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Upper Local Uniform Monotonicity in F-Normed Musielak–Orlicz Spaces

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Abstract: In this paper, the necessary and sufficient conditions for the upper strict monotonicity point and the upper local uniform monotonicity point are given in the case of Musielak–Orlicz spaces equipped with the Mazur–Orlicz F-norm. Moreover, strict monotonicity and upper local uniform monotonicity are easily deduced in the case of Musielak–Orlicz spaces endowed with the Mazur–Orlicz F-norm, and the work by Kaczmarek presented in the references is encompassed by the corollaries presented in this paper.

Keywords: Musielak–Orlicz spaces; Mazur–Orlicz F-norm; F-norm Köthe spaces; upper strict monotonicity point; upper local uniform monotonicity point

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1. Introduction and Preliminaries

It is widely known that monotonicity properties play an important role in geometric properties. For example, in the best approximation problem of a Banach lattice, monotonicity properties play a similar role to that of rotundity properties in the best approximation problem of a Banach space. Therefore, we can know that various monotonicity points play an analogous role to rotundity points (exposed points, strongly exposed points, etc.) in geometric properties. In recent years, monotonicity points have been extensively studied in Musielak–Orlicz, Orlicz–Lorentz, and Calderón–Lozanovskiĭ spaces (see [1–4]).

In this paper, we obtain the necessary and sufficient conditions for the upper strict monotonicity point and the upper local uniform monotonicity point in the case of Musielak–Orlicz spaces equipped with the Mazur–Orlicz F-norm under various conditions. Furthermore, the necessary and sufficient conditions for strict monotonicity and upper local uniform monotonicity in the case of Musielak–Orlicz spaces equipped with the Mazur–Orlicz F-norm are also obtained.

Let us denote $\mathbb{N}$ and $\mathbb{R}$ as the sets of natural and real numbers, respectively, and $\mathbb{R}_+ := [0, \infty)$.

Assume that (T, Σ, m) is a complete, finite, and non-atomic measure space. Let $L^0 = L^0(T, \Sigma, m)$ be the space of all real-valued and Σ-measurable functions on T. $L^1 = L^1(T, \Sigma, m)$ is the space of all real-valued and Σ-integrable functions on T.

Definition 1. *A function $\Phi : T \times [0, +\infty) \to [0, +\infty]$ is called a monotone Musielak–Orlicz function if the following conditions are satisfied:*

(1) $\Phi(t, 0) = 0$;

(2) $\Phi(t, u)$ *is non-decreasing and continuous with respect to u on $[0, b_\Phi(t))$ for almost every $t \in T$ and left continuous at $b_\Phi(t)$;*

i.e.,

(i) $\lim\limits_{u \to b_{\Phi(t)}} \Phi(t, u) = \Phi(t, b_\Phi(t)) \in (0, +\infty)$ *whenever $b_\Phi(t) < +\infty$;*

(ii) $\lim_{u \to b_{\Phi(t)}} \Phi(t,u) = +\infty$ *whenever* $b_\Phi(t) = +\infty$;

(iii) *For almost every* $t \in T$, *there is a* $u_t > 0$ *such that* $\Phi(t, u_t) > 0$ *and* $\Phi(t,u)$ *is* $\Sigma-$ *measurable with respect to* t *for each* $u \in \mathbb{R}_+$,

where

$$b_\Phi(t) := \sup\{u \geq 0 : \Phi(t,u) < +\infty\}.$$

Define

$$a_\Phi(t) := \sup\{u \geq 0 : \Phi(t,u) = 0\},$$

$$\operatorname{supp} x = \{t \in T : x(t) \neq 0\}$$

and

$$S_\Phi^+(t) = \{u : \Phi(t,u) < \Phi(t,v) \ for \ 0 \leq u < v\}.$$

The functions $a_\Phi(.)$ and $b_\Phi(.)$ are Σ measurable and the proofs are similar to the proof of [5]. The function $\Phi(t,u)$ is continuous on $[0, b_\Phi(t))$ in regard to u for almost every $t \in T$.

Definition 2 (see [6])**.** *We say that a monotone Musielak–Orlicz function* Φ *satisfies the* Δ_2-*condition (for brevity, we write* $\Phi \in \Delta_2$*) if there exists a set* $T_1 \in \Sigma$ *with* $m(T_1) = 0$*, a constant* $K > 0$*, and a function* $0 \leq h \in L^1(T, \Sigma, m)$ *such that* $\Phi(t, 2u) \leq K\Phi(t,u) + h(t)$ *for all* $t \in T \backslash T_1$.

If $\Phi \in \Delta_2$, then $b_\Phi(t) = +\infty$ for almost every $t \in T$.
The function $I_\Phi : L^0 \to [0, \infty]$ is defined by the formula

$$I_\Phi(x) = \int_T \Phi(t, |x(t)|)dt.$$

The space

$$L_\Phi = \{x \in L^0 : I_\Phi(\lambda x) < \infty \ \ for \ some \ \lambda > 0\}$$

is said to be a Musielak–Orlicz space (see [7–9]). Define the subspace E_Φ of L_Φ by the formula:

$$E_\Phi = \{x \in L^0 : I_\Phi(\lambda x) < \infty \ \ for \ any \ \lambda > 0\}.$$

The Mazur–Orlicz F-norm is defined by the formula (see [7–9]):

$$\|x\|_F = \inf\{\lambda > 0 : I_\Phi(\frac{x}{\lambda}) \leq \lambda\} \ \ , \forall x \in L_\Phi.$$

Definition 3 (see [10])**.** *Given any real vector space* X *the functional* $x \mapsto \|x\|_F \in \mathbb{R}_+$*, is called an F-norm if the following conditions are satisfied:*

(i) $\|x\|_F = 0$ *if and only if* $x = 0$;
(ii) $\|-x\|_F = \|x\|_F$ *for all* $x \in X$;
(iii) $\|x+y\|_F \leq \|x\|_F + \|y\|_F$ *for all* $x, y \in X$;
(iv) $\|\lambda_n x_n - \lambda x\|_F \to 0$ *whenever* $\|x_n - x\|_F \to 0$ *and* $\lambda_n \to \lambda$ *for any* $x \in X$, $(x_n)_{n=1}^\infty$ *in* X, $\lambda \in \mathbb{R}$ *and* $(\lambda_n)_{n=1}^\infty$ *in* $\mathbb{R}$.

An F-normed space $X = (X, \|\cdot\|_F)$ is an F-space under the condition that the F-normed space X is complete with regard to the F-norm topology. If Z is complete, the lattice $Z = (Z, \leq, \|\cdot\|_F)$ equipped with a monotone F-norm $\|\cdot\|_F$ is an F-lattice, where $\leq$ denotes the partial order relation.

Definition 4 (see [10])**.** *An F-space* $(X, \|\cdot\|_F)$ *is called an F-normed Köthe space if it is a linear subspace of* L^0 *satisfying the following conditions:*

(i) *If* $x \in L^0, y \in X$ *and* $|x| \leq |y|$*, then* $x \in X$ *and* $\|x\|_F \leq \|y\|_F$;

(ii) *There exists a strictly positive $x \in X$ (called a weak unit).*

A complete F-normed *Köthe* space is an F-lattice. If the measure m is non-atomic, then X is an F-normed *Köthe* function space.

Definition 5 (see [10]). *A point x in an F-normed Köthe space $(X, \|\cdot\|_F)$ is said to be an upper strict monotonicity point if for any $y \in X$ such that $0 \leq x \leq y$, the condition $\|x\|_F < \|y\|_F$ holds whenever $x \neq y$ (or equivalently $\|x+y\|_F > \|x\|_F$ whenever $0 \leq y$ and $y \neq 0$). If every point in X is an upper strict monotonicity point, then we say that X is upper strictly monotone.*

Definition 6 (see [10]). *A point x in an F-normed Köthe space $(X, \|\cdot\|_F)$ is said to be an upper local uniform monotonicity point if for any $\{x_n\}_{n=1}^{\infty}$ in X such that $0 \leq x \leq x_n$ for all $n \in \mathbb{N}$ and $\|x_n\|_F \to \|x\|_F$ as $n \to \infty$, the condition $\|x_n - x\|_F \to 0$ as $n \to \infty$ holds. If every point in X is an upper local uniform monotonicity point, we then say that X is upper locally uniformly monotone.*

If a point x is an upper local uniform monotonicity point, it must be an upper strict monotonicity point. However, an upper strict monotonicity point may not be an upper local uniform monotonicity point.

Lemma 1 (see [5]). *If $\Phi \notin \triangle_2$, then $D_\Phi = \{t \in T : b_\Phi(t) < \infty\}$ is a non-null set or for any*

$$b_1 \geq b_2 \geq \cdots \geq 1, 1 < p_1 \leq p_2 \leq \cdots, q_n > 0 \ (n \in \mathbb{N}),$$

there exist measurable functions $\{x_n(t)\}_{n=1}^{\infty}$ and mutually disjoint $\{F_n(t)\}_{n=1}^{\infty}$ in Σ such that $0 \leq x_n(t) < \infty$ on F_n and

$$\int_{F_n} \Phi(t, x_n(t))dt = q_n, \Phi(t, b_n x_n(t)) \geq p_n \Phi(t, x_n(t))(t \in F_n, n \in \mathbb{N}).$$

Lemma 2. *For every monotone Musielak–Orlicz function Φ, and $x \in L_\Phi \backslash \{0\}$ we have:*

(1) $I_\Phi\left(\frac{x}{\|x\|_F}\right) \leq \|x\|_F \Leftrightarrow I_\Phi\left(\frac{x}{\|x\|_F}\right) < +\infty$;

(2) $I_\Phi\left(\frac{x}{\|x\|_F}\right) = \|x\|_F$ *whenever* $I_\Phi\left(\lambda \frac{x}{\|x\|_F}\right) < \infty$ *for some* $\lambda > 1$;

(3) *If* $I_\Phi\left(\frac{x}{\lambda}\right) = \lambda$ *for* $\lambda > 0$, *then* $\|x\|_F = \lambda$.

Proof. The proofs of conditions (1) and (2) are similar to the proof of Lemma 6.1 from [11], and the proof of condition (3) is similar to the proof of Lemma 2.16 from [10], so they are omitted. □

Lemma 3. $\lim\limits_{n\to\infty} \|x_n\|_F = 0$ *if and only if* $\lim\limits_{n\to\infty} I_\Phi(\lambda x_n) = 0$ *for any* $\lambda > 0$.

Proof. The proof is similar to the proof of [11], so it is omitted. □

2. Results in F-Normed Musielak–Orlicz Spaces

Theorem 1.

(a) *If* $\frac{x(t)}{\|x\|_F} = b_\Phi(t)$ *and* $b_\Phi(t) < +\infty$ *for* $t \in \operatorname{supp} x$, *then* $x \in L^\Phi \backslash \{0\}$ *is upper strict monotonicity point;*

(b) *If* $m(\{t \in \operatorname{supp} x : \frac{x(t)}{\|x\|_F} < b_\Phi(t) < +\infty\}) > 0$, *then* $x \in L^\Phi \backslash \{0\}$ *is upper strict monotonicity point if and only if the following conditions are satisfied:*

  (1) $I_\Phi(\frac{x}{\|x\|_F}) = \|x\|_F$;

  (2) $\frac{x(t)}{\|x\|_F} \geq a_\Phi(t)$ *for almost every* $t \in \operatorname{supp} x$;

  (3) $\frac{x(t)}{\|x\|_F} \in S_\Phi^+(t)$ *for almost every* $t \in \operatorname{supp} x$.

(c) *If $m(\{t \in \operatorname{supp} x : \frac{x(t)}{\|x\|_F} < b_\Phi(t) < +\infty\}) = 0$, then $x \in L^\Phi \backslash \{0\}$ is an upper strict monotonicity point if and only if the following conditions are satisfied:*
   (1) $\frac{x(t)}{\|x\|_F} \geq a_\Phi(t)$ *for almost every* $t \in \operatorname{supp} x$;
   (2) $\frac{x(t)}{\|x\|_F} \in S_\Phi^+(t)$ *for almost every* $t \in \operatorname{supp} x$.

Proof. Case (a): Assume $0 \leq x(t) \leq y_1(t)$ for $t \in T$ and there exists $e \subset T, m(e) > 0$ such that $x(t) < y_1(t)$ for $t \in e$. Denote $e_n = \{t \in e : (1 + \frac{1}{n})x(t) \leq y_1(t)\}$. Then $\bigcup_{n=1}^\infty e_n = e$, and there exists $n_0 \in \mathbb{N}$ such that $m(e_{n_0}) > 0$. Hence, we have

$$\begin{aligned} I_\Phi\left(\frac{y_1}{(1+\frac{1}{n_0^2})\|x\|_F}\right) &\geq \int_{e_{n_0}} \Phi(t, \frac{(1+\frac{1}{n_0})x(t)}{(1+\frac{1}{n_0^2})\|x\|_F})dt \\ &\geq \int_{e_{n_0}} \Phi(t, \frac{(1+\frac{1}{n_0})}{(1+\frac{1}{n_0^2})} b_\Phi(t))dt \\ &= +\infty, \end{aligned}$$

which means that $\|y_1\|_F \geq (1 + \frac{1}{n_0^2})\|x\|_F > \|x\|_F$; hence, x is an upper strict monotonicity point.

Case (b): The necessity.

First let us prove the necessity of condition (1). Assume $\int_T \Phi(t, \frac{x(t)}{\|x\|_F})dt < \|x\|_F$. Put $e = \{t \in \operatorname{supp} x : \frac{x(t)}{\|x\|_F} < b_\Phi(t)\}$. Because the function $\Phi(t, \frac{x(t)+\|x\|_F b_\Phi(t)}{2\|x\|_F})\chi_e(t)$ is a measurable function and finite for almost every $t \in e$, there exists a subset $e_0 \subset e$ with $m(e_0) > 0$ such that $\Phi(t, \frac{x(t)+\|x\|_F b_\Phi(t)}{2\|x\|_F})\chi_{e_0}(t)$ is a bounded measurable function. Hence,

$$\int_{e_0} \Phi(t, \frac{x(t) + \|x\|_F b_\Phi(t)}{2\|x\|_F})dt < +\infty.$$

By absolute continuity of the Lebesgue integral, there is a subset $\tilde{e} \subset e_0$ with $m(\tilde{e}) > 0$ for which

$$\int_{\tilde{e}} \Phi(t, \frac{x(t) + \|x\|_F b_\Phi(t)}{2\|x\|_F})dt \leq \|x\|_F - \int_T \Phi(t, \frac{x(t)}{\|x\|_F})dt.$$

Set

$$y_2(t) = x(t)\chi_{T\backslash\tilde{e}}(t) + \frac{x(t) + \|x\|_F b_\Phi(t)}{2}\chi_{\tilde{e}}(t).$$

Then, $0 \leq x(t) \leq y_2(t)$ for $t \in T$ and $x(t) < y_2(t)$ for $t \in \tilde{e}$, hence $\|x\|_F < \|y_2\|_F$. However, the inequality

$$\begin{aligned} I_\Phi(\frac{y_2}{\|x\|_F}) &= \int_{T\backslash\tilde{e}} \Phi(t, \frac{x(t)}{\|x\|_F})dt + \int_{\tilde{e}} \Phi(t, \frac{x(t) + \|x\|_F b_\Phi(t)}{2\|x\|_F})dt \\ &\leq \int_{T\backslash\tilde{e}} \Phi(t, \frac{x(t)}{\|x\|_F})dt + \|x\|_F - \int_T \Phi(t, \frac{x(t)}{\|x\|_F})dt \\ &\leq \|x\|_F, \end{aligned}$$

we have that the inequality $\|y_2\|_F \leq \|x\|_F$ holds; This is a contradiction.

Next, we are going to prove that condition (2) is true. If $m(\{t \in \operatorname{supp} x : \frac{x(t)}{\|x\|_F} < a_\Phi(t)\}) > 0$. Denote $F_0 = \{t \in \operatorname{supp} x : \frac{x(t)}{\|x\|_F} < a_\Phi(t)\}$ and $y_2(t) = x\chi_{T\backslash F_0}(t) +$

$\frac{x+\|x\|_F a_\Phi}{2}\chi_{F_0}(t)$. We obtain that $0 \leq x(t) \leq y_2(t)$ for $t \in T$ and $x(t) < y_2(t)$ for any $t \in F_0$. It is known that

$$\begin{aligned}
\|x\|_F &= \int_T \Phi(t, \frac{x(t)}{\|x\|_F})dt \\
&= \int_{T\setminus F_0} \Phi(t, \frac{x(t)}{\|x\|_F})dt \\
&= \int_{T\setminus F_0} \Phi(t, \frac{y_2(t)}{\|x\|_F})dt + \int_{F_0} \Phi(t, \frac{y_2(t)}{\|x\|_F})dt \\
&= \int_T \Phi(t, \frac{y_2(t)}{\|x\|_F})dt.
\end{aligned}$$

We can discern from the above equality that $\|y_2\|_F \leq \|x\|_F$. Together with the previous conditions, we can obtain that $\|y_2\|_F = \|x\|_F$, which contradicts the fact that x is an upper strict monotonicity point.

Finally, we will prove that condition (3) holds. If $m(\{t \in T : \frac{x(t)}{\|x\|_F} \notin S^+_\Phi(t)\}) > 0$. We want to prove that there exists $a, b \in \mathbb{R}_+, a < b$ satisfying

$$\Phi(t, a) = \Phi(t, b), t \in T_{a,b},$$

where $T_{a,b} = \{t \in T : a \leq \frac{x(t)}{\|x\|_F} < b\}$. As the set of positive rational numbers is countable, assume them to be $\{r_1, r_2, \cdots\}$ and

$$A_{n,m} = \{t \in T : \Phi(t, r_n) = \Phi(t, r_m)\}.$$

We obtain that

$$A = \{t \in T : \frac{x(t)}{\|x\|_F} \notin S^+_\Phi(t)\} = \bigcup_{n,m=1}^{\infty} (A \cap A_{n,m})$$

hence, $m(A) \leq \sum_{n,m=1}^{\infty} m(A \cap A_{n,m})$. In virtue of the condition $m(A) > 0$, there exist $r_{n_0}, r_{m_0} \in Q^+$ such that $m(A \cap A_{n_0,m_0}) > 0$. Let $a = r_{n_0}, b = r_{m_0}$. Suppose that $a < b$. Thus, $m(\{t \in T : a \leq \frac{x(t)}{\|x\|_F} < b\}) > 0$. Denote

$$y_2(t) = x\chi_{T\setminus T_{a,b}}(t) + b\|x\|_F \chi_{T_{a,b}}(t).$$

We obtain that

$$\begin{aligned}
I_\Phi(\frac{y_2}{\|x\|_F}) &= \int_{T\setminus T_{a,b}} \Phi(t, \frac{x(t)}{\|x\|_F})dt + \int_{T_{a,b}} \Phi(t, b)dt \\
&= \int_{T\setminus T_{a,b}} \Phi(t, \frac{x(t)}{\|x\|_F})dt + \int_{T_{a,b}} \Phi(t, \frac{x(t)}{\|x\|_F})dt \\
&= \int_T \Phi(t, \frac{x(t)}{\|x\|_F})dt \\
&= I_\Phi(\frac{x}{\|x\|_F}) \\
&= \|x\|_F,
\end{aligned}$$

which can further yield that $\|y_2\|_F = \|x\|_F$; the equality contradicts the fact that x is upper strict monotonicity point.

The sufficiency.

Suppose $0 \leq x(t) \leq y_2(t)$ for $t \in T$ and there exist $e_1 \subset T$ and $m(e_1) > 0$ satisfying $x(t) < y_2(t)$ for $t \in e_1$. We will to prove that the inequality $\|x\|_F < \|y_2\|_F$ holds. By the condition (1) we can get that

$$
\begin{aligned}
\|x\|_F &= \int_T \Phi(t, \frac{x(t)}{\|x\|_F})dt \\
&= \int_{T\setminus e_1} \Phi(t, \frac{x(t)}{\|x\|_F})dt + \int_{e_1} \Phi(t, \frac{x(t)}{\|x\|_F})dt \\
&< \int_{T\setminus e_1} \Phi(t, \frac{y_2(t)}{\|x\|_F})dt + \int_{e_1} \Phi(t, \frac{y_2(t)}{\|x\|_F})dt \\
&= I_\Phi(\frac{y_2}{\|x\|_F}).
\end{aligned}
$$

From the above inequality we can get that $I_\Phi(\frac{y_2}{\|x\|_F}) > \|x\|_F$. By the definition of F-norm we have that $I_\Phi(\frac{y_2}{\|y_2\|_F}) \le \|y_2\|_F$. Thus, we obtain $\|x\|_F < \|y_2\|_F$.

Case (c): The proof of Case (c) is similar to that of Case (b), so we have omitted the proof. □

Corollary 1. *$x \in E_\Phi$ is upper strict monotonicity point if and only if the following conditions are satisfied:*

(1) $m(\{t \in T : 0 < \frac{x(t)}{\|x\|_F} < a_\Phi(t)\}) = 0$*;*
(2) $\frac{x(t)}{\|x\|_F} \in S^+_\Phi(t)$ *for almost every* $t \in \operatorname{supp} x$*.*

Proof. The condition $x \in E_\Phi$ implies that $b_\Phi(t) = \infty$, which means that the statement $m(\{t \in \operatorname{supp} x : \frac{x(t)}{\|x\|_F} < b_\Phi(t) < +\infty\}) = 0$ in Theorem 1 (c) is satisfied. □

Corollary 2. *L_Φ is strictly monotone if and only if the following conditions are satisfied:*

(1) $a_\Phi(t) = 0$ *for almost every* $t \in T$*;*
(2) $\Phi \in \triangle_2$*;*
(3) $\Phi(t, u)$ *is strictly monotonically increasing with respect to u for almost every* $t \in T$*.*

Proof. The necessity.

(1) If there is a set $e \subset T$ satisfying $m(e) > 0$ and $a_\Phi(t) > 0$ for $t \in e$, and let $x(t) = \frac{1}{2}a_\Phi(t)$; this yields that $x(t)$ is not an upper strict monotonicity point. Further, L_Φ is not strictly monotone.

(2) If $\Phi \notin \triangle_2$, in combination with Lemma 1 we can take $b_n = 1 + \frac{1}{n}, p_n = 2^n, q_n = \frac{1}{2^n}$ for $n \in \mathbb{N}$ such that

$$
\int_{F_n} \Phi(t, x_n(t))dt = \frac{1}{2^n}, \Phi(t, b_n x_n(t)) \ge p_n \Phi(t, x_n(t)) \quad (t \in F_n, n \in \mathbb{N}),
$$

where $\{x_n(t)\}_{n=1}^\infty$ are $\sum -$ measurable functions and $\{F_n\}$ in $\sum$ are mutually disjoint sets satisfying $0 \le x_n(t) < \infty$ on the set $\{F_n\}$.

Define

$$
x(t) = \sum_{n=1}^\infty x_n \chi_{F_n}(t), y(t) = \sum_{n=2}^\infty x_n \chi_{F_n}(t).
$$

Then

$$
\begin{aligned}
I_\Phi(x) &= \sum_{n=1}^\infty \int_{F_n} \Phi(t, x_n(t))dt \\
&= \sum_{n=1}^\infty \frac{1}{2^n} \\
&= 1.
\end{aligned}
$$

From the condition (3) in Lemma 2, we can obtain that $x, y \in L_\Phi$ and $\|y\|_F \le \|x\|_F = 1$. For any $\lambda \in (0,1)$, there exists $m \in \mathbb{N}, m \ge 2$ such that $\frac{1}{\lambda} > 1 + \frac{1}{n}$ for any $n \ge m$. Then

$$\begin{aligned} I_\Phi(\frac{y}{\lambda}) &\ge \sum_{n=m}^{\infty} \int_{F_n} \Phi(t, \frac{x_n(t)}{\lambda}) dt \\ &\ge \sum_{n=m}^{\infty} \int_{F_n} \Phi\left(t, (1+\frac{1}{n}) x_n(t)\right) dt \\ &\ge \sum_{n=m}^{\infty} 2^n \int_{F_n} \Phi(t, x_n(t)) dt \\ &= \sum_{n=m}^{\infty} 1 \\ &= \infty. \end{aligned}$$

Consequently, we have that the inequality $\|y\|_F \ge \lambda$ holds. Because λ is arbitrary, we obtain $\|y\|_F \ge 1$. Further, we conclude that $\|x\|_F = \|y\|_F = 1$ and $x(t)$ is not an upper strict monotonicity point, which means that L_Φ is not strictly monotone.

(3) If $\Phi(t,u)$ is constant function for each $t \in T_1$ and $a \le u < b$, where $T_1 \subset T$, $0 < m(T_1) < m(T)$ and $0 < a < b$. Suppose $\lim_{u \to \infty} \Phi(t,u) = +\infty$ for $t \in T$. Select $M > 0$ such that

$$M \cdot \frac{1}{3} m(T \setminus T_1) > 1 - \int_{T_1} \Phi(t,b) dt.$$

Define $\delta(t) = \inf\{u \ge 0 : \Phi(t,u) \ge M\}$ for $t \in T$. Because $\lim_{u \to \infty} \Phi(t,u) = +\infty$, we yield that $\delta(t)$ is measurable. The fact $\lim_{n \to \infty} m(\{t \in T \backslash T_1 : \delta(t) > n\}) = 0$ implies that there exists $n_0 \in \mathbb{N}$ satisfying

$$m(\{t \in T \backslash T_1 : \delta(t) > n_0\}) < \frac{1}{3} m(T \backslash T_1).$$

Define $T_2 = \{t \in T \backslash T_1 : \delta(t) > n_0\}$ and $T_3 = T \backslash (T_1 \cup T_2)$. Therefore, we obtain $m(T_3) \ge \frac{1}{3} m(T \backslash T_1)$ and

$$M \cdot m(T_3) > 1 - \int_{T_1} \Phi(t,b) dt.$$

Because $\Phi(t,n_0)\chi_{T_3}$ is an almost everywhere finite measurable function, we know that there exists $T_4 \subset T_3$ such that

$$M \cdot m(T_4) > 1 - \int_{T_1} \Phi(t,b) dt$$

and $\Phi(t,n_0)\chi_{T_4}$ is an integrable function. We can yield that

$$\int_{T_4} \Phi(t,n_0) dt > 1 - \int_{T_1} \Phi(t,b) dt.$$

There must exist $T_5 \subset T_4$ such that

$$\int_{T_5} \Phi(t,n_0) dt = 1 - \int_{T_1} \Phi(t,b) dt$$

under the condition (T, Σ, m) is a non-atomic measure space. Define $x(t) = b\chi_{T_1}(t) + n_0 \chi_{T_5}(t)$. Thus, $I_\Phi(x) = 1$, $\|x\|_F = 1$. According to the condition $m(\{t \in T : \frac{x(t)}{\|x\|_F} \notin S_\Phi^+(t)\}) > 0$, we have that x is not strict monotonicity point; hence, L_Φ is not strictly monotone.

The sufficiency.

The condition $\Phi \in \triangle_2$ implies that $m(\{t \in \operatorname{supp} x : \frac{x(t)}{\|x\|_F} < b_\Phi(t) < +\infty\}) = 0$. Thus, x is a upper strict monotonicity point, and L_Φ is strictly monotone. $\square$

Theorem 2. *$x \in L^{\Phi} \backslash \{0\}$ is upper local uniform monotonicity point if and only if the following conditions are satisfied:*

(1) $m(t \in \operatorname{supp} x : \frac{x(t)}{\|x\|_F} < a_{\Phi}(t)) = 0$;

(2) $\frac{x(t)}{\|x\|_F} \in S_{\Phi}^{+}(t)$ *for almost every* $t \in \operatorname{supp} x$;

(3) $\Phi \in \triangle_2$.

Proof. The necessity.

Because the upper local uniform monotonicity point is an upper strict monotonicity point, we only need to prove condition (3).
Case 1: Let $e = \{t \in T : b_{\Phi}(t) < +\infty\}$. Suppose that $m(e) > 0$. From the definition of the Musielak–Orlicz function, we know that $\Phi(t, b_{\Phi}(t))\chi_e(t)$ is a finite measurable function. Hence, there exists a $e_0 \subset e$ with $m(e_0) > 0$ such that

$$\int_{e_0} \Phi(t, b_{\Phi}(t))dt < +\infty.$$

Let $e_n = \{t \in T : \frac{x(t)}{\|x\|_F} < (1 - \frac{1}{n})b_{\Phi}(t)\}$. Thus, there is a $n_0 \in \mathbb{N}$ such that $m(e_{n_0}) > 0$. Take $G_n \subset e_{n_0}$ such that $m(G_n) > 0$ and $m(G_n) \to 0$. Denote

$$x_n(t) = x(t)\chi_{T\backslash G_n}(t) + \|x\|_F b_{\Phi}\chi_{G_n}(t), \ (n = 1, 2, \cdots).$$

Then $0 \le x(t) \le x_n(t)$ for $t \in T$ and

$$\begin{aligned} I_{\Phi}(\frac{x_n}{\|x\|_F}) &= \int_{T\backslash G_n} \Phi(t, \frac{x(t)}{\|x\|_F})dt + \int_{G_n} \Phi(t, b_{\Phi}(t))dt \\ &\le \|x\|_F + \int_{G_n} \Phi(t, b_{\Phi}(t))dt. \end{aligned}$$

Because $\lim\limits_{n\to\infty} \int_{G_n} \Phi(t, b_{\Phi}(t))dt = 0$, then $\|x_n\|_F \to \|x\|_F$. Moreover,

$$\begin{aligned} I_{\Phi}\left(\frac{x_n(t) - x(t)}{\frac{\|x\|_F}{2n_0}}\right) &> \int_{G_n} \Phi\left(t, \frac{\frac{\|x\|_F}{n_0} b_{\Phi}(t)}{\frac{\|x\|_F}{2n_0}}\right)dt \\ &= \int_{G_n} \Phi(t, 2b_{\Phi}(t))dt \\ &= \infty. \end{aligned}$$

The above inequality shows that $\|x_n - x\|_F > \frac{\|x\|_F}{2n_0}$; this is a contradiction.

Case 2: Let $b_{\Phi}(t) = +\infty$. Take $c > 0$ satisfying $m(\{t \in \operatorname{supp} x : 0 < x(t) \le c\}) > 0$ and denote $A = \{t \in \operatorname{supp} x : 0 < x(t) \le c\}$. Let $r_1, r_2, \cdots$ be the set of rational numbers on the interval $[0, 1]$. According to $\Phi(t, \frac{1}{r_i})$ is a measurable function on A that is finite for almost every $t \in A$. Then, there exists a $Q_i \subset A, m(Q_i) \le \frac{m(A)}{2^{i+1}}$ satisfying $\Phi(t, \frac{1}{r_i})$ is an integrable function on the interval $A\backslash Q_i$. Denote $Q = \bigcup\limits_{n=1}^{\infty} Q_i$. Then, $\Phi(t, \frac{1}{r_i})$ is an integrable function on the interval $A\backslash Q$ and

$$\begin{aligned} m(A\backslash Q) &\ge m(A) - \sum_{i=1}^{\infty} m(Q_i) \\ &\ge m(A) - \frac{1}{2}m(A) \\ &= \frac{1}{2}m(A). \end{aligned}$$

Hence, we can assume that $\Phi(t, \frac{1}{r_i})$ is an integrable function on A for each $i \in \mathbb{N}$.

Applying Lemma 1 with $b_n = 1 + \frac{1}{n}, p_n = 2^n, q_n = \frac{1}{2^n}$, where $n \in \mathbb{N}$. We can find a sequence $\{x_n(t)\}_{n=1}^{\infty}$ of $\sum -$ measurable functions and mutually disjoint measurable subsets $\{F_n\}$ in A such that $0 \le x_n(t) < \infty$,

$$\int_{F_n} \Phi(t, x_n(t))dt = \frac{1}{2^n}, \Phi(t, b_n x_n(t)) \ge p_n \Phi(t, x_n(t)) \quad (t \in F_n, n \in \mathbb{N}).$$

Let us define

$$y(t) = \sum_{n=1}^{\infty} x_n(t)\chi_{F_n}(t).$$

Then, $I_\Phi(y) = \sum_{n=1}^{\infty} \int_{F_n} \Phi(t, x_n(t))dt = \sum_{n=1}^{\infty} \frac{1}{2^n} = 1, y \in L^\Phi$ and $\|y\|_F \le 1$. Moreover, for any $\lambda \in (0,1)$, there exists an $m \in \mathbb{N}, m \ge 2$ such that $\frac{1}{\lambda} > 1 + \frac{1}{n}$ for any $n \ge m$. Then,

$$\begin{aligned} I_\Phi(\frac{y}{\lambda}) &\ge \sum_{n=m}^{\infty} \int_{F_n} \Phi(t, \frac{x_n(t)}{\lambda})dt \\ &\ge \sum_{n=m}^{\infty} \int_{F_n} \Phi\left(t, (1 + \frac{1}{n})x_n(t)\right)dt \\ &\ge \sum_{n=m}^{\infty} 2^n \int_{F_n} \Phi(t, x_n(t))dt \\ &= \sum_{n=m}^{\infty} 1 \\ &= \infty. \end{aligned}$$

By the above inequality we conclude that there exists $y(t) = \sum_{n=1}^{\infty} x_n(t)\chi_{F_n}(t)$ satisfying the following conditions:

(a) $\operatorname{supp} y \subset A$;
(b) $\| \sum_{n=m}^{\infty} x_n \chi_{F_n} \|_F = 1$ as $m \to \infty$;
(c) $I_\Phi\left(\sum_{n=m}^{\infty} x_n \chi_{F_n} \right) \to 0$;
(d) $I_\Phi\left(\frac{\sum_{n=m}^{\infty} x_n \chi_{F_n}}{\lambda} \right) = +\infty$ for any $\lambda \in (0,1)$.

Define

$$W_m(t) = x(t)\chi_{T \setminus \bigcup\limits_{n=m}^{\infty} F_n}(t) + \|x\|_F \sum_{n=m}^{\infty} x_n(t)\chi_{F_n}(t)$$

and

$$Z_m(t) = x(t) + \|x\|_F \sum_{n=m}^{\infty} x_n(t)\chi_{F_n}(t).$$

We have $0 \le x(t) \le Z_m(t)$ for $t \in T$ and want to prove that $\|W_m\|_F \to \|x\|_F$. According to inequality

$$\begin{aligned} I_\Phi(\frac{W_m}{\|x\|_F}) &\le I_\Phi\left(\sum_{n=m}^{\infty} x_n \chi_{F_n} \right) + I_\Phi(\frac{x}{\|x\|_F}) \\ &\le \|x\|_F + I_\Phi\left(\sum_{n=m}^{\infty} x_n \chi_{F_n} \right), \end{aligned}$$

we know that $\overline{\lim\limits_{m\to\infty}} \|W_m\|_F \le \|x\|_F$. For any $\lambda \in (0,1)$,

$$I_\Phi(\frac{W_m}{\lambda\|x\|_F}) \ge I_\Phi\left(\frac{\sum_{n=m}^{\infty} x_n \chi_{F_n}}{\lambda} \right) = \infty.$$

Hence, $\liminf_{n=\infty} \|W_m\|_F \geq \lambda\|x\|_F$. Further by the arbitrariness of λ, we obtain $\liminf_{n\to\infty} \|W_m\|_F \geq \|x\|_F$, thus the equality $\|W_m\|_F = \|x\|_F$ holds.

Next, we are going to prove $\lim_{m\to\infty} \|Z_m - W_m\|_F = 0$. Because

$$Z_m(t) - W_m(t) = \sum_{n=m}^{\infty} x(t)\chi_{F_n}(t)$$

and

$$\|Z_m - W_m\|_F = \|x\chi_{\bigcup_{n=m}^{\infty} F_n}\|_F \leq \|c\chi_{\bigcup_{n=m}^{\infty} F_n}\|_F,$$

we only need to prove the condition $\lim_{m\to\infty} \|\chi_{\bigcup_{n=m}^{\infty} F_n}\|_F = 0$. For any $i \in \mathbb{N}$, we have $I_\Phi\left(\frac{\chi_{\bigcup_{n=m}^{\infty} F_n}}{r_i}\right) = \int_{\bigcup_{n=m}^{\infty} F_n} \Phi(t, \frac{1}{r_i})dt$. The condition $\Phi(t, \frac{1}{r_i})$ is an integrable function and $m(\bigcup_{n=m}^{\infty} F_n) \to 0$ imply that there exists $m_0 \in \mathbb{N}$ such that $I_\Phi\left(\frac{\chi_{\bigcup_{n=m}^{\infty} F_n}}{r_i}\right) \leq r_i$ as $m \geq m_0$. Hence, $\|\chi_{\bigcup_{n=m}^{\infty} F_n}\|_F \leq r_i$. By the arbitrariness of r_i, we obtain $\lim_{m\to\infty} \|\chi_{\bigcup_{n=m}^{\infty} F_n}\|_F = 0$. The three-angle inequality implies that

$$\lim_{m\to\infty} \|Z_m\|_F = \|x\|_F,$$

$$Z_m(t) - x(t) = \|x\|_F \sum_{n=m}^{\infty} x_n(t)\chi_{F_n}(t).$$

From the above equalities we obtain

$$I_\Phi\left(\frac{Z_m - x}{\frac{1}{2}\|x\|_F}\right) = I_\Phi\left(\frac{\sum_{n=m}^{\infty} x_n\chi_{F_n}}{\frac{1}{2}}\right) = \infty,$$

$$\|Z_m - x\|_F \geq \frac{1}{2}\|x\|_F,$$

which contradicts the fact that x is an upper local uniform monotonicity point.

The sufficiency.

Suppose the conditions $0 \leq x \leq x_n$ and $\lim_{n\to\infty} \|x_n\|_F = \|x\|_F$ are satisfied; then, we only need to prove that the equality $\lim_{n\to\infty} \|x_n - x\|_F = 0$ holds. It is known that

$$\lim_{n\to\infty} I_\Phi(\frac{x}{\|x_n\|_F}) = \|x\|_F.$$

Further, we can obtain that

$$\lim_{n\to\infty} \left(\int_T \Phi(t, \frac{x_n(t)}{\|x_n\|_F})dt - \int_T \Phi(t, \frac{x(t)}{\|x_n\|_F})dt \right) = 0,$$

$$\lim_{n\to\infty} \int_T \left(\Phi(t, \frac{x_n(t)}{\|x_n\|_F}) - \Phi(t, \frac{x(t)}{\|x_n\|_F})\right)dt = 0.$$

Because $\Phi(t, \frac{x_n(t)}{\|x_n\|_F}) - \Phi(t, \frac{x(t)}{\|x_n\|_F}) \geq 0$, the equality

$$\lim_{n\to\infty} \int_T \left(\Phi(t, \frac{x_n(t)}{\|x_n\|_F}) - \Phi(t, \frac{x(t)}{\|x_n\|_F})\right)dt = 0 \quad (1)$$

holds. According to finiteness of the measure, there is a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ satisfying $\Phi(t, \frac{x_{n_k}(t)}{\|x_{n_k}\|_F}) - \Phi(t, \frac{x(t)}{\|x_{n_k}\|_F}) \to 0$ for almost every $t \in T$. Without loss of generality, suppose the equality

$$\lim_{k\to\infty}\left(\Phi(t, \frac{x_{n_k}(t)}{\|x_{n_k}\|_F}) - \Phi(t, \frac{x(t)}{\|x_{n_k}\|_F})\right) = 0 \tag{2}$$

for any $t \in T$ is true. Furthermore, there is a sequence $\{\eta_k\}_{k=1}^{\infty} \in (0,1)$ and $y \in L^1$ satisfying $\lim\limits_{k\to\infty} \eta_k = 0$ and

$$0 \le \Phi(t, \frac{x_{n_k}(t)}{\|x_{n_k}\|_F}) - \Phi(t, \frac{x(t)}{\|x_{n_k}\|_F}) \le \eta_k y \quad (k \in \mathbb{N})$$

for any $t \in T$, where

$$\Phi(t, \frac{x_{n_k}(t)}{\|x_{n_k}\|_F}) \le \eta_k y + \Phi(t, \frac{x(t)}{\|x_{n_k}\|_F}) \le y + \Phi(t, \frac{x(t)}{\|x\|_F}) \tag{3}$$

for any $t \in T$.

By the fact that $\frac{x(t)}{\|x_n\|_F} \in S_{\Phi}^{+}(t)$ for almost every $t \in \operatorname{supp} x$, there exists a set $e_0 \in \operatorname{supp} x$ and $m(e_0) = 0$ such that $\frac{x(t)}{\|x_n\|_F} \in S_{\Phi}^{+}(t)$ for $t \in \operatorname{supp} x \backslash e_0$. Let $t_0 \in \operatorname{supp} x \backslash e_0$; we can easily obtain that $\Phi(t_0, u) > \Phi(t_0, \frac{x(t)}{\|x\|_F})$ for $t_0 \in \operatorname{supp} x \backslash e_0$ under the condition $u > \frac{x(t)}{\|x\|_F}$.

The next step we need to prove is $\lim\limits_{k\to\infty} \frac{x_{n_k}(t_0)}{\|x_{n_k}\|_F} = \frac{x(t_0)}{\|x\|_F}$ for $t_0 \in \operatorname{supp} x \backslash e_0$. If the equality (1) imply that the sequence $\{\frac{x_{n_k}(t_0)}{\|x_{n_k}\|_F}\}$ is bounded. Denote $\lim\limits_{k\to\infty} \frac{x_{n_k}(t_0)}{\|x_{n_k}\|_F} - \frac{x(t_0)}{\|x\|_F} = c$. There is a maximum strict monotonicity interval of $\Phi(t, u)$ $[a, b)$ satisfying $\frac{x(t_0)}{\|x\|_F} \in [a, b)$. From the fact that $\lim\limits_{k\to\infty} \frac{x(t_0)}{\|x_{n_k}\|_F} = \frac{x(t_0)}{\|x\|_F} < b$, we can obtain that there exists an $n_0 \in \mathbb{N}$ such that $\frac{x(t_0)}{\|x_{n_k}\|_F} < b$ whenever $n_k \ge n_0$.

We have to consider two cases.

Case 1: If there is a $k_1 \in \mathbb{N}$ such that $a \le \frac{x(t_0)}{\|x_{n_k}\|_F} < b$ whenever $k \ge k_1$. We want to prove that for some $d > 0$, the inequality

$$\Phi(t_0, u) + d \le \Phi(t_0, v) \quad (*)$$

holds whenever $u \in [a, b), v - u \ge \frac{c}{2}$. Suppose to the contrary, there are the sequences $\{u_n\}_{n=1}^{\infty} \subset [a, b)$ and $\{v_n\}_{n=1}^{\infty} \subset R$ with $u_n - v_n \ge \frac{c}{2}$ satisfying

$$\Phi(t_0, u_n) + \frac{1}{n} \ge \Phi(t_0, v_n) > \Phi(t_0, u_n).$$

The sequence $\{u_n\}_{n=1}^{\infty}$ is bounded, which implies that there are $u_0, v_0 \in R$ such that $u_n \to u_0$, $v_n \to v_0$. Because $\Phi(t_0, u)$ is continuous, we have that $\Phi(t_0, u_0) = \Phi(t_0, v_0)$. However, because $u_0 \in [a, b)$ and $u_0 < v_0$, it is easy to see that $\Phi(t_0, u_0) < \Phi(t_0, v_0)$. We obtain a contradiction.

As $\lim\limits_{k\to\infty} \frac{x_{n_k}(t_0)}{\|x_{n_k}\|_F} - \frac{x(t_0)}{\|x_{n_k}\|_F} = c$, we can find a $k_2 \in \mathbb{N}$ such that $\frac{x_{n_k}(t_0)}{\|x_{n_k}\|_F} - \frac{x(t_0)}{\|x_{n_k}\|_F} \ge \frac{c}{2}$ whenever $k \ge k_2$. The inequality $(*)$ can easily yield that there exists $d > 0$ such that

$$\Phi(t_0, \frac{x(t_0)}{\|x_{n_k}\|_F}) + d \le \Phi(t_0, \frac{x_{n_k}(t_0)}{\|x_{n_k}\|_F}).$$

Combined with equality (2) we also obtain a contradiction. Thus, we have

$$\lim_{k\to\infty} \frac{x_{n_k}(t_0)}{\|x_{n_k}\|_F} = \frac{x(t_0)}{\|x\|_F}.$$

Case 2: If there exists $k_3 \in \mathbb{N}$ such that $\frac{x(t_0)}{\|x_{n_k}\|_F} < a$ whenever $k \geq k_3$.

There must exist $k_4 \in \mathbb{N}$ and $b_1 \in [a, b)$ such that $[a, b_1) \subseteq (\frac{x(t_0)}{\|x_{n_k}\|_F}, \frac{x_{n_k}(t_0)}{\|x_{n_k}\|_F})$. Then, the following proof is similar to the Case 1. There is $d' > 0$ satisfying the inequality:

$$\Phi(t_0, \frac{x(t_0)}{\|x_{n_k}\|_F}) + d' \leq \Phi(t_0, \frac{x_{n_k}(t_0)}{\|x_{n_k}\|_F}),$$

which contradicts equality (2). Then, we can yield that $\lim_{k\to\infty} \frac{x_{n_k}(t)}{\|x_{n_k}\|_F} = \frac{x(t)}{\|x\|_F}$ for almost every $t \in \operatorname{supp} x$. By the fact that $\lim_{k\to\infty} \|x_{n_k}\|_F = \|x\|_F$, we obtain that the equality

$$\lim_{k\to\infty} x_{n_k}(t) = x(t)$$

for almost every $t \in T$ holds.

Therefore, by inequality (3), we can conclude that

$$\Phi(t, \frac{x_{n_k}(t)}{\|x_{n_k}\|_F} - \frac{x(t)}{\|x_{n_k}\|_F}) \leq \Phi(t, \frac{x_{n_k}(t)}{\|x_{n_k}\|_F}) \leq y + \Phi(t, \frac{x(t)}{\|x\|_F}) \in L^1.$$

Then, for any $\lambda > 0$, the Lebesgue dominated convergence theorem concludes that $I_\Phi(\lambda(x_{n_k} - x)) \to 0$ as $k \to \infty$. Further, Lemma 3 yields that $\|x - x_{n_k}\|_F \to 0$ as $k \to \infty$. The double extract subsequence theorem fulfills the proof. $\square$

Corollary 3. *E_Φ is upper locally uniformly monotone if and only if the following conditions are satisfied:*

(1) $a_\Phi(t) = 0$ for almost every $t \in T$;
(2) $\Phi(t, u)$ is strictly monotonically increasing with respect to u for almost every $t \in T$.

Corollary 4. *L_Φ is upper locally uniformly monotone if and only if the following conditions are satisfied:*

(1) $a_\Phi(t) = 0$ for almost every $t \in T$;
(2) $\Phi(t, u)$ is strictly monotonically increasing with respect to u for almost every $t \in T$;
(3) $\Phi \in \triangle_2$.

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