

Article

Generalizations of Hardy-Type Inequalities by Montgomery Identity and New Green Functions

Kristina Krulić Himmelreich ^{1,†}, Josip Pečarić ^{2,†}, Dora Pokaz ^{3,*,†} and Marjan Praljak ^{4,†}

- ¹ Faculty of Textile Technology, University of Zagreb, Prilaz Baruna Filipovića 28a, 10000 Zagreb, Croatia
- ² Croatian Academy of Sciences and Arts, Trg Nikole Šubića Zrinskog 11, 10000 Zagreb, Croatia
- ³ Faculty of Civil Engineering, University of Zagreb, Fra Andrije Kačića Miošića 26, 10000 Zagreb, Croatia
- ⁴ Faculty of Food Technology and Biotechnology, University of Zagreb, Pierrottijeva 6, 10000 Zagreb, Croatia
- * Correspondence: dora.pokaz@grad.unizg.hr
- † All authors contributed equally to this work.

Abstract: In this paper we extend general Hardy’s inequality by appropriately combining Montgomery’s identity and Green functions. Related Grüss and Ostrowski-type inequalities are also derived.

Keywords: n-convexity; Montgomery identity; Hardy inequality; Grüss-type inequality; Ostrowsky-type inequality

MSC: 26D10, 26D15

1. Introduction

The area of mathematical inequalities is very large. There are many mathematicians that study, improve, and generalize many inequalities such as the Hardy, the Hardy–Hilbert, the Steffensen, the Opial, the Boas, etc. Here, we focus on the famous Hardy’s inequality; see [1]. It states that

$$\int_0^\infty \left(\frac{1}{x} \int_0^x f(t) dt \right)^p dx \leq \left(\frac{p}{p-1} \right)^p \int_0^\infty f^p(x) dx, \quad p > 1, \quad (1)$$

holds for all non-negative functions $f \in L^p(0, \infty)$, with the constant $\left(\frac{p}{p-1}\right)^p$ being sharp. Inequality (1) has important applications in operator theory since it can be reinterpreted as

$$\|Hf\|_p \leq \frac{p}{p-1} \|f\|_p$$

where $Hf(x) = \frac{1}{x} \int_0^x f(t) dt$ is the Hardy operator and $\|\cdot\|_p$ is the standard L^p norm. For these reasons, Hardy’s inequality, both in discrete and continuous case, has attracted a lot of interest of researchers, and there are many papers and monographs dedicated to its development; here, we mention, e.g., [2–13].

We continue with Pólya–Knopp’s inequality,

$$\int_0^\infty \exp \left(\frac{1}{x} \int_0^x \ln f(t) dt \right) dx < e \int_0^\infty f(x) dx, \quad (2)$$

which holds for positive functions $f \in L^1(\mathbb{R}_+)$. Pólya–Knopp’s inequality is a limiting case of Hardy’s inequality, since it can be obtained from (1) by rewriting it with the function f replaced with $f^{\frac{1}{p}}$ and then by letting $p \rightarrow \infty$.



Citation: Himmelreich, K.K.; Pečarić, J.; Pokaz, D.; Praljak, M. Generalizations of Hardy-Type Inequalities by Montgomery Identity and New Green Functions. *Axioms* **2023**, *12*, 434. <https://doi.org/10.3390/axioms12050434>

Academic Editor: Behzad Djafari-Rouhani

Received: 28 February 2023
 Revised: 29 March 2023
 Accepted: 24 April 2023
 Published: 27 April 2023



Copyright: © 2023 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

The starting point of our paper will be Hardy’s inequality in the general setting given in [14] (see also [15]), and we will first introduce some notation. Let $(\Sigma_i, \Omega_i, \mu_i), i = 1, 2$, be measure spaces with positive σ -finite measures, $k : \Omega_1 \times \Omega_2 \rightarrow [0, \infty)$ a measurable and non-negative kernel and

$$0 < K(x) = \int_{\Omega_2} k(x, t) d\mu_2(t), \quad x \in \Omega_1. \tag{3}$$

For a measurable $f : \Omega_2 \rightarrow \mathbb{R}$, let A_k denote the integral operator

$$A_k f(x) = \frac{1}{K(x)} \int_{\Omega_2} k(x, t) f(t) d\mu_2(t), \tag{4}$$

Theorem 1 ([14]). *Let the weight $u : \Omega_1 \rightarrow [0, \infty)$ and kernel $k : \Omega_1 \times \Omega_2 \rightarrow [0, \infty)$ be such that $\frac{k(x,y)}{K(x)} u(x)$ is locally integrable on Ω_1 for each $y \in \Omega_2$ and let v be given by*

$$v(y) = \int_{\Omega_1} \frac{k(x, y)}{K(x)} u(x) d\mu_1(x) < \infty. \tag{5}$$

If ϕ is a convex function on an interval $I \subseteq \mathbb{R}$, then the inequality

$$\int_{\Omega_1} \phi(A_k f(x)) u(x) d\mu_1(x) \leq \int_{\Omega_2} \phi(f(y)) v(y) d\mu_2(y) \tag{6}$$

holds for all measurable functions $f : \Omega_2 \rightarrow I$, where A_k is given by (4).

Inequality (6) is, indeed, a generalization of Hardy’s inequality. After taking $f(t) = g(t^{\frac{p-1}{p}}) t^{\frac{-1}{p}}$ and some straightforward calculation, we obtain that (1) is equivalent to

$$\int_0^\infty \left(\frac{1}{x} \int_0^x g(t) dt \right)^p \frac{dx}{x} \leq \int_0^\infty g^p(x) \frac{dx}{x}. \tag{7}$$

Applying (6) with $\Omega_1 = \Omega_2 = (0, \infty)$, $k(x, y) = 1$ for $0 \leq y \leq x$ and $k(x, y) = 0$ otherwise, $d\mu_1(x) = d\mu_2(x) = dx$, $u(x) = \frac{1}{x}$ (which yields $v(y) = \frac{1}{y}$) and $\phi(u) = u^p$ gives (7).

We will state our results for the class of n -convex functions, which is a more general class of functions that contains convex functions as a special case. We will now recall some basic definitions and properties of n -convex functions.

Definition 1. *The n -th order divided difference, $n \in \mathbb{N}_0$, of a function $\phi : [\alpha, \beta] \rightarrow \mathbb{R}$ at mutually distinct points $x_0, x_1, \dots, x_n \in [\alpha, \beta]$ is defined recursively by*

$$[x_i; \phi] = \phi(x_i), \quad i = 0, \dots, n$$

$$[x_0, \dots, x_n; \phi] = \frac{[x_1, \dots, x_n; \phi] - [x_0, \dots, x_{n-1}; \phi]}{x_n - x_0}.$$

The value $[x_0, \dots, x_n; \phi]$ is independent of the order of the points x_0, \dots, x_n . A function $f : [\alpha, \beta] \rightarrow \mathbb{R}$ is n -convex if all its n -th order divided differences are non-negative, i.e., $[x_0, \dots, x_n; f] \geq 0$ for all choices $x_i \in [\alpha, \beta]$. Thus, 0-convex functions are non-negative and 1-convex functions are non-decreasing, while 2-convex functions are convex in the classical sense. An n times differentiable is n -convex if and only if its n -derivative is non-negative (see [10]).

In our presentation, we will also need the following generalization of the Montgomery identity given in [16].

Theorem 2. Let $n \in \mathbb{N}$, $\phi : I \rightarrow \mathbb{R}$ be such that $\phi^{(n-1)}$ is absolutely continuous, where I is an open interval in \mathbb{R} and $\alpha, \beta \in I$, $\alpha < \beta$. Then

$$\begin{aligned} \phi(t) = & \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} \phi(s) ds + \sum_{k=0}^{n-2} \frac{\phi^{(k+1)}(\alpha)}{k!(k+2)} \frac{(t - \alpha)^{k+2}}{\beta - \alpha} - \\ & \sum_{k=0}^{n-2} \frac{\phi^{(k+1)}(\beta)}{k!(k+2)} \frac{(t - \beta)^{k+2}}{\beta - \alpha} + \frac{1}{(n-1)!} \int_{\alpha}^{\beta} T_n(t, s) \phi^{(n)}(s) ds, \end{aligned} \tag{8}$$

where

$$T_n(t, s) = \begin{cases} -\frac{(t-s)^n}{n(\beta-\alpha)} + \frac{t-\alpha}{\beta-\alpha} (t-s)^{n-1}, & \alpha \leq s \leq t, \\ -\frac{(t-s)^n}{n(\beta-\alpha)} + \frac{t-\beta}{\beta-\alpha} (t-s)^{n-1}, & t < s \leq \beta. \end{cases} \tag{9}$$

For $n = 1$, the sums $\sum_{k=0}^{n-2}$ are empty and identity (8) reduces to the well-known Montgomery identity

$$\phi(t) = \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} \phi(s) ds + \int_{\alpha}^{\beta} P(t, s) \phi'(s) ds,$$

where $P(t, s)$ is the Peano kernel, which is defined by

$$P(t, s) = \begin{cases} \frac{s-\alpha}{\beta-\alpha}, & \alpha \leq s \leq t, \\ \frac{s-\beta}{\beta-\alpha}, & t < s \leq \beta. \end{cases}$$

Hardy-type inequalities obtained by similar methods as in this paper were given in [15,17–20]. We also draw attention to two papers [21,22] about Sherman’s inequality and Montgomery identity.

2. Main Results

Throughout the paper, G_{ω} , $\omega = 1, 2, 3, 4$, will denote the following Green functions defined on $[\alpha, \beta] \times [\alpha, \beta]$ with

$$G_1(t, s) = \begin{cases} \alpha - s, & \alpha \leq s \leq t; \\ \alpha - t, & t \leq s \leq \beta. \end{cases} \tag{10}$$

$$G_2(t, s) = \begin{cases} t - \beta, & \alpha \leq s \leq t; \\ s - \beta, & t \leq s \leq \beta. \end{cases} \tag{11}$$

$$G_3(t, s) = \begin{cases} t - \alpha, & \alpha \leq s \leq t; \\ s - \alpha, & t \leq s \leq \beta. \end{cases} \tag{12}$$

$$G_4(t, s) = \begin{cases} \beta - s, & \alpha \leq s \leq t; \\ \beta - t, & t \leq s \leq \beta. \end{cases} \tag{13}$$

Note that all these functions are continuous and convex with respect to the first variable.

Lemma 1. For $\phi \in C^2([\alpha, \beta])$, the following identities hold

$$\phi(t) = \phi(\alpha) + (t - \alpha)\phi'(\beta) + \int_{\alpha}^{\beta} G_1(t, s)\phi''(s) ds, \tag{14}$$

$$\phi(t) = \phi(\beta) + (t - \beta)\phi'(\alpha) + \int_{\alpha}^{\beta} G_2(t, s)\phi''(s)ds, \tag{15}$$

$$\phi(t) = \phi(\beta) + (t - \alpha)\phi'(\alpha) - (\beta - \alpha)\phi'(\beta) + \int_{\alpha}^{\beta} G_3(t, s)\phi''(s)ds, \tag{16}$$

$$\phi(t) = \phi(\alpha) - (\beta - t)\phi'(\beta) + (\beta - \alpha)\phi'(\alpha) + \int_{\alpha}^{\beta} G_4(t, s)\phi''(s)ds, \tag{17}$$

where the functions G_{ω} , $\omega = 1, \dots, 4$, are defined by (10)–(13).

The next theorem gives our first main result.

Theorem 3. Let A_k be as in (4), $\omega \in \{1, 2, 3, 4\}$, G_{ω} as in (10)–(13) and u a weight function with v given by (5). Then, the following statements are equivalent:

(i) For every continuous convex function $\phi : [\alpha, \beta] \rightarrow \mathbb{R}$, we have

$$\int_{\Omega_1} \phi(A_k f(x))u(x)d\mu_1(x) \leq \int_{\Omega_2} \phi(f(y))v(y)d\mu_2(y) \tag{18}$$

(ii) For each $s \in [\alpha, \beta]$, we have

$$\int_{\Omega_1} G_{\omega}(A_k f(x), s)u(x)d\mu_1(x) \leq \int_{\Omega_2} G_{\omega}(f(y), s)v(y)d\mu_2(y) \tag{19}$$

Proof. (i) \Rightarrow (ii): The functions $G_{\omega}(\cdot, s)$, $s \in [\alpha, \beta]$, $\omega = 1, 2, 3, 4$, are continuous and convex, and applying these functions to (18), we obtain (19).

(ii) \Rightarrow (i): Let $\phi \in C^2([\alpha, \beta])$. Identities (14)–(17) and some simple calculations yield

$$\begin{aligned} & \int_{\Omega_2} \phi(f(y))v(y)d\mu_2(y) - \int_{\Omega_1} \phi(A_k f(x))u(x)d\mu_1(x) \\ &= \int_{\alpha}^{\beta} \left[\int_{\Omega_2} G_{\omega}(f(y), s)v(y)d\mu_2(y) - \int_{\Omega_1} G_{\omega}(A_k f(x), s)u(x)d\mu_1(x) \right] \phi''(s)ds \end{aligned} \tag{20}$$

for $\omega = 1, 2, 3, 4$. If, additionally, ϕ is convex, then $\phi''(s) \geq 0$ for $s \in [\alpha, \beta]$. Furthermore, by assumption (19), the first factor under the integral on the right-hand side is also non-negative. Therefore, the right-hand side of (20) is non-negative, so the left-hand side is as well, i.e., (18) holds. Since each continuous convex function on a segment can be attained as a uniform limit of C^2 convex functions, the claim of the theorem follows. \square

In the next theorem, we will use the Montgomery identity to obtain general identities. In turn, these identities will be used in derivation of Hardy-type inequalities.

Theorem 4. Let $n \in \mathbb{N}$, $n \geq 4$ and $\phi : I \rightarrow \mathbb{R}$ be such that $\phi^{(n-1)}$ is absolutely continuous, where I is an open interval in \mathbb{R} and $\alpha, \beta \in I$, $\alpha < \beta$. Furthermore, let A_k be as in (4), $\omega \in \{1, 2, 3, 4\}$, G_{ω} as in (10)–(13), T_n as in (9) and u a weight function with v given by (5). Then

(i)

$$\begin{aligned}
 & \int_{\Omega_2} \phi(f(y))v(y)d\mu_2(y) - \int_{\Omega_1} \phi(A_k f(x))u(x)d\mu_1(x) \\
 &= \int_{\alpha}^{\beta} \left[\int_{\Omega_2} G_{\omega}(f(y), t)v(y)d\mu_2(y) - \int_{\Omega_1} G_{\omega}(A_k f(x), t)u(x)d\mu_1(x) \right] \\
 & \times \sum_{k=1}^{n-1} \frac{k}{(k-1)!} \left(\frac{\phi^{(k)}(\alpha)(t-\alpha)^{k-1} - \phi^{(k)}(\beta)(t-\beta)^{k-1}}{\beta-\alpha} \right) dt \\
 &+ \frac{1}{(n-3)!} \int_{\alpha}^{\beta} \int_{\alpha}^{\beta} \left[\int_{\Omega_2} G_{\omega}(f(y), t)v(y)d\mu_2(y) - \int_{\Omega_1} G_{\omega}(A_k f(x), t)u(x)d\mu_1(x) \right] \\
 & \cdot \tilde{T}_{n-2}(t, s)\phi^{(n)}(s)dsdt,
 \end{aligned} \tag{21}$$

where

$$\tilde{T}_{n-2}(t, s) = \begin{cases} \frac{1}{\beta-\alpha} \left[\frac{(t-s)^{n-2}}{n-2} + (t-\alpha)(t-s)^{n-3} \right], & \alpha \leq s \leq t, \\ \frac{1}{\beta-\alpha} \left[\frac{(t-s)^{n-2}}{n-2} + (t-\beta)(t-s)^{n-3} \right], & t < s \leq \beta. \end{cases} \tag{22}$$

(ii)

$$\begin{aligned}
 & \int_{\Omega_2} \phi(f(y))v(y)d\mu_2(y) - \int_{\Omega_1} \phi(A_k f(x))u(x)d\mu_1(x) \\
 &= \frac{1}{2} \frac{\phi'(\beta) - \phi'(\alpha)}{\beta - \alpha} \left[\int_{\Omega_2} v(y)f^2(y)d\mu_2(y) - \int_{\Omega_1} u(x)A_k^2 f(x)d\mu_1(x) \right] \\
 &+ \int_{\alpha}^{\beta} \left[\int_{\Omega_2} G_{\omega}(f(y), t)v(y)d\mu_2(y) - \int_{\Omega_1} G_{\omega}(A_k f(x), t)u(x)d\mu_1(x) \right] \times \\
 & \sum_{k=3}^{n-1} \frac{k-2}{(k-1)!} \left(\frac{\phi^{(k)}(\alpha)(t-\alpha)^{k-1} - \phi^{(k)}(\beta)(t-\beta)^{k-1}}{\beta-\alpha} \right) dt \\
 &+ \frac{1}{(n-3)!} \int_{\alpha}^{\beta} \int_{\alpha}^{\beta} \left[\int_{\Omega_2} G_{\omega}(f(y), t)v(y)d\mu_2(y) - \int_{\Omega_1} G_{\omega}(A_k f(x), t)u(x)d\mu_1(x) \right] \\
 & \cdot T_{n-2}(t, s)\phi^{(n)}(s)dsdt.
 \end{aligned} \tag{23}$$

Proof.

(i) If we differentiate twice the Montgomery identity (8), we obtain

$$\begin{aligned}
 \phi''(t) &= \sum_{k=1}^{n-1} \frac{k}{(k-1)!} \left(\frac{\phi^{(k)}(\alpha)(t-\alpha)^{k-1} - \phi^{(k)}(\beta)(t-\beta)^{k-1}}{\beta-\alpha} \right) \\
 &+ \frac{1}{(n-3)!} \int_{\alpha}^{\beta} \tilde{T}_{n-2}(t, s)\phi^{(n)}(s)ds.
 \end{aligned} \tag{24}$$

Inserting (24) in (20), we derive the first identity (21).

(ii) Substituting ϕ with ϕ'' and n with $n - 2$ in (8) gives

$$\begin{aligned} \phi''(t) &= \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} \phi''(t) dt + \sum_{k=0}^{n-4} \frac{\phi^{(k+3)}(\alpha)}{k!(k+2)} \frac{(t - \alpha)^{k+2}}{\beta - \alpha} - \sum_{k=0}^{n-4} \frac{\phi^{(k+3)}(\beta)}{k!(k+2)} \frac{(t - \beta)^{k+2}}{\beta - \alpha} \\ &+ \frac{1}{(n - 3)!} \int_{\alpha}^{\beta} T_{n-2}(t, s) \phi^{(n)}(s) ds, \end{aligned}$$

so

$$\begin{aligned} \phi''(t) &= \frac{\phi'(\beta) - \phi'(\alpha)}{\beta - \alpha} + \sum_{k=3}^{n-1} \frac{k - 2}{(k - 1)!} \left(\frac{\phi^{(k)}(\alpha)(t - \alpha)^{k-1} - \phi^{(k)}(\beta)(t - \beta)^{k-1}}{\beta - \alpha} \right) \\ &+ \frac{1}{(n - 3)!} \int_{\alpha}^{\beta} T_{n-2}(t, s) \phi^{(n)}(s) ds. \end{aligned} \tag{25}$$

Furthermore, Lemma 1 applied for $\phi(t) = \frac{1}{2}t^2$ shows that

$$\int_{\alpha}^{\beta} G_{\omega}(z, t) dt = \frac{1}{2}z^2 + P(z, \alpha, \beta)$$

for $\omega = 1, 2, 3, 4$, where P is a linear polynomial in the variable z . This fact, after some straightforward calculation, yields

$$\begin{aligned} &\int_{\alpha}^{\beta} \left[\int_{\Omega_2} G_{\omega}(f(y), t) v(y) d\mu_2(y) - \int_{\Omega_1} G_{\omega}(A_k f(x), t) u(x) d\mu_1(x) \right] dt = \\ &\int_{\Omega_2} v(y) \frac{f^2(y)}{2} d\mu_2(y) - \int_{\Omega_1} u(x) \frac{A_k^2 f(x)}{2} d\mu_1(x) \end{aligned} \tag{26}$$

for $\omega = 1, 2, 3, 4$. Finally, inserting (25) in (20) and taking into account (26) gives (23). \square

In the following theorem, new Hardy-type inequalities are derived from the above identities.

Theorem 5. Suppose that all the assumptions of Theorem 4 hold with the additional assumption that n is even. If $\phi : I \rightarrow \mathbb{R}$ is n -convex, then:

(i)

$$\begin{aligned} &\int_{\Omega_2} \phi(f(y)) v(y) d\mu_2(y) - \int_{\Omega_1} \phi(A_k f(x)) u(x) d\mu_1(x) \\ &\geq \int_{\alpha}^{\beta} \left[\int_{\Omega_2} G_{\omega}(f(y), t) v(y) d\mu_2(y) - \int_{\Omega_1} G_{\omega}(A_k f(x), t) u(x) d\mu_1(x) \right] \\ &\times \sum_{k=1}^{n-1} \frac{k}{(k - 1)!} \left(\frac{\phi^{(k)}(\alpha)(t - \alpha)^{k-1} - \phi^{(k)}(\beta)(t - \beta)^{k-1}}{\beta - \alpha} \right) dt, \end{aligned} \tag{27}$$

(ii)

$$\begin{aligned}
 & \int_{\Omega_2} \phi(f(y))v(y)d\mu_2(y) - \int_{\Omega_1} \phi(A_k f(x))u(x)d\mu_1(x) \\
 & \geq \frac{1}{2} \frac{\phi'(\beta) - \phi'(\alpha)}{\beta - \alpha} \left[\int_{\Omega_2} v(y)f^2(y)d\mu_2(y) - \int_{\Omega_1} u(x)A_k^2 f(x)d\mu_1(x) \right] \\
 & + \int_{\alpha}^{\beta} \left[\int_{\Omega_2} G_{\omega}(f(y), t)v(y)d\mu_2(y) - \int_{\Omega_1} G_{\omega}(A_k f(x), t)u(x)d\mu_1(x) \right] \times \\
 & \sum_{k=3}^{n-1} \frac{k-2}{(k-1)!} \left(\frac{\phi^{(k)}(\alpha)(t-\alpha)^{k-1} - \phi^{(k)}(\beta)(t-\beta)^{k-1}}{\beta - \alpha} \right) dt. \tag{28}
 \end{aligned}$$

Proof. Since for each $\omega \in \{1, 2, 3, 4\}$ and $t \in [\alpha, \beta]$ the function $G_{\omega}(\cdot, s)$ is continuous and convex, Theorem 1 yields

$$\int_{\Omega_2} G_{\omega}(f(y), t)v(y)d\mu_2(y) - \int_{\Omega_1} G_{\omega}(A_k f(x), t)u(x)d\mu_1(x) \geq 0, \text{ for all } t \in [\alpha, \beta]. \tag{29}$$

Furthermore, for even n , the function \tilde{T}_{n-2} given by (22) is obviously non-negative, while the function T_{n-2} is also non-negative since

$$T_{n-2}(t, s) = \begin{cases} \frac{(t-s)^{n-3}}{(\beta-\alpha)} \left[t - \alpha - \frac{t-s}{n} \right], & \alpha \leq s \leq t, \\ \frac{(s-t)^{n-3}}{(\beta-\alpha)} \left[\beta - t - \frac{s-t}{n} \right], & t < s \leq \beta. \end{cases}$$

Furthermore, $\phi^{(n)} \geq 0$ since ϕ is n -convex. Finally, from identities (21) and (23), due to the positivity of $\phi^{(n)}$, \tilde{T}_{n-2} and T_{n-2} and inequality (29), we obtain inequalities (27) and (28), respectively. \square

With additional convexity assumptions, the right-hand sides of inequalities (27) and (28) can be further simplified.

Theorem 6. Suppose that all the assumptions of Theorem 5 hold and denote the functions

$$L_1(z) = \int_{\alpha}^{\beta} G_{\omega}(z, t) \times \sum_{k=1}^{n-1} \frac{k}{(k-1)!} \left(\frac{\phi^{(k)}(\alpha)(t-\alpha)^{k-1} - \phi^{(k)}(\beta)(t-\beta)^{k-1}}{\beta - \alpha} \right) dt \tag{30}$$

and

$$\begin{aligned}
 L_2(z) = & \frac{\phi'(\beta) - \phi'(\alpha)}{\beta - \alpha} \frac{z^2}{2} + \int_{\alpha}^{\beta} G_{\omega}(z, t) \times \\
 & \sum_{k=3}^{n-1} \frac{k-2}{(k-1)!} \left(\frac{\phi^{(k)}(\alpha)(t-\alpha)^{k-1} - \phi^{(k)}(\beta)(t-\beta)^{k-1}}{\beta - \alpha} \right) dt. \tag{31}
 \end{aligned}$$

If ϕ is n -convex and L_1 or L_2 are convex, then

$$\int_{\Omega_1} \phi(A_k f(x))u(x)d\mu_1(x) \leq \int_{\Omega_2} \phi(f(y))v(y)d\mu_2(y).$$

Proof. The right-hand side of inequality (27) can be rewritten as

$$\int_{\Omega_2} v(y)L_1(f(y))d\mu_2(y) - \int_{\Omega_1} u(x)L_1(A_k f(x))d\mu_1(x).$$

Since L_1 is convex, by Theorem 1, the last expression is non-negative. Therefore, the stated inequality follows from (27).

The claim for the function L_2 follows from inequality (28) in an analogous way. \square

Remark 1. Differentiating twice the identities from Lemma 1, we can conclude

$$g''(z) = \frac{\partial^2}{\partial z^2} \int_{\alpha}^{\beta} G_{\omega}(z, t)g''(t)dt. \tag{32}$$

Applying (32) for a function g such that

$$g''(t) = \sum_{k=1}^{n-1} \frac{k}{(k-1)!} \left(\frac{\phi^{(k)}(\alpha)(t-\alpha)^{k-1} - \phi^{(k)}(\beta)(t-\beta)^{k-1}}{\beta-\alpha} \right) \tag{33}$$

we can conclude

$$L_1''(z) = \frac{\partial^2}{\partial z^2} \int_{\alpha}^{\beta} G_{\omega}(z, t)g''(t)dt = g''(z).$$

Therefore, the convexity of the function L_1 is equivalent to the non-negativity of the expression (33) for each $t \in [\alpha, \beta]$.

Similarly, taking also into account (26), one can show that the convexity of the function L_2 is equivalent to

$$\frac{\phi'(\beta) - \phi'(\alpha)}{\beta - \alpha} + \sum_{k=3}^{n-1} \frac{k-2}{(k-1)!} \left(\frac{\phi^{(k)}(\alpha)(t-\alpha)^{k-1} - \phi^{(k)}(\beta)(t-\beta)^{k-1}}{\beta - \alpha} \right) \geq 0$$

for each $t \in [\alpha, \beta]$.

3. Related Grüss and Ostrowski-Type Inequalities

Consider the Čebyšev functional

$$T(h, g) := \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} h(t)g(t)dt - \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} h(t)dt \cdot \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} g(t)dt$$

for Lebesgue integrable functions $h, g : [\alpha, \beta] \rightarrow \mathbb{R}$. The next two theorems from [23] provide Grüss and Ostrowski-type inequalities involving the above functional.

Theorem 7. Let $h, g : [\alpha, \beta] \rightarrow \mathbb{R}$ be two absolutely continuous functions with $(\cdot - \alpha)(\beta - \cdot)(h')^2, (\cdot - \alpha)(\beta - \cdot)(g')^2 \in L([\alpha, \beta])$. Then

$$|T(h, g)| \leq \frac{1}{\sqrt{2}} |T(h, h)|^{\frac{1}{2}} \frac{1}{\sqrt{\beta - \alpha}} \left(\int_{\alpha}^{\beta} (s - \alpha)(\beta - s)[g'(s)]^2 ds \right)^{\frac{1}{2}}. \tag{34}$$

The constant $\frac{1}{\sqrt{2}}$ is the best possible in (34).

Theorem 8. Assume that $g : [\alpha, \beta] \rightarrow \mathbb{R}$ is monotonic non-decreasing and $h : [\alpha, \beta] \rightarrow \mathbb{R}$ is absolutely continuous with $h' \in L^{\infty}([\alpha, \beta])$. Then

$$|T(h, g)| \leq \frac{1}{2(\beta - \alpha)} \|h'\|_{\infty} \int_{\alpha}^{\beta} (s - \alpha)(\beta - s)dg(s). \tag{35}$$

The constant $\frac{1}{2}$ is the best possible in (35).

To simplify notation, for $\omega \in \{1, 2, 3, 4\}$, we introduce functions $P_{1,\omega}, P_{2,\omega} : [\alpha, \beta] \rightarrow \mathbb{R}$. We assume that all the terms appearing in $P_{1,\omega}$ and $P_{2,\omega}$ satisfy the assumptions of Theorem 4.

$$P_{1,\omega}(s) = \int_{\alpha}^{\beta} \left[\int_{\Omega_2} G_{\omega}(f(y), t)v(y)d\mu_2(y) - \int_{\Omega_1} G_{\omega}(A_k f(x), t)u(x)d\mu_1(x) \right] \cdot \tilde{T}_{n-2}(t, s)dt, \tag{36}$$

$$P_{2,\omega}(s) = \int_{\alpha}^{\beta} \left[\int_{\Omega_2} G_{\omega}(\phi(f(y), t))v(y)d\mu_2(y) - \int_{\Omega_1} G_{\omega}(A_k f(x), t)u(x)d\mu_1(x) \right] \cdot T_{n-2}(t, s)dt, \tag{37}$$

Theorem 9. Let $n \in \mathbb{N}, n \geq 4, P_{1,\omega}$ and $P_{2,\omega}$ be as in (36) and (37) and $\phi : [\alpha, \beta] \rightarrow \mathbb{R}$ be such that $\phi^{(n)}$ is absolutely continuous with $(\cdot - \alpha)(\beta - \cdot)(\phi^{(n+1)})^2 \in L([\alpha, \beta])$. Then

(i) If $(\cdot - \alpha)(\beta - \cdot)(P'_{1,\omega})^2 \in L([\alpha, \beta])$, the remainder

$$\begin{aligned} \kappa^1(\phi; \alpha, \beta) &= \int_{\Omega_2} \phi(f(y))v(y)d\mu_2(y) - \int_{\Omega_1} \phi(A_k f(x))u(x)d\mu_1(x) \\ &\quad - \int_{\alpha}^{\beta} \left[\int_{\Omega_2} G_{\omega}(f(y), t)v(y)d\mu_2(y) - \int_{\Omega_1} G_{\omega}(A_k f(x), t)u(x)d\mu_1(x) \right] \\ &\quad \times \sum_{k=1}^{n-1} \frac{k}{(k-1)!} \left(\frac{\phi^{(k)}(\alpha)(t-\alpha)^{k-1} - \phi^{(k)}(\beta)(t-\beta)^{k-1}}{\beta-\alpha} \right) dt \\ &\quad - \frac{\phi^{(n-1)}(\beta) - \phi^{(n-1)}(\alpha)}{(n-3)!(\beta-\alpha)} \int_{\alpha}^{\beta} P_{1,\omega}(s)ds \end{aligned} \tag{38}$$

is bounded by

$$|\kappa^1(\phi; \alpha, \beta)| \leq \frac{\sqrt{\beta-\alpha}}{\sqrt{2}(n-3)!} |T(P_{1,\omega}, P_{1,\omega})|^{\frac{1}{2}} \left(\int_{\alpha}^{\beta} (s-\alpha)(\beta-s)[\phi^{(n+1)}(s)]^2 ds \right)^{\frac{1}{2}}. \tag{39}$$

(ii) If $(\cdot - \alpha)(\beta - \cdot)(P'_{2,\omega})^2 \in L([\alpha, \beta])$, the remainder

$$\begin{aligned} \kappa^2(\phi; \alpha, \beta) &= \int_{\Omega_2} \phi(f(y))v(y)d\mu_2(y) - \int_{\Omega_1} \phi(A_k f(x))u(x)d\mu_1(x) \\ &\quad - \frac{1}{2} \frac{\phi'(\beta) - \phi'(\alpha)}{\beta - \alpha} \left[\int_{\Omega_2} v(y)f^2(y)d\mu_2(y) - \int_{\Omega_1} u(x)A_k^2 f(x)d\mu_1(x) \right] \\ &\quad - \int_{\alpha}^{\beta} \left[\int_{\Omega_2} G_{\omega}(f(y), t)v(y)d\mu_2(y) - \int_{\Omega_1} G_{\omega}(A_k f(x), t)u(x)d\mu_1(x) \right] \times \\ &\quad \sum_{k=3}^{n-1} \frac{k-2}{(k-1)!} \left(\frac{\phi^{(k)}(\alpha)(t-\alpha)^{k-1} - \phi^{(k)}(\beta)(t-\beta)^{k-1}}{\beta - \alpha} \right) dt \\ &\quad - \frac{\phi^{(n-1)}(\beta) - \phi^{(n-1)}(\alpha)}{(n-3)!(\beta - \alpha)} \int_{\alpha}^{\beta} P_{2,\omega}(s)ds, \end{aligned} \tag{40}$$

is bounded by

$$|\kappa^2(\phi; \alpha, \beta)| \leq \frac{\sqrt{\beta - \alpha}}{\sqrt{2}(n-3)!} |T(P_{2,\omega}, P_{2,\omega})|^{\frac{1}{2}} \left(\int_{\alpha}^{\beta} (s - \alpha)(\beta - s)[\phi^{(n+1)}(s)]^2 ds \right)^{\frac{1}{2}}. \tag{41}$$

Proof.

(i) From (21) and (38), we conclude

$$\kappa^1(\phi; \alpha, \beta) = \frac{1}{(n-3)!} \int_{\alpha}^{\beta} P_{1,\omega}(s)\phi^{(n)}(s)ds - \frac{\phi^{(n-1)}(\beta) - \phi^{(n-1)}(\alpha)}{(n-3)!(\beta - \alpha)} \int_{\alpha}^{\beta} P_{1,\omega}(s)ds. \tag{42}$$

The assumptions of Theorem 7 are satisfied for $h = P_{1,\omega}$ and $g = \phi^{(n)}$, so

$$\begin{aligned} &\left| \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} P_{1,\omega}(s)\phi^{(n)}(s)ds - \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} P_{1,\omega}(s)ds \cdot \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} \phi^{(n)}(s)ds \right| \\ &\leq \frac{1}{\sqrt{2}} |T(P_{1,\omega}, P_{1,\omega})|^{\frac{1}{2}} \frac{1}{\sqrt{\beta - \alpha}} \left(\int_{\alpha}^{\beta} (s - \alpha)(\beta - s)[\phi^{(n+1)}(s)]^2 ds \right)^{\frac{1}{2}}. \end{aligned} \tag{43}$$

Therefore, from (42) and (43), we obtain (39).

(ii) Similarly as in part (i), we obtain (41).

□

Theorem 10. Let $n \in \mathbb{N}, n \geq 4, P_{1,\omega}$ and $P_{2,\omega}$ be as in (36) and (37) and $\phi : [\alpha, \beta] \rightarrow \mathbb{R}$ be such that $\phi^{(n)}$ is monotonic non-decreasing. Then

(i) If $P_{1,\omega}$ is absolutely continuous with $P'_{1,\omega} \in L^{\infty}([\alpha, \beta])$, then the remainder $\kappa^1(\phi; \alpha, \beta)$ given by (38) is bounded by

$$\begin{aligned} |\kappa^1(\phi; \alpha, \beta)| &\leq \\ &\frac{\|P'_{1,\omega}\|_{\infty}}{(n-3)!} \left[\frac{(\beta - \alpha)(\phi^{(n-1)}(\beta) + \phi^{(n-1)}(\alpha))}{2} - \{\phi^{(n-2)}(\beta) - \phi^{(n-2)}(\alpha)\} \right]. \end{aligned} \tag{44}$$

(ii) If $P_{2,\omega}$ is absolutely continuous with $P'_{2,\omega} \in L^\infty([\alpha, \beta])$, then the remainder $\kappa^2(\phi; \alpha, \beta)$ given by (40) is bounded by

$$\begin{aligned} & \left| \kappa^2(\phi; \alpha, \beta) \right| \\ & \leq \frac{\|P'_{2,\omega}\|_\infty}{(n-3)!} \left[\frac{(\beta-\alpha)(\phi^{(n-1)}(\beta) + \phi^{(n-1)}(\alpha))}{2} - \left\{ \phi^{(n-2)}(\beta) - \phi^{(n-2)}(\alpha) \right\} \right]. \end{aligned} \tag{45}$$

Proof.

(i) Assumptions of Theorem 8 are satisfied for $h = P_{1,\omega}$ and $g = \phi^{(n)}$, so, taking into account (42), we have

$$\begin{aligned} & \left| \frac{1}{\beta-\alpha} \int_\alpha^\beta P_{1,\omega}(s) \phi^{(n)}(s) ds - \frac{1}{\beta-\alpha} \int_\alpha^\beta P_{1,\omega}(s) ds \cdot \frac{1}{\beta-\alpha} \int_\alpha^\beta \phi^{(n)}(s) ds \right| \\ & \leq \frac{1}{2(\beta-\alpha)} \|P'_{1,\omega}\|_\infty \int_\alpha^\beta (s-\alpha)(\beta-s) \phi^{(n+1)}(s) ds. \end{aligned} \tag{46}$$

Simple calculation yields

$$\begin{aligned} & \int_\alpha^\beta (s-\alpha)(\beta-s) \phi^{(n+1)}(s) ds = \int_\alpha^\beta [2s - (\alpha + \beta)] \phi^{(n)}(s) ds \\ & = (\beta-\alpha) [\phi^{(n-1)}(\beta) + \phi^{(n-1)}(\alpha)] - 2 [\phi^{(n-2)}(\beta) - \phi^{(n-2)}(\alpha)]. \end{aligned}$$

Finally, inserting the last expression in (46) and taking into account (42), we obtain (44).

(i) Similarly as in part (i), we obtain (45).

□

The last theorem gives an Ostrowki-type bound for the generalized Hardy’s inequality. The symbol $\|\cdot\|_p$ denotes the standard $L^p([\alpha, \beta])$ norm of a function, i.e.,

$$\|g\|_p = \left(\int_\alpha^\beta |g(s)|^p ds \right)^{\frac{1}{p}}$$

for $1 \leq p < \infty$, while $\|g\|_\infty$ is the essential supremum of g .

Theorem 11. Let $n \in \mathbb{N}$, $n \geq 4$, $P_{1,\omega}$ and $P_{2,\omega}$ be as in (36) and (37), $1 \leq p, q \leq \infty$, $1/p + 1/q = 1$ and $\phi : [\alpha, \beta] \rightarrow \mathbb{R}$ be such that $\|\phi^{(n)}\|_p < \infty$. Then

(i)

$$\begin{aligned} & \left| \int_{\Omega_2} \phi(f(y))v(y) d\mu_2(y) - \int_{\Omega_1} \phi(A_k f(x))u(x) d\mu_1(x) \right. \\ & \left. - \int_\alpha^\beta \left[\int_{\Omega_2} G_\omega(f(y), t)v(y) d\mu_2(y) - \int_{\Omega_1} G_\omega(A_k f(x), t)u(x) d\mu_1(x) \right] \times \right. \\ & \left. \sum_{k=1}^{n-1} \frac{k}{(k-1)!} \left(\frac{\phi^{(k)}(\alpha)(t-\alpha)^{k-1} - \phi^{(k)}(\beta)(t-\beta)^{k-1}}{\beta-\alpha} \right) dt \right| \\ & \leq \frac{1}{(n-3)!} \|\phi^{(n)}\|_p \|P_{1,\omega}\|_q. \end{aligned}$$

The constant $\|P_{1,\omega}\|_q / (n-3)!$ is sharp when $1 < p \leq \infty$ and the best possible when $p = 1$.

(ii)

$$\begin{aligned} & \left| \int_{\Omega_2} \phi(f(y))v(y)d\mu_2(y) - \int_{\Omega_1} \phi(A_k f(x))u(x)d\mu_1(x) \right. \\ & \quad \left. - \frac{1}{2} \frac{\phi'(\beta) - \phi'(\alpha)}{\beta - \alpha} \left[\int_{\Omega_2} v(y)f^2(y)d\mu_2(y) - \int_{\Omega_1} u(x)A_k^2 f(x)d\mu_1(x) \right] \right. \\ & \quad \left. - \left[\int_{\Omega_2} G_\omega(f(y), t)v(y)d\mu_2(y) - \int_{\Omega_1} G_\omega(A_k f(x), t)u(x)d\mu_1(x) \right] \times \right. \\ & \quad \left. \sum_{k=3}^{n-1} \frac{k-2}{(k-1)!} \left(\frac{\phi^{(k)}(\alpha)(t-\alpha)^{k-1} - \phi^{(k)}(\beta)(t-\beta)^{k-1}}{\beta - \alpha} \right) dt \right. \\ & \quad \left. \leq \frac{1}{(n-3)!} \|\phi^{(n)}\|_p \|P_{2,\omega}\|_q \right. \end{aligned}$$

The constant $\|P_{2,\omega}\|_q / (n-3)!$ is sharp when $1 < p \leq \infty$ and the best possible when $p = 1$.

Proof. The proof is similar to the proof of Theorem 12 in [21]. \square

4. Discussion

We have presented new results regarding Hardy’s inequality in a general setting. The main results involve Hardy-type inequalities and four new Green functions. We were motivated by the results given in papers [15–17]. These papers contain results involving the Hardy inequality and Taylor’s formula and also results involving the Hardy inequality and Abel–Gontscharoff’s interpolating polynomial. We have also derived related Grüss and Ostrowski-type inequalities. In our next papers, we plan to present new results involving Hardy’s inequality and Hermite’s and Lidstone’s interpolating polynomials. The results presented here are of theoretical nature, and any suggestions for possible applications and further research are welcome.

Author Contributions: All authors contributed equally to this work. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

- Hardy, G.H. Notes on some points in the integral calculus LX: An inequality between integrals. *Messenger Math.* **1925**, *54*, 150–156.
- Goldstein, G.R.; Goldstein, J.A.; Rhandi, A. Weighted Hardy’s inequality and the Kolmogorov equation perturbed by an inverse-square potential. *Appl. Anal.* **2012**, *91*, 2057–2071. [\[CrossRef\]](#)
- Kaijser, S.; Nikolova, L.; Persson, L.E.; Wedestig, A. Hardy-Type Inequalities via Convexity. *Math. Inequal. Appl.* **2005**, *8*, 403–417. [\[CrossRef\]](#)
- Kaijser, S.; Persson, L.-E.; Öberg, A. On Carleman and Knopp’s Inequalities. *J. Approx. Theory* **2002**, *117*, 140–151. [\[CrossRef\]](#)
- Kufner, A.; Maligranda, L.; Persson, L.-E. The prehistory of the Hardy inequality. *Am. Math. Mon.* **2006**, *113*, 715–732. [\[CrossRef\]](#)
- Kufner, A.; Maligranda, L.; Persson, L.-E. *The Hardy Inequality. About Its History and Some Related Results*; Vydavatelsky Servis Publishing House: Pilsen, IL, USA, 2007.
- Kufner, A.; Persson, L.-E. *Weighted Inequalities of Hardy Type*; World Scientific: Hackensack, NJ, USA; London, UK; Singapore; Hong Kong, China, 2003. [\[CrossRef\]](#)
- Opic, B.; Kufner, A. *Hardy Type Inequalities*; Longman: Harlow, OR, USA, 1990.
- Niculescu, C.P.; Persson, L.-E. *Convex Functions and Their Applications. A Contemporary Approach*; Springer: Berlin/Heidelberg, Germany; New York, NY, USA; Hong Kong, China; London, UK; Milan, Italy; Paris, France; Tokyo, Japan, 2005.
- Pečarić, J.E.; Proschan, F.; Tong, Y.L. *Convex Functions, Partial Orderings and Statistical Applications*; Academic Press: San Diego, CA, USA, 1992.

11. Rauf, K.; Omolehin, J.O. Some Notes Inequality Related to G.H. Hardy's Integral Inequality. *Punjab Univ. J. Math.* **2006**, *38*, 9–13.
12. Saied, A.I.; AlNemer, G.; Zakarya, M.; Cesarano, C.; Rezk, H.M. Some New Generalized Inequalities of Hardy Type Involving Several Functions on Time Scale Nabla Calculus. *Axioms* **2022**, *11*, 662. [[CrossRef](#)]
13. Sarkaya, M.Z.; Yildirim, H. Some Hardy Type Integral Inequalities. *J. Inequal. Pure Appl. Math.* **2006**, *7*, 1–5.
14. Krulić, K.; Pečarić, J.; Persson, L.-E. Some new Hardy–type inequalities with general kernels. *Math. Inequal. Appl.* **2009**, *12*, 473–485.
15. Krulić Himmelreich, K.; Pečarić, J.; Pokaz, D. *Inequalities of Hardy and Jensen*; Element: Zagreb, Croatia, 2013.
16. Aglič Aljinović, A.; Pečarić, J.; Vukelić, A. On some Ostrowski type inequalities via Montgomery identity and Taylor's formula II. *Tamkang Jour. Math.* **2005**, *36*, 279–301. [[CrossRef](#)]
17. Krulić Himmelreich, K. Generalizations of Hardy type inequalities by Taylor's formula. *Math. Slovaca Math. Slovaca* **2022**, *72*, 67–84. [[CrossRef](#)]
18. Krulić Himmelreich, K. Some new inequalities involving the generalized Hardy operator. *Math. Pannonica* **2022**, *28*, 127–132. [[CrossRef](#)]
19. Krulić Himmelreich, K.; Pečarić, J. Some new Hardy–type inequalities with general kernels II. *Math. Inequal. Appl.* **2016**, *19*, 73–84.
20. Krulić Himmelreich, K.; Pečarić, J.; Pokaz, D.; Praljak, M. Generalizations of Hardy Type Inequalities by Abel–Gontscharoff's Interpolating Polynomial. *Mathematics* **2021**, *9*, 1724. [[CrossRef](#)]
21. Khan, M.A.; Khan, J.; Pečarić, J. Generalization of Sherman's inequality by Montgomery identity and Green function. *Electron. J. Math. Anal. Appl.* **2017**, *5*, 1–17.
22. Khan, M.A.; Khan, J.; Pečarić, J. Generalization of Sherman's inequality by Montgomery identity and new Green function. *Adv. Stud. Contemp. Math. (Kyungshang)* **2017**, *27*, 495–514.
23. Cerone, P.; Dragomir, S.S. Some new Ostrowski-type bounds for the Chebyshev functional and applications. *J. Math. Inequal.* **2014**, *8*, 159–170. [[CrossRef](#)]

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.