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A New Hardy–Hilbert-Type Integral Inequality Involving One Multiple Upper Limit Function and One Derivative Function of Higher Order

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Abstract: Using weight functions and parameters, as well as applying real analytic techniques, we derive a new Hardy–Hilbert-type integral inequality with the homogeneous kernel $\frac{1}{(x+y)^{\lambda+n}}$ involving one multiple upper limit function and one derivative function of higher order. Certain equivalent statements of the optimal constant factor related to some parameters are considered. A few particular inequalities and the case of reverses are also provided.

Keywords: weight function; Hardy–Hilbert-type integral inequality; multiple upper limit function; derivative function of higher order; parameter; gamma function; reverse

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1. Introduction

Assuming that $p > 1, \frac{1}{p} + \frac{1}{q} = 1, a_m, b_n \geq 0,$

$$0 < \sum_{m=1}^{\infty} a_m^p < \infty$$

and

$$0 < \sum_{n=1}^{\infty} b_n^q < \infty,$$

the following Hardy–Hilbert inequality with the optimal constant factor $\pi/\sin(\frac{\pi}{p})$ has been proven (cf. [1], Theorem 315):

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)}\left(\sum_{m=1}^{\infty} a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty} b_n^q\right)^{\frac{1}{q}}. \tag{1}$$

If $f(x), g(y) \geq 0,$

$$0 < \int_0^{\infty} f(x)dx < \infty$$

and

$$0 < \int_0^{\infty} g(y)dy < \infty,$$

then we still have the following integral analogue of (1) known as Hardy–Hilbert integral inequality (cf. [1], Theorem 316):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}dxdy < \frac{\pi}{\sin(\pi/p)}\left(\int_0^\infty f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty g^q(y)dy\right)^{\frac{1}{q}}, \quad (2)$$

where the identical constant factor $\pi/\sin(\frac{\pi}{p})$ remains optimal. Inequalities (1) and (2) have proven to be essential in various applications of mathematical analysis (cf. [2–13]).

In 2006, applying the Euler–Maclaurin summation formula, Krnic et al. [14] established an extension of (1) with the kernel $\frac{1}{(m+n)^\lambda}(0 < \lambda \leq 4)$. Making use of the result of [14], in 2019, Adiyasuren et al. [15] considered an extension of (1), which involved two partial sums, and subsequently, in 2020, Mo et al. [16] proved an extension of (2), which involved two upper-limit functions. In 2016–2017, Hong et al. [17,18] presented several equivalent statements of the extensions of (1) and (2) with the best possible constant factors and multi-parameters. Some similar results were established in [19–27].

In the present paper, following the methods of [15,17], using weight functions and parameters, as well as applying real analytic techniques, we prove a new Hardy–Hilbert-type integral inequality with the kernel $\frac{1}{(x+y)^{\lambda+n}}$ involving one multiple upper limit function and one derivative function of higher order. Equivalent statements of the best possible constant factor related to the parameters are considered. Some particular inequalities and the case of reverses are obtained. The lemmas and theorems provide an extensive account of this type of inequalities.

2. Some Lemmas

In what follows, we suppose that $p > 0$ $(p \neq 1)$, $\frac{1}{p} + \frac{1}{q} = 1$, $m, n \in \mathbf{N} = \{1, 2, \cdots\}$, $\lambda > -\min\{m, n\}$, $-m < \lambda_1 < \lambda$, $0 < \lambda_2 < \lambda + m$, $\widehat{\lambda}_1 := \frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q}$, $\widehat{\lambda}_2 := \frac{\lambda-\lambda_1}{q} + \frac{\lambda_2}{p}$ and the following

Assumption (I):

For $F_0(x) := f(x)$, being a non-negative continuous function, except for finitely many points in $\mathbf{R}_+ = (0, \infty)$, satisfying

$$f(x) = o(e^{tx}) \ \ (t > 0; x \to \infty),$$

$$F_k(x) := \int_0^x F_{k-1}(x)dx \ \ (k = 1, \cdots, m),$$

$g^{(n)}(y)$ is a non-negative continuous function, except for finite points in $\mathbf{R}_+$, satisfying

$$g^{(n)}(y) = o(e^{ty}) \ \ (t > 0; y \to \infty),$$

$$g^{(0)}(y) = g(y),$$

$g^{(j-1)}(y)$ is a non-negative differentiable function in $\mathbf{R}_+$ with

$$g^{(j-1)}(0^+) = 0 \ \ (j = 1, \cdots, n).$$

We also assume that

$$0 < \int_0^\infty x^{p(1-\widehat{\lambda}_1-m)-1}F_m^p(x)dx < \infty \ \text{ and } \ 0 < \int_0^\infty y^{q(1-\widehat{\lambda}_2)-1}(g^{(n)}(y))^q dy < \infty.$$

Note: According to Assumption (I), since $f(x) \geq 0$, $F_k(x)$ is increasing and $F_k(\infty) = \infty$ or constant. If there exists a last constant $k_0 \leq k$ such that $F_{k_0}(\infty)$ =constant, then

$$\lim_{x\to\infty}\frac{F_k(x)}{e^{tx}} = \cdots = \frac{1}{t^{k-k_0}}\lim_{x\to\infty}\frac{F_{k_0}(x)}{e^{tx}} = 0;$$

otherwise, we still have

$$\lim_{x\to\infty}\frac{F_k(x)}{e^{tx}}=\cdots=\frac{1}{t^k}\lim_{x\to\infty}\frac{f(x)}{e^{tx}}=0,$$

namely,

$$F_k(x)=o(e^{tx})\ \ (t>0;x\to\infty)\ \ (k=1,\cdots,m).$$

In the same way, we still can show that

$$g^{(j-1)}(y)=o(e^{ty})\ \ (t>0;\ y\to\infty)\ \ (j=1,\cdots,n).$$

We define the gamma function as follows:

$$\Gamma(\alpha):=\int_0^\infty e^{-t}t^{\alpha-1}dt\ (\alpha>0), \tag{3}$$

satisfying $\Gamma(\alpha+1)=\alpha\Gamma(\alpha)(\alpha>0)$, and define the following beta function (cf. [28]):

$$B(u,v):=\int_0^\infty \frac{t^{u-1}}{(1+t)^{u+v}}dt=\frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}(u,v>0). \tag{4}$$

According to (3), for $\lambda,x,y>0$, we still have the following formula related to the gamma function:

$$\frac{1}{(x+y)^\lambda}=\frac{1}{\Gamma(\lambda)}\int_0^\infty e^{-(x+y)t}t^{\lambda-1}dt. \tag{5}$$

Lemma 1. *For $t>0,m,n\in\mathbf{N}$, we have the following expressions:*

$$\int_0^\infty e^{-tx}f(x)dx = t^m\int_0^\infty e^{-tx}F_m(x)dx, \tag{6}$$

$$\int_0^\infty e^{-ty}g(y)dy = t^{-n}\int_0^\infty e^{-ty}g^{(n)}(y)dy. \tag{7}$$

Proof. According to Assumption (I), for $k=1,\cdots,m$, we get that $e^{-tx}F_k(x)|_0^\infty=0$, then integration by parts,

$$\begin{aligned}&\int_0^\infty e^{-tx}F_{k-1}(x)dx\\ =&\int_0^\infty e^{-tx}dF_k(x)=e^{-tx}F_k(x)|_0^\infty-\int_0^\infty F_k(x)de^{-tx}\\ =&\ t\int_0^\infty e^{-tx}F_k(x)(x)dx.\end{aligned}$$

Substituting $k=1,\ldots,m$ in the above expression, by simplification, we obtain (6).

According to Assumption (I), for $j=1,\cdots,n$, we get that $e^{-ty}g^{(j-1)}(y)|_0^\infty=0$, and

$$\begin{aligned}&\int_0^\infty e^{-ty}g^{(j)}(y)dy\\ =&\int_0^\infty e^{-ty}dg^{(j-1)}(y)=e^{-ty}g^{(j-1)}(y)|_0^\infty-\int_0^\infty g^{(j-1)}(y)de^{-ty}\\ =&\ t\int_0^\infty e^{-ty}g^{(j-1)}(y)dy.\end{aligned}$$

Substituting $j=1,\cdots,n$ in the above expression, we obtain (7).

This completes the proof of the lemma. □

Note: (6) (resp. (7)) is naturally the value for $m=0$ (resp. $n=0$).

Lemma 2. *For $0 < s_1, s_2 < s$, define the following weight functions:*

$$\omega_s(s_2, x) \quad : \quad = x^{s-s_2} \int_0^\infty \frac{t^{s_2-1}}{(x+t)^s} dt (x \in \mathbf{R}_+), \tag{8}$$

$$\omega_s(s_1, y) \quad : \quad = y^{s-s_1} \int_0^\infty \frac{t^{s_1-1}}{(x+t)^s} dt (y \in \mathbf{R}_+). \tag{9}$$

We have the following expressions:

$$\omega_s(s_2, x) \quad := \quad B(s_2, s - s_2)(x \in \mathbf{R}_+), \tag{10}$$

$$\omega_s(s_1, y) \quad := \quad B(s_1, s - s_1)(y \in \mathbf{R}_+). \tag{11}$$

Proof. Setting $u = \frac{t}{x}$, we derive

$$\begin{aligned} \omega_s(s_2, x) &= x^{s-s_2} \int_0^\infty \frac{(ux)^{s_2-1}}{(x+ux)^s} x du \\ &= \int_0^\infty \frac{u^{s_2-1} du}{(1+u)^s} = B(s_2, s - s_2), \end{aligned}$$

namely, (10) follows. In the same way, we obtain (11).

This completes the proof of the lemma. □

Lemma 3. *Suppose that $\lambda > -m, -m < \lambda_1 < \lambda, 0 < \lambda_2 < \lambda + m$.*
(i) For $p > 1$, we have the following extended Hardy–Hilbert integral inequality:

$$\begin{aligned} I_{\lambda+m} &:= \int_0^\infty \int_0^\infty \frac{F_m(x) g^{(n)}(y)}{(x+y)^{\lambda+m}} dx dy \\ &< B^{\frac{1}{p}}(\lambda_2, \lambda + m - \lambda_2) B^{\frac{1}{q}}(\lambda_1 + m, \lambda - \lambda_1) \\ &\quad \times \left[\int_0^\infty x^{p(1-\widehat{\lambda}_1 - m)-1} F_m^p(x) dx \right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\widehat{\lambda}_2)-1} (g^{(n)}(y))^q dy \right]^{\frac{1}{q}}; \end{aligned} \tag{12}$$

(ii) for $0 < p < 1$, we have the reverse of (13).

Proof. (i) By Hölder's inequality (cf. [29]), we obtain

$$\begin{aligned} I_{\lambda+m} &= \int_0^\infty \int_0^\infty \frac{1}{(x+y)^{\lambda+m}} \left[\frac{y^{(\lambda_2-1)/p}}{x^{(\lambda_1+m-1)/q}} F_m(x) \right] \\ &\quad \times \left[\frac{x^{(\lambda_1+m-1)/q}}{y^{(\lambda_2-1)/p}} g^{(n)}(y) \right] dx dy \\ &\leq \left\{ \int_0^\infty \left[\int_0^\infty \frac{1}{(x+y)^{\lambda+m}} \frac{y^{\lambda_2-1}}{x^{(\lambda_1+m-1)(p-1)}} dy \right] F_m^p(x) dx \right\}^{\frac{1}{p}} \\ &\quad \times \left\{ \int_0^\infty \left[\int_0^\infty \frac{1}{(x+y)^{\lambda+m}} \frac{x^{\lambda_1+m-1}}{y^{(\lambda_2-1)(q-1)}} dx \right] (g^{(n)}(y))^q dy \right\}^{\frac{1}{q}} \\ &= \left[\int_0^\infty \omega_{\lambda+m}(\lambda_2, x) x^{p(1-\widehat{\lambda}_1-m)-1} F_m^p(x) dx \right]^{\frac{1}{p}} \\ &\quad \times \left[\int_0^\infty \omega_{\lambda+m}(\lambda_1 + m, y) y^{q(1-\widehat{\lambda}_2)-1} (g^{(n)}(y))^q dy \right]^{\frac{1}{q}}. \end{aligned} \tag{13}$$

If (14) retains the form of equality, then, there exist constants A and B such that they are not both zero, satisfying

$$A\frac{y^{\lambda_2-1}}{x^{(\lambda_1+m-1)(p-1)}}F_m^p(x)=B\frac{x^{\lambda_1+m-1}}{y^{(\lambda_2-1)(q-1)}}(g^{(n)}(y))^q \ \ a.e. \ \text{in} \ \mathbf{R}_+^2.$$

Assuming that $A\neq 0$, for fixed $a,e,y\in\mathbf{R}_+$, we have

$$x^{p(1-\widehat{\lambda}_1-m)-1}F_m^p(x)=[\frac{B}{A}y^{q(1-\widehat{\lambda}_2)}(g^{(n)}(y))^q]x^{-1-(\lambda-\lambda_1-\lambda_2)} \ \ a.e. \ \text{in} \ \mathbf{R}_+.$$

Since for any $a:=\lambda-\lambda_1-\lambda_2\in\mathbf{R}$, $\int_0^\infty x^{-1-a}dx=\infty$, the above expression contradicts the fact that

$$0<\int_0^\infty x^{p(1-\widehat{\lambda}_1-m)-1}F_m^p(x)dx<\infty.$$

Therefore, by (10) and (11), setting $s=\lambda+m\ (>0), s_1=\lambda_1+m\ (\in(0,\lambda+m))$, $s_2=\lambda_2\ (\in(0,\lambda+m))$, in view of (14), we have (13).

(ii) Similarly, according to the reverse Hölder inequality, we obtain the reverse of (13).

This completes the proof of the lemma. □

3. Main Results

Theorem 1. *Suppose that* $\lambda>-\min\{m,n\}$, $-m<\lambda_1<\lambda, 0<\lambda_2<\lambda+m$.
(i) For $p>1$, *we have the following Hardy–Hilbert-type integral inequality involving one multiple upper limit function and one derivative function of higher order:*

$$\begin{aligned} I &:= \int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x+y)^{\lambda+n}}dxdy \\ &< \frac{\Gamma(\lambda+m)}{\Gamma(\lambda+n)}B^{\frac{1}{p}}(\lambda_2,\lambda+m-\lambda_2)B^{\frac{1}{q}}(\lambda_1+m,\lambda-\lambda_1) \\ &\quad\times\left[\int_0^\infty x^{p(1-\widehat{\lambda}_1-m)-1}F_m^p(x)dx\right]^{\frac{1}{p}}\left[\int_0^\infty y^{q(1-\widehat{\lambda}_2)-1}(g^{(n)}(y))^q dy\right]^{\frac{1}{q}}. \end{aligned} \tag{14}$$

(ii) For $0<p<1$, *we obtain the reverse of* (14).

In particular, for $\lambda_1+\lambda_2=\lambda\ (0<\lambda_1,\lambda_2<\lambda)$, we reduce (14) to the following:

$$\begin{aligned} I &= \int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x+y)^{\lambda+n}}dxdy<\frac{\Gamma(\lambda+m)}{\Gamma(\lambda+n)}B(\lambda_1+m,\lambda_2) \\ &\quad\times\left[\int_0^\infty x^{p(1-\lambda_1-m)-1}F_m^p(x)dx\right]^{\frac{1}{p}}\left[\int_0^\infty y^{q(1-\lambda_2)-1}(g^{(n)}(y))^q dy\right]^{\frac{1}{q}}, \end{aligned} \tag{15}$$

where the constant factor $\frac{\Gamma(\lambda+m)}{\Gamma(\lambda+n)}B(\lambda_1+m,\lambda_2)$ is the best possible.

Proof. (i) In view of (6), (7) and Fubini's theorem (cf. [30]), we obtain

$$\begin{aligned} I &= \frac{1}{\Gamma(\lambda+n)}\int_0^\infty\int_0^\infty f(x)g(y)\left[\int_0^\infty t^{\lambda+n-1}e^{-(x+y)t}dt\right]dxdy \\ &= \frac{1}{\Gamma(\lambda+n)}\int_0^\infty t^{\lambda+n-1}\left(\int_0^\infty e^{-xt}f(x)dx\right)\left(\int_0^\infty g(y)e^{-yt}dy\right)dt \\ &= \frac{1}{\Gamma(\lambda+n)}\int_0^\infty t^{\lambda+n-1}\left(t^m\int_0^\infty e^{-xt}F_m(x)dx\right)\left(t^{-n}\int_0^\infty g^{(n)}(y)e^{-yt}dy\right)dt \\ &= \frac{1}{\Gamma(\lambda+n)}\int_0^\infty\int_0^\infty F_m(x)g^{(n)}(y)\left[\int_0^\infty t^{\lambda+m-1}e^{-(x+y)t}dt\right]dxdy \\ &= \frac{\Gamma(\lambda+m)}{\Gamma(\lambda+n)}\int_0^\infty\int_0^\infty \frac{F_m(x)g^{(n)}(y)}{(x+y)^{\lambda+m}}dxdy=\frac{\Gamma(\lambda+m)I_{\lambda+m}}{\Gamma(\lambda+n)}. \end{aligned} \tag{16}$$

Then, according to (13), we obtain (14).

(ii) According to (17) and the reverse of (13), we derive the reverse of (14).
For $\lambda_1 + \lambda_2 = \lambda$ $(0 < \lambda_1, \lambda_2 < \lambda)$ in (14), we deduce (15).
For any $0 < \varepsilon < \min\{p\lambda_1, q\lambda_2\}$, we set the following functions:

$$\begin{aligned}\widetilde{F}_0(x) &= \widetilde{f}(x) := \begin{cases} 0, 0 < x < 1, \\ x^{\lambda_1 - \frac{\varepsilon}{p} - 1}, x \geq 1 \end{cases}, \\ \widetilde{F}_k(x) &= \int_0^x \left(\int_0^{t_{k-1}} \cdots \int_0^{t_2} \widetilde{f}(t_1) dt_1 \cdots dt_{k-1} \right) dt_k \; (k = 1, \cdots, m),\end{aligned}$$

and then $\widetilde{F}_m(x) = 0, 0 < x < 1$,

$$\begin{aligned}\widetilde{F}_m(x) &= \int_1^x \left(\int_1^{t_{m-1}} \cdots \int_1^{t_2} t_1{}^{\lambda_1 - \frac{\varepsilon}{p} - 1} dt_1 \cdots dt_{m-1} \right) dt_m \\ &= [\prod_{i=0}^{m-1} (\lambda_1 - \frac{\varepsilon}{p} + i)]^{-1} (x^{\lambda_1 - \frac{\varepsilon}{p} + m - 1} - O_1(x^{m-1})) \\ &\leq [\prod_{i=0}^{m-1} (\lambda_1 - \frac{\varepsilon}{p} + i)]^{-1} x^{\lambda_1 - \frac{\varepsilon}{p} + m - 1} (x \geq 1),\end{aligned}$$

where $O_1(x^{m-1})$ $(x \geq 1)$ is a positive polynomial of $(m-1)$-degree. We also set

$$\begin{aligned}\widetilde{g}^{(n)}(y) &= \begin{cases} 0, 0 < y \leq 1, \\ \prod_{i=0}^{n-1} (\lambda_2 + j - \frac{\varepsilon}{q}) y^{\lambda_2 - \frac{\varepsilon}{q} - 1}, y > 1 \end{cases}, \\ \widetilde{g}^{(l)}(y) &= \prod_{j=0}^{l-1} (\lambda_2 + j - \frac{\varepsilon}{q}) \int_0^y \left(\int_0^{t_{n-l-1}} \cdots \int_0^{t_2} \widetilde{g}^{(n)}(t_1) dt_1 \cdots dt_{n-l-1} \right) dt_{n-l} \\ (l &= 1, \cdots, n),\end{aligned}$$

and $\widetilde{g}(y) = 0$ $(0 < y \leq 1)$,

$$\begin{aligned}\widetilde{g}(y) &= \prod_{j=0}^{n-1} (\lambda_2 + j - \frac{\varepsilon}{q}) \int_1^y \left(\int_1^{t_{n-1}} \cdots \int_1^{t_2} t_1{}^{\lambda_2 - \frac{\varepsilon}{q} - 1} dt_1 \cdots dt_{n-1} \right) dt_n \\ &= y^{\lambda_2 - \frac{\varepsilon}{q} + n - 1} - O_2(y^{n-1}) \leq y^{\lambda_2 - \frac{\varepsilon}{q} + n - 1} \; (y > 1),\end{aligned}$$

where $O_2(y^{n-1})$ $(y > 1)$ is a positive polynomial of $n-1$-degree.

We observe that $\widetilde{F}_k(x)$ $(k = 0, \cdots, m)$ and $\widetilde{g}^{(l)}(y)$ $(l = 0, \cdots, n)$ all satisfy Assumption (I) on $F_k, g^{(l)}$.

If there exists a positive constant

$$M \leq \frac{\Gamma(\lambda + m)}{\Gamma(\lambda + n)} B(\lambda_1 + m, \lambda_2)$$

such that (15) is valid when we replace $\frac{\Gamma(\lambda+m)}{\Gamma(\lambda+n)} B(\lambda_1 + m, \lambda_2)$ with M, then, in particular, since

$$\begin{aligned}\widetilde{J} \; : \; &= \left[\int_0^\infty x^{p(1-\lambda_1-m)-1} \widetilde{F}_m^p(x)) dx \right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\lambda_2)-1} (\widetilde{g}^{(n)}(y))^q dy \right]^{\frac{1}{q}} \\ &\leq [\prod_{i=0}^{m-1} (\lambda_1 + i - \frac{\varepsilon}{p})]^{-1} \prod_{j=0}^{n-1} (\lambda_2 + j - \frac{\varepsilon}{q}) \int_1^\infty x^{-\varepsilon-1} dx \\ &= \frac{1}{\varepsilon} [\prod_{i=0}^{m-1} (\lambda_1 + i - \frac{\varepsilon}{p})]^{-1} \prod_{j=0}^{n-1} (\lambda_2 + j - \frac{\varepsilon}{q}),\end{aligned}$$

we have

$$\begin{aligned}\widetilde{I} &:= \int_0^\infty \int_0^\infty \frac{\widetilde{f}(x)\widetilde{g}(y)}{(x+y)^{\lambda+n}}dxdy \\ &< M\widetilde{J} = \frac{M}{\varepsilon}[\prod_{i=0}^{m-1}(\lambda_1 + i - \frac{\varepsilon}{p})]^{-1}\prod_{j=0}^{n-1}(\lambda_2 + j - \frac{\varepsilon}{q}).\end{aligned}$$

In view of Fubini's theorem (cf. [30]), it follows that

$$\widetilde{I} = \int_1^\infty \left[\int_1^\infty \frac{y^{\lambda_2+n-\frac{\varepsilon}{q}-1} - O_2(y^{n-1})}{(x+y)^{\lambda+n}}dy\right]x^{\lambda_1-\frac{\varepsilon}{p}-1}dx = I_0 - I_1,$$

where

$$\begin{aligned}I_0 &= \int_1^\infty \left[\int_1^\infty \frac{y^{\lambda_2+n-\frac{\varepsilon}{q}-1}}{(x+y)^{\lambda+n}}dy\right]x^{\lambda_1+m-\frac{\varepsilon}{p}-1}dx \\ &= \int_1^\infty x^{-\varepsilon-1}\left[\int_{\frac{1}{x}}^\infty \frac{u^{\lambda_2+n-\frac{\varepsilon}{q}-1}}{(1+u)^{\lambda+n}}du\right]dx \\ &= \int_1^\infty x^{-\varepsilon-1}\left[\int_{\frac{1}{x}}^1 \frac{u^{\lambda_2+n-\frac{\varepsilon}{q}-1}}{(1+u)^{\lambda+n}}du\right]dx + \frac{1}{\varepsilon}\int_1^\infty \frac{u^{\lambda_2+n-\frac{\varepsilon}{q}-1}}{(1+u)^{\lambda+n}}du \\ &= \int_0^1 \left(\int_{\frac{1}{u}}^1 x^{-\varepsilon-1}dx\right)\frac{u^{\lambda_2+n-\frac{\varepsilon}{q}-1}}{(1+u)^{\lambda+n}}du + \frac{1}{\varepsilon}\int_1^\infty \frac{u^{\lambda_2+n-\frac{\varepsilon}{q}-1}}{(1+u)^{\lambda+n}}du \\ &= \frac{1}{\varepsilon}\left[\int_0^1 \frac{u^{\lambda_2+n-\frac{\varepsilon}{q}-1}}{(1+u)^{\lambda+n}}du + \int_1^\infty \frac{u^{\lambda_2+n-\frac{\varepsilon}{q}-1}}{(1+u)^{\lambda+n}}du\right],\end{aligned}$$

$$\begin{aligned}0 < I_1 &:= \int_1^\infty \left[\int_1^\infty \frac{O_2(y^{n-1})}{(x+y)^{\lambda+n}}dy\right]x^{\lambda_1-\frac{\varepsilon}{p}-1}dx \\ &= \int_1^\infty \left[\int_1^\infty \frac{O_2(y^{n-1})}{(x+y)^{(\lambda_2/2)+n}}dy\right]\frac{x^{\lambda_1-\frac{\varepsilon}{p}-1}}{(x+y)^{\lambda_1+(\lambda_2/2)}}dx \\ &\le \int_1^\infty \left[\int_1^\infty \frac{O_2(y^{n-1})}{y^{(\lambda_2/2)+n}}dy\right]\frac{x^{\lambda_1-\frac{\varepsilon}{p}-1}}{x^{\lambda_1+(\lambda_2/2)}}dx \\ &= \int_1^\infty x^{-\frac{\lambda_2}{2}-\frac{\varepsilon}{p}-1}dx\left[\int_1^\infty O(y^{-\frac{\lambda_2}{2}-1})dy\right] \le M_1 < \infty.\end{aligned}$$

According to the above results, it follows that

$$\begin{aligned}&\int_0^1 \frac{u^{\lambda_2+n-\frac{\varepsilon}{q}-1}}{(1+u)^{\lambda+n}}du + \int_1^\infty \frac{u^{\lambda_2+n-\frac{\varepsilon}{q}-1}}{(1+u)^{\lambda+n}}du - \varepsilon I_1 \\ &= \varepsilon\widetilde{I} < M[\prod_{i=0}^{m-1}(\lambda_1 + i - \frac{\varepsilon}{p})]^{-1}\prod_{j=0}^{n-1}(\lambda_2 + j - \frac{\varepsilon}{q}).\end{aligned}$$

Letting $\varepsilon \to 0^+$ in the above inequality, in view of the continuity of the beta function, we obtain

$$B(\lambda_1, \lambda_2 + n) \le M[\prod_{i=0}^{m-1}(\lambda_1 + i)]^{-1}\prod_{j=0}^{n-1}(\lambda_2 + j),$$

namely,

$$\frac{\Gamma(\lambda+m)}{\Gamma(\lambda+n)}B(\lambda_1+m,\lambda_2)=B(\lambda_1,\lambda_2+n)\prod_{i=0}^{m-1}(\lambda_1+i)[\prod_{j=0}^{n-1}(\lambda_2+j)]^{-1}\leq M$$

and then

$$M=\frac{\Gamma(\lambda+m)}{\Gamma(\lambda+n)}B(\lambda_1+m,\lambda_2)$$

is the best possible constant factor of (15).

This completes the proof of the theorem. □

Remark 1. *For* $\widehat{\lambda}_1=\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}$, $\widehat{\lambda}_2=\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p}$, *it follows that* $\widehat{\lambda}_1+\widehat{\lambda}_2=\lambda$. *We find* $0<\widehat{\lambda}_1<\frac{\lambda}{p}+\frac{\lambda}{q}=\lambda$, *then* $0<\widehat{\lambda}_2=\lambda-\widehat{\lambda}_1<\lambda$. *According to Hölder's inequality* (cf. [29]), *it follows that*

$$\begin{aligned}
0 &< B(\widehat{\lambda}_1+m,\widehat{\lambda}_2)=\int_0^\infty\frac{u^{\frac{\lambda+m-\lambda_2}{p}+\frac{\lambda_1+m}{q}-1}}{(1+u)^{\lambda+m}}du\\
&= \int_0^\infty\frac{1}{(1+u)^{\lambda+m}}\left(u^{\frac{\lambda+m-\lambda_2-1}{p}}\right)\left(u^{\frac{\lambda_1+m-1}{q}}\right)du\\
&\leq \left[\int_0^\infty\frac{u^{\lambda+m-\lambda_2-1}}{(1+u)^{\lambda+m}}du\right]^{\frac{1}{p}}\left[\int_0^\infty\frac{u^{\lambda_1+m-1}}{(1+u)^{\lambda+m}}du\right]^{\frac{1}{q}}\\
&= B^{\frac{1}{p}}(\lambda_2,\lambda+m-\lambda_2)B^{\frac{1}{q}}(\lambda_1+m,\lambda-\lambda_1)<\infty.
\end{aligned} \tag{17}$$

Theorem 2. *For* $p>1$, *if the constant factor*

$$\frac{\Gamma(\lambda+m)}{\Gamma(\lambda+n)}B^{\frac{1}{p}}(\lambda_2,\lambda+m-\lambda_2)B^{\frac{1}{q}}(\lambda_1+m,\lambda-\lambda_1)$$

in (14) *is the best possible, then for* $0<\lambda_1,\lambda_2<\lambda$, *we have* $\lambda_1+\lambda_2=\lambda$.

Proof. According to (15) (for $\lambda_i=\widehat{\lambda}_i(i=1,2)$), since

$$\frac{\Gamma(\lambda+m)}{\Gamma(\lambda+n)}B^{\frac{1}{p}}(\lambda_2,\lambda+m-\lambda_2)B^{\frac{1}{q}}(\lambda_1+m,\lambda-\lambda_1)$$

is the best possible constant factor in (14), we have

$$\begin{aligned}
&\frac{\Gamma(\lambda+m)}{\Gamma(\lambda+n)}B^{\frac{1}{p}}(\lambda_2,\lambda+m-\lambda_2)B^{\frac{1}{q}}(\lambda_1+m,\lambda-\lambda_1)\\
\leq\ &\frac{\Gamma(\lambda+m)}{\Gamma(\lambda+n)}B(\widehat{\lambda}_1+m,\widehat{\lambda}_2)\ (\in\mathbf{R}_+),
\end{aligned}$$

namely

$$B(\widehat{\lambda}_1+m,\widehat{\lambda}_2)\geq B^{\frac{1}{p}}(\lambda_2,\lambda+m-\lambda_2)B^{\frac{1}{q}}(\lambda_1+m,\lambda-\lambda_1).$$

It follows that (18) retains the form of equality.

We observe that (18) retains the form of equality if and only if there exist constants A and B such that they are not both zero and $Au^{\lambda+m-\lambda_2-1}=Bu^{\lambda_1+m-1}$ *a.e.* in $\mathbf{R}_+$ (cf. [29]). Assuming that $A\neq0$, it follows that $u^{\lambda-\lambda_2-\lambda_1}=B/A$ *a.e.* in $\mathbf{R}_+$, namely, $\lambda-\lambda_2-\lambda_1=0$. Hence, we have $\lambda_1+\lambda_2=\lambda$.

This completes the proof of the theorem. □

Theorem 3. *The following statements ((i), (ii), (iii) and (iv)) are equivalent:*

(i) *Both*

$$B^{\frac{1}{p}}(\lambda_2, \lambda + m + -\lambda_2) B^{\frac{1}{q}}(\lambda_1 + m, \lambda - \lambda_1)$$

and

$$B\left(\frac{\lambda - \lambda_2}{p} + \frac{\lambda_1}{q} + m, \frac{\lambda - \lambda_1}{q} + \frac{\lambda_2}{p}\right)$$

are independent of p, q*;*

(ii) $B^{\frac{1}{p}}(\lambda_2, \lambda + m - \lambda_2) B^{\frac{1}{q}}(\lambda_1 + m, \lambda - \lambda_1) = B(\frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q} + m, \frac{\lambda-\lambda_1}{q} + \frac{\lambda_2}{p})$*;*

(iii) *For* $0 < \lambda_1, \lambda_2 < \lambda$*, we have* $\lambda_1 + \lambda_2 = \lambda$*;*

(iv) *The constant factor*

$$\frac{\Gamma(\lambda + m)}{\Gamma(\lambda + n)} B^{\frac{1}{p}}(\lambda_2, \lambda + m - \lambda_2) B^{\frac{1}{q}}(\lambda_1 + m, \lambda - \lambda_1)$$

is the best possible in (14).

Proof. $(i) \Rightarrow (ii)$. In view of the continuity of the beta function, we derive

$$\begin{aligned}
& B^{\frac{1}{p}}(\lambda_2, \lambda + m - \lambda_2) B^{\frac{1}{q}}(\lambda_1 + m, \lambda - \lambda_1) \\
= & \lim_{p \to \infty} \lim_{q \to 1^+} B^{\frac{1}{p}}(\lambda_2, \lambda + m - \lambda_2) B^{\frac{1}{q}}(\lambda_1 + m, \lambda - \lambda_1) \\
= & B(\lambda_1 + m, \lambda - \lambda_1), \\
& B\left(\frac{\lambda - \lambda_2}{p} + \frac{\lambda_1}{q} + m, \frac{\lambda - \lambda_1}{q} + \frac{\lambda_2}{p}\right) \\
= & \lim_{p \to \infty} \lim_{q \to 1^+} B\left(\frac{\lambda - \lambda_2}{p} + \frac{\lambda_1}{q} + m, \frac{\lambda - \lambda_1}{q} + \frac{\lambda_2}{p}\right) \\
= & B(\lambda_1 + m, \lambda - \lambda_1).
\end{aligned}$$

Hence, we have

$$\begin{aligned}
& B^{\frac{1}{p}}(\lambda_2, \lambda + m - \lambda_2) B^{\frac{1}{q}}(\lambda_1 + m, \lambda - \lambda_1) \\
= & B\left(\frac{\lambda - \lambda_2}{p} + \frac{\lambda_1}{q} + m, \frac{\lambda - \lambda_1}{q} + \frac{\lambda_2}{p}\right).
\end{aligned}$$

$(ii) \Rightarrow (iii)$. In view of

$$B^{\frac{1}{p}}(\lambda_2, \lambda + m - \lambda_2) B^{\frac{1}{q}}(\lambda_1 + m, \lambda - \lambda_1) = B(\widehat{\lambda}_1, \widehat{\lambda}_2),$$

(17) retains the form of equality. In view of the proof of Theorem 2, we have

$$\lambda_1 + \lambda_2 = \lambda.$$

$(iii) \Rightarrow (iv)$. If $\lambda_1 + \lambda_2 = \lambda$ $(0 < \lambda_1, \lambda_2 < \lambda)$, then according to Theorem 1, the constant factor

$$\frac{\Gamma(\lambda + m)}{\Gamma(\lambda + n)} B^{\frac{1}{p}}(\lambda_2, \lambda + m - \lambda_2) B^{\frac{1}{q}}(\lambda_1 + m, \lambda - \lambda_1)$$

in (14) is the best possible.

$(iv) \Rightarrow (i)$. According to Theorem 2, we have $\lambda_1 + \lambda_2 = \lambda$; then,

$$\begin{aligned}
B^{\frac{1}{p}}(\lambda_2, \lambda + m - \lambda_2) B^{\frac{1}{q}}(\lambda_1 + m, \lambda - \lambda_1) &= B(\lambda_1 + m, \lambda_2), \\
B\left(\frac{\lambda - \lambda_2}{p} + \frac{\lambda_1}{q} + m, \frac{\lambda - \lambda_1}{q} + \frac{\lambda_2}{p}\right) &= B(\lambda_1 + m, \lambda_2),
\end{aligned}$$

both of which are independent of p, q.

Hence, statements (i), (ii), (iii) and (iv) are equivalent.

This completes the proof of the theorem. □

For $n = m$, we have:

Corollary 1. *For $p > 1$, we have the following Hardy–Hilbert-type integral inequality involving one multiple upper limit function and one derivative function of m-order:*

$$\begin{aligned} &\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^{\lambda+m}} dxdy \\ < \ & B^{\frac{1}{p}}(\lambda_2, \lambda+m-\lambda_2) B^{\frac{1}{q}}(\lambda_1+m, \lambda-\lambda_1) \\ &\times \left[\int_0^\infty x^{p(1-\widehat{\lambda}_1-m)-1} F_m^p(x) dx\right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\widehat{\lambda}_2)-1} (g^{(m)}(y))^q dy\right]^{\frac{1}{q}}. \end{aligned} \quad (18)$$

Moreover, for $0 < \lambda_1, \lambda_2 < \lambda$, the constant factor

$$B^{\frac{1}{p}}(\lambda_2, \lambda+m-\lambda_2) B^{\frac{1}{q}}(\lambda_1+m, \lambda-\lambda_1)$$

in (19) is the best possible if and only if $\lambda_1 + \lambda_2 = \lambda$. For

$$\lambda_1 + \lambda_2 = \lambda \ \ (0 < \lambda_1, \lambda_2 < \lambda),$$

we reduce (19) to the following:

$$\begin{aligned} &\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^{\lambda+m}} dxdy < B(\lambda_1+m, \lambda_2) \\ \times \ & \left[\int_0^\infty x^{p(1-\lambda_1-m)-1} F_m^p(x) dx\right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\lambda_2)-1} (g^{(m)}(y))^q dy\right]^{\frac{1}{q}}, \end{aligned} \quad (19)$$

where the constant factor $B(\lambda_1 + m, \lambda_2)$ is the best possible factor.

Remark 2. *(i) For $\lambda_1 + \lambda_2 = \lambda$ in (13), we have*

$$\begin{aligned} I_{\lambda+m} &= \int_0^\infty \int_0^\infty \frac{F_m(x) g^{(n)}(y)}{(x+y)^{\lambda+m}} dxdy < B(\lambda_1+m, \lambda_2) \\ &\times \left[\int_0^\infty x^{p(1-\lambda_1-m)-1} F_m^p(x) dx\right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\lambda_2)-1} (g^{(n)}(y))^q dy\right]^{\frac{1}{q}}. \end{aligned} \quad (20)$$

We confirm that the constant factor $B(\lambda_1 + m, \lambda_2)$ in (20) is the best possible. Otherwise, we would reach a contradiction according to (17) (for $\lambda_1 + \lambda_2 = \lambda$) that the constant factor in (15) is not the best possible.

(ii) In view of the note of Lemma 1, Theorem 1, Theorem 2 and Theorem 3 are valid for $m = 0$ or $n = 0$. For $m = n = 0, \lambda = 1, \lambda_1 = \frac{1}{q}, \lambda_2 = \frac{1}{p}$, both (15) and (20) reduce to (2).

Remark 3. *(i) For $\lambda = 1, \lambda_1 = \frac{1}{q}, \lambda_2 = \frac{1}{p}$ in (15), we have*

$$\begin{aligned} &\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^{1+n}} dxdy < \frac{\pi}{n! \sin(\frac{\pi}{p})} \prod_{i=0}^{m-1} (\frac{1}{q} + i) \\ \times \ & \left(\int_0^\infty x^{-pm} F_m^p(x) dx\right)^{\frac{1}{p}} \left[\int_0^\infty (g^{(n)}(y))^q dy\right]^{\frac{1}{q}}; \end{aligned} \quad (21)$$

(ii) For $\lambda = 1, \lambda_1 = \frac{1}{p}, \lambda_2 = \frac{1}{q}$ *in* (15), *we have the dual form of* (21) *as follows:*

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^{1+n}} dxdy < \frac{\pi}{n! \sin(\frac{\pi}{p})} \prod_{i=0}^{m-1} (\frac{1}{p} + i)$$
$$\times \left(\int_0^\infty x^{p(1-m)-2} F_m^p(x) dx \right)^{\frac{1}{p}} \left[\int_0^\infty y^{p-2} (g^{(n)}(y))^q dy \right]^{\frac{1}{q}}; \tag{22}$$

(iii) For $p = q = 2$, *both* (15) *and* (21) *reduce to*

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^{1+n}} dxdy$$
$$< \frac{\pi(2m-1)!!}{n!2^m} \left[\int_0^\infty x^{-2m} F_m^2(x) dx \int_0^\infty (g^{(n)}(y))^2 dy \right]^{\frac{1}{2}}. \tag{23}$$

The constant factors in the above inequalities are all the best possible.

4. The Reverses

According to Lemma 3 and Theorem 1 (ii), we have:

Theorem 4. *For* $0 < p < 1$ $(q < 0)$, *we have the following reverse Hardy–Hilbert-type integral inequality involving one multiple upper limit function and one derivative function of higher order:*

$$\begin{aligned} I &= \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^{\lambda+n}} dxdy \\ &> \frac{\Gamma(\lambda+m)}{\Gamma(\lambda+n)} B^{\frac{1}{p}}(\lambda_2, \lambda+m-\lambda_2) B^{\frac{1}{q}}(\lambda_1+m, \lambda-\lambda_1) \\ &\quad \times \left[\int_0^\infty x^{p(1-\widehat{\lambda}_1-m)-1} F_m^p(x) dx \right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\widehat{\lambda}_2)-1} (g^{(n)}(y))^q dy \right]^{\frac{1}{q}}. \end{aligned} \tag{24}$$

In particular, for $\lambda_1 + \lambda_2 = \lambda$ $(0 < \lambda_1, \lambda_2 < \lambda)$, we reduce (24) to the following:

$$\begin{aligned} I &= \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^{\lambda+n}} dxdy > \frac{\Gamma(\lambda+m)}{\Gamma(\lambda+n)} B(\lambda_1+m, \lambda_2) \\ &\quad \times \left[\int_0^\infty x^{p(1-\lambda_1-m)-1} F_m^p(x) dx \right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\lambda_2)-1} (g^{(n)}(y))^q dy \right]^{\frac{1}{q}}, \end{aligned} \tag{25}$$

where the constant factor

$$\frac{\Gamma(\lambda+m)}{\Gamma(\lambda+n)} B(\lambda_1+m, \lambda_2)$$

is the best possible.

Proof. We only prove that the constant factor $\frac{\Gamma(\lambda+m)}{\Gamma(\lambda+n)} B(\lambda_1+m, \lambda_2)$ in (25) is the best possible.

For any $0 < \varepsilon < \lambda_1 \min\{p, |q|\}$, we consider the functions $\widetilde{F}_k(x)$ $(k = 0, \cdots, m)$ and $\widetilde{g}^{(n-l)}(y)$ $(l = 0, \cdots, n)$ as in Theorem 1. If there exists a positive constant

$$M \geq \frac{\Gamma(\lambda+m)}{\Gamma(\lambda+n)} B(\lambda_1+m, \lambda_2),$$

such that (25) is valid when we replace $\frac{\Gamma(\lambda+m)}{\Gamma(\lambda+n)} B(\lambda_1+m, \lambda_2)$ with M, then in particular, since

$$\begin{aligned}\widetilde{J} &:= \left[\int_0^\infty x^{p(1-\lambda_1-m)-1}\widetilde{F}_m^p(x)dx\right]^{\frac{1}{p}}\left[\int_0^\infty y^{q(1-\lambda_2)-1}(\widetilde{g}^{(n)}(y))^p dy\right]^{\frac{1}{q}} \\ &= \left[\int_1^\infty x^{-\varepsilon-1}dx - \int_1^\infty O_1(x^{-p\lambda_1-1})dx\right]^{\frac{1}{p}}\left(\int_1^\infty y^{-\varepsilon-1}dy\right)^{\frac{1}{q}} \\ &= \frac{1}{\varepsilon}[\prod_{i=0}^{m-1}(\lambda_1+i-\frac{\varepsilon}{p})]^{-1}\prod_{j=0}^{n-1}(\lambda_2+j-\frac{\varepsilon}{q})(1-\varepsilon O(1))^{\frac{1}{p}},\end{aligned}$$

we have

$$\begin{aligned}\widetilde{I} &:= \int_0^\infty\int_0^\infty \frac{\widetilde{f}(x)\widetilde{g}(y)}{(x+y)^\lambda}dxdy \\ &> M\widetilde{J} = \frac{M}{\varepsilon}[\prod_{i=0}^{m-1}(\lambda_1+i-\frac{\varepsilon}{p})]^{-1}\prod_{j=0}^{n-1}(\lambda_2+j-\frac{\varepsilon}{q})(1-\varepsilon O(1))^{\frac{1}{p}}.\end{aligned}$$

In view of the proof of the results of Theorem 1, it follows that

$$\widetilde{I} = \int_1^\infty\left[\int_1^\infty \frac{y^{\lambda_2+n-\frac{\varepsilon}{q}-1} - O_2(y^{n-1})}{(x+y)^{\lambda+n}}dy\right]x^{\lambda_1-\frac{\varepsilon}{p}-1}dx = I_0 - I_1,$$

where

$$\begin{aligned}I_0 &= \int_1^\infty\left[\int_1^\infty \frac{y^{\lambda_2+n-\frac{\varepsilon}{q}-1}}{(x+y)^{\lambda+n}}dy\right]x^{\lambda_1-\frac{\varepsilon}{p}-1}dx \\ &= \frac{1}{\varepsilon}\left[\int_0^1 \frac{u^{\lambda_2+n-\frac{\varepsilon}{q}-1}}{(1+u)^{\lambda+n}}du + \int_1^\infty \frac{u^{\lambda_2+n-\frac{\varepsilon}{q}-1}}{(1+u)^{\lambda+n}}du\right].\end{aligned}$$

$$0 < I_1 = \int_1^\infty\left[\int_1^\infty \frac{O_2(y^{n-1})}{(x+y)^{\lambda+n}}dy\right]x^{\lambda_1-\frac{\varepsilon}{p}-1}dx \le M_2 < \infty,$$

According to the above results, it follows that

$$\begin{aligned}&\int_0^1 \frac{u^{\lambda_2+n-\frac{\varepsilon}{q}-1}}{(1+u)^{\lambda+n}}du + \int_1^\infty \frac{u^{\lambda_2+n-\frac{\varepsilon}{q}-1}}{(1+u)^{\lambda+n}}du - \varepsilon I_1 \\ =\ & \varepsilon\widetilde{I} > M[\prod_{i=0}^{m-1}(\lambda_1+i-\frac{\varepsilon}{p})]^{-1}\prod_{j=0}^{n-1}(\lambda_2+j-\frac{\varepsilon}{q})(1-\varepsilon O(1))^{\frac{1}{p}}.\end{aligned}$$

Letting $\varepsilon \to 0^+$ in the above inequality, in view of the continuity of the beta function, we obtain

$$B(\lambda_1,\lambda_2+n) \ge M[\prod_{i=0}^{m-1}(\lambda_1+i)]^{-1}\prod_{j=0}^{n-1}(\lambda_2+j),$$

namely

$$\frac{\Gamma(\lambda+m)}{\Gamma(\lambda+n)}B(\lambda_1+m,\lambda_2) = B(\lambda_1,\lambda_2+n)\prod_{i=0}^{m-1}(\lambda_1+i)[\prod_{j=0}^{n-1}(\lambda_2+j)]^{-1} \ge M$$

and then, $M = \frac{\Gamma(\lambda+m)}{\Gamma(\lambda+n)}B(\lambda_1+m,\lambda_2)$ is the best possible constant factor of (25).

This completes the proof of the theorem. □

Remark 4. *For*

$$\widehat{\lambda}_1 = \frac{\lambda - \lambda_2}{p} + \frac{\lambda_1}{q} = \frac{\lambda - \lambda_1 - \lambda_2}{p} + \lambda_1,$$
$$\widehat{\lambda}_2 = \frac{\lambda - \lambda_1}{q} + \frac{\lambda_2}{p} = \frac{\lambda - \lambda_1 - \lambda_2}{q} + \lambda_2,$$

it follows that $\widehat{\lambda}_1 + \widehat{\lambda}_2 = \lambda$. *We find that for*

$$-p\lambda_1 < \lambda - \lambda_1 - \lambda_2 < p(\lambda - \lambda_1),$$
$$0 < \widehat{\lambda}_1 < \lambda;$$

then,

$$0 < \widehat{\lambda}_2 = \lambda - \widehat{\lambda}_1 < \lambda.$$

According to the reverse Hölder inequality (cf. [29]), *we obtain*

$$\begin{aligned} \infty &> B(\widehat{\lambda}_1 + m, \widehat{\lambda}_2) = \int_0^\infty \frac{u^{\frac{\lambda+m-\lambda_2}{p} + \frac{\lambda_1+m}{q} - 1}}{(1+u)^{\lambda+m}} du \\ &= \int_0^\infty \frac{1}{(1+u)^{\lambda+m}} \left(u^{\frac{\lambda+m-\lambda_2-1}{p}} \right) \left(u^{\frac{\lambda_1+m-1}{q}} \right) du \\ &\geq \left[\int_0^\infty \frac{u^{\lambda+m-\lambda_2-1}}{(1+u)^{\lambda+m}} du \right]^{\frac{1}{p}} \left[\int_0^\infty \frac{u^{\lambda_1+m-1}}{(1+u)^{\lambda+m}} du \right]^{\frac{1}{q}} \\ &= B^{\frac{1}{p}}(\lambda_2, \lambda + m - \lambda_2) B^{\frac{1}{q}}(\lambda_1 + m, \lambda - \lambda_1) > 0. \end{aligned} \tag{26}$$

Theorem 5. *For* $0 < p < 1$, $0 < \lambda_1, \lambda_2 < \lambda$, *if the constant factor*

$$\frac{\Gamma(\lambda+m)}{\Gamma(\lambda+n)} B^{\frac{1}{p}}(\lambda_2, \lambda + m - \lambda_2) B^{\frac{1}{q}}(\lambda_1 + m, \lambda - \lambda_1)$$

in (24) *is the best possible, then for* $-p\lambda_1 < \lambda - \lambda_1 - \lambda_2 < p(\lambda - \lambda_1)$, *we have* $\lambda_1 + \lambda_2 = \lambda$.

Proof. For $-p\lambda_1 < \lambda - \lambda_1 - \lambda_2 < p(\lambda - \lambda_1)$, by (25) (for $\lambda_i = \widehat{\lambda}_i (i = 1, 2)$), since

$$\frac{\Gamma(\lambda+m)}{\Gamma(\lambda+n)} B^{\frac{1}{p}}(\lambda_2, \lambda + m - \lambda_2) B^{\frac{1}{q}}(\lambda_1 + m, \lambda - \lambda_1)$$

is the best possible constant factor in (24), we have

$$\begin{aligned} & \frac{\Gamma(\lambda+m)}{\Gamma(\lambda+n)} B^{\frac{1}{p}}(\lambda_2, \lambda + m - \lambda_2) B^{\frac{1}{q}}(\lambda_1 + m, \lambda - \lambda_1) \\ \geq \; & \frac{\Gamma(\lambda+m)}{\Gamma(\lambda+n)} B(\widehat{\lambda}_1 + m, \widehat{\lambda}_2) \; (\in \mathbf{R}_+), \end{aligned}$$

namely,

$$B(\widehat{\lambda}_1 + m, \widehat{\lambda}_2) \leq B^{\frac{1}{p}}(\lambda_2, \lambda + m - \lambda_2) B^{\frac{1}{q}}(\lambda_1 + m, \lambda - \lambda_1).$$

It follows that (27) retains the form of equality.

We observe that (27) retains the form of equality if and only if there exist constants A and B such that they are not both zero and $Au^{\lambda+m-\lambda_2-1} = Bu^{\lambda_1+m-1}$ *a.e.* in $\mathbf{R}_+$ (cf. [29]). Assuming that $A \neq 0$, it follows that $u^{\lambda-\lambda_2-\lambda_1} = B/A$ *a.e.* in $\mathbf{R}_+$, namely $\lambda - \lambda_2 - \lambda_1 = 0$; then, $\lambda_1 + \lambda_2 = \lambda$.

This completes the proof of the theorem. □

For $n = m$, we have:

Corollary 2. *For $0 < p < 1$, we have the following reverse Hardy–Hilbert-type integral inequality involving one multiple upper limit function and one derivative function of m-order:*

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^{\lambda+m}} dxdy$$
$$> B^{\frac{1}{p}}(\lambda_2, \lambda+m-\lambda_2) B^{\frac{1}{q}}(\lambda_1+m, \lambda-\lambda_1) \tag{27}$$
$$\times \left[\int_0^\infty x^{p(1-\widehat{\lambda}_1-m)-1} F_m^p(x) dx\right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\widehat{\lambda}_2)-1} (g^{(m)}(y))^q dy\right]^{\frac{1}{q}}.$$

Moreover, if $0 < \lambda_1, \lambda_2 < \lambda$, the constant factor

$$B^{\frac{1}{p}}(\lambda_2, \lambda+m-\lambda_2) B^{\frac{1}{q}}(\lambda_1+m, \lambda-\lambda_1)$$

in (28) is the best possible, then for

$$-p\lambda_1 < \lambda - \lambda_1 - \lambda_2 < p(\lambda - \lambda_1),$$

we have $\lambda_1 + \lambda_2 = \lambda$. For $\lambda_1 + \lambda_2 = \lambda$ $(0 < \lambda_1, \lambda_2 < \lambda)$, we reduce (19) to the following:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^{\lambda+m}} dxdy > B(\lambda_1+m, \lambda_2)$$
$$\times \left[\int_0^\infty x^{p(1-\lambda_1-m)-1} F_m^p(x) dx\right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\lambda_2)-1} (g^{(m)}(y))^q dy\right]^{\frac{1}{q}}, \tag{28}$$

where the constant factor $B(\lambda_1 + m, \lambda_2)$ is the best possible.

Remark 5. *For $r > 1, \frac{1}{r} + \frac{1}{s} = 1, \lambda_1 = \frac{\lambda}{r}, \lambda_2 = \frac{\lambda}{s}$ in (25), we have the following reverse inequality*

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^{\lambda+n}} dxdy$$
$$> \prod_{i=0}^{m-1} (\frac{\lambda}{r} + i) [\prod_{j=0}^{n-1} (\lambda + j)]^{-1} B(\frac{\lambda}{r}, \frac{\lambda}{s}) \tag{29}$$
$$\times \left[\int_0^\infty x^{p(1-\frac{\lambda}{r}-m)-1} F_m^p(x) dx\right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\frac{\lambda}{s})-1} (g^{(n)}(y))^q dy\right]^{\frac{1}{q}},$$

where the constant factor

$$\prod_{i=0}^{m-1} (\frac{\lambda}{r} + i) [\prod_{j=0}^{n-1} (\lambda + j)]^{-1} B(\frac{\lambda}{r}, \frac{\lambda}{s})$$

is the best possible. In particular, for $\lambda = 1$, we have

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^{1+n}} dxdy$$
$$> \frac{\pi}{n! \sin(\frac{\pi}{r})} \left[\int_0^\infty x^{p(\frac{1}{s}-m)-1} F_m^p(x) dx\right]^{\frac{1}{p}} \left[\int_0^\infty y^{\frac{q}{r}-1} (g^{(n)}(y))^q dy\right]^{\frac{1}{q}}, \tag{30}$$

where the constant factor $\frac{\pi}{n! \sin(\frac{\pi}{r})}$ is still the best possible.

5. Conclusions

In the present paper, we followed the methods of [15,17], used weight functions and introduced parameters in order to prove a new Hardy–Hilbert-type integral inequality with the kernel $\frac{1}{(x+y)^{\lambda+n}}$ involving one multiple upper limit function and one derivative function of higher order. In this study, we also considered equivalent statements of the best possible constant factor related to the parameters and obtained some particular inequalities, in addition to considering the case of reverses. The lemmas and theorems presented in this work provide an extensive account of this type of inequalities.

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