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Bombieri-Type Inequalities and Their Applications in Semi-Hilbert Spaces

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Abstract: This paper presents new results related to Bombieri's generalization of Bessel's inequality in a semi-inner product space induced by a positive semidefinite operator A. Specifically, we establish new inequalities that generalize the classical Bessel inequality and extend previous results in this area. Furthermore, our findings have applications to the study of operators on positive semidefinite inner product spaces, also known as semi-Hilbert spaces, and contribute to a deeper understanding of their properties and applications. Our work has implications for various fields, including functional analysis and operator theory.

Keywords: positive semidefinite operator; bombieri inequality; joint A-numerical radius; euclidean A-seminorm; inequalities

MSC: 47B65; 46C20; 47A12; 47A30; 46C05

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1. Introduction

Inequalities play a fundamental role in analysis and have widespread applications in various branches of mathematics. Among the most classical inequalities are Bessel's inequality, Bombieri's inequality, Selberg's inequality, and Heilbronn's inequality, which have been extensively studied and applied in many areas, including harmonic analysis, probability theory, and number theory. These inequalities are also widely used in the study of operators on Hilbert spaces.

Recently, there has been growing interest in the study of operators on positive semidefinite inner product spaces, also known as semi-Hilbert spaces. Semi-Hilbert spaces are a more general class of inner product spaces that are not necessarily complete, but satisfy certain axioms that allow for the development of a useful theory. In this paper, we focus on a positive semidefinite inner product space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ induced by a positive semi-definite operator A, and denote it as $(\mathcal{H}, \langle\cdot,\cdot\rangle_A)$.

Semi-Hilbert spaces provide a natural framework for studying various mathematical problems, especially those involving singular or unbounded operators. Our paper contributes to the theory of semi-Hilbert spaces by establishing new Bombieri-type inequalities that generalize the classical Bessel inequality and several related results. Our proposed inequalities are novel and have the potential to be applied in various areas of analysis. In addition, our results provide deeper insight into the properties of operators on semi-Hilbert spaces. We begin by introducing the notation, recalling the definition of semi-Hilbert spaces, and presenting our main contributions.

Throughout this paper, we work with a complex Hilbert space $\mathcal{H}$ equipped with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$. The set of all bounded linear operators on $\mathcal{H}$ is denoted by $\mathbb{B}(\mathcal{H})$, and for a bounded linear operator T on $\mathcal{H}$, we use $\mathcal{R}(T)$ to denote the range of T, $\mathcal{N}(T)$ to denote its null space, and T^* to denote its adjoint. We define $\mathbb{B}(\mathcal{H})^+$ as the set of all bounded linear operators A on $\mathcal{H}$ such that $\langle Ax,x\rangle \geq 0$ for all $x \in \mathcal{H}$. The elements of $\mathbb{B}(\mathcal{H})^+$ are called positive operators on $\mathcal{H}$. In this paper, the term "operator" specifically refers to an element of the set $\mathbb{B}(\mathcal{H})$, and we assume that A is a non-zero operator in $\mathbb{B}(\mathcal{H})^+$. For any such A, we define a positive semidefinite sesquilinear form $\langle\cdot,\cdot\rangle_A : \mathcal{H}\times\mathcal{H} \to \mathbb{C}$ by setting

$$\langle x,y\rangle_A = \langle Ax,y\rangle,$$

for all $x,y \in \mathcal{H}$. We use the notation $\|\cdot\|_A$ to represent the seminorm induced by the positive semidefinite sesquilinear form $\langle\cdot,\cdot\rangle_A$. This seminorm is defined on every vector $x \in \mathcal{H}$ as $\|x\|_A = \sqrt{\langle x,x\rangle_A}$. We observe that the seminorm $\|\cdot\|_A$ vanishes on a vector $x \in \mathcal{H}$ if and only if x belongs to $\mathcal{N}(A)$. In addition, the seminorm $\|\cdot\|_A$ induces a norm on $\mathcal{H}$ if and only if A is one-to-one. It follows that the semi-Hilbert space $(\mathcal{H},\|\cdot\|_A)$ is complete if and only if $\mathcal{R}(A)$ is closed in $\mathcal{H}$.

Now, we recall several well-known inequalities which hold true in inner product spaces that are real or complex. However, in this paper, we assume without loss of generality that $\mathcal{H}$ is always a complex Hilbert space. We begin with Bessel's inequality, which is a fundamental result in functional analysis and has numerous important applications in various areas of mathematics and engineering. More precisely, Bessel's inequality states that for any orthonormal vectors $e_1, e_2, \cdots, e_d$ in $\mathcal{H}$, meaning that they satisfy $\langle e_i, e_j\rangle = \delta_{ij}$ (where δ_{ij} is the Kronecker delta symbol) for all $i,j \in \{1,\ldots,d\}$, the following inequality holds:

$$\sum_{i=1}^{d} |\langle x,e_i\rangle|^2 \leq \|x\|^2, \tag{1}$$

for every $x \in \mathcal{H}$. For additional results related to Bessel's inequality, readers are referred to [1,2] and Chapter XV of the book [3].

In 1971, E. Bombieri [4] proposed a generalization of Bessel's inequality that applies to any set of vectors in the inner product space $\mathcal{H}$. This generalization is known as Bombieri's inequality, and it extends the applicability of Bessel's inequality beyond orthonormal sets of vectors. When the vectors y_i are orthonormal, Bombieri's inequality reduces to Bessel's inequality (1). To state Bombieri's inequality, we first consider the set of vectors y_i. Then, for any vector $x \in \mathcal{H}$, the inequality can be written as follows:

$$\sum_{i=1}^{d} |\langle x,y_i\rangle|^2 \leq \|x\|^2 \max_{i\in\{1,\ldots,d\}} \left\{ \sum_{j=1}^{d} |\langle y_i,y_j\rangle| \right\}, \tag{2}$$

This inequality has important applications in the theory of Fourier series and Fourier transforms. For more information about Bombieri's inequality and its applications, refer to [3,4].

A further generalization of Bessel's inequality was discovered by A. Selberg (see, e.g., [3] (p. 394)). Suppose that $x, y_1, \ldots, y_d$ are vectors in $\mathcal{H}$, where $y_i \neq 0$ for all $i \in \{1,\ldots,d\}$. Then Selberg's inequality states that:

$$\sum_{i=1}^{d} \frac{|\langle x,y_i\rangle|^2}{\sum_{j=1}^{d}|\langle y_i,y_j\rangle|} \leq \|x\|^2, \quad \forall\, x \in \mathcal{H}. \tag{3}$$

Selberg's inequality is a generalization of Bessel's inequality and applies to any set of vectors $(x, y_1, \ldots, y_d)$ in $\mathcal{H}$. When the vectors $(y_i)_{i\in\{1,\ldots,d\}}$ are orthonormal, inequality (3) reduces to Bessel's inequality (1). Selberg's inequality has important applications in harmonic analysis and mathematical physics, and has been extensively studied in the literature (see, e.g., [5,6]).

H. Heilbronn discovered a type of inequality related to Bessel's result in 1958 [7] (see also [3] (p. 395)). Let x be a vector in a Hilbert space $\mathcal{H}$, and let $y_1, \ldots, y_d$ be vectors in $\mathcal{H}$. Then the following inequality, known as Heilbronn's inequality, holds:

$$\sum_{i=1}^{d} |\langle x, y_i \rangle| \leq \|x\| \left(\sum_{i,j=1}^{d} |\langle y_i, y_j \rangle| \right)^{\frac{1}{2}}. \tag{4}$$

In the special case where $y_1, \ldots, y_d$ are orthonormal, Heilbronn's inequality (4) reduces to the inequality (2) of Bessel's inequality. Heilbronn's inequality has important applications in analysis and geometry, and has been studied extensively in the literature.

In 1992, J.E. Pečari'c [8] (see also [3] (p. 394)) derived a general inequality in inner product spaces. Let $x, y_1, \ldots, y_d \in \mathcal{H}$ and $\gamma_1, \ldots, \gamma_d \in \mathbb{C}$. Then, the following inequality holds:

$$\left| \sum_{i=1}^{d} \gamma_i \langle x, y_i \rangle \right|^2 \leq \|x\|^2 \sum_{i=1}^{d} |\gamma_i|^2 \left(\sum_{j=1}^{d} |\langle y_i, y_j \rangle| \right). \tag{5}$$

From this, we can conclude that,

$$\left| \sum_{i=1}^{d} \gamma_i \langle x, y_i \rangle \right|^2 \leq \|x\|^2 \sum_{i=1}^{d} |\gamma_i|^2 \max_{k \in \{1,\ldots,d\}} \left\{ \sum_{j=1}^{d} |\langle y_k, y_j \rangle| \right\}.$$

Pečari'c showed that the Bombieri inequality (2) can be derived from (5) by choosing $\gamma_i = \overline{\langle x, y_i \rangle}$ (using the second inequality). The Selberg inequality (3) can be obtained from the first part of (5) by choosing $\gamma_i = \frac{\overline{\langle x, y_i \rangle}}{\sum_{j=1}^{d} |\langle y_i, y_j \rangle|}$, for every $i \in \{1, \ldots, d\}$ and the Heilbronn inequality (4) can be obtained from the first part of (5) by choosing $\gamma_i = \frac{\overline{\langle x, y_i \rangle}}{|\langle x, y_i \rangle|}$ for any $i \in \{1, \ldots, d\}$. Additional results related to the above bounds can be found in [2,9].

Very recently, the authors of this work (referenced as [10]) have extended several well-known inequalities to the context of semi-Hilbert spaces in order to establish important bounds for the joint A-numerical radius of semi-Hilbert space operators. In particular, Bombieri's well-known inequality has been extended to the context of semi-Hilbert spaces. Specifically, the following inequality holds:

$$\sum_{i=1}^{d} \left| \langle x, y_i \rangle_A \right|^2 \leq \|x\|_A^2 \max_{i \in \{1,\ldots,d\}} \left\{ \sum_{j=1}^{d} \left| \langle y_i, y_j \rangle_A \right| \right\} \tag{6}$$

This study builds upon prior research conducted in [10] and introduces various forms of inequality (6) as its primary contribution. By obtaining these various forms, we can obtain a better understanding of the characteristics and connections between operators in semi-Hilbert spaces. These different expressions provide a more complete exploration and examination of the inequality, allowing for a broader and more detailed comprehension.

We conclude this section by providing a brief overview of the content covered in this paper. Specifically, our aim is to introduce and investigate several forms of the aforementioned inequalities within the setting of semi-Hilbert spaces. The focus of our study is on applying these inequalities to explore operator tuples within this context. We will present a set of inequalities that establish connections between the joint A-numerical radius and the Euclidean A-seminorm of operator tuples. Through the examination of these relationships, we seek to enhance our understanding of the behavior exhibited by operator tuples in semi-Hilbert spaces.

2. Preliminary Results

This section provides some preliminary results that will serve as building blocks for proving the main theorems. Specifically, we start by introducing an intriguing lemma that has its own value.

Lemma 1. *Suppose we have vectors $\xi_1, \dots, \xi_d$ in $\mathcal{H}$ and complex numbers $\mu_1, \dots, \mu_d$ in $\mathbb{C}$. Then the inequality below holds true:*

$$\left\| \sum_{i=1}^{d} \mu_i \xi_i \right\|_A^2 \leq \Theta, \tag{7}$$

where

$$\Theta = \begin{cases} \max\limits_{k \in \{1,\dots,d\}} |\mu_k|^2 \sum\limits_{i,j=1}^{d} \left| \langle \xi_i, \xi_j \rangle_A \right|; \\ or \\ \max\limits_{k \in \{1,\dots,d\}} |\mu_k| \left(\sum\limits_{i=1}^{d} |\mu_i|^r \right)^{\frac{1}{r}} \left(\sum\limits_{i=1}^{d} \left(\sum\limits_{j=1}^{d} \left| \langle \xi_i, \xi_j \rangle_A \right| \right)^s \right)^{\frac{1}{s}}, \quad r > 1,\ \frac{1}{r} + \frac{1}{s} = 1; \\ or \\ \max\limits_{k \in \{1,\dots,d\}} |\mu_k| \sum\limits_{k=1}^{d} |\mu_k| \max\limits_{1 \leq i \leq n} \left(\sum\limits_{j=1}^{d} \left| \langle \xi_i, \xi_j \rangle_A \right| \right); \\ or \\ \left(\sum\limits_{k=1}^{d} |\mu_k|^p \right)^{\frac{1}{p}} \max\limits_{i \in \{1,\dots,d\}} |\mu_i| \left(\sum\limits_{i=1}^{d} \left(\sum\limits_{j=1}^{d} \left| \langle \xi_i, \xi_j \rangle_A \right| \right)^q \right)^{\frac{1}{q}}, \quad p > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\ or \\ \left(\sum\limits_{k=1}^{d} |\mu_k|^p \right)^{\frac{1}{p}} \left(\sum\limits_{i=1}^{d} |\mu_i|^t \right)^{\frac{1}{t}} \left[\sum\limits_{i=1}^{d} \left(\sum\limits_{j=1}^{d} \left| \langle \xi_i, \xi_j \rangle_A \right|^q \right)^{\frac{u}{q}} \right]^{\frac{1}{u}}, \quad p > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\ \qquad t > 1,\ \frac{1}{t} + \frac{1}{u} = 1; \\ or \\ \left(\sum\limits_{k=1}^{d} |\mu_k|^p \right)^{\frac{1}{p}} \sum\limits_{i=1}^{d} |\mu_i| \max\limits_{i \in \{1,\dots,d\}} \left\{ \left(\sum\limits_{j=1}^{d} \left| \langle \xi_i, \xi_j \rangle_A \right|^q \right)^{\frac{1}{q}} \right\}, \quad p > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\ or \\ \sum\limits_{k=1}^{d} |\mu_k| \max\limits_{i \in \{1,\dots,d\}} |\mu_i| \sum\limits_{i=1}^{d} \left[\max\limits_{j \in \{1,\dots,d\}} \left| \langle \xi_i, \xi_j \rangle_A \right| \right]; \\ or \\ \sum\limits_{k=1}^{d} |\mu_k| \left(\sum\limits_{i=1}^{d} |\mu_i|^m \right)^{\frac{1}{m}} \left(\sum\limits_{i=1}^{d} \left[\max\limits_{j \in \{1,\dots,d\}} \left| \langle \xi_i, \xi_j \rangle_A \right| \right]^l \right)^{\frac{1}{l}}, \quad m > 1,\ \frac{1}{m} + \frac{1}{l} = 1; \\ or \\ \left(\sum\limits_{k=1}^{d} |\mu_k| \right)^2 \max\limits_{j \in \{1,\dots,d\}} \left| \langle \xi_i, \xi_j \rangle_A \right|. \end{cases}$$

Proof. Let $\xi_k \in \mathcal{H}$ and $\mu_k \in \mathbb{C}$ for all $k \in 1, \dots, d$. Then, we have:

$$\begin{aligned} \left\| \sum_{i=1}^{d} \mu_i \xi_i \right\|_A^2 &= \langle \sum_{i=1}^{d} \mu_i \xi_i, \sum_{j=1}^{d} \mu_j \xi_j \rangle_A \\ &= \sum_{i=1}^{d} \sum_{j=1}^{d} \mu_i \overline{\mu_j} \langle \xi_i, \xi_j \rangle_A = \left| \sum_{i=1}^{d} \sum_{j=1}^{d} \mu_i \overline{\mu_j} \langle \xi_i, \xi_j \rangle_A \right|. \end{aligned}$$

This implies that

$$\begin{aligned}\left\|\sum_{i=1}^{d}\mu_i\xi_i\right\|_A^2 &\le \sum_{i=1}^{d}\sum_{j=1}^{d}|\mu_i||\mu_j|\left|\langle\xi_i,\xi_j\rangle_A\right| \\ &= \sum_{i=1}^{d}|\mu_i|\left(\sum_{j=1}^{d}|\mu_j|\left|\langle\xi_i,\xi_j\rangle_A\right|\right) := \Omega.\end{aligned}$$

By applying Hölder's inequality, we obtain multiple inequalities. Specifically, for any $i \in \{1,\dots,d\}$, we have:

$$\sum_{j=1}^{d}|\mu_j|\left|\langle\xi_i,\xi_j\rangle_A\right| \le \begin{cases} \max\limits_{k\in\{1,\dots,d\}}|\mu_k|\sum\limits_{j=1}^{d}\left|\langle\xi_i,\xi_j\rangle_A\right| \\ \left(\sum\limits_{k=1}^{d}|\mu_k|^p\right)^{\frac{1}{p}}\left(\sum\limits_{j=1}^{d}\left|\langle\xi_i,\xi_j\rangle_A\right|^q\right)^{\frac{1}{q}}, \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \sum\limits_{k=1}^{d}|\mu_k|\max\limits_{j\in\{1,\dots,d\}}\left|\langle\xi_i,\xi_j\rangle_A\right|. \end{cases}$$

As a result,

$$\Omega \le \begin{cases} \max\limits_{k\in\{1,\dots,d\}}|\mu_k|\sum\limits_{i=1}^{d}|\mu_i|\sum\limits_{j=1}^{d}\left|\langle\xi_i,\xi_j\rangle_A\right| =: \theta_1; \\ \left(\sum\limits_{k=1}^{d}|\mu_k|^p\right)^{\frac{1}{p}}\sum\limits_{i=1}^{d}|\mu_i|\left(\sum\limits_{j=1}^{d}\left|\langle\xi_i,\xi_j\rangle_A\right|^q\right)^{\frac{1}{q}} := \theta_p, \text{ with } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \sum\limits_{k=1}^{d}|\mu_k|\sum\limits_{i=1}^{d}|\mu_i|\max\limits_{j\in\{1,\dots,d\}}\left|\langle\xi_i,\xi_j\rangle_A\right| =: \theta_\infty. \end{cases}$$

We can also obtain the following using Hölder's inequality:

$$\sum_{i=1}^{d}|\mu_i|\left(\sum_{j=1}^{d}\left|\langle\xi_i,\xi_j\rangle_A\right|\right) \le \begin{cases} \max\limits_{i\in\{1,\dots,d\}}|\mu_i|\sum\limits_{i,j=1}^{d}\left|\langle\xi_i,\xi_j\rangle_A\right|; \\ \left(\sum\limits_{i=1}^{d}|\mu_i|^r\right)^{\frac{1}{r}}\left(\sum\limits_{i=1}^{d}\left(\sum\limits_{j=1}^{d}\left|\langle\xi_i,\xi_j\rangle_A\right|\right)^s\right)^{\frac{1}{s}}, \quad r>1,\ \frac{1}{r}+\frac{1}{s}=1; \\ \sum\limits_{i=1}^{d}|\mu_i|\max\limits_{i\in\{1,\dots,d\}}\left(\sum\limits_{j=1}^{d}\left|\langle\xi_i,\xi_j\rangle_A\right|\right). \end{cases}$$

Based on the previous inequalities, it can be deduced that

$$\theta_1 \le \begin{cases} \max\limits_{k\in\{1,\dots,d\}} |\mu_k|^2 \sum\limits_{i,j=1}^{d} \left|\langle \xi_i, \xi_j \rangle_A\right|; \\ \max\limits_{k\in\{1,\dots,d\}} |\mu_k| \left(\sum\limits_{i=1}^{d} |\mu_i|^r\right)^{\frac{1}{r}} \left(\sum\limits_{i=1}^{d} \left(\sum\limits_{j=1}^{d} \left|\langle \xi_i, \xi_j \rangle_A\right|\right)^s\right)^{\frac{1}{s}}, \quad r > 1,\ \frac{1}{r} + \frac{1}{s} = 1; \\ \max\limits_{k\in\{1,\dots,d\}} |\mu_k| \sum\limits_{i=1}^{d} |\mu_i| \max\limits_{i\in\{1,\dots,d\}} \left(\sum\limits_{j=1}^{d} \left|\langle \xi_i, \xi_j \rangle_A\right|\right). \end{cases}$$

Thus, the first three inequalities in (7) were derived.

Additionally, it can be shown by applying Hölder's inequality again that:

$$\theta_p \le \left(\sum_{k=1}^{d} |\mu_k|^p\right)^{\frac{1}{p}} \times \begin{cases} \max\limits_{i\in\{1,\dots,d\}} |\mu_i| \sum\limits_{i=1}^{d} \left(\sum\limits_{j=1}^{d} \left|\langle \xi_i, \xi_j \rangle_A\right|^q\right)^{\frac{1}{q}}; \\ \left(\sum\limits_{i=1}^{d} |\mu_i|^t\right)^{\frac{1}{t}} \left(\sum\limits_{i=1}^{d} \left(\sum\limits_{j=1}^{d} \left|\langle \xi_i, \xi_j \rangle_A\right|^q\right)^{\frac{u}{q}}\right)^{\frac{1}{u}}, \quad t > 1,\ \frac{1}{t} + \frac{1}{u} = 1; \\ \sum\limits_{i=1}^{d} |\mu_i| \max\limits_{i\in\{1,\dots,d\}} \left\{\left(\sum\limits_{j=1}^{d} \left|\langle \xi_i, \xi_j \rangle_A\right|^q\right)^{\frac{1}{q}}\right\}. \end{cases}$$

Consequently, the next three inequalities in (7) are proven.

Furthermore, we can use the same Hölder inequality to assert that:

$$\theta_\infty \le \sum_{k=1}^{d} |\mu_k| \times \begin{cases} \max\limits_{i\in\{1,\dots,d\}} |\mu_i| \sum\limits_{i=1}^{d} \left(\max\limits_{j\in\{1,\dots,d\}} \left|\langle \xi_i, \xi_j \rangle_A\right|\right); \\ \left(\sum\limits_{i=1}^{d} |\mu_i|^m\right)^{\frac{1}{m}} \left(\sum\limits_{i=1}^{d} \left(\max\limits_{j\in\{1,\dots,d\}} \left|\langle \xi_i, \xi_j \rangle_A\right|\right)^l\right)^{\frac{1}{l}}, \\ \qquad m > 1,\ \frac{1}{m} + \frac{1}{l} = 1; \\ \sum\limits_{i=1}^{d} |\mu_i| \max\limits_{i,j\in\{1,\dots,d\}} \left|\langle \xi_i, \xi_j \rangle_A\right|. \end{cases}$$

Therefore, the proof of the lemma is complete as the last three inequalities stated in Equation (7) have been demonstrated. □

In case we desire to establish certain bounds for $\left\|\sum_{i=1}^{d} \mu_i \xi_i\right\|_A^2$ based on $\sum_{i=1}^{d} |\mu_i|^2$, we can make use of the following corollaries.

Corollary 1. *Suppose that the conditions of Lemma 1 hold, where $\xi_1, \dots, \xi_d$ and $\mu_1, \dots, \mu_d$ are involved. Provided that $1 < p \le 2$ and $1 < t \le 2$, the following inequality is valid:*

$$\left\|\sum_{i=1}^{d} \mu_i \xi_i\right\|_A^2 \le d^{\frac{1}{p}+\frac{1}{t}-1} \sum_{k=1}^{d} |\mu_k|^2 \left[\sum_{i=1}^{d} \left(\sum_{j=1}^{d} \left|\langle \xi_i, \xi_j \rangle_A\right|^q\right)^{\frac{u}{q}}\right]^{\frac{1}{u}}, \tag{8}$$

where $\frac{1}{p}+\frac{1}{q}=1$, $\frac{1}{t}+\frac{1}{u}=1$.

Proof. The proof proceeds by observing the monotonicity of power means. Firstly, we can express that

$$\left(\frac{\sum_{k=1}^{d}|\mu_k|^p}{d}\right)^{\frac{1}{p}} \leq \left(\frac{\sum_{k=1}^{d}|\mu_k|^2}{d}\right)^{\frac{1}{2}};\ 1<p\leq 2,$$
$$\left(\frac{\sum_{k=1}^{d}|\mu_k|^t}{d}\right)^{\frac{1}{t}} \leq \left(\frac{\sum_{k=1}^{d}|\mu_k|^2}{d}\right)^{\frac{1}{2}};\ 1<t\leq 2.$$

Hence, we can derive that

$$\left(\sum_{k=1}^{d}|\mu_k|^p\right)^{\frac{1}{p}} \leq d^{\frac{1}{p}-\frac{1}{2}}\left(\sum_{k=1}^{d}|\mu_k|^2\right)^{\frac{1}{2}},$$
$$\left(\sum_{k=1}^{d}|\mu_k|^t\right)^{\frac{1}{t}} \leq d^{\frac{1}{t}-\frac{1}{2}}\left(\sum_{k=1}^{d}|\mu_k|^2\right)^{\frac{1}{2}}.$$

We can obtain (8) by utilizing the fifth inequality in (7). Hence, the proof is finished □

Remark 1. *A noteworthy special case occurs when* $p=q=t=u=2$, *which yields the following result:*

$$\left\|\sum_{i=1}^{d}\mu_i\xi_i\right\|_A^2 \leq \sum_{k=1}^{d}|\mu_k|^2\left(\sum_{i,j=1}^{d}\left|\langle\xi_i,\xi_j\rangle_A\right|^2\right)^{\frac{1}{2}}.$$

Corollary 2. *Under the conditions of Lemma 1, if* $1<p\leq 2$, *the following corollary holds:*

$$\left\|\sum_{i=1}^{d}\mu_i\xi_i\right\|_A^2 \leq d^{\frac{1}{p}}\sum_{k=1}^{d}|\mu_k|^2 \max_{i\in\{1,\ldots,d\}}\left[\left(\sum_{j=1}^{d}\left|\langle\xi_i,\xi_j\rangle_A\right|^q\right)^{\frac{1}{q}}\right], \tag{9}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Since

$$\left(\sum_{k=1}^{d}|\mu_k|^p\right)^{\frac{1}{p}} \leq d^{\frac{1}{p}-\frac{1}{2}}\left(\sum_{k=1}^{d}|\mu_k|^2\right)^{\frac{1}{2}},$$

and

$$\sum_{k=1}^{d}|\mu_k| \leq d^{\frac{1}{2}}\left(\sum_{k=1}^{d}|\mu_k|^2\right)^{\frac{1}{2}},$$

then, we apply the sixth inequality in (7) to derive (9). □

Similarly, we can demonstrate the next two corollaries using analogous techniques.

Corollary 3. *If* $1<m\leq 2$ *and under the assumptions of Lemma 1, we have the following inequality:*

$$\left\|\sum_{i=1}^{d}\mu_i\xi_i\right\|_A^2 \leq d^{\frac{1}{m}}\sum_{k=1}^{d}|\mu_k|^2\left(\sum_{i=1}^{d}\left[\max_{j\in\{1,\ldots,d\}}\left|\langle\xi_i,\xi_j\rangle_A\right|\right]^l\right)^{\frac{1}{l}},$$

where $\frac{1}{m}+\frac{1}{l}=1$.

Corollary 4. *Assuming the conditions of Lemma 1, we can conclude that*

$$\left\|\sum_{i=1}^{d}\mu_i\xi_i\right\|_A^2 \le d\sum_{k=1}^{d}|\mu_k|^2 \max_{i,j\in\{1,\dots,d\}}\left|\langle\xi_i,\xi_j\rangle_A\right|.$$

One may also find the following lemma to be of interest.

Lemma 2. *Assuming the conditions of Lemma 1, the following inequalities hold:*

$$\left\|\sum_{i=1}^{d}\mu_i\xi_i\right\|_A^2 \le \sum_{i=1}^{d}|\mu_i|^2\sum_{j=1}^{d}\left|\langle\xi_i,\xi_j\rangle_A\right| \le \begin{cases} \sum_{i=1}^{d}|\mu_i|^2 \max_{i\in\{1,\dots,d\}}\left[\sum_{j=1}^{d}\left|\langle\xi_i,\xi_j\rangle_A\right|\right]; \\ \left(\sum_{i=1}^{d}|\mu_i|^{2p}\right)^{\frac{1}{p}}\left(\left(\sum_{j=1}^{d}\left|\langle\xi_i,\xi_j\rangle_A\right|\right)^q\right)^{\frac{1}{q}}; \\ \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \max_{i\in\{1,\dots,d\}}|\mu_i|^2\sum_{i,j=1}^{d}\left|\langle\xi_i,\xi_j\rangle_A\right|. \end{cases}$$

Proof. Based on Lemma 1, it is established that

$$\left\|\sum_{i=1}^{d}\mu_i\xi_i\right\|_A^2 \le \sum_{i=1}^{d}\sum_{j=1}^{d}|\mu_i||\mu_j|\left|\langle\xi_i,\xi_j\rangle_A\right|.$$

By making a simple observation (also referenced in [3] (p. 394)), it can be inferred that for any $i,j\in\{1,\dots,d\}$, the inequality

$$|\mu_i||\mu_j| \le \frac{1}{2}\left(|\mu_i|^2+|\mu_j|^2\right)$$

holds. Therefore, we can conclude that

$$\begin{aligned}\sum_{i=1}^{d}\sum_{j=1}^{d}|\mu_i||\mu_j|\left|\langle\xi_i,\xi_j\rangle_A\right| &\le \frac{1}{2}\sum_{i,j=1}^{d}\left(|\mu_i|^2+|\mu_j|^2\right)\left|\langle\xi_i,\xi_j\rangle_A\right| \\ &= \frac{1}{2}\left[\sum_{i,j=1}^{d}|\mu_i|^2\left|\langle\xi_i,\xi_j\rangle_A\right| + \sum_{i,j=1}^{d}|\mu_j|^2\left|\langle\xi_i,\xi_j\rangle_A\right|\right] \\ &= \sum_{i,j=1}^{d}|\mu_i|^2\left|\langle\xi_i,\xi_j\rangle_A\right|.\end{aligned}$$

Therefore, we have established the validity of the first inequality in the Lemma.

The second part of the Lemma can be obtained by utilizing Hölder's inequality, but we will not provide further elaboration on this. □

Based on the Lemma mentioned above, we are now in a position to state the following theorem as an application.

Theorem 1. *For any vectors $x,y_1,\dots,y_d$ in $\mathcal{H}$ and complex numbers $\gamma_1,\dots,\gamma_d\in\mathbb{C}$, the following inequalities hold:*

$$\left|\sum_{i=1}^{d}\gamma_i\langle x,y_i\rangle_A\right|^2 \le \Gamma\|x\|_A^2, \tag{10}$$

where

$$\Gamma = \begin{cases} \max\limits_{k\in\{1,\ldots,d\}} |\gamma_k|^2 \sum\limits_{i,j=1}^{d} \left|\left\langle y_i, y_j \right\rangle_A\right|; \\ or \\ \max\limits_{k\in\{1,\ldots,d\}} |\gamma_k| \left(\sum\limits_{i=1}^{d} |\gamma_i|^r \right)^{\frac{1}{r}} \left[\sum\limits_{i=1}^{d} \left(\sum\limits_{j=1}^{d} \left|\left\langle y_i, y_j \right\rangle_A\right| \right)^s \right]^{\frac{1}{s}}, \quad r > 1,\ \frac{1}{r}+\frac{1}{s}=1; \\ or \\ \max\limits_{k\in\{1,\ldots,d\}} |\gamma_k| \sum\limits_{k=1}^{d} |\gamma_k| \max\limits_{i\in\{1,\ldots,d\}} \left(\sum\limits_{j=1}^{d} \left|\left\langle y_i, y_j \right\rangle_A\right| \right); \\ or \\ \left(\sum\limits_{k=1}^{d} |\gamma_k|^p \right)^{\frac{1}{p}} \max\limits_{i\in\{1,\ldots,d\}} |\gamma_i| \left(\sum\limits_{i=1}^{d} \left(\sum\limits_{j=1}^{d} \left|\left\langle y_i, y_j \right\rangle_A\right| \right)^q \right)^{\frac{1}{q}}, \quad p > 1,\ \frac{1}{p}+\frac{1}{q}=1; \\ or \\ \left(\sum\limits_{k=1}^{d} |\gamma_k|^p \right)^{\frac{1}{p}} \left(\sum\limits_{i=1}^{d} |\gamma_i|^t \right)^{\frac{1}{t}} \left[\sum\limits_{i=1}^{d} \left(\sum\limits_{j=1}^{d} \left|\left\langle y_i, y_j \right\rangle_A\right|^q \right)^{\frac{u}{q}} \right]^{\frac{1}{u}}, \quad p > 1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \qquad t > 1,\ \frac{1}{t}+\frac{1}{u}=1; \\ or \\ \left(\sum\limits_{k=1}^{d} |\gamma_k|^p \right)^{\frac{1}{p}} \sum\limits_{i=1}^{d} |\gamma_i| \max\limits_{i\in\{1,\ldots,d\}} \left\{ \left(\sum\limits_{j=1}^{d} \left|\left\langle y_i, y_j \right\rangle_A\right|^q \right)^{\frac{1}{q}} \right\}, \quad p > 1,\ \frac{1}{p}+\frac{1}{q}=1; \\ or \\ \sum\limits_{k=1}^{d} |\gamma_k| \max\limits_{i\in\{1,\ldots,d\}} |\gamma_i| \sum\limits_{i=1}^{d} \left[\max\limits_{j\in\{1,\ldots,d\}} \left|\left\langle y_i, y_j \right\rangle_A\right| \right]; \\ or \\ \sum\limits_{k=1}^{d} |\gamma_k| \left(\sum\limits_{i=1}^{d} |\gamma_i|^m \right)^{\frac{1}{m}} \left(\sum\limits_{i=1}^{d} \left[\max\limits_{j\in\{1,\ldots,d\}} \left|\left\langle y_i, y_j \right\rangle_A\right| \right]^l \right)^{\frac{1}{l}}, \quad m > 1,\ \frac{1}{m}+\frac{1}{l}=1; \\ or \\ \left(\sum\limits_{k=1}^{d} |\gamma_k| \right)^2 \max\limits_{i,j\in\{1,\ldots,d\}} \left|\left\langle y_i, y_j \right\rangle_A\right|. \end{cases}$$

Proof. First, we observe that:

$$\sum_{i=1}^{d} \gamma_i \langle x, y_i \rangle_A = \left\langle x, \sum_{i=1}^{d} \overline{\gamma_i} y_i \right\rangle_A.$$

We then apply Schwarz's inequality for inner product spaces, resulting in:

$$\left| \sum_{i=1}^{d} \gamma_i \langle x, y_i \rangle_A \right|^2 \leq \|x\|_A^2 \left\| \sum_{i=1}^{d} \overline{\gamma_i} y_i \right\|_A^2.$$

Finally, Lemma 1 is utilized with $\mu_i = \overline{\gamma_i}$, $\xi_i = y_i$ $(i = 1, \ldots, d)$, to obtain the desired inequality (10). Further details have been omitted. □

If one requires bounds in terms of $\sum_{i=1}^{d} |\gamma_i|^2$, the following corollaries may be of use:

Corollary 5. *Under the assumptions of Theorem 1, and for* $1 < p \leq 2$, $1 < t \leq 2$, $\frac{1}{p} + \frac{1}{q} = 1$, $\frac{1}{t} + \frac{1}{u} = 1$, *the inequality*

$$\left| \sum_{i=1}^{d} \gamma_i \langle x, y_i \rangle_A \right|^2 \leq d^{\frac{1}{p}+\frac{1}{t}-1} \|x\|_A^2 \sum_{i=1}^{d} |\gamma_i|^2 \left[\sum_{i=1}^{d} \left(\sum_{j=1}^{d} \left| \langle y_i, y_j \rangle_A \right|^q \right)^{\frac{u}{q}} \right]^{\frac{1}{u}}, \tag{11}$$

holds, and in particular, for $p = q = t = u = 2$,

$$\left| \sum_{i=1}^{d} \gamma_i \langle x, y_i \rangle_A \right|^2 \leq \|x\|_A^2 \sum_{i=1}^{d} |\gamma_i|^2 \left(\sum_{i,j=1}^{d} \left| \langle y_i, y_j \rangle_A \right|^2 \right)^{\frac{1}{2}}.$$

Proof. The proof for this corollary is analogous to the one presented in Corollary 1, and therefore, we omit it. □

Corollary 6. *Assuming the conditions stated in Theorem 1 and for* $1 < p \leq 2$, *the following inequality holds:*

$$\left| \sum_{i=1}^{d} \gamma_i \langle x, y_i \rangle_A \right|^2 \leq d^{\frac{1}{p}} \|x\|_A^2 \sum_{k=1}^{d} |\gamma_k|^2 \max_{i \in \{1,\ldots,d\}} \left[\sum_{j=1}^{d} \left| \langle y_i, y_j \rangle_A \right|^q \right]^{\frac{1}{q}}, \tag{12}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. The proof of this statement follows a similar approach to that of Corollary 2. □

The following two corollaries are additional results that can be derived from the preceding theorem. For the sake of brevity, we present them without providing their proofs.

Corollary 7. *Assuming the same conditions hold for* x, y_i, *and* γ_i *as mentioned above, and for* $1 < m \leq 2$, *we have the following:*

$$\left| \sum_{i=1}^{d} \gamma_i \langle x, y_i \rangle_A \right|^2 \leq d^{\frac{1}{m}} \|x\|_A^2 \sum_{k=1}^{d} |\gamma_k|^2 \left(\sum_{i=1}^{d} \left[\max_{j \in \{1,\ldots,d\}} \left| \langle y_i, y_j \rangle_A \right| \right]^l \right)^{\frac{1}{l}}, \tag{13}$$

where $\frac{1}{m} + \frac{1}{l} = 1$.

Corollary 8. *Based on the conditions mentioned earlier for* x, y_i, *and* γ_i, *we can derive the following inequality:*

$$\left| \sum_{i=1}^{d} \gamma_i \langle x, y_i \rangle_A \right|^2 \leq d \|x\|_A^2 \sum_{k=1}^{d} |\gamma_k|^2 \max_{i,j \in \{1,\ldots,d\}} \left| \langle y_i, y_j \rangle_A \right|. \tag{14}$$

To conclude this section, we would like to highlight an important observation. Specifically, leveraging Lemma 2 enables us to derive the following set of inequalities under the assumptions of Theorem 1, which we present in the form of the following remark:

Remark 2. *By utilizing Lemma 2, we can demonstrate that assuming Theorem 1, the following inequalities hold:*

$$\left|\sum_{i=1}^{d}\gamma_i\langle x,y_i\rangle_A\right|^2 \le \|x\|_A^2\sum_{i=1}^{d}|\gamma_i|^2\sum_{j=1}^{d}\left|\langle y_i,y_j\rangle_A\right|$$

$$\le \|x\|_A^2\times\begin{cases}\displaystyle\sum_{i=1}^{d}|\gamma_i|^2\max_{i\in\{1,\dots,d\}}\left[\sum_{j=1}^{d}\left|\langle y_i,y_j\rangle_A\right|\right];\\[2ex] \displaystyle\left(\sum_{i=1}^{d}|\gamma_i|^{2p}\right)^{\frac{1}{p}}\left(\sum_{i=1}^{d}\left(\sum_{j=1}^{d}\left|\langle y_i,y_j\rangle_A\right|\right)^q\right)^{\frac{1}{q}},\\ \qquad p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2ex] \displaystyle\max_{i\in\{1,\dots,d\}}|\gamma_i|^2\sum_{i,j=1}^{d}\left|\langle y_i,y_j\rangle_A\right|.\end{cases}$$

These inequalities provide alternative results to Pečarić's inequality (5).

3. Some Inequalities of Bombieri Type

In this section, we discuss inequalities of Bombieri type which can be derived from (10) by setting $\gamma_i=\overline{\langle x,y_i\rangle_A}$ for $i=1,\dots,d$. By making this choice in the first inequality of (10), the following inequality can be obtained:

$$\left(\sum_{i=1}^{d}\left|\langle x,y_i\rangle_A\right|^2\right)^2 \le \|x\|_A^2\max_{i\in\{1,\dots,d\}}\left|\langle x,y_i\rangle_A\right|^2\sum_{i,j=1}^{d}\left|\langle y_i,y_j\rangle_A\right|.$$

This implies that

$$\sum_{i=1}^{d}\left|\langle x,y_i\rangle_A\right|^2 \le \|x\|_A\max_{i\in\{1,\dots,d\}}\left|\langle x,y_i\rangle_A\right|\left(\sum_{i,j=1}^{d}\left|\langle y_i,y_j\rangle_A\right|\right)^{\frac{1}{2}},\quad x\in\mathcal{H}. \tag{15}$$

Similarly, by choosing $\gamma_i=\overline{\langle x,y_i\rangle_A}$ for $i=1,\dots,d$ in the second inequality of (10), we obtain the following result:

$$\left(\sum_{i=1}^{d}\left|\langle x,y_i\rangle_A\right|^2\right)^2 \le \|x\|_A^2\max_{i\in\{1,\dots,d\}}\left|\langle x,y_i\rangle_A\right|\left(\sum_{i=1}^{d}\left|\langle x,y_i\rangle_A\right|^r\right)^{\frac{1}{r}}\left[\sum_{i=1}^{d}\left(\sum_{j=1}^{d}\left|\langle y_i,y_j\rangle_A\right|\right)^s\right]^{\frac{1}{s}},$$

which implies that

$$\sum_{i=1}^{d}\left|\langle x,y_i\rangle_A\right|^2 \le \|x\|_A\max_{1\le i\le n}\left|\langle x,y_i\rangle_A\right|^{\frac{1}{2}}\left(\sum_{i=1}^{d}\left|\langle x,y_i\rangle_A\right|^r\right)^{\frac{1}{2r}}\left[\sum_{i=1}^{d}\left(\sum_{j=1}^{d}\left|\langle y_i,y_j\rangle_A\right|\right)^s\right]^{\frac{1}{2s}}, \tag{16}$$

where $\frac{1}{r}+\frac{1}{s}=1$, $s>1$.

By using the same method of choosing $\gamma_i=\overline{\langle x,y_i\rangle_A}$ for $i=1,\dots,d$ in the third to ninth inequalities in (10), we can obtain the following results:

The third inequality in (10) gives

$$\sum_{i=1}^{d}\left|\langle x,y_i\rangle_A\right|^2 \le \|x\|_A\max_{1\le i\le n}\left|\langle x,y_i\rangle_A\right|^{\frac{1}{2}}\left(\sum_{i=1}^{d}\left|\langle x,y_i\rangle_A\right|\right)^{\frac{1}{2}}\left[\max_{i\in\{1,\dots,d\}}\left(\sum_{j=1}^{d}\left|\langle y_i,y_j\rangle_A\right|\right)\right]. \tag{17}$$

The fourth inequality in (10) leads to

$$\sum_{i=1}^{d}\left|\langle x,y_i\rangle_A\right|^2 \le \|x\|_A \max_{i\in\{1,\dots,d\}}\left|\langle x,y_i\rangle_A\right|^{\frac{1}{2}}\left(\sum_{i=1}^{d}\left|\langle x,y_i\rangle_A\right|^p\right)^{\frac{1}{2p}}\left[\sum_{i=1}^{d}\left(\sum_{j=1}^{d}\left|\langle y_i,y_j\rangle_A\right|^q\right)^{\frac{1}{q}}\right]^{\frac{1}{2}}, \tag{18}$$

where $p>1$, $\frac{1}{p}+\frac{1}{q}=1$.

The fifth inequality in (10) implies

$$\sum_{i=1}^{d}\left|\langle x,y_i\rangle_A\right|^2 \le \|x\|_A \left(\sum_{i=1}^{d}\left|\langle x,y_i\rangle_A\right|^p\right)^{\frac{1}{2p}}\left(\sum_{i=1}^{d}\left|\langle x,y_i\rangle_A\right|^t\right)^{\frac{1}{2t}}\left[\sum_{i=1}^{d}\left(\sum_{j=1}^{d}\left|\langle y_i,y_j\rangle_A\right|^q\right)^{\frac{u}{q}}\right]^{\frac{1}{2u}}, \tag{19}$$

where $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $t>1$, $\frac{1}{t}+\frac{1}{u}=1$.

The sixth inequality in (10) results in

$$\sum_{i=1}^{d}\left|\langle x,y_i\rangle_A\right|^2 \le \|x\|_A \left(\sum_{i=1}^{d}\left|\langle x,y_i\rangle_A\right|^p\right)^{\frac{1}{2p}}\left(\sum_{i=1}^{d}\left|\langle x,y_i\rangle_A\right|\right)^{\frac{1}{2}}\max_{i\in\{1,\dots,d\}}\left\{\left(\sum_{j=1}^{d}\left|\langle y_i,y_j\rangle_A\right|^q\right)^{\frac{1}{2q}}\right\}, \tag{20}$$

where $p>1$, $\frac{1}{p}+\frac{1}{q}=1$.

The seventh inequality in (10) provides

$$\sum_{i=1}^{d}\left|\langle x,y_i\rangle_A\right|^2 \le \|x\|_A \left[\sum_{i=1}^{d}\left|\langle x,y_i\rangle_A\right|\right]^{\frac{1}{2}}\max_{i\in\{1,\dots,d\}}\left|\langle x,y_i\rangle_A\right|^{\frac{1}{2}}\left(\sum_{i=1}^{d}\left[\max_{j\in\{1,\dots,d\}}\left|\langle y_i,y_j\rangle_A\right|\right]\right)^{\frac{1}{2}}. \tag{21}$$

The eighth inequality in (10) yields

$$\sum_{i=1}^{d}\left|\langle x,y_i\rangle_A\right|^2 \le \|x\|_A \left[\sum_{i=1}^{d}\left|\langle x,y_i\rangle_A\right|^m\right]^{\frac{1}{2m}}\left[\sum_{i=1}^{d}\left[\max_{j\in\{1,\dots,d\}}\left|\langle y_i,y_j\rangle_A\right|^l\right]\right]^{\frac{1}{2l}}, \tag{22}$$

where $m>1$, $\frac{1}{m}+\frac{1}{l}=1$.

Finally, the ninth inequality in (10) produces

$$\sum_{i=1}^{d}\left|\langle x,y_i\rangle_A\right|^2 \le \|x\|_A \sum_{i=1}^{d}\left|\langle x,y_i\rangle_A\right|\max_{i,j\in\{1,\dots,d\}}\left|\langle y_i,y_j\rangle_A\right|^{\frac{1}{2}}. \tag{23}$$

By setting $(y_i)_{i\in\{1,\dots,d\}}=(e_i)_{i\in\{1,\dots,d\}}$, where $(e_i)_{i\in\{1,\dots,d\}}$ are A-orthonormal vectors in $\mathcal{H}$, i.e., $\langle e_i,e_j\rangle_A=\delta_{ij}$ for all $i,j\in 1,\dots,d$, we can derive a set of inequalities that resemble Bessel's inequality from the nine equalities mentioned above. More precisely, we have the following bounds:

$$\sum_{i=1}^{d}\left|\langle x,e_i\rangle_A\right|^2 \le \sqrt{d}\|x\|_A \max_{i\in\{1,\dots,d\}}\left\{\left|\langle x,e_i\rangle_A\right|\right\};$$

$$\sum_{i=1}^{d}\left|\langle x,e_i\rangle_A\right|^2 \le d^{\frac{1}{2s}}\|x\|_A \max_{i\in\{1,\dots,d\}}\left\{\left|\langle x,e_i\rangle_A\right|^{\frac{1}{2}}\right\}\left(\sum_{i=1}^{d}\left|\langle x,e_i\rangle_A\right|^r\right)^{\frac{1}{2r}},$$

where $r>1$, $\frac{1}{r}+\frac{1}{s}=1$;

$$\sum_{i=1}^{d}\left|\langle x,e_i\rangle_A\right|^2 \le \|x\|_A \max_{i\in\{1,\dots,d\}}\left\{\left|\langle x,e_i\rangle_A\right|^{\frac{1}{2}}\right\}\left(\sum_{i=1}^{d}\left|\langle x,e_i\rangle_A\right|\right)^{\frac{1}{2}};$$

$$\sum_{i=1}^{d}\left|\langle x,e_i\rangle_A\right|^2 \le \sqrt{d}\|x\|_A \max_{i\in\{1,\ldots,d\}}\left\{\left|\langle x,e_i\rangle_A\right|^{\frac{1}{2}}\right\}\left(\sum_{i=1}^{d}\left|\langle x,e_i\rangle_A\right|^p\right)^{\frac{1}{2p}},$$

where $p>1$;

$$\sum_{i=1}^{d}\left|\langle x,e_i\rangle_A\right|^2 \le d^{\frac{1}{2u}}\|x\|_A\left(\sum_{i=1}^{d}\left|\langle x,e_i\rangle_A\right|^p\right)^{\frac{1}{2p}}\left(\sum_{i=1}^{d}\left|\langle x,e_i\rangle_A\right|^t\right)^{\frac{1}{2t}},$$

where $p>1$, $t>1$, $\frac{1}{t}+\frac{1}{u}=1$;

$$\sum_{i=1}^{d}\left|\langle x,e_i\rangle_A\right|^2 \le \|x\|_A\left(\sum_{i=1}^{d}\left|\langle x,e_i\rangle_A\right|^p\right)^{\frac{1}{2p}}\left(\sum_{i=1}^{d}\left|\langle x,e_i\rangle_A\right|\right)^{\frac{1}{2}},\ \ p>1;$$

$$\sum_{i=1}^{d}\left|\langle x,e_i\rangle_A\right|^2 \le \sqrt{d}\|x\|_A\left(\sum_{i=1}^{d}\left|\langle x,e_i\rangle_A\right|\right)^{\frac{1}{2}}\max_{i\in\{1,\ldots,d\}}\left\{\left|\langle x,e_i\rangle_A\right|^{\frac{1}{2}}\right\};$$

$$\sum_{i=1}^{d}\left|\langle x,e_i\rangle_A\right|^2 \le d^{\frac{1}{2l}}\|x\|_A\left[\sum_{i=1}^{d}\left|\langle x,e_i\rangle_A\right|^m\right]^{\frac{1}{m}},$$

where $m>1$, $\frac{1}{m}+\frac{1}{l}=1$. Finally, we have

$$\sum_{i=1}^{d}\left|\langle x,e_i\rangle_A\right|^2 \le \|x\|_A\sum_{i=1}^{d}\left|\langle x,e_i\rangle_A\right|.$$

The Corollaries 5–8 yield the following results. Specifically, if we set $\gamma_i=\overline{\langle x,y_i\rangle_A}$ in (11), then

$$\left(\sum_{i=1}^{d}\left|\langle x,y_i\rangle_A\right|^2\right)^2 \le d^{\frac{1}{p}+\frac{1}{t}-1}\|x\|_A^2\sum_{i=1}^{d}\left|\langle x,y_i\rangle_A\right|^2\left[\sum_{i=1}^{d}\left(\sum_{j=1}^{d}\left|\langle y_i,y_j\rangle_A\right|^q\right)^{\frac{u}{q}}\right]^{\frac{1}{u}}.$$

We can readily obtain the following inequality of Bombieri type:

$$\sum_{i=1}^{d}\left|\langle x,y_i\rangle_A\right|^2 \le d^{\frac{1}{p}+\frac{1}{t}-1}\|x\|_A^2\left[\sum_{i=1}^{d}\left(\sum_{j=1}^{d}\left|\langle y_i,y_j\rangle_A\right|^q\right)^{\frac{u}{q}}\right]^{\frac{1}{u}},$$

where $1<p\le 2$, $1<t\le 2$, $\frac{1}{p}+\frac{1}{q}=1$, $\frac{1}{t}+\frac{1}{u}=1$.

By choosing $p=q=t=u=2$ in the inequality, we can obtain the following Bombieri-type inequality:

$$\sum_{i=1}^{d}\left|\langle x,y_i\rangle_A\right|^2 \le \|x\|_A^2\left(\sum_{i,j=1}^{d}\left|\langle y_i,y_j\rangle_A\right|^2\right)^{\frac{1}{2}}. \tag{24}$$

A different proof of (24) for the special case $A=I$ can also be found in [2].

We can apply a similar approach for (13) by choosing $\gamma_i=\overline{\langle x,y_i\rangle_A}$, which yields:

$$\sum_{i=1}^{d}\left|\langle x,y_i\rangle_A\right|^2 \le d^{\frac{1}{m}}\|x\|_A^2\left(\sum_{i=1}^{d}\left[\max_{j\in\{1,\ldots,d\}}\left|\langle y_i,y_j\rangle_A\right|\right]^l\right)^{\frac{1}{l}},$$

where $m>1$, $\frac{1}{m}+\frac{1}{l}=1$.

In conclusion, by setting $\gamma_i = \overline{\langle x, y_i \rangle_A}$ for $i = 1, \dots, d$ in (14), we obtain the following inequality:

$$\sum_{i=1}^{d} \left| \langle x, y_i \rangle_A \right|^2 \leq d \|x\|_A^2 \max_{i,j \in \{1,\dots,d\}} \left| \langle y_i, y_j \rangle_A \right|.$$

Remark 3. *To compare the generalized Bombieri's inequality (6) to our result presented in the subsequent inequality:*

$$\sum_{i=1}^{d} \left| \langle x, y_i \rangle_A \right|^2 \leq \|x\|_A^2 \left\{ \sum_{i,j=1}^{d} \left| \langle y_i, y_j \rangle_A \right|^2 \right\}^{\frac{1}{2}}, \tag{25}$$

we define the following two quantities:

$$M_1 := \max_{i \in \{1,\dots,d\}} \left\{ \sum_{j=1}^{d} \left| \langle y_i, y_j \rangle_A \right| \right\} \quad \text{and} \quad M_2 := \left[\sum_{i,j=1}^{d} \left| \langle y_i, y_j \rangle_A \right|^2 \right]^{\frac{1}{2}}.$$

If $(y_i)_{i \in \{1,\dots,d\}}$ are A-orthonormal vectors with $d \geq 2$, then $M_1 = 1$, $M_2 = \sqrt{d}$, indicating that in this case, the inequality (6) provides a better bound than (25).

On the other hand, let's consider the case where $d = 2$ and A is the identity operator on the real Hilbert space $\mathcal{H} = \mathbb{R}$ with the inner product $\langle x, y \rangle := xy$. Let $y_1 = m$ and $y_2 = r$ be two positive real numbers. In this case, we have

$$M_1 = \max\left\{ m^2 + mr, mr + r^2 \right\} = (m + r) \max(m, r),$$

$$M_2 = \left(m^4 + m^2 r^2 + m^2 r^2 + r^4 \right)^{\frac{1}{2}} = m^2 + r^2.$$

Assuming that $m \geq r$, we have $M_1 = m^2 + mr \geq m^2 + r^2 = M_2$. This shows that, in this case, the bound given by inequality (25) is better than the one given by inequality (6).

Thus, it can be concluded that in general, the two bounds given by inequalities (6) and (25) are incomparable.

4. Inequalities for Operators

In this section, we will apply the inequalities obtained in the previous section to establish several inequalities for operators acting on semi-Hilbert spaces. Specifically, we will use the Bombieri-type inequalities in the context of semi-Hilbert spaces to obtain bounds for the joint A-numerical radius and the Euclidean A-seminorm of operator tuples.

To begin, we recall several terminologies and facts related to operator theory in the context of semi-Hilbert spaces. We start with the notion of A-adjoint. For $T \in \mathbb{B}(\mathcal{H})$, an operator $R \in \mathbb{B}(\mathcal{H})$ is called an A-adjoint operator of T if for every $x, y \in \mathcal{H}$, we have $\langle Tx, y \rangle_A = \langle x, Ry \rangle_A$, that is, $AR = T^*A$ (see [11]). Note that the existence of an A-adjoint operator is not guaranteed for every operator. The set of all operators that admit A-adjoints is denoted by $\mathbb{B}_A(\mathcal{H})$.

By Douglas theorem [12], we have $T \in \mathbb{B}_A(\mathcal{H})$ if and only if $\mathcal{R}(T^*A) \subseteq \mathcal{R}(A)$. If $T \in \mathbb{B}_A(\mathcal{H})$, then the "reduced" solution of the equation $AX = T^*A$ is a distinguished A-adjoint operator of T, which is denoted by $T^{\sharp_A}$. Moreover, if $T \in \mathbb{B}_A(\mathcal{H})$, then $T^{\sharp_A} \in \mathbb{B}_A(\mathcal{H})$ and $(T^{\sharp_A})^{\sharp_A} = P_{\overline{\mathcal{R}(A)}} T P_{\overline{\mathcal{R}(A)}}$, where $P_{\overline{\mathcal{R}(A)}}$ is the orthogonal projection onto the closure of the range of A.

An important observation, as an application of Douglas theorem, is that operators in $\mathbb{B}_{A^{1/2}}(\mathcal{H})$, called A-bounded operators, are characterized by the existence of a constant $c > 0$ such that $\|Tx\|_A < c\|x\|_A$ for all $x \in \mathcal{H}$. It is important to note that both $\mathbb{B}_A(\mathcal{H})$ and $\mathbb{B}_{A^{1/2}}(\mathcal{H})$ are subalgebras of $\mathbb{B}(\mathcal{H})$. However, they are neither closed nor dense in $\mathbb{B}(\mathcal{H})$, and the inclusions $\mathbb{B}_A(\mathcal{H}) \subseteq \mathbb{B}_{A^{1/2}}(\mathcal{H}) \subseteq \mathbb{B}(\mathcal{H})$ are generally strict. Nevertheless, if A is one-to-one and has a closed range, the inclusions hold with equality. For more information on results related to operator theory in semi-Hilbert spaces, we recommend referring to [11,13–16].

For the sequel, $\mathbb{B}(\mathcal{H})^d$ denotes the set of all d-tuples of operators. Let $\mathbf{T} = (T_1, \ldots, T_d) \in \mathbb{B}(\mathcal{H})^d$ be a d-tuple of operators. The following two quantities

$$\omega_A(\mathbf{T}) := \sup_{x \in \mathbb{S}_{\mathcal{H}}^A} \sqrt{\sum_{k=1}^{d} \left|\langle T_k x, x\rangle_A\right|^2} \quad \text{and} \quad \|\mathbf{T}\|_A = \sup_{x \in \mathbb{S}_{\mathcal{H}}^A} \sqrt{\sum_{k=1}^{d} \|T_k x\|_A^2} \tag{26}$$

are defined in [17]. Here, $\mathbb{S}_{\mathcal{H}}^A$ is the unit sphere of $\mathcal{H}$ with respect to the seminorm $\|\cdot\|_A$, which is defined as the set of all vectors $x \in \mathcal{H}$ such that $\|x\|_A = 1$.

It is worth noting that both $\omega_A(\mathbf{T})$ and $\|\mathbf{T}\|_A$ may be equal to $+\infty$ even for $d = 1$ (see [18]). However, if $\mathbf{T} = (T_1, \ldots, T_d) \in \mathbb{B}_{A^{1/2}}(\mathcal{H})^d$, then they define two equivalent seminorms (see [17]). In this case, $\omega_A(\mathbf{T})$ is called the joint A-numerical radius of $\mathbf{T}$, and $\|\mathbf{T}\|_A$ is called the joint operator A-seminorm of $\mathbf{T}$.

When $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$, we can obtain the definitions of the A-numerical radius and the operator A-seminorm of T by setting $d = 1$ in (26). Specifically, we have

$$\omega_A(T) = \sup_{x \in \mathbb{S}_{\mathcal{H}}^A} |\langle Tx, x\rangle_A| \quad \text{and} \quad \|T\|_A = \sup_{x \in \mathbb{S}_{\mathcal{H}}^A} \|Tx\|_A.$$

The investigation of these quantities has been the subject of extensive research in the existing literature, as demonstrated by numerous studies including [14,15] and the references cited therein.

In [14], a different joint A-seminorm for $\mathbf{T} = (T_1, \ldots, T_d) \in \mathbb{B}_{A^{1/2}}(\mathcal{H})^d$, called as Euclidean A-seminorm, was introduced as

$$\|\mathbf{T}\|_{e,A} = \sup_{(\rho_1, \ldots, \rho_d) \in \mathbb{B}_d} \|\rho_1 T_1 + \ldots + \rho_d T_d\|_A,$$

where $\mathbb{B}_d$ denotes the open unit ball of $\mathbb{C}^d$ given by

$$\mathbb{B}_d := \left\{ \rho = (\rho_1, \ldots, \rho_d) \in \mathbb{C}^d\,;\ \|\rho\|_2^2 := \sum_{k=1}^{d} |\rho_k|^2 < 1 \right\}.$$

Our initial outcome in this section is described below:

Theorem 2. *Suppose $\mathbf{T} = (T_1, \ldots, T_d) \in \mathbb{B}_A(\mathcal{H})^d$. Then, for all $1 < p \leq 2$, $1 < t \leq 2$, $q > 1$, and $u > 1$, satisfying $\frac{1}{p} + \frac{1}{q} = 1$ and $\frac{1}{t} + \frac{1}{u} = 1$, the following holds:*

$$\|\mathbf{T}\|_{e,A}^2 \leq d^{\frac{1}{p}+\frac{1}{t}-1} \left[\sum_{i=1}^{d} \left(\sum_{j=1}^{d} \omega_A^q \left(T_j^{\sharp_A} T_i \right) \right)^{\frac{u}{q}} \right]^{\frac{1}{u}}.$$

Proof. Let $\mu_1, \ldots, \mu_d \in \mathbb{C}$ and $x \in \mathcal{H}$. If we apply (8) to $\xi_i = T_i x$ for every $i \in 1, \ldots, d$, where $x \in \mathcal{H}$, we can infer that:

$$\left\| \sum_{i=1}^{d} \mu_i T_i x \right\|_A^2 \leq d^{\frac{1}{p}+\frac{1}{t}-1} \sum_{k=1}^{d} |\mu_k|^2 \left[\sum_{i=1}^{d} \left(\sum_{j=1}^{d} \left| \langle T_i x, T_j x \rangle_A \right|^q \right)^{\frac{u}{q}} \right]^{\frac{1}{u}}$$

for $1 < p \leq 2$, $1 < t \leq 2$, where $\frac{1}{p} + \frac{1}{q} = 1$, $\frac{1}{t} + \frac{1}{u} = 1$.

This is equivalent to

$$\left\|\sum_{i=1}^{d}\mu_i T_i x\right\|_A^2 \leq d^{\frac{1}{p}+\frac{1}{t}-1}\sum_{k=1}^{d}|\mu_k|^2\left[\sum_{i=1}^{d}\left(\sum_{j=1}^{d}\left|\left\langle T_j^{\sharp_A}T_i x, x\right\rangle_A\right|^q\right)^{\frac{u}{q}}\right]^{\frac{1}{u}}. \tag{27}$$

By considering the supremum of (27) over $x \in \mathbb{S}_{\mathcal{H}}^A$, we obtain:

$$\begin{aligned}
\left\|\sum_{i=1}^{d}\mu_i T_i\right\|_A^2 &= \sup_{x\in\mathbb{S}_{\mathcal{H}}^A}\left\|\sum_{i=1}^{d}\mu_i T_i x\right\|_A^2 \\
&\leq d^{\frac{1}{p}+\frac{1}{t}-1}\sum_{k=1}^{d}|\mu_k|^2 \sup_{x\in\mathbb{S}_{\mathcal{H}}^A}\left[\sum_{i=1}^{d}\left(\sum_{j=1}^{d}\left|\left\langle T_j^{\sharp_A}T_i x, x\right\rangle_A\right|^q\right)^{\frac{u}{q}}\right]^{\frac{1}{u}} \\
&\leq d^{\frac{1}{p}+\frac{1}{t}-1}\sum_{k=1}^{d}|\mu_k|^2\left[\sum_{i=1}^{d}\sup_{x\in\mathbb{S}_{\mathcal{H}}^A}\left(\sum_{j=1}^{d}\left|\left\langle T_j^{\sharp_A}T_i x, x\right\rangle_A\right|^q\right)^{\frac{u}{q}}\right]^{\frac{1}{u}} \\
&\leq d^{\frac{1}{p}+\frac{1}{t}-1}\sum_{k=1}^{d}|\mu_k|^2\left[\sum_{i=1}^{d}\left(\sum_{j=1}^{d}\sup_{x\in\mathbb{S}_{\mathcal{H}}^A}\left|\left\langle T_j^{\sharp_A}T_i x, x\right\rangle_A\right|^q\right)^{\frac{u}{q}}\right]^{\frac{1}{u}} \\
&= d^{\frac{1}{p}+\frac{1}{t}-1}\sum_{k=1}^{d}|\mu_k|^2\left[\sum_{i=1}^{d}\left(\sum_{j=1}^{d}\omega_A^q\left(T_j^{\sharp_A}T_i\right)\right)^{\frac{u}{q}}\right]^{\frac{1}{u}},
\end{aligned}$$

which proves

$$\left\|\sum_{i=1}^{d}\mu_i T_i\right\|_A^2 \leq d^{\frac{1}{p}+\frac{1}{t}-1}\sum_{k=1}^{d}|\mu_k|^2\left[\sum_{i=1}^{d}\left(\sum_{j=1}^{d}\omega_A^q\left(T_j^{\sharp_A}T_i\right)\right)^{\frac{u}{q}}\right]^{\frac{1}{u}}. \tag{28}$$

Using (28), we can conclude that:

$$\begin{aligned}
\|\mathbf{T}\|_{e,A}^2 &= \sup_{(\mu_1,\ldots,\mu_d)\in\mathbb{B}_d}\left\|\sum_{i=1}^{d}\mu_i T_i\right\|_A^2 \\
&\leq d^{\frac{1}{p}+\frac{1}{t}-1}\sup_{(\lambda_1,\ldots,\lambda_d)\in\mathbb{B}_d}\left(\sum_{k=1}^{d}|\mu_k|^2\left[\sum_{i=1}^{d}\left(\sum_{j=1}^{d}\omega_A^q\left(T_j^{\sharp_A}T_i\right)\right)^{\frac{u}{q}}\right]^{\frac{1}{u}}\right) \\
&= d^{\frac{1}{p}+\frac{1}{t}-1}\left[\sum_{i=1}^{d}\left(\sum_{j=1}^{d}\omega_A^q\left(T_j^{\sharp_A}T_i\right)\right)^{\frac{u}{q}}\right]^{\frac{1}{u}}.
\end{aligned}$$

Thus, we have shown that the desired inequality holds. □

Remark 4. *A noteworthy special case arises when $p = q = t = u = 2$, which yields:*

$$\left\|\sum_{i=1}^{d}\mu_i T_i\right\|_A^2 \leq \sum_{k=1}^{d}|\mu_k|^2\left[\sum_{i=1}^{d}\sum_{j=1}^{d}\omega_A^2\left(T_j^{\sharp_A}T_i\right)\right]^{\frac{1}{2}},$$

for every $\mu_1, \ldots, \mu_d \in \mathbb{C}$. *Also,*

$$\|\mathbf{T}\|^2_{e,A} \leq \left[\sum_{i=1}^{d}\sum_{j=1}^{d}\omega_A^2\left(T_j^{\sharp_A}T_i\right)\right]^{\frac{1}{2}}.$$

Corollary 9. *Let* $\mathbf{T} = (T_1, \ldots, T_d) \in \mathbb{B}_A(\mathcal{H})^d$. *Then for all* $\mu_1, \ldots, \mu_d \in \mathbb{C}$ *and for* $1 < p \leq 2$, $q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$, *we have*

$$\|\mathbf{T}\|^2_{e,A} \leq d^{\frac{1}{p}} \max_{i\in\{1,\ldots,d\}} \left(\sum_{j=1}^{d}\omega_A^q\left(T_j^{\sharp_A}T_i\right)\right)^{\frac{1}{q}}. \quad (29)$$

Proof. Let $1 < p \leq 2$ and $q > 1$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. For any $\mu_1, \ldots, \mu_d \in \mathbb{C}$ and $x \in \mathcal{H}$, we can utilize (9) with $\xi_i = T_i x$ for every $i \in \{1, \ldots, d\}$ to obtain:

$$\left\|\sum_{i=1}^{d}\mu_i T_i x\right\|_A^2 \leq d^{\frac{1}{p}}\sum_{k=1}^{d}|\mu_k|^2 \max_{i\in\{1,\ldots,d\}}\left[\left(\sum_{j=1}^{d}\left|\left\langle T_i x, T_j x\right\rangle_A\right|^q\right)^{\frac{1}{q}}\right].$$

Employing arguments similar to those used in the proof of Theorem 2, we conclude that:

$$\left\|\sum_{i=1}^{d}\mu_i T_i\right\|_A^2 \leq d^{\frac{1}{p}}\sum_{k=1}^{d}|\mu_k|^2 \max_{i\in\{1,\ldots,d\}}\left(\sum_{j=1}^{d}\omega_A^q\left(T_j^{\sharp_A}T_i\right)\right)^{\frac{1}{q}}.$$

The desired result follows directly from taking the supremum over all $(\mu_1, \ldots, \mu_d) \in \mathbb{B}_d$ in the last inequality. □

Remark 5. *When we set* $p = q = 2$ *in Equation* (29), *we obtain*

$$\|\mathbf{T}\|^2_{e,A} \leq d^{\frac{1}{2}} \max_{i\in\{1,\ldots,d\}} \left(\sum_{j=1}^{d}\omega_A^2\left(T_j^{\sharp_A}T_i\right)\right)^{\frac{1}{2}}.$$

Additionally, we can apply Corollary 3 to obtain the next corollary.

Corollary 10. *If we assume the conditions of Corollary 9 and* $1 < m \leq 2$, *then*

$$\|\mathbf{T}\|^2_{e,A} \leq d^{\frac{1}{m}}\left(\sum_{i=1}^{d}\left[\max_{j\in\{1,\ldots,d\}}\omega_A\left(T_j^{\sharp_A}T_i\right)\right]^l\right)^{\frac{1}{l}},$$

where $\frac{1}{m} + \frac{1}{l} = 1$.

Remark 6. *Substituting* $m = l = 2$ *into the corollary above, we obtain*

$$\|\mathbf{T}\|^2_{e,A} \leq d^{\frac{1}{2}}\left(\sum_{i=1}^{d}\left[\max_{j\in\{1,\ldots,d\}}\omega_A\left(T_j^{\sharp_A}T_i\right)\right]^2\right)^{\frac{1}{2}}.$$

Utilizing Lemma 2, we can demonstrate in a similar fashion as previously stated that:

Theorem 3. *Consider* $\mathbf{T} = (T_1, \ldots, T_d) \in \mathbb{B}_A(\mathcal{H})^d$. *Then, for any* $\mu_1, \ldots, \mu_d \in \mathbb{C}$, *it holds that:*

$$\left\| \sum_{i=1}^{d} \mu_i T_i \right\|_A^2 \leq \sum_{i=1}^{d} |\mu_i|^2 \sum_{j=1}^{d} \omega_A\left(T_j^{\sharp_A} T_i\right) \leq \sum_{i=1}^{d} |\mu_i|^2 \max_{i \in \{1,\ldots,d\}} \left[\sum_{j=1}^{d} \omega_A\left(T_j^{\sharp_A} T_i\right) \right]$$

and

$$\|\mathbf{T}\|_{e,A}^2 \leq \max_{i \in \{1,\ldots,d\}} \left[\sum_{j=1}^{d} \omega_A\left(T_j^{\sharp_A} T_i\right) \right].$$

Additionally, we obtain the following inequalities for the joint A-numerical radius:

Theorem 4. *Let* $\mathbf{T} = (T_1, \ldots, T_d) \in \mathbb{B}_A(\mathcal{H})^d$. *Then*

$$\omega_A^2(\mathbf{T}) \leq \max_{i \in \{1,\ldots,d\}} \{\omega_A(T_i)\} \left(\sum_{i,j=1}^{d} \omega_A\left(T_j^{\sharp_A} T_i\right) \right)^{\frac{1}{2}},$$

and

$$\omega_A^2(\mathbf{T}) \leq \max_{i \in \{1,\ldots,d\}} \left\{ \omega_A^{\frac{1}{2}}(T_i) \right\} \left(\sum_{i=1}^{d} \omega_A^r(T_i) \right)^{\frac{1}{2r}} \left[\sum_{i=1}^{d} \left(\sum_{j=1}^{d} \omega_A\left(T_j^{\sharp_A} T_i\right) \right)^s \right]^{\frac{1}{2s}},$$

where $\frac{1}{r} + \frac{1}{s} = 1$, $s > 1$.

Also,

$$\omega_A^2(\mathbf{T}) \leq \max_{i \in \{1,\ldots,d\}} \left\{ \omega_A^{\frac{1}{2}}(T_i) \right\} \left(\sum_{i=1}^{d} \omega_A(T_i) \right)^{\frac{1}{2}} \max_{i \in \{1,\ldots,d\}} \left\{ \sum_{j=1}^{d} \omega_A\left(T_j^{\sharp_A} T_i\right) \right\};$$

$$\omega_A^2(\mathbf{T}) \leq \max_{i \in \{1,\ldots,d\}} \left\{ \omega_A^{\frac{1}{2}}(T_i) \right\} \left(\sum_{i=1}^{d} \omega_A^p(T_i) \right)^{\frac{1}{2p}} \times \left[\sum_{i=1}^{d} \left(\sum_{j=1}^{d} \omega_A^q\left(T_j^{\sharp_A} T_i\right) \right)^{\frac{1}{q}} \right]^{\frac{1}{2}},$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$;

$$\omega_A^2(\mathbf{T}) \leq \left(\sum_{i=1}^{d} \omega_A^p(T_i) \right)^{\frac{1}{2p}} \left(\sum_{i=1}^{d} \omega_A^t(T_i) \right)^{\frac{1}{2t}} \left[\sum_{i=1}^{d} \left(\sum_{j=1}^{d} \omega_A^q\left(T_j^{\sharp_A} T_i\right) \right)^{\frac{u}{q}} \right]^{\frac{1}{2u}},$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $t > 1$, $\frac{1}{t} + \frac{1}{u} = 1$;

$$\omega_A^2(\mathbf{T}) \leq \left(\sum_{i=1}^{d} \omega_A^p(T_i) \right)^{\frac{1}{2p}} \left(\sum_{i=1}^{d} \omega_A(T_i) \right)^{\frac{1}{2}} \max_{i \in \{1,\ldots,d\}} \left\{ \left(\sum_{j=1}^{d} \omega_A^q\left(T_j^{\sharp_A} T_i\right) \right)^{\frac{1}{2q}} \right\},$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$;

$$\omega_A^2(\mathbf{T}) \leq \left[\sum_{i=1}^{d} \omega_A(T_i) \right]^{\frac{1}{2}} \max_{i \in \{1,\ldots,d\}} \left\{ \omega_A^{\frac{1}{2}}(T_i) \right\} \left(\sum_{i=1}^{d} \max_{j \in \{1,\ldots,d\}} \left\{ \omega_A\left(T_j^{\sharp_A} T_i\right) \right\} \right)^{\frac{1}{2}};$$

$$\omega_A^2(\mathbf{T}) \leq \left[\sum_{i=1}^{d} \omega_A^m(T_i) \right]^{\frac{1}{2m}} \left[\sum_{i=1}^{d} \max_{j \in \{1,\ldots,d\}} \left\{ \omega_A^l\left(T_j^{\sharp_A} T_i\right) \right\} \right]^{\frac{1}{2l}},$$

where $m > 1$, $\frac{1}{m} + \frac{1}{l} = 1$; *and*

$$\omega_A^2(\mathbf{T}) \leq \sum_{i=1}^{d} \omega_A(T_i) \max_{j\in\{1,\ldots,d\}} \left\{ \omega_A^{\frac{1}{2}}\left(T_j^{\sharp_A} T_i\right) \right\}.$$

Proof. Let $x \in \mathcal{H}$. By applying (15) to $y_i = T_i x$, we obtain:

$$\sum_{i=1}^{d} \left| \langle x, T_i x \rangle_A \right|^2 \leq \|x\|_A \max_{i\in\{1,\ldots,d\}} \left| \langle x, T_i x \rangle_A \right| \left(\sum_{i,j=1}^{d} \left| \left\langle T_j^{\sharp_A} T_i x, x \right\rangle_A \right| \right)^{\frac{1}{2}}.$$

When the supremum is taken over $x \in \mathbb{S}_{\mathcal{H}}^A$, the resulting value is obtained as

$$\begin{aligned}
\omega_A^2(\mathbf{T}) &= \sup_{x\in\mathbb{S}_{\mathcal{H}}^A} \sum_{i=1}^{d} \left| \langle x, T_i x \rangle_A \right|^2 \\
&\leq \sup_{x\in\mathbb{S}_{\mathcal{H}}^A} \left\{ \|x\|_A \max_{i\in\{1,\ldots,d\}} \left| \langle x, T_i x \rangle_A \right| \left(\sum_{i,j=1}^{d} \left| \left\langle T_j^{\sharp_A} T_i x, x \right\rangle_A \right| \right)^{\frac{1}{2}} \right\} \\
&\leq \sup_{x\in\mathbb{S}_{\mathcal{H}}^A} \left\{ \max_{i\in\{1,\ldots,d\}} \left| \langle x, T_i x \rangle_A \right| \right\} \sup_{x\in\mathbb{S}_{\mathcal{H}}^A} \left\{ \left(\sum_{i,j=1}^{d} \left| \left\langle T_j^{\sharp_A} T_i x, x \right\rangle_A \right| \right)^{\frac{1}{2}} \right\} \\
&= \left\{ \max_{i\in\{1,\ldots,d\}} \sup_{x\in\mathbb{S}_{\mathcal{H}}^A} \left| \langle x, T_i x \rangle_A \right| \right\} \sup_{x\in\mathbb{S}_{\mathcal{H}}^A} \left\{ \left(\sum_{i,j=1}^{d} \left| \left\langle T_j^{\sharp_A} T_i x, x \right\rangle_A \right| \right)^{\frac{1}{2}} \right\} \\
&\leq \max_{i\in\{1,\ldots,d\}} \sup_{x\in\mathbb{S}_{\mathcal{H}}^A} \left| \langle x, T_i x \rangle_A \right| \left(\sum_{i,j=1}^{d} \sup_{x\in\mathbb{S}_{\mathcal{H}}^A} \left| \left\langle T_j^{\sharp_A} T_i x, x \right\rangle_A \right| \right)^{\frac{1}{2}} \\
&= \max_{i\in\{1,\ldots,d\}} \{\omega_A(T_i)\} \left(\sum_{i,j=1}^{d} \omega_A\left(T_j^{\sharp_A} T_i\right) \right)^{\frac{1}{2}}.
\end{aligned}$$

Thus, the first inequality in Theorem 4 has been proven.

Similarly, by using (16), we obtain that

$$\sum_{i=1}^{d} \left| \langle x, T_i x \rangle_A \right|^2 \leq \|x\|_A \max_{1\leq i\leq d} \left| \langle x, T_i x \rangle_A \right|^{\frac{1}{2}} \left(\sum_{i=1}^{d} \left| \langle x, T_i x \rangle_A \right|^r \right)^{\frac{1}{2r}} \left[\sum_{i=1}^{d} \left(\sum_{j=1}^{d} \left| \left\langle T_j^{\sharp_A} T_i x, x \right\rangle_A \right| \right)^s \right]^{\frac{1}{2s}}.$$

The above condition, i.e., $\frac{1}{r} + \frac{1}{s} = 1$ and $s > 1$, as mentioned earlier, leads to the derivation of the second inequality in Theorem 4.

By applying the inequalities (17)–(23), we can derive the subsequent expression for $y_i = T_i x$:

$$\sum_{i=1}^{d} \left| \langle x, T_i x \rangle_A \right|^2 \leq \|x\|_A \max_{1\leq i\leq d} \left| \langle x, T_i x \rangle_A \right|^{\frac{1}{2}} \left(\sum_{i=1}^{d} \left| \langle x, T_i x \rangle_A \right| \right)^{\frac{1}{2}} \left[\max_{i\in\{1,\ldots,d\}} \left(\sum_{j=1}^{d} \left| \left\langle T_j^{\sharp_A} T_i x, x \right\rangle_A \right| \right) \right];$$

$$\sum_{i=1}^{d} \left| \langle x, T_i x \rangle_A \right|^2 \leq \|x\|_A \max_{1\leq i\leq d} \left| \langle x, T_i x \rangle_A \right|^{\frac{1}{2}} \left(\sum_{i=1}^{d} \left| \langle x, T_i x \rangle_A \right|^p \right)^{\frac{1}{2p}} \left[\sum_{i=1}^{d} \left(\sum_{j=1}^{d} \left| \left\langle T_j^{\sharp_A} T_i x, x \right\rangle_A \right|^q \right)^{\frac{1}{q}} \right]^{\frac{1}{2}},$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$;

$$\sum_{i=1}^{d} \left| \left\langle x, T_i x \right\rangle_A \right|^2 \leq \|x\|_A \left(\sum_{i=1}^{d} \left| \left\langle x, T_i x \right\rangle_A \right|^p \right)^{\frac{1}{2p}} \left(\sum_{i=1}^{d} \left| \left\langle x, T_i x \right\rangle_A \right|^t \right)^{\frac{1}{2t}} \left[\sum_{i=1}^{d} \left(\sum_{j=1}^{d} \left| \left\langle T_j^{\sharp_A} T_i x, x \right\rangle_A \right|^q \right)^{\frac{u}{q}} \right]^{\frac{1}{2u}},$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $t > 1$, $\frac{1}{t} + \frac{1}{u} = 1$;

$$\sum_{i=1}^{d} \left| \left\langle x, T_i x \right\rangle_A \right|^2 \leq \|x\|_A \left(\sum_{i=1}^{d} \left| \left\langle x, T_i x \right\rangle_A \right|^p \right)^{\frac{1}{2p}} \left(\sum_{i=1}^{d} \left| \left\langle x, T_i x \right\rangle_A \right| \right)^{\frac{1}{2}} \max_{i \in \{1,\ldots,d\}} \left\{ \left(\sum_{j=1}^{d} \left| \left\langle T_j^{\sharp_A} T_i x, x \right\rangle_A \right|^q \right)^{\frac{1}{2q}} \right\},$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$;

$$\sum_{i=1}^{d} \left| \left\langle x, T_i x \right\rangle_A \right|^2 \leq \|x\|_A \left[\sum_{i=1}^{d} \left| \left\langle x, T_i x \right\rangle_A \right| \right]^{\frac{1}{2}} \max_{i \in \{1,\ldots,d\}} \left| \left\langle x, T_i x \right\rangle_A \right|^{\frac{1}{2}} \left(\sum_{i=1}^{d} \left[\max_{j \in \{1,\ldots,d\}} \left| \left\langle T_j^{\sharp_A} T_i x, x \right\rangle_A \right| \right] \right)^{\frac{1}{2}};$$

$$\sum_{i=1}^{d} \left| \left\langle x, T_i x \right\rangle_A \right|^2 \leq \|x\|_A \left[\sum_{i=1}^{d} \left| \left\langle x, T_i x \right\rangle_A \right|^m \right]^{\frac{1}{2m}} \left[\sum_{i=1}^{d} \left[\max_{j \in \{1,\ldots,d\}} \left| \left\langle T_j^{\sharp_A} T_i x, x \right\rangle_A \right|^l \right] \right]^{\frac{1}{2l}},$$

where $m > 1$, $\frac{1}{m} + \frac{1}{l} = 1$; and

$$\sum_{i=1}^{d} \left| \left\langle x, T_i x \right\rangle_A \right|^2 \leq \|x\|_A \sum_{i=1}^{d} \left| \left\langle x, T_i x \right\rangle_A \right| \max_{j \in \{1,\ldots,d\}} \left| \left\langle T_j^{\sharp_A} T_i x, x \right\rangle_A \right|^{\frac{1}{2}}.$$

When we take the supremum over $x \in \mathbb{S}_{\mathcal{H}}^A$ in the above inequalities, we obtain the desired inequalities of Theorem 4. □

The following is the final result of this paper:

Theorem 5. *For $\mathbf{T} = (T_1, \ldots, T_d) \in \mathbb{B}_A(\mathcal{H})^d$, and for any $\mu_1, \ldots, \mu_d \in \mathbb{C}$, and $1 < p \leq 2$, $1 < t \leq 2$, $q > 1$, $u > 1$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$, $\frac{1}{t} + \frac{1}{u} = 1$, we have:*

$$\|\mathbf{T}\|_A^2 \leq d^{\frac{1}{2}\left(\frac{1}{p}+\frac{1}{t}\right)} \left[\sum_{i=1}^{d} \left(\sum_{j=1}^{d} \omega_A^q \left(T_j^{\sharp_A} T_j T_i^{\sharp_A} T_i \right) \right)^{\frac{u}{q}} \right]^{\frac{1}{2u}}. \tag{30}$$

Proof. If we choose $\gamma_i = 1$ and $y_i = T_i^{\sharp_A} T_i x$ in the inequality (11), where $x \in \mathcal{H}$, then we obtain:

$$\left| \sum_{i=1}^{d} \left\langle x, T_i^{\sharp_A} T_i x \right\rangle_A \right|^2 \leq \|x\|_A^2 d^{\frac{1}{p}+\frac{1}{t}} \left[\sum_{i=1}^{d} \left(\sum_{j=1}^{d} \left| \left\langle T_i^{\sharp_A} T_i x, T_j^{\sharp_A} T_j x \right\rangle_A \right|^q \right)^{\frac{u}{q}} \right]^{\frac{1}{u}},$$

where $1 < p \leq 2$, $1 < t \leq 2$, $\frac{1}{p} + \frac{1}{q} = 1$, $\frac{1}{t} + \frac{1}{u} = 1$. This can be restated as

$$\sum_{i=1}^{d} \|T_i x\|_A^2 \leq \|x\|_A d^{\frac{1}{2}\left(\frac{1}{p}+\frac{1}{t}\right)} \left[\sum_{i=1}^{d} \left(\sum_{j=1}^{d} \left| \left\langle T_j^{\sharp_A} T_j T_i^{\sharp_A} T_i x, x \right\rangle_A \right|^q \right)^{\frac{u}{q}} \right]^{\frac{1}{2u}}.$$

Taking the supremum over $x \in \mathbb{S}_{\mathcal{H}}^A$ in the inequality above yields

$$\begin{aligned}
\|\mathbf{T}\|_A^2 &:= \sup_{x\in\mathbb{S}_{\mathcal{H}}^A}\left(\sum_{i=1}^{d}\|T_i x\|_A^2\right) \\
&\leq \sup_{x\in\mathbb{S}_{\mathcal{H}}^A}\left(\|x\|_A d^{\frac{1}{2}\left(\frac{1}{p}+\frac{1}{t}\right)}\left[\sum_{i=1}^{d}\left(\sum_{j=1}^{d}\left|\left\langle T_j^{\sharp_A}T_jT_i^{\sharp_A}T_i x, x\right\rangle_A\right|^q\right)^{\frac{u}{q}}\right]^{\frac{1}{2u}}\right) \\
&\leq d^{\frac{1}{2}\left(\frac{1}{p}+\frac{1}{t}\right)}\left[\sup_{x\in\mathbb{S}_{\mathcal{H}}^A}\sum_{i=1}^{d}\left(\sum_{j=1}^{d}\left|\left\langle T_j^{\sharp_A}T_jT_i^{\sharp_A}T_i x, x\right\rangle_A\right|^q\right)^{\frac{u}{q}}\right]^{\frac{1}{2u}} \\
&\leq d^{\frac{1}{2}\left(\frac{1}{p}+\frac{1}{t}\right)}\left[\sum_{i=1}^{d}\left(\sup_{x\in\mathbb{S}_{\mathcal{H}}^A}\sum_{j=1}^{d}\left|\left\langle T_j^{\sharp_A}T_jT_i^{\sharp_A}T_i x, x\right\rangle_A\right|^q\right)^{\frac{u}{q}}\right]^{\frac{1}{2u}} \\
&\leq d^{\frac{1}{2}\left(\frac{1}{p}+\frac{1}{t}\right)}\left[\sum_{i=1}^{d}\left(\sum_{j=1}^{d}\sup_{x\in\mathbb{S}_{\mathcal{H}}^A}\left|\left\langle T_j^{\sharp_A}T_jT_i^{\sharp_A}T_i x, x\right\rangle_A\right|^q\right)^{\frac{u}{q}}\right]^{\frac{1}{2u}}.
\end{aligned}$$

Consequently, inequality (30) is established promptly. □

Remark 7. *As a special case of Theorem 5, when we set $p = q = t = u = 2$, we obtain the inequality*

$$\|\mathbf{T}\|_A^2 \leq d^{\frac{1}{2}}\left[\sum_{i=1}^{d}\sum_{j=1}^{d}\omega_A^2\left(T_j^{\sharp_A}T_jT_i^{\sharp_A}T_i\right)\right]^{\frac{1}{4}}.$$

Remark 8. *For $1 < p \leq 2$, using (12) we can derive a similar inequality:*

$$\|\mathbf{T}\|_A^2 \leq d^{\frac{1}{2}\left(\frac{1}{p}+1\right)}\left[\max_{i\in\{1,\ldots,d\}}\sum_{j=1}^{d}\omega_A^q\left(T_j^{\sharp_A}T_jT_i^{\sharp_A}T_i\right)\right]^{\frac{1}{2q}},$$

where $\frac{1}{p}+\frac{1}{q}=1$. On the other hand, from (13) we obtain:

$$\|\mathbf{T}\|_A^2 \leq d^{\frac{1}{2}\left(\frac{1}{m}+1\right)}\left(\sum_{i=1}^{d}\left[\max_{j\in\{1,\ldots,d\}}\omega_A\left(T_j^{\sharp_A}T_jT_i^{\sharp_A}T_i\right)\right]^{l}\right)^{\frac{1}{l}},$$

where $1 < m \leq 2$ and $\frac{1}{m}+\frac{1}{l}=1$.

5. Conclusions

In conclusion, this paper introduces new findings about Bombieri's generalization of Bessel's inequality in positive semidefinite inner product spaces. These findings extend the classical Bessel inequality and contribute to our understanding of operators in positive semidefinite inner product spaces, also known as semi-Hilbert spaces.

This work provides a starting point for future research and opens up possibilities for exploring new results on Boas-Bellman type inequalities in semi-Hilbert spaces. By investigating these topics, researchers can advance the field and gain insights into functional analysis and operator theory.

Overall, this study has paved the way for further investigations and has the potential to impact various fields. It serves as a foundation for future studies and encourages exploration of new results and applications in semi-Hilbert spaces.

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