, and say $w \in N_Q(u_1) \setminus \{u\}$. We first consider $N_Q(u_1) \cap N_Q(u_3) = \emptyset$. The following edge-coloring θ with two colors makes G strongly properly connected: $\theta(u_1u) = 1$, $\theta(u_1w) = \theta(u_2u) = 2$, color all edges of $E(Q)$ with 1 and all edges incident with u_3 in $E[V(F), V(Q)]$ with 2, and color the remaining edges with 1. Thus, $pc(G) = spc(G) = 2$.

Next, we consider $N_Q(u_1) \cap N_Q(u_3) \neq \emptyset$ and say $w_1 \in N_Q(u_1) \cap N_Q(u_3)$. Let $d_Q(u_1) = 2$ and $d_Q(u_2) = d_Q(u_3) = 1$. The following edge-coloring θ with two colors makes G properly connected: color all edges of $E(Q)$ with 1 and the remaining edges with 2. Hence, $pc(G) = 2$. If there exists a strong proper coloring θ of G with two colors, then $\theta(u_1u) \neq \theta(u_2u)$. Without a loss of generality, let $\theta(u_1u) = 1$ and $\theta(u_2u) = 2$. Since $u_2uw_1u_3$ is the unique $u_2 - u_3$ geodesic, it follows that $\theta(uw_1) = 1$ and $\theta(u_3w_1) = 2$. Note that $u_1w_1u_3$ is the unique $u_1 - u_3$ geodesic, and thus $\theta(u_1w_1) = 1$. Since u_2uv is the unique $u_2 - v$ geodesic and u_3w_1v is the unique $u_3 - v$ geodesic, we obtain $\theta(uv) = \theta(w_1v) = 1$, where $v \in V(Q) \setminus \{u, w_1\}$. There is no proper geodesic connecting u_1 and v, which is a contradiction. Hence, $spc(G) \geq 3$. On the other hand, we assign a strong proper coloring θ' with three colors to G as follows: $\theta'(u_1u) = \theta'(u_3w_1) = 1$, $\theta'(u_2u) = \theta'(u_1w_1) = 2$, and color all edges of $E(Q)$ with 3. Therefore, $spc(G) = 3$. Let $d_Q(u_1) \geq 3$. The following edge-coloring θ of G with two colors makes G strongly properly connected: $\theta(u_1u) = \theta(u_1w_1) = 1$, $\theta(u_2u) = \theta(u_3w_1) = \theta(u_1w) = 2$, where $w \in N_Q(u_1) \setminus \{u, w_1\}$, and color the remaining edges with 1. Thus, $pc(G) = spc(G) = 2$. Let $max\{d_Q(u_2), d_Q(u_3)\} \geq 2$. Without a loss of generality, we consider $d_Q(u_2) \geq 2$. Define an edge-coloring θ of G with two colors as follows: $\theta(u_1u) = \theta(u_3w_1) = \theta(u_2z) = 1$, where $z \in N_Q(u_2)$, $\theta(u_1w_1) = \theta(u_2u) = 2$, and color all edges of $E(Q)$ with 2 and the remaining edges with 1. Obviously, θ is a strong proper coloring of G, and so $pc(G) = spc(G) = 2$.

Case 2. $diam(G) \geq 4$. Since $diam(G) \geq 4$, it follows that $F \cong P_3$ or $F \cong K_2 + K_1$. Assume that $F \cong P_3$, where $u_1u_2, u_2u_3 \in E(G)$. Since $diam(G) \geq 4$, we have $min\{d_Q(u_1), d_Q(u_3)\} = 0$, $max\{d_Q(u_1), d_Q(u_3)\} \geq 1$, and $d_Q(u_2) = 0$. Without a loss of generality, let $d_Q(u_1) \geq 1$ and $d_Q(u_2) = d_Q(u_3) = 0$. Note that G is traceable, and we have $pc(G) = 2$ by Proposition 1. The following edge-coloring θ with two colors makes G strongly properly connected: color u_1u_2 and all edges of $E(Q)$ with 2, and color the remaining edges with 1. Thus, $spc(G) = 2$.

Assume that $F \cong K_2 + K_1$, where $u_1u_2 \in E(G)$. Since $diam(G) \geq 4$, we have $min\{d_Q(u_1), d_Q(u_2)\} = 0$, $max\{d_Q(u_1), d_Q(u_2)\} \geq 1$, and $d_Q(u_3) \geq 1$. Without a loss of generality, let $d_Q(u_1) \geq 1$, $d_Q(u_2) = 0$ and $N_Q(u_1) \cap N_Q(u_3) = \emptyset$. Observe that G is traceable, and we obtain $pc(G) = 2$ by Proposition 1. Assign a strong proper coloring θ with two colors to G as follows: color u_1u_2 and all edges of $E(Q)$ with 2, and color the remaining edges with 1. Hence, $spc(G) = 2$. □

3. Rainbow Connection and Clique Number

Kemnitz and Schiermeyer [18] considered the rainbow connection number of graph G of order n, $diam(G) = 2$, and $\omega(G) = n - i$ for $2 \leqslant i \leqslant 3$. In this section, we investigate the rainbow connection number of graph G of order n, $diam(G) \geq 3$, and $\omega(G) = n - i$ for $2 \leqslant i \leqslant 3$.

Theorem 5. *Let G be a connected graph of order n, $diam(G) \geq 3$, and $\omega(G) = n - 2$. Let Q be a maximum clique of G and $V(G) \setminus V(Q) = \{u_1, u_2\}$. Then, $rc(G) = 3$.*

Proof. Let $F = G[V(G) \setminus V(Q)]$ and let θ be an edge-coloring of G. Since $diam(G) \geq 3$, we have $rc(G) \geq diam(G) \geq 3$. Assume that $F \cong K_2$. Since $diam(G) \geq 3$, we obtain $max\{d_Q(u_1), d_Q(u_2)\} \geq 1$ and $min\{d_Q(u_1), d_Q(u_2)\} = 0$. The following edge-coloring θ with three colors makes G rainbow-connected: color u_1u_2 with 1 and all edges of $E[V(F), V(Q)]$ with 2, and color all edges of $E(Q)$ with 3. Thus, $rc(G) = 3$.

Assume that $F \cong 2K_1$. Since G is a connected graph with $diam(G) \geq 3$, it follows that $d_Q(u_1) \geq 1$, $d_Q(u_2) \geq 1$ and $N_Q(u_1) \cap N_Q(u_2) = \emptyset$. Assign an edge-coloring θ with three colors to G as follows: assign 1 to all edges that are incident with u_1, assign 2 to all edges that are incident with u_2, and assign 3 to all edges of $E(Q)$. It is not difficult to check that G is rainbow-connected with the above edge-coloring, and so $rc(G) = 3$. □

Theorem 6. *Let G be a connected graph of order n, $diam(G) \geq 3$, and $\omega(G) = n - 3$. Let Q be a maximum clique of G and $V(G) \setminus V(Q) = \{u_1, u_2, u_3\}$. Then, either $rc(G) = 3$, or $rc(G) = 4$ if and only if one of the following holds.*

(i) $G[V(G) \setminus V(Q)] \cong P_3$, *where* $u_1u_2, u_2u_3 \in E(G)$, $d_Q(u_1) = d_Q(u_3) = 0$, *and* $d_Q(u_2) = 1$.
(ii) $G[V(G) \setminus V(Q)] \cong K_2 + K_1$, *where* $u_1u_2 \in E(G)$, $d_Q(u_2) = 0$, $d_Q(u_1) = d_Q(u_3) = 1$, *and* $N_Q(u_1) \cap N_Q(u_3) \neq \emptyset$.
(iii) $G[V(G) \setminus V(Q)] \cong 3K_1$, $N_Q(u_1) \cap N_Q(u_2) \cap N_Q(u_3) = \emptyset$, $|N_Q(u_1) \cap N_Q(u_2)| = 1$, *and* $d_Q(u_1) = d_Q(u_2) = d_Q(u_3) = 1$.
(iv) $G[V(G) \setminus V(Q)] \cong P_3$, *where* $u_1u_2, u_2u_3 \in E(G)$, $d_Q(u_1) \geq 1$, *and* $d_Q(u_2) = d_Q(u_3) = 0$.
(v) $G[V(G) \setminus V(Q)] \cong K_2 + K_1$, *where* $u_1u_2 \in E(G)$, $d_Q(u_1) \geq 1$, $d_Q(u_2) = 0$, $d_Q(u_3) \geq 1$, *and* $N_Q(u_1) \cap N_Q(u_3) = \emptyset$.

Proof. Let $F = G[V(G) \setminus V(Q)]$, and let θ be an edge-coloring of G. We prove this theorem by the following two cases.

Case 1. $diam(G) = 3$. We have $rc(G) \geq diam(G) = 3$. We distinguish the following four subcases by analyzing the structure of F.

Subcase 1.1. $F \cong K_3$. The following edge-coloring θ with three colors makes G rainbow-connected: $\theta(u_1u_2) = \theta(u_2u_3) = \theta(u_1u_3) = 1$, and color all edges of $E(Q)$ with 3 and all edges of $E[V(F), V(Q)]$ with 2. Thus, $rc(G) = 3$.

Subcase 1.2. $F \cong P_3$, where $u_1u_2, u_2u_3 \in E(G)$. Assume that $d_Q(u_1) = d_Q(u_3) = 0$. Suppose $d_Q(u_2) = 1$, and say $N_Q(u_2) = \{u\}$. If an edge-coloring θ is a rainbow coloring of G, then $\theta(u_1u_2) \neq \theta(u_2u_3) \neq \theta(u_2u) \neq \theta(uv)$, where $v \in V(Q) \setminus \{u\}$. Hence, $rc(G) \geq 4$. Allocate a rainbow coloring θ' with four colors to G as follows: $\theta'(u_1u_2) = 1$, $\theta'(u_2u_3) = 2$, $\theta'(u_2u) = 3$, and color all edges of $E(Q)$ with 4. Thus, $rc(G) = 4$. Suppose $d_Q(u_2) \geq 2$, and say $u, v \in N_Q(u_2)$. The following edge-coloring θ with three colors makes G rainbow-connected: $\theta(u_1u_2) = \theta(u_2u) = 1$, $\theta(u_2u_3) = \theta(u_2v) = 2$, and color the remaining edges with 3. Hence, $rc(G) = 3$.

Assume that $min\{d_Q(u_1), d_Q(u_3)\} = 0$ and $max\{d_Q(u_1), d_Q(u_3)\} \geq 1$. Without a loss of generality, let $d_Q(u_3) = 0$ and $d_Q(u_1) \geq 1$. Since $diam(G) = 3$, we have $d_Q(u_2) \geq 1$. Define an edge-coloring θ of G with three colors as follows: $\theta(u_1u_2) = 1$, $\theta(u_2u_3) = 2$, and color all edges of $E[V(F), V(Q)]$ with 1 and all edges of $E(Q)$ with 3. We can check that G is rainbow-connected with the above edge-coloring, and so $rc(G) = 3$.

Assume that $d_Q(u_1) \geq 1$ and $d_Q(u_3) \geq 1$. Since $diam(G) = 3$, it follows that $d_Q(u_2) = 0$ and $N_Q(u_1) \cup N_Q(u_3) \neq V(Q)$. The following edge-coloring θ with three colors makes

G rainbow-connected: $\theta(u_1u_2) = 1$, $\theta(u_2u_3) = 2$, assign 3 to all edges of $E(Q)$, assign 2 to the edges of $E[V(F), V(Q)]$ which are incident with u_1, and assign 1 to the edges of $E[V(F), V(Q)]$ which are incident with u_3. Thus, $rc(G) = 3$.

Subcase 1.3. $F \cong K_2 + K_1$, where $u_1u_2 \in E(G)$. Since G is connected, we obtain $d_Q(u_3) \geq 1$ and $max\{d_Q(u_1), d_Q(u_2)\} \geq 1$. Without a loss of generality, let $d_Q(u_1) \geq 1$.

Assume that $d_Q(u_2) = 0$. Since $diam(G) = 3$, we have $N_Q(u_1) \cap N_Q(u_3) \neq \emptyset$, and say $u \in N_Q(u_1) \cap N_Q(u_3)$. Suppose $d_Q(u_1) = d_Q(u_3) = 1$. If there exists a rainbow coloring θ of G with three colors, then $\theta(u_2u_1) \neq \theta(u_1u) \neq \theta(uu_3)$. Without a loss of generality, let $\theta(u_2u_1) = 1, \theta(u_1u) = 2$ and $\theta(uu_3) = 3$. In order to have a rainbow path connecting u_2 and v for any $v \in V(Q) \setminus \{u\}$, let $\theta(uv) = 3$. There is no rainbow path between u_3 and v, which is a contradiction. Thus, $rc(G) \geq 4$. On the other hand, the following edge-coloring θ' with four colors makes G rainbow-connected: $\theta'(u_2u_1) = 1$, $\theta'(u_1u) = 2$, $\theta'(uu_3) = 3$, and color all edges of $E(Q)$ with 4. Hence, $rc(G) = 4$. Suppose $max\{d_Q(u_1), d_Q(u_3)\} \geq 2$. We first consider $d_Q(u_1) \geq 2$, and say $v \in N_Q(u_1) \setminus \{u\}$. Assign an edge-coloring θ with three colors to G as follows: $\theta(u_2u_1) = 1$, $\theta(u_1u) = 2$, $\theta(u_3u) = \theta(u_1v) = 3$, and color the remaining edges with 2. It is obvious that G is rainbow-connected with the above edge-coloring, and so $rc(G) = 3$. Next, we consider $d_Q(u_3) \geq 2$, and say $w \in N_Q(u_3) \setminus \{u\}$. Define a rainbow coloring θ of G with three colors as follows: $\theta(u_2u_1) = 1$, $\theta(u_1u) = \theta(u_3w) = 2$, $\theta(u_3u) = 3$, and color all edges of $E(Q)$ with 3 and the remaining edges with 2. Thus, $rc(G) = 3$.

Assume that $d_Q(u_2) \geq 1$. Since $diam(G) = 3$, we obtain $min\{|N_Q(u_1) \cap N_Q(u_3)|, |N_Q(u_2) \cap N_Q(u_3)|\} = 0$. Suppose $max\{|N_Q(u_1) \cap N_Q(u_3)|, |N_Q(u_2) \cap N_Q(u_3)|\} \geq 1$. Without a loss of generality, let $|N_Q(u_1) \cap N_Q(u_3)| \geq 1$ and $|N_Q(u_2) \cap N_Q(u_3)| = 0$. Let $u \in N_Q(u_1) \cap N_Q(u_3)$ and $v \in N_Q(u_2)$. The following edge-coloring θ with three colors makes G rainbow-connected: $\theta(u_1u) = \theta(u_2v) = 1$, $\theta(u_3u) = 2$, and color the remaining edges with 3. Hence, $rc(G) = 3$. Suppose $max\{|N_Q(u_1) \cap N_Q(u_3)|, |N_Q(u_2) \cap N_Q(u_3)|\} = 0$. Let $w \in N_Q(u_1), v \in N_Q(u_2)$ and $u \in N_Q(u_3)$, where $w = v$ is possible. Allocate an edge-coloring θ with three colors to G: $\theta(u_1u_2) = \theta(u_3u) = 1$, $\theta(u_1w) = \theta(u_2v) = 2$, and color the remaining edges with 3. We can verify that G is rainbow-connected with the above edge-coloring, and so $rc(G) = 3$.

Subcase 1.4. $F \cong 3K_1$. Since $diam(G) = 3$, it follows that $min\{|N_Q(u_1) \cap N_Q(u_2)|, |N_Q(u_1) \cap N_Q(u_3)|, |N_Q(u_2) \cap N_Q(u_3)|\} = 0$. Assume that $max\{|N_Q(u_1) \cap N_Q(u_2)|, |N_Q(u_1) \cap N_Q(u_3)|, |N_Q(u_2) \cap N_Q(u_3)|\} = 0$. Let $u \in N_Q(u_1), v \in N_Q(u_2)$ and $w \in N_Q(u_3)$. The following edge-coloring θ with three colors makes G rainbow-connected: $\theta(u_1u) = \theta(vw) = \theta(vz) = 1$; $\theta(uv) = \theta(u_3w) = \theta(uz) = 2$; $\theta(u_2v) = \theta(uw) = \theta(wz) = 3$ for any $z \in V(Q) \setminus \{u, v, w\}$; and color the remaining edges with 1. Thus, $rc(G) = 3$.

Assume that $max\{|N_Q(u_1) \cap N_Q(u_2)|, |N_Q(u_1) \cap N_Q(u_3)|, |N_Q(u_2) \cap N_Q(u_3)|\} \geq 1$. Without a loss of generality, let $|N_Q(u_1) \cap N_Q(u_2)| \geq 1$, and say $u \in N_Q(u_1) \cap N_Q(u_2)$. Suppose $d_Q(u_1) = d_Q(u_2) = d_Q(u_3) = 1$. If an edge-coloring θ is a rainbow coloring of G, then $\theta(u_1u) \neq \theta(u_2u) \neq \theta(uv) \neq \theta(u_3v)$, where $\{v\} = N_Q(u_3)$. Thus, $rc(G) \geq 4$. On the other hand, we define a rainbow coloring θ' of G with four colors as follows: $\theta'(u_1u) = 1, \theta'(u_2u) = 2, \theta'(u_3v) = 3$, and color all edges of $E(Q)$ with 4. Hence, $rc(G) = 4$. Suppose $max\{d_Q(u_1), d_Q(u_2)\} \geq 2$. Without a loss of generality, let $d_Q(u_1) \geq 2$, and say $w \in N_Q(u_1) \setminus \{u\}$. Assign an edge-coloring θ with three colors to G: $\theta(u_1u) = \theta(u_2u) = 1$; $\theta(u_1w) = \theta(u_3v) = 2$, where $v \in N_Q(u_3)$ and $v = w$ is possible; and color the remaining edges with 3. Obviously, the edge-coloring θ is a rainbow coloring of G, and so $rc(G) = 3$. Suppose $d_Q(u_3) \geq 2$, and say $v_1, v_2 \in N_Q(u_3)$. The following edge-coloring θ with three colors makes G rainbow-connected: $\theta(u_1u) = \theta(u_3v_1) = 1$, $\theta(u_2u) = \theta(u_3v_2) = 2$, and color the remaining edges with 3. Thus, $rc(G) = 3$.

Case 2. $diam(G) \geq 4$. We obtain $rc(G) \geq diam(G) \geq 4$. Since $diam(G) \geq 4$, it follows that $F \cong P_3$ or $F \cong K_2 + K_1$. Assume that $F \cong P_3$, where $u_1u_2, u_2u_3 \in E(G)$. Since $diam(G) \geq 4$, we have $min\{d_Q(u_1), d_Q(u_3)\} = 0$, $max\{d_Q(u_1), d_Q(u_3)\} \geq 1$, and $d_Q(u_2) = 0$. Without a loss of generality, let $d_Q(u_1) \geq 1$ and $d_Q(u_2) = d_Q(u_3) = 0$. Allocate a rainbow coloring θ with four colors to G as follows: color u_1u_2 with 2 and u_2u_3 with 1, and color all edges of $E[V(F), V(Q)]$ with 3 and all edges of $E(Q)$ with 4. Therefore, $rc(G) = 4$.

Assume that $F \cong K_2 + K_1$, where $u_1u_2 \in E(G)$. Since $diam(G) \geq 4$, it follows that $min\{d_Q(u_1), d_Q(u_2)\} = 0$, $max\{d_Q(u_1), d_Q(u_2)\} \geq 1$, and $d_Q(u_3) \geq 1$. Without a loss of generality, let $d_Q(u_1) \geq 1$, $d_Q(u_2) = 0$, and $N_Q(u_1) \cap N_Q(u_3) = \emptyset$. The following edge-coloring θ with four colors makes G rainbow-connected: $\theta(u_1u_2) = 1$, $\theta(u_1u) = 2$, and $\theta(u_3v) = 3$, where $u \in N_Q(u_1)$ and $v \in N_Q(u_3)$, and color the remaining edges with 4. Hence, $rc(G) = 4$. □

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References

1. Bondy, J.A.; Murty, U.S.R. *Graph Theory*; Springer: Berlin, Germany, 2008.
2. Chartrand, G.; Johns, G.L.; McKeon, K.A.; Zhang, P. Rainbow connection in graphs. *Math. Bohem.* **2008**, *133*, 85–98. [CrossRef]
3. Krivelevich, M.; Yuster, R. The rainbow connection of a graph is (at most) reciprocal to its minimum degree. *J. Graph Theory* **2010**, *63*, 185–191. [CrossRef]
4. Liu, H.; Mestre, Â.; Sousa, T. Total rainbow *k*-connection in graphs. *Discrete Appl. Math.* **2014**, *174*, 92–101. [CrossRef]
5. Schiermeyer, I. Rainbow connection and minimum degree. *Discrete Appl. Math.* **2013**, *161*, 1784–1787. [CrossRef]
6. Huang, X.; Li, X.; Shi, Y.; Yue, J.; Zhao, Y. Rainbow connections for outerplanar graphs with diameter 2 and 3. *Appl. Math. Comput.* **2014**, *242*, 277–280. [CrossRef]
7. Li, H.; Ma, Y.; Li, X. The vertex-rainbow connection number of some graph operations. *Discuss. Math. Graph Theory* **2021**, *41*, 513–530. [CrossRef]
8. Ma, Y.; Nie, K.; Jin, F.; Wang, C. Total rainbow connection numbers of some special graphs. *Appl. Math. Comput.* **2019**, *360*, 213–220. [CrossRef]
9. Li, X.; Shi, Y.; Sun, Y. Rainbow connections of graphs: A survey. *Graphs Combin.* **2013**, *29*, 1–38. [CrossRef]
10. Li, X.; Sun, Y. *Rainbow Connections of Graphs*; SpringerBriefs in Math.; Springer: New York, NY, USA, 2012.
11. Borozan, V.; Fujita, S.; Gerek, A.; Magnant, C.; Manoussakis, Y.; Montero, L.; Tuza, Z. Proper connection of graphs. *Discrete Math.* **2012**, *312*, 2550–2560. [CrossRef]
12. Andrews, E.; Lumduanhom, C.; Laforge, E.; Zhang, P. On proper-path colorings in graphs. *J. Combin. Math. Combin. Comput.* **2016**, *97*, 189–207.
13. Huang, F.; Li, X.; Wang, S. Upper bounds of proper connection number of graphs. *J. Comb. Optim.* **2017**, *34*, 165–173. [CrossRef]
14. Huang, F.; Li, X.; Wang, S. Proper Connection Numbers of Complementary Graphs. *Bull. Malays. Math. Sci. Soc.* **2018**, *41*, 1199–1209. [CrossRef]
15. Li, X.; Wei, M.; Yue, J. Proper connection number and connected dominating sets. *Theoret. Comput. Sci.* **2015**, *607*, 480–487. [CrossRef]
16. Ma, Y.; Zhang, X. Graphs with (strong) proper connection numbers $m-3$ and $m-4$. *Appl. Math. Comput.* **2023**, *445*, 127843. [CrossRef]
17. Li, X.; Magnant, C. Properly Colored Notions of Connectivity—A Dynamic Survey. *Theory Appl. Graphs* **2015**, *0*, 2. [CrossRef]
18. Kemnitz, A.; Schiermeyer, I. Graphs with rainbow connection number two. *Discuss. Math. Graph Theory* **2011**, *31*, 313–320. [CrossRef]
19. Lu, Z.; Ma, Y. Graphs with vertex rainbow connection number two. *Sci. China Math.* **2015**, *58*, 1803–1810. [CrossRef]