

Article

On the Construction of Quantum and LCD Codes from Cyclic Codes over the Finite Commutative Rings

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Abstract: Let \mathbb{F}_q be a field of order q , where q is a power of an odd prime p , and α and β are two non-zero elements of \mathbb{F}_q . The primary goal of this article is to study the structural properties of cyclic codes over a finite ring $R = \mathbb{F}_q[u_1, u_2]/\langle u_1^2 - \alpha^2, u_2^2 - \beta^2, u_1u_2 - u_2u_1 \rangle$. We decompose the ring R by using orthogonal idempotents $\Delta_1, \Delta_2, \Delta_3$, and Δ_4 as $R = \Delta_1R \oplus \Delta_2R \oplus \Delta_3R \oplus \Delta_4R$, and to construct quantum-error-correcting (QEC) codes over R . As an application, we construct some optimal LCD codes.

Keywords: cyclic code; Gray map; quantum code; LCD code

MSC: 94B05; 94B15; 94B60



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1. Introduction

Throughout this paper, unless indicated otherwise, \mathbb{F}_q (where q is an odd prime power) denotes the field of order q , and α and β are non-zero elements of \mathbb{F}_q . Next, let us consider the finite ring $R = \mathbb{F}_q[u_1, u_2]/\langle u_1^2 - \alpha^2, u_2^2 - \beta^2, u_1u_2 - u_2u_1 \rangle$. It is straightforward to check that R is a non-chain semi-local ring of order q^4 . Cyclic codes are very useful for the construction of quantum-error-correcting (QEC) codes. QEC codes are different from classical-error-correcting (CEC) codes. A significant breakthrough happened in 1998, when Calderbank et al. [1] solved the problem of obtaining QEC codes with the help of CEC codes over GF(4). Calderbank et al. [1] also introduced a method to construct QEC codes from CEC codes. Over finite fields, cyclic codes have been extensively investigated (see, for example, [2–5] and references therein). In 2015, from the cyclic codes over $\mathbb{F}_q + v\mathbb{F}_q + v^2\mathbb{F}_q + v^3\mathbb{F}_q$ (where $q = p^m$, p is a prime such that $3|(p-1)$, $v^4 = v$, and m is a positive integer), Gao et al. [6] constructed new quantum codes over \mathbb{F}_q . Afterwards, Ozen et al. [7] constructed many ternary quantum codes from cyclic codes over $\mathbb{F}_3 + u\mathbb{F}_3 + v\mathbb{F}_3 + uv\mathbb{F}_3$. In 2021, Ashraf et al. [8] found better quantum and LCD codes over the ring $\mathbb{F}_{p^m} + v\mathbb{F}_{p^m}$ with $v^2 = 1$, where m is a positive integer. In this article, we discuss the structural properties of cyclic codes over the ring R . On this ring R , we construct a Gray map that provides better parameters and contributes to the finding of better quantum codes over R than presented in [8–13] (and references therein).

In this paper, our main aim is to study the structural properties of cyclic codes over the finite ring R , and to construct quantum-error-correcting (QEC) codes over R . Moreover, we also study LCD codes. The major contributions of this paper are as follows:

1. This paper provides superior quantum codes to those presented in recent references [8–13], see Table 1.

2. This paper provides some new quantum codes, see Table 2.
3. This paper investigates some optimal LCD codes over the ring R , see Table 3.

Table 1. Quantum codes from cyclic codes over R .

n	$h_1(z)$	$h_2(z)$	$h_3(z)$	$h_4(z)$	$\eta(\mathcal{C})$	$[[n, k, d]]_q$	$[[n', k', d']]_q$
10	$z + 1$	$z + 1$	$z + 4$	$z + 4$	$[40, 36, 2]$	$[[40, 32, 2]]_5$	$[[40, 24, 2]]_5$ [9]
20	$(z + 2)^2$ $(z + 4)$	$(z + 2)^2$ $(z + 4)$	$(z + 2)^2$ $(z + 4)$	$(z + 2)^2$ $(z + 4)$	$[80, 68, 3]$	$[[80, 56, 3]]_5$	$[[80, 54, 3]]_5$ [11]
22	$z + 1$	$z + 1$	$z + 4$	$z + 4$	$[88, 84, 2]$	$[[88, 80, 2]]_5$	$[[88, 48, 2]]_5$ [9]
28	$z + 2$	$z + 2$	$z + 3$	$z + 3$	$[112, 108, 2]$	$[[112, 104, 2]]_5$	$[[112, 64, 2]]_5$ [9]
30	$(z + 1)^2$ $(z^2 + z + 1)$	$(z + 1)^2$ $(z^2 + z + 1)$	$(z + 1)^2$ $(z^2 + z + 1)$	$(z + 1)^2$ $(z^2 + z + 1)$	$[120, 104, 3]$	$[[120, 88, 3]]_5$	$[[120, 32, 3]]_5$ [12]
31	$z + 4$	$z + 4$	$z + 4$	$(z^3 + z^2 + z + 4)$ $(z^3 + z^2 + 3z + 4)$	$[124, 115, 4]$	$[[124, 106, 4]]_5$	$[[124, 100, 4]]_5$ [8]
35	$z + 4$	$z + 4$	$(z + 4)^2$	$(z + 4)$ $(z^6 + z^5 + z^4 + z^3 + z^2 + z + 1)$	$[140, 129, 3]$	$[[140, 118, 3]]_5$	$[[140, 112, 2]]_5$ [9]
42	$(z + 4)$ $(z^6 + 2z^4 + 3z^3 + 2z^2 + 1)$	$(z + 4)$ $(z^6 + 2z^4 + 3z^3 + 2z^2 + 1)$	$(z + 4)$ $(z^6 + 2z^4 + 3z^3 + 2z^2 + 1)$	$(z + 4)$ $(z^6 + 2z^4 + 3z^3 + 2z^2 + 1)$	$[168, 140, 4]$	$[[168, 112, 4]]_5$	$[[168, 96, 2]]_5$ [10]
24	$z + 3$	$z + 3$	$z + 3$	$(z + 3)$ $(z^2 + z + 4)$	$[96, 90, 3]$	$[[96, 84, 3]]_7$	$[[96, 80, 3]]_7$ [8]
78	$(z + 3)^2$ $(z + 12)$	$(z + 3)^2$ $(z + 12)$	$(z + 3)^2$ $(z + 12)$	$(z + 3)^2$ $(z + 12)$	$[312, 300, 3]$	$[[312, 288, 3]]_{13}$	$[[312, 282, 3]]_{13}$ [13]
12	1	$z + 1$	$z + 1$	$(z + 1)$ $(z^2 + 4z + 16)$ $(z^2 + z + 1)$	$[48, 41, 4]$	$[[48, 34, 4]]_{17}$	$[[48, 32, 4]]_{17}$ [8]
19	$z + 18$	$z + 18$	$(z + 18)^2$	$(z + 18)^{14}$	$[76, 58, 4]$	$[[76, 40, 4]]_{19}$...

Table 2. New quantum codes from cyclic codes over R .

n	$h_1(z)$	$h_2(z)$	$h_3(z)$	$h_4(z)$	$\eta(\mathcal{C})$	$[[n, k, d]]_q$	
9	$(z + 2)^4$	$(z + 2)^2$	$z + 2$	1	$[36, 29, 3]$	$[[36, 22, 3]]_3$	<i>New quantum code</i>
25	$(z + 4)^6$	$z + 4$	$z + 4$	1	$[100, 92, 3]$	$[[100, 84, 3]]_5$	<i>New quantum code</i>
15	$(z + 4)^2(z^2 + z + 1)$	$z + 4$	$z^2 + z + 1$	1	$[60, 53, 3]$	$[[60, 46, 3]]_5$	<i>New quantum code</i>
14	$(z + 1)(z + 6)^3$	$z + 1$	$z + 6$	1	$[56, 50, 4]$	$[[56, 44, 4]]_7$	<i>New quantum code</i>
11	$(z + 10)^5$	$z + 10$	$z + 10$	1	$[44, 37, 4]$	$[[44, 30, 4]]_{11}$	<i>New quantum code</i>

Table 3. Gray images of LCD codes of length n over R .

n	$h_1(z)$	$h_2(z)$	$h_3(z)$	$h_4(z)$	$\eta(\mathcal{C})$	
4	1	$z + 1$	$z + 1$	$(z + 1)(z^2 + 1)$	$[16, 11, 4]_3$	<i>Optimal</i>
22	$z + 1$	$z + 1$	$z + 1$	$z + 1$	$[88, 84, 2]_3$	<i>Optimal</i>
6	1	$z + 1$	$z + 1$	$(z + 1)(z^2 + z + 1)$ $(z^2 + 4z + 1)$	$[24, 17, 4]_5$...
8	1	$z + 1$	$z + 1$	$(z + 1)(z^2 + 4z + 1)$	$[32, 27, 4]_7$	<i>Optimal</i>

Table 3. Cont.

n	$h_1(z)$	$h_2(z)$	$h_3(z)$	$h_4(z)$	$\eta(\mathcal{C})$
37	$z^6 + 5z^5 + 5z^4 + 4z^3 + 5z^2 + 5z + 1$	$z^6 + 5z^5 + 5z^4 + 4z^3 + 5z^2 + 5z + 1$	$z^6 + 4z^5 + 3z^4 + 7z^3 + 3z^2 + 4z + 1$	$z^6 + 4z^5 + 3z^4 + 7z^3 + 3z^2 + 4z + 1$	[148, 124, 5] ₁₁
39	$(z^2 + z + 1)(z^{12} + z^{11} + z^{10} + z^9 + z^8 + z^7 + z^6 + z^5 + z^4 + z^3 + z^2 + z + 1)$	$(z^2 + z + 1)(z^{12} + z^{11} + z^{10} + z^9 + z^8 + z^7 + z^6 + z^5 + z^4 + z^3 + z^2 + z + 1)$	$(z^2 + z + 1)(z^{12} + z^{11} + z^{10} + z^9 + z^8 + z^7 + z^6 + z^5 + z^4 + z^3 + z^2 + z + 1)$	$(z^2 + z + 1)(z^{12} + z^{11} + z^{10} + z^9 + z^8 + z^7 + z^6 + z^5 + z^4 + z^3 + z^2 + z + 1)$	[156, 100, 4] ₁₁
11	$z^{10} + z^9 + z^8 + z^7 + z^6 + z^5 + z^4 + z^3 + z^2 + z + 1$	$z^{10} + z^9 + z^8 + z^7 + z^6 + z^5 + z^4 + z^3 + z^2 + z + 1$	$z^{10} + z^9 + z^8 + z^7 + z^6 + z^5 + z^4 + z^3 + z^2 + z + 1$	$z^{10} + z^9 + z^8 + z^7 + z^6 + z^5 + z^4 + z^3 + z^2 + z + 1$	[44, 4, 11] ₁₉
28	$(z + 1)(z^6 + z^5 + z^4 + z^3 + z^2 + z + 1)(z^6 + 11z^5 + 3z^4 + 11z^3 + 3z^2 + 11z + 1)$	$(z + 1)(z^6 + z^5 + z^4 + z^3 + z^2 + z + 1)(z^6 + 11z^5 + 3z^4 + 11z^3 + 3z^2 + 11z + 1)$	$(z + 1)(z^6 + z^5 + z^4 + z^3 + z^2 + z + 1)(z^6 + 8z^5 + 3z^4 + 8z^3 + 3z^2 + 8z + 1)$	$(z + 1)(z^6 + z^5 + z^4 + z^3 + z^2 + z + 1)(z^6 + 8z^5 + 3z^4 + 8z^3 + 3z^2 + 8z + 1)$	[112, 60, 8] ₁₉
34	$(z + 1)(z^8 + 13z^7 + 15z^6 + 16z^5 + 8z^4 + 16z^3 + 15z^2 + 13z + 1)$	$(z + 1)(z^8 + 13z^7 + 15z^6 + 16z^5 + 8z^4 + 16z^3 + 15z^2 + 13z + 1)$	$(z + 1)(z^8 + 13z^7 + 15z^6 + 16z^5 + 8z^4 + 16z^3 + 15z^2 + 13z + 1)$	$(z + 1)(z^8 + 13z^7 + 15z^6 + 16z^5 + 8z^4 + 16z^3 + 15z^2 + 13z + 1)$	[136, 100, 4] ₁₉

2. Preliminary Results

In this section, we deal with the study of some preliminaries and describe the Gray map over the ring R . Moreover, we establish some important results which are needed in the subsequent discussions. If a code \mathcal{C} is an R -submodule of R^n (where n is a positive integer), then \mathcal{C} is linear. The elements of \mathcal{C} are called codewords. The size of \mathcal{C} refers to the total number of codewords in \mathcal{C} , which is indicated by $|\mathcal{C}|$. We recall some basic definitions as following:

- (i) The Hamming distance between two vectors $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and $\mathbf{y} = (y_1, y_2, \dots, y_n)$ is the number of places where they differ, and is denoted by $d(\mathbf{x}, \mathbf{y})$.
- (ii) The Hamming weight of a vector $\mathbf{x} = (x_1, x_2, \dots, x_n)$ is the number of non-zero x_i and is denoted by $wt(\mathbf{x})$.
- (iii) The Euclidean inner product of any two vectors $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and $\mathbf{y} = (y_1, y_2, \dots, y_n)$ is defined as $\mathbf{x} \cdot \mathbf{y} = x_0y_0 + x_1y_1 + \dots + x_{n-1}y_{n-1}$ and the dual of linear code \mathcal{C} is $\mathcal{C}^\perp = \{\mathbf{x} \in R^n \mid \mathbf{x} \cdot \mathbf{y} = 0 \forall \mathbf{y} \in \mathcal{C}\}$.
- (iv) A code \mathcal{C} is said to be self-dual if $\mathcal{C} = \mathcal{C}^\perp$, self-orthogonal if $\mathcal{C} \subseteq \mathcal{C}^\perp$, and dual containing if $\mathcal{C}^\perp \subseteq \mathcal{C}$.

Clearly, the ring R can be expressed as $R = \mathbb{F}_q + u_1\mathbb{F}_q + u_2\mathbb{F}_q + u_1u_2\mathbb{F}_q$ such that $u_1^2 = \alpha^2$, $u_2^2 = \beta^2$ and $u_1u_2 = u_2u_1$; here \mathbb{F}_q is the finite field of order q , where $q = p^m$ for odd prime p and $m \geq 1$. It is a commutative non-chain semi-local ring with four maximal ideals. An element z of R is of the form $z = a_1 + a_2u_1 + a_3u_2 + a_4u_1u_2$, where $a_i \in \mathbb{F}_q$ and

$1 \leq i \leq 4$. With the help of a set of orthogonal idempotents, every element of this ring can be represented:

$$\begin{aligned} \Delta_1 &= \frac{(\alpha + u_1)(\beta + u_2)}{4\alpha\beta}, \\ \Delta_2 &= \frac{(\alpha + u_1)(\beta - u_2)}{4\alpha\beta}, \\ \Delta_3 &= \frac{(\alpha - u_1)(\beta + u_2)}{4\alpha\beta}, \end{aligned}$$

and

$$\Delta_4 = \frac{(\alpha - u_1)(\beta - u_2)}{4\alpha\beta}.$$

It is straightforward to show that $\Delta_i^2 = \Delta_i$, $0 = \Delta_i\Delta_j$, and $\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 = 1$, where $1 \leq i, j \leq 4$, and $i \neq j$. In view of the Chinese Remainder Theorem, we obtain $R = \Delta_1R \oplus \Delta_2R \oplus \Delta_3R \oplus \Delta_4R = \Delta_1\mathbb{F}_q \oplus \Delta_2\mathbb{F}_q \oplus \Delta_3\mathbb{F}_q \oplus \Delta_4\mathbb{F}_q$. Thus, we can express every element z of R as $z = a_1 + a_2u_1 + a_3u_2 + a_4u_1u_2 = \Delta_1z_1 + \Delta_2z_2 + \Delta_3z_3 + \Delta_4z_4$, where $a_i, z_i \in \mathbb{F}_q$ and $1 \leq i \leq 4$.

The Gray map $\eta : R \rightarrow \mathbb{F}_q^4$ is defined by

$$\eta(\Delta_1z_1 + \Delta_2z_2 + \Delta_3z_3 + \Delta_4z_4) = (z_1, z_2, z_3, z_4)A, \tag{1}$$

where $A \in GL_4(\mathbb{F}_q)$ is a fixed matrix and $GL_4(\mathbb{F}_q)$ is the linear group of all 4×4 invertible matrices over the field \mathbb{F}_q such that $AA^T = \epsilon I_{4 \times 4}$, where A^T is the transpose of A and $\epsilon \in \mathbb{F}_q \setminus \{0\}$.

The above Gray map is linear, and we can extend it component-wise from R^n to \mathbb{F}_q^{4n} , where n is a positive integer. For any element $z = \Delta_1z_1 + \Delta_2z_2 + \Delta_3z_3 + \Delta_4z_4 \in R$, the Lee weight of z is defined as $w_L(z) = w_H(\eta(z))$, where w_H represents the Hamming weight over \mathbb{F}_q . We begin our discussion with the following result related to the Gray map (1):

Proposition 1. *The map $\eta : R \rightarrow \mathbb{F}_q^4$ defined in (1) is an \mathbb{F}_q -linear and distance-preserving map from (R^n, d_L) to (\mathbb{F}_q^{4n}, d_H) , where $d_L = d_H$.*

Proof. Let $z, z' \in R^n$ such that

$$\begin{aligned} z &= \Delta_1z_1 + \Delta_2z_2 + \Delta_3z_3 + \Delta_4z_4 \\ z' &= \Delta_1z'_1 + \Delta_2z'_2 + \Delta_3z'_3 + \Delta_4z'_4 \end{aligned}$$

and $z_i, z'_i \in \mathbb{F}_q^n$ for $1 \leq i \leq 4$. Then, we have

$$\begin{aligned} \eta(z + z') &= \eta(\Delta_1z_1 + \Delta_1z'_1 + \Delta_2z_2 + \Delta_2z'_2 + \Delta_3z_3 + \Delta_3z'_3 + \Delta_4z_4 + \Delta_4z'_4) \\ &= \eta(\Delta_1(z_1 + z'_1) + \Delta_2(z_2 + z'_2) + \Delta_3(z_3 + z'_3) + \Delta_4(z_4 + z'_4)) \\ &= (z_1 + z'_1, z_2 + z'_2, z_3 + z'_3, z_4 + z'_4)A \\ &= (z_1, z_2, z_3, z_4)A + (z'_1, z'_2, z'_3, z'_4)A \\ &= \eta(z) + \eta(z') \text{ for all } z, z' \in R^n. \end{aligned}$$

Furthermore, for any $\alpha \in \mathbb{F}_q$, we have

$$\begin{aligned} \eta(\alpha z) &= \eta(\Delta_1\alpha z_1 + \Delta_2\alpha z_2 + \Delta_3\alpha z_3 + \Delta_4\alpha z_4) \\ &= (\alpha z_1, \alpha z_2, \alpha z_3, \alpha z_4)A \\ &= \alpha(z_1, z_2, z_3, z_4)A \\ &= \alpha\eta(z) \text{ for all } z \in R^n. \end{aligned}$$

Hence, η is an \mathbb{F}_q -linear. As for the second part, we know that

$$\begin{aligned} d_L(z, z') &= w_L(z - z') \\ &= w_H(\eta(z - z')) \\ &= w_H(\eta(z) - \eta(z')) \\ &= d_H(\eta(z), \eta(z')). \end{aligned}$$

Therefore, η is a distance-preserving map. \square

Define $\Theta_1 \otimes \Theta_2 \otimes \Theta_3 \otimes \Theta_4 = \{(\theta_1, \theta_2, \theta_3, \theta_4) \mid \theta_i \in \Theta_i : 1 \leq i \leq 4\}$ and $\Theta_1 \oplus \Theta_2 \oplus \Theta_3 \oplus \Theta_4 = \{(\theta_1 + \theta_2 + \theta_3 + \theta_4) \mid \theta_i \in \Theta_i : 1 \leq i \leq 4\}$. Let C be a linear code of length n over R . We define that

$$C_1 = \{z_1 \in \mathbb{F}_q^n \mid \Delta_1 z_1 + \Delta_2 z_2 + \Delta_3 z_3 + \Delta_4 z_4 \in C, \text{ where } z_2, z_3, z_4 \in \mathbb{F}_q^n\},$$

$$C_2 = \{z_2 \in \mathbb{F}_q^n \mid \Delta_1 z_1 + \Delta_2 z_2 + \Delta_3 z_3 + \Delta_4 z_4 \in C, \text{ where } z_1, z_3, z_4 \in \mathbb{F}_q^n\},$$

$$C_3 = \{z_3 \in \mathbb{F}_q^n \mid \Delta_1 z_1 + \Delta_2 z_2 + \Delta_3 z_3 + \Delta_4 z_4 \in C, \text{ where } z_1, z_2, z_4 \in \mathbb{F}_q^n\},$$

and

$$C_4 = \{z_4 \in \mathbb{F}_q^n \mid \Delta_1 z_1 + \Delta_2 z_2 + \Delta_3 z_3 + \Delta_4 z_4 \in C, \text{ where } z_1, z_2, z_3 \in \mathbb{F}_q^n\}.$$

Now, each C_i is a linear code of length n over \mathbb{F}_q for $1 \leq i \leq 4$. Hence, any linear code of length n can be represented as $C = \Delta_1 C_1 \oplus \Delta_2 C_2 \oplus \Delta_3 C_3 \oplus \Delta_4 C_4$ and $|C| = |C_1||C_2||C_3||C_4|$ over R . A matrix is called a generator matrix of C if the rows of the matrix generate C . If M_i are the generator matrices of the linear code C_i , for $i = 1, 2, 3, 4$, respectively, then a generator matrix of C is

$$M = \begin{pmatrix} \Delta_1 M_1 \\ \Delta_2 M_2 \\ \Delta_3 M_3 \\ \Delta_4 M_4 \end{pmatrix}$$

and a generator matrix of $\eta(C)$ is

$$\eta(M) = \begin{pmatrix} \eta(\Delta_1 M_1) \\ \eta(\Delta_2 M_2) \\ \eta(\Delta_3 M_3) \\ \eta(\Delta_4 M_4) \end{pmatrix}.$$

Proposition 2. Let $C = \Delta_1 C_1 \oplus \Delta_2 C_2 \oplus \Delta_3 C_3 \oplus \Delta_4 C_4$ be a linear code of length n over R . Then, $\eta(C)$ is a $[4n, \sum_{i=1}^4 k_i, d]$ linear code over \mathbb{F}_q for $1 \leq i \leq 4$, where each C_i is an $[n, k_i, d]$ code.

Proof. The proof is obvious with the help of the Gray map. \square

Proposition 3. If C is a linear code of length n over R , then $\eta(C) = C_1 \otimes C_2 \otimes C_3 \otimes C_4$.

Proof. The proof is similar to the one in [14]. \square

Theorem 1. Let C be a self-orthogonal linear code of length n over R and A be a 4×4 non-singular matrix over \mathbb{F}_q which has the property $AA^T = \epsilon I_4$, where I_4 is the identity matrix, $0 \neq \epsilon \in \mathbb{F}_q$, and A^T is the transpose of matrix A . Then, the Gray image $\eta(C)$ is a self-orthogonal linear code of length $4n$ over \mathbb{F}_q .

Proof. Suppose C is a self-orthogonal linear code of length n over R , i.e., $C \subseteq C^\perp$ and let $P, Q \in \eta(C)$ such that $P = \eta(p) = (p_0A, p_1A, \dots, p_{n-1}A)$ and $Q = \eta(q) = (q_0A, q_1A,$

$\dots, q_{n-1}A)$. We have to show that $\eta(C)$ is self-orthogonal, that is, $P \cdot Q = 0$. Since C is self-orthogonal, $p \cdot q = \sum_{j=0}^{n-1} p_j \cdot q_j = 0$. Therefore, $P \cdot Q = PQ^\perp = \sum_{j=0}^{n-1} p_j AA^T q_j^\perp = m \sum_{j=0}^{n-1} p_j \cdot q_j = 0$. Suppose P and Q are arbitrary, then $\eta(C) \subseteq \eta(C^\perp)$. Thus, $\eta(C)$ is a self-orthogonal linear code of length $4n$ over \mathbb{F}_q . \square

3. Structural Properties of Cyclic Codes over R

On ring R , as described, we shall explore various structural properties of cyclic codes and prove some results. We begin with the following definition:

Definition 1. A linear code C of length n over R is said to be a cyclic code if every cyclic shift of a codeword c in C is again a codeword in C , i.e., if $c = (c_0, c_1, c_2, \dots, c_{n-1}) \in C$, then its cyclic shift $\zeta(c) = (c_{n-1}, c_0, \dots, c_{n-2}) \in C$, where the operator ζ is known as cyclic shift.

Theorem 2. Let $C = \Delta_1 C_1 \oplus \Delta_2 C_2 \oplus \Delta_3 C_3 \oplus \Delta_4 C_4$ be a linear code of length n over R . Then, C is a cyclic code over R if, and only if, each C_i is a cyclic code over \mathbb{F}_q , where $1 \leq i \leq 4$.

Proof. Suppose s is any codeword in C such that $s = (s_0, s_1, \dots, s_{n-1})$. We can write its components as $s_i = \Delta_1 z_{1,i} + \Delta_2 z_{2,i} + \Delta_3 z_{3,i} + \Delta_4 z_{4,i}$, where $z_{1,i}, z_{2,i}, z_{3,i}, z_{4,i} \in \mathbb{F}_q$ and $1 \leq i \leq n - 1$. Let

$$\begin{aligned} z_1 &= (z_{0,1}, z_{1,1}, \dots, z_{n-1,1}), \\ z_2 &= (z_{0,2}, z_{1,2}, \dots, z_{n-1,2}), \\ z_3 &= (z_{0,3}, z_{1,3}, \dots, z_{n-1,3}), \\ z_4 &= (z_{0,4}, z_{1,4}, \dots, z_{n-1,4}), \end{aligned}$$

where $z_i \in C_i$ and $1 \leq i \leq 4$. Now, let us assume that every C_i is a cyclic code over \mathbb{F}_q , where $1 \leq i \leq 4$. This implies that

$$\begin{aligned} \zeta(z_1) &= (z_{n-1,1}, z_{0,1}, \dots, z_{n-2,1}) \in C_1, \\ \zeta(z_2) &= (z_{n-1,2}, z_{0,2}, \dots, z_{n-2,2}) \in C_2, \\ \zeta(z_3) &= (z_{n-1,3}, z_{0,3}, \dots, z_{n-2,3}) \in C_3, \\ \zeta(z_4) &= (z_{n-1,4}, z_{0,4}, \dots, z_{n-2,4}) \in C_4, \end{aligned}$$

Thus, $\Delta_1 \zeta(z_1) + \Delta_2 \zeta(z_2) + \Delta_3 \zeta(z_3) + \Delta_4 \zeta(z_4) \in C$. It can easily be seen that $\Delta_1 \zeta(z_1) + \Delta_2 \zeta(z_2) + \Delta_3 \zeta(z_3) + \Delta_4 \zeta(z_4) = \zeta(s)$. Hence, $\zeta(s) \in C$. We can conclude that C is a cyclic code over R .

On the other hand, let us assume that C is a cyclic code over R . Next, let us consider $s_i = \Delta_1 z_{1,i} + \Delta_2 z_{2,i} + \Delta_3 z_{3,i} + \Delta_4 z_{4,i}$, where $z_1 = (z_{0,1}, z_{1,1}, \dots, z_{n-1,1})$, $z_2 = (z_{0,2}, z_{1,2}, \dots, z_{n-1,2})$, $z_3 = (z_{0,3}, z_{1,3}, \dots, z_{n-1,3})$, and $z_4 = (z_{0,4}, z_{1,4}, \dots, z_{n-1,4})$. Then, $z_1 \in C_1$, $z_2 \in C_2$, $z_3 \in C_3$, and $z_4 \in C_4$. Again, $s = (s_0, s_1, \dots, s_{n-1}) \in C$, and by this hypothesis $\zeta(s) \in C$. We have $\Delta_1 \zeta(z_1) + \Delta_2 \zeta(z_2) + \Delta_3 \zeta(z_3) + \Delta_4 \zeta(z_4) \in C$. Here, $\zeta(z_i) \in C_i$, where $1 \leq i \leq 4$. Consequently, every C_i is a cyclic code of length n over \mathbb{F}_q , where $1 \leq i \leq 4$. \square

Theorem 3. Let $C = \Delta_1 C_1 \oplus \Delta_2 C_2 \oplus \Delta_3 C_3 \oplus \Delta_4 C_4$ be a cyclic code of length n over R and $h_i(z)$ be the standard generator polynomial of C_i . Then, $C = \langle h(z) \rangle$ and $|C| = q^{4n - \sum_{i=0}^4 h_i(z)}$, where $h(z) = \Delta_1 h_1(z) + \Delta_2 h_2(z) + \Delta_3 h_3(z) + \Delta_4 h_4(z)$ and $1 \leq i \leq 4$.

Proof. Given $C_i = \langle h_i(z) \rangle$, where $1 \leq i \leq 4$ and $C = \Delta_1 C_1 \oplus \Delta_2 C_2 \oplus \Delta_3 C_3 \oplus \Delta_4 C_4$. Let $c \in C$ be such that $c = \{c(z) \mid \Delta_1 h_1(z) + \Delta_2 h_2(z) + \Delta_3 h_3(z) + \Delta_4 h_4(z) \text{ for } h_i(z) \in C_i\}$. Therefore, $C \subseteq \langle \Delta_1 h_1(z), \Delta_2 h_2(z), \Delta_3 h_3(z), \Delta_4 h_4(z) \rangle \subseteq R[z]/\langle z^n - 1 \rangle$. For any $\Delta_1 t_1(z) h_1(z) +$

$\Delta_2 t_2(z)h_2(z) + \Delta_3 t_3(z)h_3(z) + \Delta_4 t_4(z)h_4(z) \in \langle \Delta_1 h_1(z) + \Delta_2 h_2(z) + \Delta_3 h_3(z) + \Delta_4 h_4(z) \rangle \subseteq R[z]/\langle z^n - 1 \rangle$, where $t_1(z), t_2(z), t_3(z)$, and $t_4(z) \in R[z]/\langle z^n - 1 \rangle$, there exist $s_1(z), s_2(z), s_3(z)$, and $s_4(z) \in \mathbb{F}_q[z]$ such that

$$\Delta_i t_i(z) = \Delta_i s_i(z),$$

where $1 \leq i \leq 4$. Hence, $\langle \Delta_1 h_1(z), \Delta_2 h_2(z), \Delta_3 h_3(z), \Delta_4 h_4(z) \rangle \subseteq \mathcal{C}$. This implies $\langle \Delta_1 h_1(z), \Delta_2 h_2(z), \Delta_3 h_3(z), \Delta_4 h_4(z) \rangle = \mathcal{C}$. Since $|\mathcal{C}| = |\mathcal{C}_1||\mathcal{C}_2||\mathcal{C}_3||\mathcal{C}_4|$, we have

$$|\mathcal{C}| = q^{4n - \sum_{i=1}^4 h_i(z)}.$$

□

Theorem 4. Let $\mathcal{C} = \Delta_1 \mathcal{C}_1 \oplus \Delta_2 \mathcal{C}_2 \oplus \Delta_3 \mathcal{C}_3 \oplus \Delta_4 \mathcal{C}_4$ be a cyclic code of length n over R . Then, there exists a unique monic polynomial $h(z) \in R[z]$ such that $\mathcal{C} = \langle h(z) \rangle$ and $h(z)$ divides $(z^n - 1)$. If $h_i(z)$ is the standard generator polynomial of $\mathcal{C}_i, 1 \leq i \leq 4$, then $h(z) = \Delta_1 h_1(z) + \Delta_2 h_2(z) + \Delta_3 h_3(z) + \Delta_4 h_4(z)$.

Proof. By Theorem 3, $\mathcal{C} = \langle \Delta_1 h_1(z), \Delta_2 h_2(z), \Delta_3 h_3(z), \Delta_4 h_4(z) \rangle$, where $h_i(z)$ is the generator polynomial of \mathcal{C}_i and $1 \leq i \leq 4$. Let $h(z) = \Delta_1 h_1(z) + \Delta_2 h_2(z) + \Delta_3 h_3(z) + \Delta_4 h_4(z)$. From here, $\langle h(z) \rangle \subseteq \mathcal{C}$. Now, $\Delta_i h_i(z) = \Delta_i h(z)$ and $1 \leq i \leq 4$, so $\mathcal{C} \subseteq \langle h(z) \rangle$, hence $\mathcal{C} = \langle h(z) \rangle$. Since $h_i(z)$ is a monic right divisor of $(z^n - 1)$, there are $s_i(z) \in \mathbb{F}_q[z]/\langle z^n - 1 \rangle$, where $1 \leq i \leq 4$, such that $z^n - 1 = s_1(z)h_1(z) = s_2(z)h_2(z) = s_3(z)h_3(z) = s_4(z)h_4(z)$. This shows that $z^n - 1 = [\Delta_1 s_1(z) + \Delta_2 s_2(z) + \Delta_3 s_3(z) + \Delta_4 s_4(z)]h(z)$, i.e., $h(z)|(z^n - 1)$. Here, each $h_i(z)$ is unique, and hence $h(z)$ is unique. □

Theorem 5. Let $\mathcal{C} = \Delta_1 \mathcal{C}_1 \oplus \Delta_2 \mathcal{C}_2 \oplus \Delta_3 \mathcal{C}_3 \oplus \Delta_4 \mathcal{C}_4$ be a cyclic code of length n over R . Then, $\mathcal{C}^\perp = \Delta_1 \mathcal{C}_1^\perp \oplus \Delta_2 \mathcal{C}_2^\perp \oplus \Delta_3 \mathcal{C}_3^\perp \oplus \Delta_4 \mathcal{C}_4^\perp$ is also a cyclic code of length n over R .

Proof. \mathcal{C}^\perp is a cyclic code of length n over R , since \mathcal{C} is a cyclic code of length n over R . Now, we will show that $\mathcal{C}^\perp = \Delta_1 \mathcal{C}_1^\perp \oplus \Delta_2 \mathcal{C}_2^\perp \oplus \Delta_3 \mathcal{C}_3^\perp \oplus \Delta_4 \mathcal{C}_4^\perp$. Here, \mathcal{C} is a cyclic code of length n over R . This implies \mathcal{C} is a linear code of length n over R . Let $T_1 = \{t_1 \in \mathbb{F}_q^n \mid \exists t_2, t_3, t_4$ such that $\sum_{i=1}^4 t_i \Delta_i \in \mathcal{C}^\perp\}$, for $1 \leq i \leq 4$. Hence, \mathcal{C}^\perp is uniquely expressed as $\mathcal{C}^\perp = \oplus_{i=1}^4 \Delta_i T_i$. Therefore, $T_1 \subseteq \mathcal{C}_1^\perp$. Conversely, let $q \in \mathcal{C}_1^\perp$. This implies $q \cdot s_1 = 0 \forall s_1 \in \mathcal{C}_1$. Consider $y = \sum_{i=1}^4 \Delta_i s_i \in \mathcal{C}$. Now, $\Delta_1 q \cdot y = \Delta_1 s_1 \cdot q = 0$. This shows that $\Delta_1 q \in \mathcal{C}_1^\perp$. From the specific expression of \mathcal{C}^\perp , we obtain $q \in T_1$. From here, $\mathcal{C}^\perp \subseteq T_1$. Therefore, $\mathcal{C}_1^\perp = T_1$. In the same manner, $\mathcal{C}_i^\perp = T_i$ for $1 \leq i \leq 4$. Hence, $\mathcal{C}^\perp = \Delta_1 \mathcal{C}_1^\perp \oplus \Delta_2 \mathcal{C}_2^\perp \oplus \Delta_3 \mathcal{C}_3^\perp \oplus \Delta_4 \mathcal{C}_4^\perp$. □

Lemma 1 ([1]). Let $\mathcal{C} = \langle h(z) \rangle$ be a cyclic code of length n over \mathbb{F}_q ; $h(z)$ be the generator polynomial of \mathcal{C} . Then $\mathcal{C}^\perp \subseteq \mathcal{C}$ if, and only if,

$$z^n - 1 \equiv 0 \pmod{h(z)h^*(z)},$$

where the reciprocal polynomial of $h(z)$ is denoted by $h^*(z)$.

Theorem 6. Let $\mathcal{C} = \Delta_1 \mathcal{C}_1 \oplus \Delta_2 \mathcal{C}_2 \oplus \Delta_3 \mathcal{C}_3 \oplus \Delta_4 \mathcal{C}_4$ be a cyclic code of length n over R and $\mathcal{C} = \langle h(z) \rangle = \langle \sum_{i=1}^4 \Delta_i h_i(z) \rangle$, where $h_i(z)$ be the generator polynomial of \mathcal{C}_i . Then, $\mathcal{C}^\perp \subseteq \mathcal{C}$ if, and only, if

$$z^n - 1 \equiv 0 \pmod{h_i(z)h_i^*(z)},$$

where the reciprocal polynomial of $h_i(z)$ is denoted by $h_i^*(z)$ and $1 \leq i \leq 4$.

Proof. Suppose $z^n - 1 \equiv 0 \pmod{h_i(z)h_i^*(z)}$ for $1 \leq i \leq 4$. Hence, by Lemma 1, we have $C_i^\perp \subseteq C_i$. From here, we can write $\Delta_i C_i^\perp \subseteq \Delta_i C_i$ for $1 \leq i \leq 4$. Similarly, $C^\perp = \sum_{i=0}^4 \Delta_i C_i^\perp \subseteq \sum_{i=0}^4 \Delta_i C_i = C$. Conversely, assume $C^\perp \subseteq C$ and $\sum_{i=0}^4 \Delta_i C_i^\perp \subseteq \sum_{i=0}^4 \Delta_i C_i$, but each C_i is a cyclic code over \mathbb{F}_q , such that $\Delta_i C_i \equiv C \pmod{\Delta_i}$. This implies that $C_i^\perp \subseteq C_i$, where $1 \leq i \leq 4$. By Lemma 1, we obtain

$$z^n - 1 \equiv 0 \pmod{h_i(z)h_i^*(z)},$$

where the reciprocal polynomial of $h_i(z)$ is denoted by $h_i^*(z)$ for $1 \leq i \leq 4$. \square

Corollary 1. Let $C = \Delta_1 C_1 \oplus \Delta_2 C_2 \oplus \Delta_3 C_3 \oplus \Delta_4 C_4$ be a cyclic code of length n over R . Then, $C^\perp \subseteq C$ if, and only if, $C_i^\perp \subseteq C_i$ and $1 \leq i \leq 4$.

4. Quantum and LCD Codes

This section deals with the study of quantum and LCD codes over the ring R . We begin with the following definition: Let p be a prime and $q = p^m$ for a positive integer m . Let $H(\mathbb{C})$ be a q -dimensional Hilbert space over the complex field \mathbb{C} . Then, the set of n -folded tensor products $H(\mathbb{C})^n = \underbrace{H \otimes H \otimes \dots \otimes H}_{n\text{-times}}$ is also a q^n -dimensional Hilbert space.

Definition 2 ([15]). A quantum code represented by $[[n, k, d]]_q$ is defined as a subspace of $H(\mathbb{C})^n$ with dimension q^k and minimum distance d . Moreover, we consider $[[n, k, d]]_q$ to be better than $[[n', k', d']]_q$ if either or both of the following conditions hold:

- (i) $d > d'$ whenever the code rate $\frac{k}{n} = \frac{k'}{n'}$ (larger distance).
- (ii) $\frac{k}{n} > \frac{k'}{n'}$, whenever the distance $d = d'$ (larger code rate).

Lemma 2 ([2]). (Theorem 3) (CSS Construction) Let $C_1 = [n, k_1, d_1]_q$ and $C_2 = [n, k_2, d_2]_q$ be two linear codes over $GF(q)$ with $C_2^\perp \subseteq C_1$. Furthermore, let $d = \min\{\text{wgt}(v) : v \in (C_1 \setminus C_2^\perp) \cup (C_2 \setminus C_1^\perp)\} \geq \min(d_1, d_2)$. Then, there exists a QEC code with the parameters $[[n, k_1 + k_2 - n, d]]_q$. In particular, if $C_1^\perp \subseteq C_1$, then there exists a QEC code with the parameters $[[n, 2k_1 - n, d_1]]_q$, where $d_1 = \min\{\text{wgt}(v) : v \in (C_1 \setminus C_1^\perp)\}$.

Theorem 7. Let C be a cyclic code of length n over R , and let the parameters of its Gray image be $[4n, k, d_H]$. If $C^\perp \subseteq C$, then there exists a QECC $[[4n, 2k - 4n, d_H]]_q$ over \mathbb{F}_q .

Proof. Let us consider $\mathbf{x} = (x_0, x_1, \dots, x_{n-1}) \in C, \mathbf{y} = (y_0, y_1, \dots, y_{n-1}) \in C^\perp$, where $x_i = a_i + u_1 b_i + u_2 c_i + u_1 u_2 d_i$ and $y_i = a'_i + u_1 b'_i + u_2 c'_i + u_1 u_2 d'_i, a_i, b_i, c_i, d_i, a'_i, b'_i, c'_i, d'_i \in \mathbb{F}_q$ for $0 \leq i \leq n - 1$. Since $\mathbf{x} \cdot \mathbf{y} = 0$. This gives

$$\begin{aligned} & \sum_{i=0}^{n-1} (a_i + u_1 b_i + u_2 c_i + u_1 u_2 d_i)(a'_i + u_1 b'_i + u_2 c'_i + u_1 u_2 d'_i) = 0 \\ & \sum_{i=0}^{n-1} (a_i a'_i + u_1 a_i b'_i + u_2 a_i c'_i + u_1 u_2 a_i d'_i + u_1 b_i a'_i + \alpha^2 b_i b'_i + u_1 u_2 b_i c'_i + \alpha^2 u_2 b_i d'_i + u_2 c_i a'_i + \\ & \quad u_1 u_2 c_i b'_i + \beta^2 c_i c'_i + \beta^2 u_1 c_i d'_i + u_1 u_2 d_i a'_i + \alpha^2 u_2 d_i b'_i + \beta^2 u_1 d_i c'_i + \alpha^2 \beta^2 d_i d'_i) = 0 \\ & \sum_{i=0}^{n-1} [(a_i a'_i + \alpha^2 b_i b'_i + \beta^2 c_i c'_i + \alpha^2 \beta^2 d_i d'_i) + u_1 (a_i b'_i + b_i a'_i + \beta^2 c_i d'_i + \beta^2 d_i c'_i) + \\ & \quad u_2 (a_i c'_i + \alpha^2 b_i d'_i + c_i a'_i + \alpha^2 d_i b'_i) + u_1 u_2 (a_i d'_i + b_i c'_i + c_i b'_i + d_i a'_i)] = 0. \end{aligned}$$

The above relation yields

$$\begin{aligned} \sum_{i=0}^{n-1} (a_i a'_i + \alpha^2 b_i b'_i + \beta^2 c_i c'_i + \alpha^2 \beta^2 d_i d'_i) &= 0 \\ \sum_{i=0}^{n-1} (a_i b'_i + b_i a'_i + \beta^2 c_i d'_i + \beta^2 d_i c'_i) &= 0 \\ \sum_{i=0}^{n-1} (a_i c'_i + \alpha^2 b_i d'_i + c_i a'_i + \alpha^2 d_i b'_i) &= 0 \\ \sum_{i=0}^{n-1} (a_i d'_i + b_i c'_i + c_i b'_i + d_i a'_i) &= 0. \end{aligned}$$

Additionally, $\eta(\mathbf{x}) \cdot \eta(\mathbf{y}) = 0$. Therefore, $\eta(C^\perp) \subseteq \eta(C)^\perp$. Since η is bijective, $|\eta(C^\perp)| = |\eta(C)^\perp|$. Hence, $\eta(C^\perp) = \eta(C)^\perp$. Moreover, $C^\perp \subseteq C$ implies $\eta(C)^\perp \subseteq \eta(C)$. Hence, $\eta(C)$ is a dual-containing linear code with parameters $[4n, k, d_H]$. Thus by Lemma 2, there exists a quantum-error-correcting code with the parameters $[[4n, 2k - 4n, d_H]]_q$ over \mathbb{F}_q . \square

Definition 3 ([16]). A linear code C of length n over R is said to be linear complementary dual (LCD) if $C \cap C^\perp = \{0\}$.

Lemma 3 ([17]). Let C be a cyclic code of length n over \mathbb{F}_q generated by a polynomial $h(z)$ such that $n = p^{k_1}t$, where p and t are relatively prime and $k_1 \geq 0$. Then, C is an LCD code if, and only if, $h(z)$ is a self-reciprocal and all the monic irreducible factors of $h(z)$ have the same multiplicity in $h(z)$ and in $z^n - 1$.

Definition 4. A linear code C of length n over R is said to be reversible if $(c_{n-1}, c_{n-2}, \dots, c_1, c_0) \in C$, for all $(c_0, c_1, c_2, \dots, c_{n-1}) \in C$.

Lemma 4 ([17]). Let C be a cyclic code of length n over \mathbb{F}_q such that $\gcd(n, p) = 1$. Then, C is a reversible code if, and only if, C is an LCD code.

The proofs of Theorems 8–10, Corollaries 2 and 3, and Lemma 5 are similar to those in [18].

Theorem 8. Let $C = \Delta_1 C_1 \oplus \Delta_2 C_2 \oplus \Delta_3 C_3 \oplus \Delta_4 C_4$ be a cyclic code of length n over R . Then, C is an LCD code if, and only if, C_1, C_2, C_3 , and C_4 are LCD codes of length n over \mathbb{F}_q .

Corollary 2. Let $n = p^t m$ and $\gcd(m, p) = 1$. Let $C = \Delta_1 C_1 \oplus \Delta_2 C_2 \oplus \Delta_3 C_3 \oplus \Delta_4 C_4$ be a cyclic code of length n over R , where $C_i = \langle h_i(z) \rangle$ such that $h_i(z) \in \mathbb{F}_q$ and $h_i(z) | (z^n - 1)$ for $i = 1, 2, 3, 4$. Then, C is an LCD code if, and only if, $h_i(z)$ is self-reciprocal and each monic irreducible factor of $h_i(z)$ has the same multiplicity in $h_i(z)$ and in $z^n - 1$ for $i = 1, 2, 3, 4$.

Theorem 9. Let $C = \Delta_1 C_1 \oplus \Delta_2 C_2 \oplus \Delta_3 C_3 \oplus \Delta_4 C_4$ be a cyclic code of length n over R with $\gcd(n, p) = 1$. Then, C is an LCD code if, and only if, C_1, C_2, C_3 , and C_4 are reversible codes of length n over \mathbb{F}_q .

Corollary 3. For $\gcd(n, p) = 1$, let $C = \Delta_1 C_1 \oplus \Delta_2 C_2 \oplus \Delta_3 C_3 \oplus \Delta_4 C_4$ be a cyclic code of length n over R , where C_1, C_2, C_3 , and C_4 are cyclic codes of length n over \mathbb{F}_q . Then, C is an LCD code if, and only if, $h_i(z)$ is a self-reciprocal polynomial in \mathbb{F}_q for $i = 1, 2, 3, 4$.

Lemma 5. Let C be a linear code of length n over R . Then, $\eta(C \cap C^\perp) = \eta(C) \cap \eta(C)^\perp$.

Theorem 10. Let C be a linear code of length n over R . Then, C is an LCD code if, and only if, its Gray image $\eta(C)$ is an LCD code of length $4n$ over \mathbb{F}_q .

5. Applications

In this section, we present some applications of the results proven in the previous sections. Example 3 and Table 1 demonstrate that our results provide several quantum codes which are better than the existing quantum codes that have been reported [8–13]. Moreover, we obtained new quantum codes in Example 1 and in Table 2. All of the computations involved in these examples were accomplished by using the Magma computation system [19]. We begin our discussion with the following example:

Example 1. Let $R = \mathbb{F}_3[u_1, u_2] / \langle u_1^2 - 1, u_2^2 - 1, u_1u_2 - u_2u_1 \rangle$ be a finite commutative ring, $n = 9$, and $\alpha = \beta = 1$. Then,

$$z^9 - 1 = (z + 2)^9 \in \mathbb{F}_3[x].$$

Take

$$h_1(z) = (z + 2)^4$$

$$h_2(z) = (z + 2)^2$$

$$h_3(z) = (z + 2)$$

$$h_4(z) = 1$$

and

$$A = \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \end{bmatrix}.$$

Here, matrix A satisfies the condition $AA^T = I_{4 \times 4}$, where $A \in GL_4(\mathbb{F}_3)$ and $I_{4 \times 4}$ is an identity matrix. The cyclic code $C = \langle \sum_{i=0}^4 \Delta_i h_i(z) \rangle$ is of length 9 over R and its Gray image is of length 36, dimension 29, and distance 3 over \mathbb{F}_3 , i.e., $[36, 29, 3]_3$. Moreover

$$z^9 - 1 \equiv 0 \pmod{h_i(z)h_i^*(z)},$$

for $1 \leq i \leq 4$. Thus, $C^\perp \subseteq C$ by Theorem 6. In view of Theorem 7, we conclude that there exists a quantum code $[[36, 22, 3]]_3$. This quantum code is a new quantum code (see [20] for details).

Example 2. Let $R = \mathbb{F}_{19}[u_1, u_2] / \langle u_1^2 - 4, u_2^2 - 1, u_1u_2 - u_2u_1 \rangle$ be a finite commutative ring, $n = 19$, and $\alpha = 2, \beta = 1$. Then,

$$z^{19} - 1 = (z + 18)^{19} \in \mathbb{F}_{19}[x].$$

Take

$$h_1(z) = (z + 18)$$

$$h_2(z) = (z + 18)$$

$$h_3(z) = (z + 18)^2$$

$$h_4(z) = (z + 18)^{14}$$

and

$$A = \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \end{bmatrix}.$$

Here, matrix A satisfies the condition $AA^T = 5I_{4 \times 4}$, where $A \in GL_4(\mathbb{F}_{19})$ and $I_{4 \times 4}$ is an identity matrix. The cyclic code $C = \langle \sum_{i=0}^4 \Delta_i h_i(z) \rangle$ is of length 19 over R and its Gray image is of length 76, dimension 58, and distance 4 over \mathbb{F}_{19} , i.e., $[76, 58, 4]_{19}$. Moreover

$$z^{19} - 1 \equiv 0 \pmod{h_i(z)h_i^*(z)},$$

for $1 \leq i \leq 4$. Application of Theorem 6 yields $C^\perp \subseteq C$. By Theorem 7, we conclude that there exists a quantum code $[[76, 40, 4]_{19}$.

Example 3. Let $R = \mathbb{F}_5[u_1, u_2] / \langle u_1^2 - 1, u_2^2 - 1, u_1 u_2 - u_2 u_1 \rangle$ be a finite commutative ring, $n = 30$, and $\alpha = \beta = 1$. Then,

$$z^{30} - 1 = (z + 1)^5(z + 4)^5(z^2 + z + 1)^5(z^2 + 4z + 1)^5 \in \mathbb{F}_5[x].$$

Take

$$h_1(z) = h_2(z) = h_3(z) = h_4(z) = (z + 1)^2(z^2 + z + 1)$$

and

$$A = \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \end{bmatrix}.$$

Here, matrix A satisfies the condition $AA^T = 4I_{4 \times 4}$, where $A \in GL_4(\mathbb{F}_5)$ and $I_{4 \times 4}$ is an identity matrix. The cyclic code $C = \langle \sum_{i=0}^4 \Delta_i h_i(z) \rangle$ is of length 30 over R and its Gray image is of length 120, dimension 104, and distance 3 over \mathbb{F}_5 , i.e., $[120, 104, 3]_5$. Moreover

$$z^{30} - 1 \equiv 0 \pmod{h_i(z)h_i^*(z)},$$

for $1 \leq i \leq 4$. This implies that $C^\perp \subseteq C$ by Theorem 6. In view of Theorem 7, we conclude that there exists a quantum code $[[120, 88, 3]_5$, which has the same minimum distance but a larger code rate than the best previously known quantum code $[[120, 32, 3]_5$ (see [12] for details). Therefore, our quantum code $[[120, 88, 3]_5$ is better than the best previously known quantum code $[[120, 32, 3]_5$ reported in [12].

Example 4. Let $R = \mathbb{F}_5[u_1, u_2] / \langle u_1^2 - 1, u_2^2 - 4, u_1 u_2 - u_2 u_1 \rangle$ be a finite commutative ring, $n = 6$, $\alpha = 1$, and $\beta = 2$. Then,

$$z^6 - 1 = (z + 1)(z + 4)(z^2 + z + 1)(z^2 + 4z + 1) \in \mathbb{F}_5[x].$$

Take

$$h_1(z) = 1$$

$$h_2(z) = (z + 1)$$

$$h_3(z) = (z + 1)$$

$$h_4(z) = (z + 1)(z^2 + z + 1)(z^2 + 4z + 1)$$

and

$$A = \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \end{bmatrix}.$$

Matrix A satisfies the condition $AA^T = 4I_{4 \times 4}$, where $A \in GL_4(\mathbb{F}_5)$ and $I_{4 \times 4}$ is an identity matrix. Here, $h_1(z)$, $h_2(z)$, $h_3(z)$, and $h_4(z)$ are self-reciprocal polynomials. By Corollary 3,

$\mathcal{C} = \langle \sum_{i=0}^4 \Delta_i h_i(z) \rangle$ is an LCD code of length 6 over the ring R . Hence, by Theorem 10, its Gray image $\eta(\mathcal{C})$ is also an LCD code with the parameters $[24, 17, 4]_5$ over \mathbb{F}_5 .

In Table 1 we present QEC codes obtained from cyclic codes $\mathcal{C} = \langle \sum_{i=0}^4 \Delta_i h_i(z) \rangle$ of length n over R , where $\mathcal{C}_i = \langle h_i(z) \rangle$, such that $z^n - 1 \equiv 0 \pmod{h_i(z)h_i^*(z)}$ for $i = 1, 2, 3, 4$. It is noted that our QEC codes $[[n, k, d]]_q$ are better than the existing quantum codes $[[n', k', d']]_q$ collected from the different references mentioned in this article. In Table 2, we obtain new quantum codes, and in Table 3 we construct LCD codes $\mathcal{C} = \langle \sum_{i=0}^4 \Delta_i h_i(z) \rangle$ of length n over R , where $\gcd(n, p) = 1$, $\mathcal{C}_i = \langle h_i(z) \rangle$, and $h_i(z)$ is the self-reciprocal divisor of $z^n - 1$ in $\mathbb{F}_q[z]$ for $i = 1, 2, 3, 4$.

6. Conclusions

In this article, we discuss some of the structural properties of cyclic codes over the ring $R = \mathbb{F}_q[u_1, u_2] / \langle u_1^2 - \alpha^2, u_2^2 - \beta^2, u_1 u_2 - u_2 u_1 \rangle$, where α and β are non-zero elements of \mathbb{F}_q . Furthermore, we obtain better quantum codes than presented in [8–13]. As an application, we obtain LCD codes over the ring R . This study can be generalized to a product of finite rings. We hope that this study will encourage readers to investigate these codes over other finite rings to explore new and better quantum codes in the future.

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References

1. Calderbank, A.R.; Rains, E.M.; Shor, P.M.; Sloane, N.J.A. Quantum error-correction via codes over GF(4). *IEEE Trans. Inf. Theory* **1998**, *44*, 1369–1387. [[CrossRef](#)]
2. Grassl, M.; Beth, T. On optimal quantum codes. *Int. J. Quantum Inf.* **2004**, *2*, 55–64. [[CrossRef](#)]
3. Qian, J.; Ma, W.; Gou, W. Quantum codes from cyclic codes over finite ring. *Int. J. Quantum Inf.* **2009**, *7*, 1277–1283. [[CrossRef](#)]
4. Kai, X.; Zhu, S. Quaternary construction of quantum codes from cyclic codes over $F_4 + uF_4$. *Int. J. Quantum Inf.* **2011**, *9*, 689–700. [[CrossRef](#)]
5. Li, R.; Xu, Z.; Li, X. Binary construction of quantum codes of minimum distance three and four. *IEEE Trans. Inf. Theory* **2004**, *50*, 1331–1335. [[CrossRef](#)]
6. Gao, J. Quantum codes from cyclic codes over $F_q + vF_q + v^2F_q + v^3F_q$. *Int. J. Quantum Inf.* **2015**, *13*, 1550063. [[CrossRef](#)]
7. Özen, M.; Özzaim, N.T.; Ince, H. Quantum codes from cyclic codes over $F_3 + uF_3 + vF_3 + uvF_3$. *Int. Conf. Quantum Sci. Appl. J. Phys. Conf. Ser.* **2016**, *766*, 012020-1–012020-6.
8. Ashraf, M.; Khan, N.; Mohammad, G. New Quantum and LCD Codes Over the Finite Field of Odd Characteristic. *Int. J. Theor. Phys.* **2021**, *60*, 2322–2332. [[CrossRef](#)]
9. Ashraf, M.; Mohammad, G. Quantum codes from cyclic codes over $F_q + uF_q + vF_q + uvF_q$. *Quantum Inf. Process.* **2016**, *15*, 4089–4098. [[CrossRef](#)]
10. Ashraf, M.; Mohammad, G. Quantum codes over F_0p from cyclic codes over $F_p[u, v] / \langle u^2 - 1, v^3 - v, uv - vu \rangle$. *Cryptogr. Commun.* **2019**, *11*, 325–335. [[CrossRef](#)]
11. Bag, T.; Dinh, H.Q.; Upadhyay, A.K.; Yamaka, W. New non binary quantum codes from cyclic codes over product ring. *IEEE Commun. Lett.* **2019**, *24*, 486–490. [[CrossRef](#)]

12. Islam, H.; Prakash, O. Quantum codes from the cyclic codes over $F_p[u, v, w]/\langle u^2 - 1, v^2 - 1, w^2 - 1, uv - vu, vw - wv, wu - uw \rangle$. *J. Appl. Math. Comput.* **2019**, *60*, 625–635. [[CrossRef](#)]
13. Dinh, H.Q.; Bag, T.; Upadhyay, A.K.; Ashraf, M.; Mohammad, G.; Chinnakum, W. Quantum codes from a class of constacyclic codes over finite commutative rings. *J. Algebra Appl.* **2020**, *19*, 2150003. [[CrossRef](#)]
14. Bag, T.; Upadhyay, A.K. Study on negacyclic codes over the ring $Z_p[u]/\langle u^{k+1} - u \rangle$. *J. Appl. Math. Comput.* **2019**, *59*, 693–700. [[CrossRef](#)]
15. Islam, H.; Prakash, O. New quantum and LCD codes over the finite field of even characteristic. *Def. Sci. J.* **2020**, *71*, 656–661. [[CrossRef](#)]
16. Massey, J.L. Linear codes with complementary duals. *Discret. Math.* **1992**, *106*, 337–342. [[CrossRef](#)]
17. Yang, X.; Massey, J.L. The condition for a cyclic code to have a complementary dual. *Discret. Math.* **1994**, *126*, 391–393. [[CrossRef](#)]
18. Islam, H.; Prakash, O. Construction of LCD and new quantum codes from cyclic codes over a finite non chain ring. *Cryptogr. Commun.* **2022**, *14*, 59–73. [[CrossRef](#)]
19. Bosma, W.; Cannon, J. *Handbook of Magma Functions*; University of Sydney: Sydney, Australia, 1995.
20. Aydin, N.; Liu, P.; Yoshino, B. A Database of Quantum Codes. 2021. Available online: <http://quantumcodes.info/> (accessed on 7 August 2021).

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