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# Relational Contractions Involving Shifting Distance Functions with Applications to Boundary Value Problems

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**Abstract:** This manuscript includes certain results on fixed points under a generalized contraction involving a pair of shifting distance functions in the framework of metric space endowed with a class of transitive relation. The results presented herein are illustrated by an example. Finally, we apply our result to compute a unique solution of certain first order boundary value problems.

**Keywords:** shifting distance functions; binary relations; $\mathfrak{S}$-preserving sequence

**MSC:** 47H10; 06A75; 54H25

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## 1. Introduction

In the hypotheses of the Banach contraction principle (abbreviated as: BCP), underlying mapping remains a class of continuous mapping, which is known as 'contraction'. In recent years, various types of functions have been introduced to generalize the contraction condition such as control functions, comparison functions, (c)-comparison functions, altering distance functions, shifting distance function, Geraghty functions, simulation functions, etc. On the other hand, Alam and Imdad [1] established a novel generalization of BCP, where the metric space was equipped with a relation, and the involved mapping preserved this relation. The result of Alam and Imdad [1] was further extended and improved by various authors, e.g., ([2–19]). Indeed the relation-theoretic contraction condition remains weaker as compared to the Banach contraction, as this holds for only those elements which are related with respect to the given relation.

Following Khan et al. [20], a function $\theta : [0,\infty) \to [0,\infty)$ is called an altering distance function if

(i) $\theta(s) = 0$ if and only if $s = 0$,
(ii) $\theta$ is increasing and continuous.

Employing the idea of an altering distance function, Khan et al. [20] obtained a generalization of the BCP, which runs as follows:

**Theorem 1** ([20])**.** *Let $(\mathcal{D},\sigma)$ be a metric space and $\mathcal{H} : \mathcal{D} \to \mathcal{D}$ a function. If $\exists$ an altering distance function $\theta$ and a constant $c \in [0,1)$ satisfying*

$$\theta(\sigma(\mathcal{H}q,\mathcal{H}s)) \le c\theta(\sigma(q,s)),\ \forall\, q,s \in \mathcal{D}, \tag{1}$$

*then $\mathcal{H}$ has a unique fixed point.*

Under the restriction $\theta = I$, with the identity map on $[0,\infty)$, Theorem 1 reduces to the BCP. Berzig [21] generalized the concept of an altering distance function by introducing the idea of a pair $(\theta,\eta)$ of shifting distance functions and utilized the same to extend Theorem 1, wherein the authors replaced the function $c.\theta$ (in the right hand side of (1)) with another appropriate mapping $\eta$.

The intent of this manuscript is to establish the results of the existence and uniqueness of fixed points under a contractivity condition employing a pair of shifting distance functions in the setting of relational metric space. In proving our results, we employ a locally $\mathcal{H}$-transitive binary relation. To demonstrate our main results, some illustrative examples are also provided. As an application of our results, we present a result on the existence and uniqueness of a certain boundary value problem (abbreviated as, B.V.P.).

As mentioned earlier, a relatively weaker contraction condition is utilized compared with those in the recent literature. Owing to the restrictive nature, the results proved herein and similar results in future works can be applied in fields of matrix equations, Fredholm integral equations, nonlinear elliptic problems, fractional differential equations, and delayed hematopoiesis models in addition to the B.V.P.

## 2. Preliminaries

Throughout this article, $\mathbb{N}$, $\mathbb{N}_0$, and $\mathbb{R}$ will denote the set of: natural numbers, whole numbers, and real numbers, respectively. By a relation (or more precisely, a binary relation) $\Gamma$ on a set $\mathcal{D}$, we mean any subset of $\mathcal{D}^2$. In what follows, $\mathcal{D}$ is a set, $\sigma$ is a metric on $\mathcal{D}$, $\Gamma$ is a relation on $\mathcal{D}$, and $\mathcal{H} : \mathcal{D} \to \mathcal{D}$ is a map.

**Definition 1** ([1])**.** *Two elements $q, s \in \mathcal{D}$ are said to be $\Gamma$-comparative, denoted by $[q, s] \in \Gamma$, if $(q, s) \in \Gamma$ or $(s, q) \in \Gamma$.*

**Definition 2** ([22])**.** $\Gamma^{-1} := \{(q, s) \in \mathcal{D}^2 : (s, q) \in \Gamma\}$ *is referred to as the transpose of* $\Gamma$.

**Definition 3** ([22])**.** *By the symmetric closure of* $\Gamma$*, one means the relation* $\Gamma^s := \Gamma \cup \Gamma^{-1}$.

**Proposition 1** ([1])**.** $(q, s) \in \Gamma^s \iff [q, s] \in \Gamma$.

**Definition 4** ([22])**.** *A relation on* $\mathcal{C} \subseteq \mathcal{D}$ *defined by*

$$\Gamma|_{\mathcal{C}} := \Gamma \cap \mathcal{C}^2,$$

*is called restriction of* $\Gamma$ *on* $\mathcal{C}$.

**Definition 5** ([1])**.** $\Gamma$ *is referred to as* $\mathcal{H}$*-closed, if it satisfies*

$$(\mathcal{H}q, \mathcal{H}s) \in \Gamma,$$

*for each pair* $q, s \in \mathcal{D}$ *verifying* $(q, s) \in \Gamma$.

**Proposition 2** ([5])**.** $\Gamma$ *is* $\mathcal{H}^n$*-closed provided* $\Gamma$ *remains* $\mathcal{H}$*-closed.*

**Definition 6** ([1])**.** *A sequence* $\{q_n\} \subset \mathcal{D}$ *verifying* $(q_n, q_{n+1}) \in \Gamma \ \forall \ n \in \mathbb{N}_0$ *is said to be* $\Gamma$*-preserving.*

**Definition 7** ([2])**.** *If each* $\Gamma$*-preserving Cauchy sequence in the metric space* $(\mathcal{D}, \sigma)$ *remains convergent, then one can state that* $(\mathcal{D}, \sigma)$ *is* $\Gamma$*-complete.*

**Definition 8** ([2])**.** $\mathcal{H}$ *is called* $\Gamma$*-continuous at* $q \in \mathcal{D}$ *if it satisfies*

$$\mathcal{H}(q_n) \xrightarrow{\sigma} \mathcal{H}(q)$$

*for any* $\Gamma$*-preserving sequence* $\{q_n\} \subset \mathcal{D}$ *verifying* $q_n \xrightarrow{\sigma} q$*. Further, by a* $\Gamma$*-continuous function, we mean* $\Gamma$*-continuous at all points of* $\mathcal{D}$.

**Definition 9** ([1])**.** $\Gamma$ *is* $\sigma$*-self-closed, if each* $\Gamma$*-preserving convergent sequence in* $(\mathcal{D}, \sigma)$ *admits a subsequence whose terms are* $\Gamma$*-comparative with the convergence limit.*

**Definition 10** ([23]). *Given $q, s \in \mathcal{D}$, a finite sequence $\{p_0, p_1, \ldots, p_l\} \subset \mathcal{D}$ is called a path of length $l \in \mathbb{N}$ in $\Gamma$ from $q$ to $s$ if the following hold:*

(i) $p_0 = q$ *and* $p_l = s$,
(ii) $(p_i, p_{i+1}) \in \Gamma, \quad 0 \leq i \leq l-1.$

**Definition 11** ([5]). *A subset $\mathcal{C} \subseteq \mathcal{D}$ is said to be a $\Gamma$-connected set, if $\exists$ a path between each pair of elements of $\mathcal{C}$.*

**Definition 12** ([24]). *Given $k \in \mathbb{N}_0$, $k \geq 2$, $\Gamma$ is termed k-transitive, if for any $q_0, q_1, ..., q_k \in \mathcal{D}$ satisfying $(q_{j-1}, q_j) \in \Gamma$, for $1 \leq j \leq k$, one has*

$$(q_0, q_k) \in \Gamma.$$

**Definition 13** ([25,26]). *$\Gamma$ is said to be a finitely transitive relation if it is k-transitive, for some $k \geq 2$.*

**Definition 14** ([10]). *$\Gamma$ is termed as locally finitely $\mathcal{H}$-transitive, if for each enumerable subset $\mathcal{C} \subseteq \mathcal{H}(\mathcal{D})$, there exists $k = k(\mathcal{C}) \geq 2$, such that $\Gamma|_{\mathcal{C}}$ remains k-transitive.*

The following notations are utilized in the upcoming text.

- $F(\mathcal{H})$:=the set of all fixed points of $\mathcal{H}$,
- $\mathcal{D}(\mathcal{H}, \Gamma) := \{q \in \mathcal{D} : (q, \mathcal{H}q) \in \Gamma\}$.

The following result investigated by Alam and Imdad [1] is known as the *relation-theoretic contraction principle*.

**Theorem 2** ([1,2,19]). *Assume that $(\mathcal{D}, \sigma)$ is a metric space, and $\Gamma$ is a relation on $\mathcal{D}$, while $\mathcal{H} : \mathcal{D} \to \mathcal{D}$ is a function. Moreover,*

(i) $(\mathcal{D}, \sigma)$ *is* $\Gamma$*-complete,*
(ii) $\mathcal{D}(\mathcal{H}, \Gamma)$ *is nonempty,*
(iii) $\Gamma$ *is* $\mathcal{H}$*-closed,*
(iv) $\mathcal{H}$ *is* $\Gamma$*-continuous or* $\Gamma$ *is* $\sigma$*-self-closed,*
(v) *there exists* $c \in [0, 1)$ *verifying*

$$\sigma(\mathcal{H}q, \mathcal{H}s) \leq c\sigma(q, s), \ \forall \, q, s \in \mathcal{D} \text{ with } (q, s) \in \Gamma.$$

*Then, $\mathcal{H}$ admits a fixed point. Moreover, if $\mathcal{H}(\mathcal{D})$ is $\Gamma^s$-connected, then $\mathcal{H}$ admits a unique fixed point.*

Finally, we indicate the following two known results, which are desirable to prove our main results.

**Lemma 1** ([27]). *Let $\{q_n\}$ be a sequence in a metric space $(\mathcal{D}, \sigma)$. If $\{q_n\}$ is not Cauchy, then there exists an $\varepsilon > 0$ and two subsequences $\{q_{n_\alpha}\}$ and $\{q_{m_\alpha}\}$ of $\{q_n\}$ verifying*

(i) $\alpha \leq m_\alpha < n_\alpha \ \forall \, \alpha \in \mathbb{N}$,
(ii) $\sigma(q_{m_\alpha}, q_{n_\alpha}) \geq \varepsilon$,
(iii) $\sigma(q_{m_\alpha}, q_{s_\alpha}) < \varepsilon, \forall \, s_\alpha \in \{m_\alpha + 1, m_\alpha + 2, ..., n_\alpha - 2, n_\alpha - 1\}$.

*Further, if $\lim_{n \to \infty} \sigma(q_n, q_{n+1}) = 0$, then*

$$\lim_{\alpha \to \infty} \sigma(q_{m_\alpha}, q_{n_\alpha + s}) = \varepsilon, \ \forall \, s \in \mathbb{N}_0.$$

**Lemma 2** ([25]). *Let $\Gamma$ be a relation on a set $\mathcal{D}$ and $\{q_n\} \subset \mathcal{D}$ be an $\Gamma$-preserving sequence. If $\Gamma$ is k-transitive on $\mathcal{C} = \{q_n : n \in \mathbb{N}_0\}$, then*

$$(q_n, q_{n+1+s(k-1)}) \in \Gamma, \ \forall \, n, s \in \mathbb{N}_0.$$

## 3. Main Results

Let $\theta, \eta : [0, \infty) \to [0, \infty)$ be two functions. Following Berzig [21], one says that the pair $(\theta, \eta)$ forms a pair of shifting distance functions, if they enjoy the following properties:

(i) for $w, z \in [0, \infty)$ with $\theta(w) \le \eta(z) \Longrightarrow w \le z$.

(ii) for $\{w_n\}, \{z_n\} \subset [0, \infty)$ with $\theta(w_n) \le \eta(z_n), \forall\, n \in \mathbb{N}$, and $\lim\limits_{n\to\infty} w_n = \lim\limits_{n\to\infty} z_n = l \Longrightarrow l = 0$.

**Proposition 3.** *Suppose that $(\mathcal{D}, \sigma)$ is a metric space, $\Gamma$ is a relation on $\mathcal{D}$, while $\mathcal{H} : \mathcal{D} \to \mathcal{D}$ is a function. If $(\theta, \eta)$ are shifting distance functions, then the following are equivalent:*

(I) $\theta(\sigma(\mathcal{H}q, \mathcal{H}s)) \le \eta(\sigma(q, s)), \quad \forall\, q, s \in \mathcal{D}$ *with* $(q, s) \in \Gamma$,

(II) $\theta(\sigma(\mathcal{H}q, \mathcal{H}s)) \le \eta(\sigma(q, s)), \quad \forall\, q, s \in \mathcal{D}$ *with* $[q, s] \in \Gamma$.

**Proof.** If (II) holds, then so does (I) trivially. Conversely, we assume that (I) holds. We take $q, s \in \mathcal{D}$ with $[q, s] \in \Gamma$. In the case where $(q, s) \in \Gamma$, (I) implies (II). Otherwise, in the case where $(s, q) \in \Gamma$, due to the symmetric property of $\sigma$ and (I), we obtain

$$\theta(\sigma(\mathcal{H}q, \mathcal{H}s)) = \theta(\sigma(\mathcal{H}s, \mathcal{H}q)) \le \eta(\sigma(s, q)) = \eta(\sigma(q, s)).$$

This verifies that (I)$\Rightarrow$(II). □

**Theorem 3.** *Suppose that $(\mathcal{D}, \sigma)$ is a metric space, and $\Gamma$ is a relation on $\mathcal{D}$, while $\mathcal{H} : \mathcal{D} \to \mathcal{D}$ is a function. Moreover,*

(i) *$(\mathcal{D}, \sigma)$ is $\Gamma$-complete,*

(ii) *$\mathcal{D}(\mathcal{H}, \Gamma)$ is nonempty,*

(iii) *$\Gamma$ is $\mathcal{H}$-closed and locally finitely $\mathcal{H}$-transitive,*

(iv) *$\mathcal{H}$ is $\Gamma$-continuous or $\Gamma$ is $\sigma$-self-closed,*

(v) *there exist shifting distance functions $(\theta, \eta)$ verifying*

$$\theta(\sigma(\mathcal{H}q, \mathcal{H}s)) \quad \le \quad \eta(\sigma(q, s)),\ \forall\, q, s \in \mathcal{D} \text{ with } (q, s) \in \Gamma.$$

*Then, $\mathcal{H}$ admits a fixed point.*

**Proof.** By hypothesis (ii), if $q_0 \in \mathcal{D}(\mathcal{H}, \Gamma)$, then we have $(q_0, \mathcal{H}q_0) \in \Gamma$. We construct a sequence $\{q_n\} \subset \mathcal{D}$ verifying

$$q_n = \mathcal{H}^n(q_0) = \mathcal{H}(q_{n-1}), \quad \forall\, n \in \mathbb{N}. \tag{2}$$

By assumption (iii) and Proposition 2, we obtain

$$(\mathcal{H}^n q_0, \mathcal{H}^{n+1} q_0) \in \Gamma.$$

Using (2), the above becomes

$$(q_n, q_{n+1}) \in \Gamma \quad \forall\, n \in \mathbb{N}_0, \tag{3}$$

so that $\{q_n\}$ is $\Gamma$-preserving.

If there exists $n_0 \in \mathbb{N}_0$ satisfying $\sigma(q_{n_0}, q_{n_0+1}) = 0$, then using (2), we find that $q_{n_0}$ is a fixed point of $\mathcal{H}$. Otherwise, in the case where $\sigma_n := \sigma(q_n, q_{n+1}) > 0, \forall\, n \in \mathbb{N}_0$, one uses hypothesis (v) to obtain

$$\theta(\sigma(q_{n+1}, q_{n+2})) = \theta(\sigma(\mathcal{H}q_n, \mathcal{H}q_{n+1})) \le \eta(\sigma(q_n, q_{n+1}))$$

so that

$$\theta(\sigma_{n+1}) \le \eta(\sigma_n),\ \ n \in \mathbb{N}.$$

By property (i) of the shifting distance functions, $\{\sigma_n\} \subset [0,\infty)$ is a decreasing sequence. Therefore, $\exists\, \delta \geq 0$ satisfying $\lim_{n\to\infty} \sigma_n = \delta$. Further, by property (ii) of the shifting distance functions, one obtains $\delta = 0$, *i.e.*,

$$\lim_{n\to\infty} \sigma_n = \lim_{n\to\infty} \sigma(q_n, q_{n+1}) = 0. \tag{4}$$

Employing the contradiction method, we show that $\{q_n\}$ is a Cauchy sequence. If $\{q_n\}$ is not Cauchy, then Lemma 1 ensures the existence of $\varepsilon > 0$ and two subsequences $\{q_{n_\alpha}\}$ and $\{q_{m_\alpha}\}$ of $\{q_n\}$ verifying $\alpha \leq m_\alpha < n_\alpha$, $\sigma(q_{m_\alpha}, q_{n_\alpha}) \geq \varepsilon$ and $\sigma(q_{m_\alpha}, q_{s_\alpha}) < \varepsilon$ wherein $s_\alpha \in \{m_\alpha + 1, m_\alpha + 2, ..., n_\alpha - 2, n_\alpha - 1\}$. Moreover, by (4), one obtains

$$\lim_{n\to\infty} \sigma(q_{m_\alpha}, q_{n_\alpha+s}) = \varepsilon \quad \forall s \in \mathbb{N}_0. \tag{5}$$

Since $\{q_n\} \subset \mathcal{H}(\mathcal{D})$, the range $\mathcal{C} = \{q_n : n \in \mathbb{N}_0\}$ is an enumerable subset of $\mathcal{H}(\mathcal{D})$; therefore, the locally finitely $\mathcal{H}$-transitivity of $\Gamma$ ensures the existence of a natural number $k = k(\mathcal{C}) \geq 2$, for which $\Gamma|_{\mathcal{C}}$ is $k$-transitive.

Now, $m_\alpha < n_\alpha$ and $k - 1 > 0$; therefore, by the division algorithm, one obtains

$$\begin{gathered} n_\alpha - m_\alpha = (k-1)(\mu_\alpha - 1) + (k - \eta_\alpha) \\ \mu_\alpha - 1 \geq 0, 0 \leq k - \eta_\alpha < k - 1 \end{gathered}$$

$$\Longleftrightarrow \begin{cases} n_\alpha + \eta_\alpha = m_\alpha + 1 + (k-1)\mu_\alpha \\ \mu_\alpha \geq 1, 1 < \eta_\alpha \leq k. \end{cases}$$

Here, $\mu_\alpha$ and $\eta_\alpha$ are suitable numbers so that $1 < \eta_\alpha \leq k$. Thus, we are able to choose subsequences $\{q_{n_\alpha}\}$ and $\{q_{m_\alpha}\}$ of $\{q_n\}$(satisfying (5)); so, $\eta_\alpha$ is a constant $\eta$. One has

$$m'_\alpha = n_\alpha + \eta = m_\alpha + 1 + (k-1)\mu_\alpha. \tag{6}$$

Using (5) and (6), one obtains

$$\lim_{\alpha\to\infty} \sigma(q_{m_\alpha}, q_{m'_\alpha}) = \lim_{\alpha\to\infty} \sigma(q_{m_\alpha}, q_{n_\alpha+\eta}) = \varepsilon. \tag{7}$$

Using the triangular inequality, we have

$$\sigma(q_{m_\alpha+1}, q_{m'_\alpha+1}) \leq \sigma(q_{m_\alpha+1}, q_{m_\alpha}) + \sigma(q_{m_\alpha}, q_{m'_\alpha}) + \sigma(q_{m'_\alpha}, q_{m'_\alpha+1}) \tag{8}$$

and

$$\sigma(q_{m_\alpha}, q_{m'_\alpha}) \leq \sigma(q_{m_\alpha}, q_{m_\alpha+1}) + \sigma(q_{m_\alpha+1}, q_{m'_\alpha+1}) + \sigma(q_{m'_\alpha+1}, q_{m'_\alpha})$$

or

$$\sigma(q_{m_\alpha}, q_{m'_\alpha}) - \sigma(q_{m_\alpha}, q_{m_\alpha+1}) - \sigma(q_{m'_\alpha+1}, q_{m'_\alpha}) \quad \leq \quad \sigma(q_{m_\alpha+1}, q_{m'_\alpha+1}). \tag{9}$$

Letting $\alpha \to \infty$ in (8) and (9) and using (4) and (7), we obtain

$$\lim_{\alpha\to\infty} \sigma(q_{m_\alpha+1}, q_{m'_\alpha+1}) = \varepsilon. \tag{10}$$

Due to the availability of (6) and Lemma 2, we obtain $\sigma(q_{m_\alpha}, q_{m'_\alpha}) \in \Gamma$. Further, by assumption (v), one obtains

$$\theta(\sigma(q_{m_\alpha+1}, q_{m'_\alpha+1})) = \theta(\sigma(\mathcal{H}q_{m_\alpha}, \mathcal{H}q_{m'_\alpha})) \leq \eta(\sigma(q_{m_\alpha}, q_{m'_\alpha})).$$

Using property (ii) of the shifting distance functions for $\{w_\alpha = \sigma(q_{m_\alpha}, q_{m'_\alpha})\}$, $\{z_\alpha = \sigma(q_{m_\alpha+1}, q_{m'_\alpha+1})$ and $l = \varepsilon$, one finds that $\varepsilon = 0$, which is a contradiction. Thus, $\{q_n\}$ is Cauchy; hence, the $\Gamma$-completeness of $\mathcal{D}$ provides the existence of $r \in \mathcal{D}$ verifying $q_n \xrightarrow{\sigma} q$.

Finally by (iv), one can verify that $q$ is a fixed point of $\mathcal{H}$. Firstly, we assume that $\mathcal{H}$ is $\Gamma$-continuous; then, we have

$$\mathcal{H}(q_n) \xrightarrow{\sigma} \mathcal{H}(q),$$

which by using (2), reduces to $q_{n+1} \xrightarrow{\sigma} q$ implying thereby $\mathcal{H}(q) = q$. Otherwise, we suppose that $\Gamma$ is $\sigma$-self-closed. As $\{q_n\}$ is a $\Gamma$-preserving sequence satisfying $q_n \xrightarrow{\sigma} q$, by the $\sigma$-self-closedness of $\Gamma$, there exists a subsequence of $\{q_{n_k}\}$ of $\{q_n\}$ satisfying $[q_{n_k}, q] \in \Gamma$, $\forall\, k \in \mathbb{N}_0$. Making use of assumption (v), we obtain

$$\theta(\sigma(q_{n_k+1}, \mathcal{H}q)) = \theta(\sigma(\mathcal{H}q_{n_k}, \mathcal{H}q)) \leq \eta(\sigma(q_{n_k}, q))$$

which, using axiom (i) of the shifting distance functions, gives rise to

$$\sigma(q_{n_k+1}, \mathcal{H}q) \leq \sigma(q_{n_k}, q), \ \ \forall\, k \in \mathbb{N}_0. \tag{11}$$

Due to the fact that $q_{n_k} \xrightarrow{\sigma} q$ and by the continuity of $\sigma$, one obtains $\sigma(q_{n_k}, q) \to 0$ as $k \to \infty$. Letting $k \to \infty$ in (11), one has

$$\sigma(q_{n_k+1}, \mathcal{H}q) \to 0 \text{ as } k \to \infty;$$

so,

$$q_{n_k+1} \xrightarrow{\sigma} \mathcal{H}(q).$$

Using the uniqueness of the convergence limit, we obtain $\mathcal{H}(q) = q$. □

Now, the corresponding uniqueness result is presented.

**Theorem 4.** *Along with the hypothesis of Theorem 3, if $\mathcal{H}(\mathcal{D})$ is $\Gamma^s$-connected, then $\mathcal{H}$ admits a unique fixed point.*

**Proof.** By Theorem 3, there exists at least one fixed point of $\mathcal{H}$. If $q$ and $s$ remain two fixed points of $\mathcal{H}$, then

$$\mathcal{H}^n(q) = q \text{ and } \mathcal{H}^n(s) = s \ \forall\, n \in \mathbb{N}_0.$$

Clearly $q, s \in \mathcal{H}(\mathcal{D})$. By the $\Gamma^s$-connectedness of $\mathcal{H}(\mathcal{D})$, there exists a path $\{p_0, p_1, p_2, ..., p_l\}$ between $q$ to $s$; so,

$$p_0 = q, p_l = s \text{ and } [p_i, p_{i+1}] \in \Gamma, \ \forall\, i = 0, 1, \ldots, l-1. \tag{12}$$

As $\Gamma$ is $\mathcal{H}$-closed, we have

$$[\mathcal{H}^n p_i, \mathcal{H}^n p_{i+1}] \in \Gamma, \ \ \forall\, n \in \mathbb{N}_0 \text{ and } \forall\, i = 0, 1, \ldots, l-1. \tag{13}$$

We denote

$$\delta_n^i := \sigma(\mathcal{H}^n p_i, \mathcal{H}^n p_{i+1}) \qquad \forall\, n \in \mathbb{N}_0 \text{ and } \forall\, i = 0, 1, \ldots, l-1.$$

We show that

$$\lim_{n\to\infty} \delta_n^i \;\; = \;\; 0. \tag{14}$$

For each fixed $i$, two cases arise. Firstly, one can assume that

$$\delta_{n_0}^i = \sigma(\mathcal{H}^{n_0} p_i, \mathcal{H}^{n_0} p_{i+1}) = 0, \text{ for some } n_0 \in \mathbb{N}_0,$$

which implies that $\mathcal{H}^{n_0}(p_i) = \mathcal{H}^{n_0}(p_{i+1})$. Using (2), one obtains $\mathcal{H}^{n_0+1}(p_i) = \mathcal{H}^{n_0+1}(p_{i+1})$; so, $\delta_{n_0+1}^i = 0$. Thus, by induction, one finds $\delta_n^i = 0 \ \forall\, n \geq n_0$, implying thereby $\lim\limits_{n\to\infty} \delta_n^i = 0$.

Secondly, one may assume that $\delta_n^i > 0, \forall\, n \in \mathbb{N}_0$. Using (13) along with assumption (v), we obtain

$$\begin{aligned}
\theta(\delta_{n+1}^i) &= \theta(\sigma(\mathcal{H}^{n+1}p_i, \mathcal{H}^{n+1}p_{i+1})) \\
&= \theta(\sigma(\mathcal{H}(\mathcal{H}^n p_i), \mathcal{H}(\mathcal{H}^n p_{i+1}))) \\
&\leq \eta(\sigma(\mathcal{H}^n p_i, \mathcal{H}^n p_{i+1})) \\
&= \eta(\delta_n^i);
\end{aligned}$$

so,

$$\theta(\delta_{n+1}^i) \leq \eta(\delta_n^i).$$

Applying the property (i) of shifting distance functions, the above inequality yields

$$\lim_{n\to\infty} \delta_n^i = 0.$$

Hence, in both the cases, (14) has been proved. By the triangle inequality, one obtains

$$\begin{aligned}
\sigma(q,s) &= \sigma(\mathcal{H}^n p_0, \mathcal{H}^n p_k) \\
&\leq \delta_n^0 + \delta_n^1 + \cdots + \delta_n^{k-1} \\
&\to 0 \text{ as } n \to \infty;
\end{aligned}$$

so, $q = s$. Thus, $\mathcal{H}$ admits a unique fixed point. □

Under the universal relation (i.e., $\Gamma = \mathcal{D}^2$), Theorem 4 deduces to the following fixed point result.

**Corollary 1.** *Assume that $(\mathcal{D}, \sigma)$ is a complete metric space and $\mathcal{H} : \mathcal{D} \to \mathcal{D}$ is a mapping. If there exists a pair $(\theta, \eta)$ of shifting distance functions verifying*

$$\theta(\sigma(\mathcal{H}q, \mathcal{H}s)) \leq \eta(\sigma(q,s)),\ \forall\, q,s \in \mathcal{D},$$

*then $\mathcal{H}$ admits a unique fixed point.*

## 4. Illustrative Examples

To demonstrate the earlier results, let us consider the following examples.

**Example 1.** *Consider the set $\mathcal{D} = [0,1] \cup \mathbb{N}$ with a metric $\sigma$ defined by*

$$\sigma(q,s) = \begin{cases} |q-s|, & \text{if } q,s \in [0,1] \text{ and } q \neq s; \\ q+s, & \text{if } (q,s) \notin [0,1]\times[0,1] \text{ and } q \neq s; \\ 0, & \text{if } q = s. \end{cases}$$

*On $\mathcal{D}$, we define a relation $\Gamma$ by*

$$\Gamma = \{(q,s) \in \mathcal{D}^2 : q > s\}.$$

*Notice that $(\mathcal{D}, \sigma)$ is isometric to a closed subset $A$ of the space $l^1$ of the absolutely summable sequences, whereas the set $A$ consists of the sequences $(q, 0, 0, \cdots)$ for $q \in [0,1]$ together with the sequences with $m$ $(m = 2, 3, \cdots)$ in the mth coordinate place and zeros elsewhere. It follows that the metric space $(\mathcal{D}, \sigma)$ is complete; hence, it is also $\Gamma$-complete.*

*We define the test functions $\theta, \eta : [0,\infty) \to [0,\infty)$ by*

$$\theta(z) = \begin{cases} \ln\left(\frac{1}{13} + \frac{5z}{13}\right), & \text{if } 0 \leq z \leq 1 \\ \ln\left(\frac{1}{13} + \frac{4z}{13}\right), & \text{if } z > 1 \end{cases}$$

*and*

$$\eta(z) = \begin{cases} \ln\left(\frac{1}{13} + \frac{3z}{13}\right), & \text{if } 0 \leq z \leq 1 \\ \ln\left(\frac{1}{13} + \frac{2z}{13}\right), & \text{if } z > 1. \end{cases}$$

*Next, we verify that* $(\theta, \eta)$ *are shifting distance functions. We take* $w, z \in [0, \infty)$ *with* $\theta(w) \leq \eta(z)$*. If* $0 \leq w \leq 1$ *and* $0 \leq z \leq 1$*, then* $\ln\left(\frac{1}{13} + \frac{5w}{13}\right) \leq \ln\left(\frac{1}{13} + \frac{3z}{13}\right)$*; so,* $\ln\left(\frac{1+3z}{1+5w}\right) \geq 0$ *implying thereby* $w \leq \frac{3}{5}z < z$*. If* $w > 1$ *and* $z > 1$*, then similar to the previous case, we obtain* $w \leq \frac{2}{4}z < z$*. In the case where* $0 \leq w \leq 1$ *and* $z > 1$*, the conclusion is trivial. For the case* $w > 1$ *and* $0 \leq z \leq 1$*, the inequality* $\theta(w) \leq \eta(z)$ *does not hold. Hence, in each of the cases, one has* $w \leq z$*. Again, if* $\{w_n\}, \{z_n\} \subset [0, \infty)$ *are sequences with* $\theta(w_n) \leq \eta(z_n)$*,* $\forall\, n \in \mathbb{N}$ *and* $\lim_{n\to\infty} w_n = \lim_{n\to\infty} z_n = l$*, then the continuity of the logarithm function gives rise to* $l = 0$*. Thus, we conclude that* $(\theta, \eta)$ *forms a pair of shifting distance functions.*

*We assume that* $\mathcal{H} : \mathcal{D} \to \mathcal{D}$ *is a map defined by*

$$\mathcal{H}(q) = \begin{cases} q/6, & \text{if } q \in [0, 1) \\ 1/48, & \text{if } q \in \mathbb{N}. \end{cases}$$

*We take* $q, s \in \mathcal{D}$ *with* $(q, s) \in \Gamma$*; then,* $q > s$*. Then, the following cases arise:*

***Case-I:*** *If* $q \in [0, 1]$*, then one has*

$$\begin{aligned} \theta(\sigma(\mathcal{H}q, \mathcal{H}s)) &= \ln\left(\frac{1}{13} + \frac{6}{13}\sigma(\mathcal{H}q, \mathcal{H}s)\right) \\ &= \ln\left(\frac{1}{13} + \frac{6}{13}|\mathcal{H}q - \mathcal{H}s|\right) \\ &= \ln\left(\frac{1}{13} + \frac{1}{13}|q - s|\right) \\ &\leq \eta(\sigma(q, s)). \end{aligned}$$

***Case-II:*** *If* $q \in \mathbb{N} - \{1\}$*, then for* $s \in [0, 1)$*, one has*

$$\begin{aligned} \theta(\sigma(\mathcal{H}q, \mathcal{H}s)) &= \ln\left(\frac{1}{13} + \frac{6}{13}\sigma(\mathcal{H}q, \mathcal{H}s)\right) \\ &= \ln\left(\frac{1}{13} + \frac{6}{13}|\mathcal{H}q - \mathcal{H}s|\right) \\ &\leq \ln\left(\frac{1}{13} + \frac{6}{13}\left(\frac{1}{48} + \frac{s}{6}\right)\right) \\ &\leq \ln\left(\frac{1}{13} + \frac{1}{104} + \frac{s}{13}\right) \\ &\leq \eta(\sigma(q, s)), \qquad \left(\text{as } \frac{1}{104} + \frac{s}{13} \leq \frac{1}{13}(q + s)\right). \end{aligned}$$

*Otherwise, when* $s \in \mathbb{N}$*, one obtains*

$$\begin{aligned} \theta(\sigma(\mathcal{H}q, \mathcal{H}s)) &= \ln\left(\frac{1}{13} + \frac{6}{13}\sigma(\mathcal{H}q, \mathcal{H}s)\right) \\ &= \ln\left(\frac{1}{13}\right) \\ &\leq \eta(\sigma(q, s)). \end{aligned}$$

*Therefore,* $\mathcal{H}$ *satisfies assumption* (v) *of Theorem 3. Moreover,* $\mathcal{H}$ *is* $\Gamma$*-continuous while* $\Gamma$ *is locally finitely* $\mathcal{H}$*-transitive as well as* $\mathcal{H}$*-closed. The rest of the conditions of Theorems 3 and 4 are easily verified. Hence,* $\mathcal{H}$ *possesses a unique fixed point (namely:* $q = 0$*).*

**Example 2.** *Consider the set* $\mathcal{D} = [2,4]$ *with Euclidean metric* $\sigma$ *and a relation* $\Gamma = \{(2,2),(2,3),(3,2),(3,3),(0,4)\}$. *Then,* $(\mathcal{D},\sigma)$ *is a* $\Gamma$*-complete metric space. Assume that* $\mathcal{H} : \mathcal{D} \to \mathcal{D}$ *is a map defined by*

$$\mathcal{H}(q) = \begin{cases} 2 & \text{if } 2 \leq q \leq 3 \\ 3 & \text{if } 3 < q \leq 4. \end{cases}$$

*Then,* $\Gamma$ *is* $\mathcal{H}$*-closed. Suppose that* $\{q_n\} \subset \mathcal{D}$ *is a* $\Gamma$*-preserving sequence satisfying* $q_n \xrightarrow{\sigma} q$ *so that* $(q_n, q_{n+1}) \in \Gamma$, *for each* $n \in \mathbb{N}$. *Note that* $(q_n, q_{n+1}) \notin \{(2,4)\}$, *implying thereby* $(q_n, q_{n+1}) \in \{(2,2),(2,3),(3,2),(3,3)\}$, $\forall\, n \in \mathbb{N}$; *so,* $\{q_n\} \subset \{2,3\}$. *As* $\{2,3\}$ *is closed, we have* $[q_n, q] \in \Gamma$. *It follows that* $\Gamma$ *is* $\sigma$*-self-closed.*

*We define the test functions* $\theta, \eta : [0,\infty) \to [0,\infty)$ *by*

$$\theta(z) = z^2 \quad \text{and} \quad \eta(z) = \frac{z^2}{z^2+1}.$$

*Then,* $(\theta,\eta)$ *remains as the pair of shifting distance functions. Moreover, the contractivity condition* (v) *of Theorem 3 holds for the pair* $(\theta,\eta)$. *The rest of the assumptions of Theorems 3 and 4 are also satisfied. Consequently,* $\mathcal{H}$ *possesses a unique fixed point (namely:* $q = 2$*).*

## 5. An Application to Boundary Value Problems

In the sequel, $\mathcal{C}[0,a]$ denotes the class of all real valued continuous functions on $[0,a]$ (where $a > 0$), and $\mathcal{C}^1[0,a]$ denotes the class of all real valued continuously differentiable functions on $[0,a]$. Let us consider the following BVP:

$$\begin{cases} \mu'(s) = f(s,\mu(s)), & s \in [0,a] \\ \mu(0) = \mu(a) \end{cases} \tag{15}$$

where $f : [0,a] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

**Definition 15** ([28])**.** *We say that* $\mu_0 \in \mathcal{C}^1[0,a]$ *is a lower solution of* (15)*, if*

$$\begin{cases} \mu_0'(s) \leq f(s,\mu_0(s)), & s \in [0,a] \\ \mu_0(0) \leq \mu_0(a). \end{cases}$$

Now, we present the main result of this section.

**Theorem 5.** *In addition to Problem* (15)*, suppose that there exists* $\beta, \lambda > 0$ *verifying*

$$\beta \leq \sqrt{\frac{2\lambda(e^{\lambda a}-1)}{a(e^{\lambda a}+1)}}, \tag{16}$$

*such that* $\forall\, r,t \in \mathbb{R}$ *with* $r \leq t$, *one has*

$$0 \leq [f(s,t) + \lambda t] - [f(s,r) + \lambda r] \leq \beta\sqrt{\frac{(r-s)^2}{(r-s)^2+1}}. \tag{17}$$

*Further, if Problem* (15) *admits a lower solution, then it has a unique solution.*

**Proof.** Problem (15) can be rewritten as

$$\begin{cases} \mu'(s) + \lambda\mu(s) = f(s,\mu(s)) + \lambda\mu(s), & \forall\, s \in [0,a] \\ \mu(0) = \mu(a). \end{cases} \tag{18}$$

Further, (18) reduces to an integral equation of the form

$$\mu(s) = \int_0^a \mathcal{L}(s,\xi)[f(\xi,\mu(\xi)) + \lambda\mu(\xi)]d\xi. \tag{19}$$

Herein, $\mathcal{L}(s,\xi)$ is the Green function of the integral equation, defined by

$$\mathcal{L}(s,\xi) = \begin{cases} \frac{e^{\lambda(a+\xi-s)}}{e^{\lambda a}-1}, & 0 \le \xi < s \le a \\ \frac{e^{\lambda(\xi-s)}}{e^{\lambda a}-1}, & 0 \le s < \xi \le a. \end{cases}$$

We denote $\mathcal{D} := \mathcal{C}[0,a]$ and define a mapping $\mathcal{H} : \mathcal{D} \to \mathcal{D}$ by

$$(\mathcal{H}\mu)(s) = \int_0^a \mathcal{L}(s,\xi)[f(\xi,\mu(\xi)) + \lambda\mu(\xi)]d\xi, \quad \forall\, s \in [0,a]. \tag{20}$$

Therefore, $\mu \in \mathcal{D}$ is a fixed point of $\mathcal{H}$, if and only if $\mu \in \mathcal{C}^1[0,a]$ is a solution of (19) and hence of (15). On $\mathcal{D}$, we define a metric $\sigma$ and a relation $\Gamma$ as follows:

$$\sigma(\mu,\vartheta) = \sup_{s\in[0,a]} |\mu(s) - \vartheta(s)|, \quad \forall\, \mu,\vartheta \in \mathcal{D} \tag{21}$$

and

$$\Gamma = \{(\mu,\vartheta) \in \mathcal{D}\times\mathcal{D} : \mu(s) \le \vartheta(s),\ \forall\, s \in [0,a]\}. \tag{22}$$

Now, we verify all the conditions mentioned in Theorems 3 and 4.

(i) As the metric space $(\mathcal{D},\sigma)$ is complete, it is also $\Gamma$-complete.

(ii) Let $\mu_0 \in \mathcal{C}^1[0,a]$ be a lower solution of (15), then we have

$$\mu_0'(s) + \lambda\mu_0(s) \le f(s,\mu_0(s)) + \lambda\mu_0(s), \quad \forall\, s \in [0,a].$$

Multiplying both sides by $e^{ks}$, we obtain

$$(\mu_0(s)e^{ks})' \le [f(s,\mu_0(s)) + \lambda\mu_0(s)]e^{ks}, \quad \forall\, s \in [0,a],$$

which yields

$$\mu_0(s)e^{ks} \le \mu_0(0) + \int_0^s [f(\xi,\mu_0(\xi)) + \lambda\mu_0(\xi)]e^{\lambda\xi}d\xi, \quad \forall\, s \in [0,a]. \tag{23}$$

Due to $\mu_0(0) \le \mu_0(a)$, we obtain

$$\mu_0(0)e^{\lambda a} \le \mu_0(a)e^{\lambda a} \le \mu_0(0) + \int_0^a [f(\xi,\mu_0(\xi)) + \lambda\mu_0(\xi)]e^{\lambda\xi}d\xi;$$

so,

$$\mu_0(0) \le \int_0^a \frac{e^{\lambda\xi}}{e^{\lambda a}-1}[f(\xi,\mu_0(\xi)) + \lambda\mu_0(\xi)]d\xi. \tag{24}$$

Using (23) and (24), we obtain

$$\begin{aligned} \mu_0(s)e^{ks} &\le \int_0^a \frac{e^{\lambda\xi}}{e^{\lambda a}-1}[f(\xi,\mu_0(\xi)) + \lambda\mu_0(\xi)]d\xi + \int_0^s e^{\lambda\xi}[f(\xi,\mu_0(\xi)) + \lambda\mu_0(\xi)]d\xi \\ &= \int_0^s \frac{e^{\lambda(a+\xi)}}{e^{\lambda a}-1}[f(\xi,\mu_0(\xi)) + \lambda\mu_0(\xi)]d\xi + \int_s^a \frac{e^{\lambda\xi}}{e^{\lambda a}-1}[f(\xi,\mu_0(\xi)) + \lambda\mu_0(\xi)]d\xi; \end{aligned}$$

so,

$$\begin{aligned}\mu_0(s) &\le \int_0^s \frac{e^{\lambda(a+\xi-s)}}{e^{\lambda a}-1}[f(\xi,\mu_0(\xi))+\lambda\mu_0(\xi)]d\xi + \int_s^a \frac{e^{\lambda(\xi-s)}}{e^{\lambda a}-1}[f(\xi,\mu_0(\xi))+\lambda\mu_0(\xi)]d\xi \\ &= \int_0^a \mathcal{L}(s,\xi)[f(\xi,\mu_0(\xi))+\lambda\mu_0(\xi)]d\xi \\ &= (\mathcal{H}\mu_0)(s), \quad \forall\, s\in[0,a].\end{aligned}$$

It follows that $(\mu_0,\mathcal{H}\mu_0)\in\Gamma$; so, $\mathcal{D}(\mathcal{H},\Gamma)$ is nonempty.

(iii) Let $\mu,\vartheta\in\mathcal{D}$ such that $(\mu,\vartheta)\in\Gamma$. Using (17), we obtain

$$f(s,\mu(s))+\lambda\mu(s)\le f(s,\vartheta(s))+\lambda\vartheta(s), \quad \forall\, s\in[0,a]. \tag{25}$$

Making use of (20) and (25), and owing to $\mathcal{L}(s,\xi)>0$, $\forall\, s,\xi\in[0,a]$, we obtain

$$\begin{aligned}(\mathcal{H}\mu)(s) &= \int_0^a \mathcal{L}(s,\xi)[f(\xi,\mu(\xi))+\lambda\mu(\xi)]d\xi \\ &\le \int_0^a \mathcal{L}(s,\xi)[f(\xi,\vartheta(\xi))+\lambda\vartheta(\xi)]d\xi \\ &= (\mathcal{H}\vartheta)(s), \quad \forall\, s\in[0,a],\end{aligned}$$

which making use of (22) reduces to $(\mathcal{H}\mu,\mathcal{H}\vartheta)\in\Gamma$. Therefore, $\Gamma$ is $\mathcal{H}$-closed. Moreover, $\Gamma$ is locally finitely $\mathcal{H}$-transitive.

(iv) We take an $\Gamma$-preserving sequence $\{\mu_n\}\subset\mathcal{D}$ converging to $\mu\in\mathcal{D}$. Then, for every $s\in[0,a]$, $\{\mu_n(s)\}\uparrow\mu(s)$ in $\mathbb{R}$. Thus, $\forall\, n\in\mathbb{N}$ and $\forall\, s\in[0,a]$, and we obtain $\mu_n(s)\le\mu(s)$. Now, by (22), we have $(\mu_n,\mu)\in\Gamma$, $\forall\, n\in\mathbb{N}$; hence, $\Gamma$ is $\sigma$-self-closed.

(v) Let $\mu,\vartheta\in\mathcal{D}$ be two elements such that $(\mu,\vartheta)\in\Gamma$. Then using (17), (20), and (21), we obtain

$$\begin{aligned}\sigma(\mathcal{H}\mu,\mathcal{H}\vartheta) &= \sup_{s\in[0,a]}|(\mathcal{H}\mu)(s)-(\mathcal{H}\vartheta)(s)| = \sup_{s\in[0,a]}\big((\mathcal{H}\vartheta)(s)-(\mathcal{H}\mu)(s)\big) \\ &\le \sup_{s\in[0,a]}\int_0^a \mathcal{L}(s,\xi)[f(\xi,\vartheta(\xi))+\lambda\vartheta(\xi)-f(\xi,\mu(\xi))-\lambda\mu(\xi)]d\xi \\ &\le \sup_{s\in[0,a]}\int_0^a \mathcal{L}(s,\xi)\beta\sqrt{\frac{[\vartheta(\xi)-\mu(\xi)]^2}{[\vartheta(\xi)-\mu(\xi)]^2+1}}d\xi.\end{aligned}$$

Making use of the Cauchy–Schwarz inequality in the last integral, the above inequality reduces to

$$\sigma(\mathcal{H}\mu,\mathcal{H}\vartheta)\le\sup_{s\in[0,a]}\left[\int_0^a \mathcal{L}(s,\xi)^2 d\xi\right]^{1/2}\cdot\left[\int_0^a \beta^2\frac{[\vartheta(\xi)-\mu(\xi)]^2}{[\vartheta(\xi)-\mu(\xi)]^2+1}d\xi\right]^{1/2}. \tag{26}$$

The first integral in the right hand side of (26) gives rise to

$$\begin{aligned}\int_0^a \mathcal{L}(s,\xi)^2 d\xi &= \int_0^s \mathcal{L}(s,\xi)^2 d\xi+\int_s^a \mathcal{L}(s,\xi)^2 d\xi \\ &= \int_0^s \frac{e^{2\lambda(a+\xi-s)}}{(e^{\lambda a}-1)^2}d\xi+\int_s^a \frac{e^{2\lambda(\xi-s)}}{(e^{\lambda a}-1)^2}d\xi \\ &= \frac{1}{2\lambda(e^{\lambda a}-1)^2}\cdot(e^{2\lambda a}-1) \\ &= \frac{e^{\lambda a}+1}{2\lambda(e^{\lambda a}-1)}.\end{aligned} \tag{27}$$

The second integral in the right hand side of (26) provides the following estimate:

$$\int_0^a \beta^2 \frac{[\vartheta(\xi)-\mu(\xi)]^2}{[\vartheta(\xi)-\mu(\xi)]^2+1} d\xi \leq \beta^2 \sup_{s\in[0,a]} \frac{|\vartheta(s)-\mu(s)|^2}{|\vartheta(s)-\mu(s)|^2+1} \cdot a = \beta^2 \frac{\sigma(\mu,\vartheta)^2}{\sigma(\mu,\vartheta)^2+1} \cdot a. \quad (28)$$

Using (27) and (28), inequality (26) becomes

$$\sigma(\mathcal{H}\mu,\mathcal{H}\vartheta) \leq \sup_{s\in[0,a]} \left[\frac{e^{\lambda a}+1}{2\lambda(e^{\lambda a}-1)}\right]^{1/2} \cdot \left[\beta^2 \frac{\sigma(\mu,\vartheta)^2}{\sigma(\mu,\vartheta)^2+1} \cdot a\right]^{1/2} = \left[\frac{e^{\lambda a}+1}{2\lambda(e^{\lambda a}-1)}\right]^{1/2} \cdot \beta \cdot \sqrt{a} \cdot \left[\frac{\sigma(\mu,\vartheta)^2}{\sigma(\mu,\vartheta)^2+1}\right]^{1/2};$$

so,

$$\sigma(\mathcal{H}\mu,\mathcal{H}\vartheta)^2 \leq \frac{e^{\lambda a}+1}{2\lambda(e^{\lambda a}-1)} \cdot \beta^2 \cdot a \cdot \frac{\sigma(\mu,\vartheta)^2}{\sigma(\mu,\vartheta)^2+1},$$

or equivalently,

$$2\lambda(e^{\lambda a}-1)[\sigma(\mu,\vartheta)^2+1]\sigma(\mathcal{H}\mu,\mathcal{H}\vartheta)^2 \leq (e^{\lambda a}+1)\beta^2 \cdot a \cdot \sigma(\mu,\vartheta)^2. \quad (29)$$

Using assumption (17), inequality (29) reduces to

$$2\lambda(e^{\lambda a}-1)[\sigma(\mu,\vartheta)^2+1]\sigma(\mathcal{H}\mu,\mathcal{H}\vartheta)^2 \leq (e^{\lambda a}+1)\frac{2\lambda(e^{\lambda a}-1)}{a(e^{\lambda a}+1)} \cdot a \cdot \sigma(\mu,\vartheta)^2,$$

i.e.,

$$\sigma(\mathcal{H}\mu,\mathcal{H}\vartheta)^2 \leq \frac{\sigma(\mu,\vartheta)^2}{\sigma(\mu,\vartheta)^2+1}. \quad (30)$$

We define $\theta,\eta : [0,\infty) \to [0,\infty)$ as follows:

$$\theta(z) = z^2 \quad \text{and} \quad \eta(z) = \frac{z^2}{z^2+1}.$$

Then, $(\theta,\eta)$ are shifting distance functions. Therefore, (30) becomes

$$\theta(\sigma(\mathcal{H}\mu,\mathcal{H}\vartheta) \leq \eta(\sigma(\mu,\vartheta))\theta(z), \quad \forall\, \mu,\vartheta \in \mathcal{D}, \text{verifying } (\mu,\vartheta) \in \Gamma.$$

Thus, the assumptions (i)–(v) of Theorem 3 have been satisfied. Now, we verify the hypotheses of Theorem 4.

Let $\mu,\vartheta \in \mathcal{D}$ be arbitrary. We denote $\varpi := \max\{\mathcal{H}\mu,\mathcal{H}\vartheta\} \in \mathcal{D}$. As $(\mathcal{H}\mu,\varpi) \in \Gamma$ and $(\mathcal{H}\vartheta,\varpi) \in \Gamma$, $\{\mathcal{H}\mu,\varpi,\mathcal{H}\vartheta\}$ forms a path in $\Gamma^s$ between $\mathcal{H}(\mu)$ and $\mathcal{H}(\vartheta)$. Therefore, $\mathcal{H}(\mathcal{D})$ is $\Gamma^s$-connected. Consequently, by Theorem 4, $\mathcal{H}$ has a unique fixed point, which is indeed the unique solution to Problem (15). □

## 6. Conclusions

We have proved fixed point theorems for a relation-theoretic contraction mapping using shifting distance functions. As future work, one can extend such results for a pair of self-mappings by proving coincidence and common fixed point theorems.

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