On Green’s Function of the Dirichlet Problem for the Polyharmonic Equation in the Ball

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Abstract: The paper gives an explicit representation of the Green’s function of the Dirichlet boundary value problem for the polyharmonic equation in the unit ball. The solution of the homogeneous Dirichlet problem is found. An example of solving the homogeneous Dirichlet problem with the simplest polynomial right-hand side of the polyharmonic equation is given.

Keywords: Dirichlet problem; polyharmonic equation; Green’s function

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1. Introduction

The Green’s functions of the Dirichlet and Neumann boundary value problems for the biharmonic and polyharmonic equations are considered in many papers. For example, in the two-dimensional case, in the work [1], on the basis of the known harmonic Green’s function, the Green’s functions of various biharmonic problems are presented. The explicit form of the Green’s function for the Robin boundary value problem is found in refs. [2,3], and the Green’s functions in the sector for the biharmonic and 3-harmonic equations are presented in [4,5]. The papers [6–8] are devoted to the construction of the Green’s function to the Dirichlet problem for the polyharmonic equation in the unit ball, and the paper [9] contains solutions to the Dirichlet and Neumann [10] problems for the homogeneous polyharmonic equation. The papers [11,12] give an explicit form of the Green’s functions for the biharmonic and 3-harmonic equations in the unit ball. As for the biharmonic equation, we can note the recent paper [13] devoted to the solvability conditions for some nonstandard problems in the ball. For the most general results on the generalized Neumann problem containing powers of normal derivatives in the boundary conditions, we note the paper [14].

In ref. [15], based on the integral representation of functions from the class \( u \in C^4(D) \cap C^3(\bar{D}) \), integral representations of solutions to the Navier problem [16] and Riquier–Neumann problem for the biharmonic equation in the unit ball are presented, and the Green’s functions of these problems are also constructed. In ref. [17], these results are extended to the polyharmonic equation. The Green’s function is also used to study nonlocal equations. We note some recently published papers on the construction of the Green’s function for various boundary value problems [18–21]. Applications of Green’s functions in problems of mechanics and physics can be found in refs. [22–27].

In ref. [11], an elementary solution of the biharmonic equation is defined as

\[
E_4(x, \xi) = \begin{cases} 
\frac{1}{2(n-2)(n-4)}|x - \xi|^{4-n}, \quad n > 4, \ n = 3 \\
-\frac{1}{4} \ln |x - \xi|, \quad n = 4 \\
\frac{|x - \xi|^2}{4} \left( \ln |x - \xi| - 1 \right), \quad n = 2
\end{cases}
\] (1)
It is proved that, for \( n \geq 3 \), the function of the form
\[
G_4(x, \xi) = E_4(x, \xi) - E_4\left( \frac{x}{|x|}, |x|\xi \right) - \frac{|x|^2 - 1}{2} |\xi|^2 - 1 \frac{1}{2} E_4\left( \frac{x}{|x|}, |x|\xi \right),
\]
where \( E(x, \xi) \) is an elementary solution of the Laplace equation [28], is the Green’s function of the Dirichlet problem for biharmonic equation in \( S \), satisfying the equalities \( G_4(x, \xi)|_{x \in \partial S} = \partial G_4(x, \xi)/\partial \nu|_{x \in \partial S} = 0 \) for \( \xi \in S \). Furthermore, in the paper [12], the function
\[
E_6(x, \xi) = \begin{cases} 
|\xi - \xi|^{6-n} & n \geq 3, n \neq 4, 6, \\
-\frac{1}{64} \ln |\xi - \xi|, & n = 6 \\
-\frac{32}{64} |\xi - \xi|^2 \ln |\xi - \xi| - \frac{3}{4}, & n = 4 \\
-\frac{1}{64} |\xi - \xi|^4 \ln |\xi - \xi| - \frac{3}{2}, & n = 2 
\end{cases}
\]
by analogy with the function \( E_4(x, \xi) \) from (1), is called an elementary solution of 3-harmonic equation. For \( x \neq \xi \), the equality \( \Delta_n E_6(x, \xi) = -E_6(x, \xi) \) holds true. The Green’s function in this case for \( n \geq 3 \) and \( n \neq 4 \) is represented as
\[
G_6(x, \xi) = E_6(x, \xi) - E_6\left( \frac{x}{|x|}, |x|\xi \right) - \frac{1}{2} \frac{|x|^2 - 1}{2} |\xi|^2 - 1 \frac{1}{2} E_4\left( \frac{x}{|x|}, |x|\xi \right) - \frac{1}{4} \frac{|x|^2 - 1}{2} \left( |\xi|^2 - 1 \right)^2 \frac{1}{4} E_4\left( \frac{x}{|x|}, |x|\xi \right).
\]

In the resulting formula, the singularity is contained only in the first term \( E_6(x, \xi) \), similarly in (2), and all other terms are 3-harmonic functions in \( S \).

In this paper, we study the representation of solutions to the homogeneous Dirichlet boundary value problem for an \( m \)-harmonic equation in the unit ball \( S = \{x \in \mathbb{R}^n : |x| < 1 \} \)
\[
\Delta^m u(x) = f(x), \quad x \in S, \\
u|_{\partial S} = 0, \quad \frac{\partial u}{\partial \nu}|_{\partial S} = 0, \ldots, \quad \frac{\partial^{m-1} u}{\partial \nu^{m-1}}|_{\partial S} = 0.
\]

In ref. [6], it is shown that the Green’s function \( G_{2m}(x, \xi) \) of this problem has the form (Boggio’s formula)
\[
G_{2m}(x, \xi) = k_m |x - \xi|^{2m-n} \int_1^g(x, \xi) (t^2 - 1)^{m-1} t^{1-n} dt,
\]
where
\[
g(x, \xi) = \frac{1}{|x - \xi|} \left( \frac{x}{|x|} - |x|\xi \right), \quad k_m = \frac{1}{\omega_n (2m - 2)!},
\]
and in ref. [29], the function \( G_{2m}(x, \xi) \) in the case \( n = 2 \) is constructed. In ref. [8], an explicit representation of the function \( G_{2m}(x, \xi) \) depending on the parity of \( n \) and the positiveness of \( 2m - n \) is found. In refs. [30,31], using Boggio’s formula, some estimates for the Green’s function \( G_{2m}(x, \xi) \) are obtained. In the present paper, another representation of the Green’s function \( G_{2m}(x, \xi) \) is given, which coincides, in particular, with Formulas (2) and (4). We use the notion of a fundamental solution instead of the previously used notion of an elementary solution \( E_{2m}(x, \xi) \) for the Riquier–Neumann problem [17].

2. Fundamental Solution

Let \( m \in \mathbb{N} \). Then, the set \( \mathbb{N} \setminus \{1\} \) can be divided into two disjoint sets: \( \mathbb{N}_m = \{ n \in \mathbb{N} : n > 2m > 1 \} \cup (2\mathbb{N} + 1) \) and its complement \( \mathbb{N}^c_m = \{ 2, 4, \ldots, 2m \} \). Because the set \( \mathbb{N}^c_m \),
is finite, then the set $\mathbb{N}_m$ is infinite. It is clear that $\mathbb{N}^c_{m-1} \subset \mathbb{N}_{m'}$ and therefore $\mathbb{N}_m \subset \mathbb{N}_{m-1}$.

Consider the fundamental solution of the $m$-harmonic equation $\Delta^m u = 0$ in the form

$$E_{2m}(x, \xi) = \begin{cases} \frac{(-1)^m |x - \xi|^{2m-n}}{(2-n,2)_m (2,2)_{m-1}}, & n \in \mathbb{N}_m, \\ \frac{(-1)^m |x - \xi|^{2m-n}}{(2-n,2)_m (2,2)_{m-1}} \left( \ln |x - \xi| - \sum_{k=1}^{m-n/2} \frac{1}{k!} \right), & n \in \mathbb{N}^c_m. \end{cases} \quad (7)$$

where $(a,b)_k = a(a+b) \ldots (a+kb-b)$ is a generalized Pochhammer symbol with the convention $(a,b)_0 = 1$, and $(a,b)_k^+$ means that if there is 0 among the factors $a, (a+b), \ldots (a+kb-b)$ included in $(a,b)_k$, then it should be replaced with 1. For example, $(-2,2)^+_3 = (-2) \cdot 1 \cdot 2 = -4$. In addition, if, in the sum included in (7), the upper index becomes smaller than the lower one, then the sum is assumed to be equal to zero. It should be borne in mind that the designation $\mathbb{N}_0 = \mathbb{N} \cup \{ 0 \}$ will also be used below, which is not associated with the function $E_{2m}(x, \xi)$.

**Remark 1.** From the equality (7), it follows that $E_2(x, \xi)$ coincides with the elementary solution of the Laplace equation $E(x, \xi)$ [28], and moreover $E_4(x, \xi) = E_4(x, \xi)$ (1) for $n \geq 3$ and $E_6(x, \xi) = E_6(x, \xi)$ (3) for $n \geq 3, n \neq 4$.

**Remark 2.** The fundamental solution $E_{2m}(x, \xi)$ differs slightly from the fundamental solution of the polyharmonic equation $G_{m,n}(x)$ considered by Sobolev in [32]. For $n \in \mathbb{N}_m$, the difference is in the factor $(-1)^m$, while for $n \in \mathbb{N}^c_m$ the difference is more noticeable: instead of $\ln(C|x|)$, as in Sobolev, the expression $\ln |x| - \sum_{k=1}^{m-n/2} \frac{1}{k!}$ is taken.

Let us introduce the notations

$$E_{2m}^*(x, \xi) = E_{2m}(x/|x| - |x| \xi, \xi), \quad h(x, \xi) = \big| x - \xi \big|^2, \quad h^*(x, \xi) = \big| x/|x| - |x| \xi \big|^2. \quad (8)$$

It is clear that $h(x, \xi)$ and $h^*(x, \xi)$ are second-order polynomials. Regarding the functions $E_{2m}(x, \xi)$ and $E_{2m}^*(x, \xi)$, some assertions are necessary.

**Lemma 1.** (a). The symmetric function $E_{2m}(x, \xi)$ is $m$-harmonic in $x \in S$ for $x \neq \xi$.

(b). The symmetric function $E_{2m}^*(x, \xi)$ is $m$-harmonic in $x \in S$ for $\xi \in \bar{S}$.

**Proof.** (a). The symmetry of the function $E_{2m}(x, \xi)$ is evident from the equality (7). In ref. [17], a function $E_{2m}(x, \xi)$ was considered, which is related to the function $E_{2m}(x, \xi)$ by the following equality

$$E_{2m}(x, \xi) = \begin{cases} E_{2m}(x, \xi), & n \in \mathbb{N}_m, \\ E_{2m}(x, \xi) - \frac{(-1)^m |x - \xi|^{2m-n}}{(2-n,2)_m (2,2)_{m-1}} \sum_{k=n/2}^{m-1} \frac{1}{k!}, & n \in \mathbb{N}^c_m. \end{cases} \quad (9)$$

In Lemma 2 [17], it is proved that the function $E_{2m}(x, \xi)$ is $m$-harmonic in $x \in S$ for $x \neq \xi$. Therefore, for $n \in \mathbb{N}_m$, the function $E_{2m}(x, \xi)$ is also $m$-harmonic in $x \in S$ for $x \neq \xi$. Next, for $n \in \mathbb{N}^c_m$, consider the expression $E_{2m}(x, \xi) - E_{2m}(x, \xi)$. Because the inclusion $2m - n \in 2\mathbb{N}_0$ holds true for $n \in \mathbb{N}^c_m = \{ 2, 4, \ldots, 2m \}$, the function $C|x - \xi|^{2m-n}$ is a polynomial of degree less or equal to $2m - 2$, which means that it is an $m$-harmonic function everywhere. Here, $C$ is the corresponding numeric coefficient from (9). Therefore, the function $E_{2m}^*(x, \xi) = E_{2m}(x, \xi) + C|x - \xi|^{2m-n}$ is $m$-harmonic in $x \in S$ for $x \neq \xi$.

Assertion (a) is proved.

(b). It is easy to see that

$$|x/|x| - |x|\xi| \geq |x/|x|| - |x||\xi| = 1 - |x||\xi| \geq 1 - |x| > 0$$
for \( x \in S, \zeta \in \bar{S} \), and hence, the function \( \mathcal{E}_{2m}^*(x, \zeta) \) is defined and differentiable for the given values of \( x \) and \( \zeta \). The symmetry of \( \mathcal{E}_{2m}^*(x, \zeta) \) follows from the equality \( |x/|x| - |\zeta|\xi|^2 = 1 - 2x \cdot \zeta + |x|^2|\zeta|^2 = |\zeta/|\zeta| - |\zeta|x|^2 \). Furthermore, it is easy to see that
\[
\frac{\partial}{\partial x_i} F(-x|\zeta| + \zeta/|\zeta|) = -|\zeta|^2 F(-x|\zeta| + \zeta/|\zeta|),
\]
where \( F_i(x) \) is the derivative of the function \( F(x) \) with respect to the \( i \)-th variable. For \( F(x) = |x|^{2m-n}(c_1 \ln |x| + c_2) \), the derivative \( F_i(x) \) exists if \( |x| > 0 \). In our case, the argument is \( |x/|x| - |x|\xi|^2 > 0 \), which means that there are derivatives of the function \( f(x) \) of any order. Therefore, by virtue of the first assertion of this lemma, we have
\[
\Lambda^m \mathcal{E}_{2m}^*(x, \zeta) = \Delta \mathcal{E}_{2m}^*(x, \zeta) = (-|\zeta|^2)^2m \mathcal{E}_{2m}^*(-x|\zeta| + \zeta/|\zeta|) = 0.
\]

The second assertion of the lemma is proved. \( \square \)

**Remark 3.** In ref. [17], an elementary solution \( E_{2m}(x, \zeta) \) of the polyharmonic equation is used, which is related to the function \( \mathcal{E}_{2m}(x, \zeta) \) by the equality (9). This function \( E_{2m}(x, \zeta) \) has the property \( \Delta_n E_{2m}(x, \zeta) = -E_{2m-2}(x, \zeta) \) for \( x \neq \zeta \). This equality was applied in the construction of the Green’s functions of the Riquier–Neumann problem.

We study the product of polyharmonic functions. It is known that the product of harmonic functions can be both a harmonic function and a function that is not \( m \)-harmonic for any \( m \in \mathbb{N} \). For example, such functions can be \( u \cdot v \) and \( u^2 \), where \( u(x_1, x_2) = e^{x_1} \cos x_2 \) and \( v(x_1, x_2) = e^{x_1} \sin x_2 \). Moreover, according to the well-known Almansi formula, the function \( |x|^{2m-2}u(x) \) is an \( m \)-harmonic function if the function \( u(x) \) is a harmonic one. We introduce the operator
\[
\Lambda u(x) = \sum_{i=1}^{n} x_i \mu_{x_i}
\]
for which, as it is easy to check, the equality \( \Delta \Lambda u = (\Lambda + 2) \Delta u \) holds true. From this equality it follows that \( \Delta^m u = (\Lambda + 2m) \Delta^m u \), and therefore, if the function \( u(x) \) is \( m \)-harmonic, then the function \( \Lambda u \) is also \( m \)-harmonic, although its representation contains terms of the form \( x_i \mu_{x_i} \).

For the following result, a slight modification of the method of mathematical induction is necessary.

(*) Let statements \( A_{n,m} \) for \( n, m \in \mathbb{N} \) be given and the following conditions be satisfied for them: (1) statements \( A_{1,1} \) and \( A_{2,m} \) are true for \( n, m \in \mathbb{N} \); (2) the validity of the statements \( A_{n-1,m} \) and \( A_{n,m-1} \), whose index belongs to \( \mathbb{N}^2 \), implies the validity of \( A_{n,m} \). Then, the statements \( A_{n,m} \) are true for all \( n, m \in \mathbb{N} \).

**Proof.** Suppose that, under conditions (1) and (2), there are statements \( A_{n,m} \) that are not true. Among them, we choose \( A_{n^*, m^*} \), whose sum of indices \( n^* + m^* \) is the smallest, and \( (n^*, m^*) \in \mathbb{N}^2 \). By condition (1), for \( A_{n^*, m^*} \), there can be neither \( n^* = 1 \) nor \( m^* = 1 \). Then, by condition (2), either the statement \( A_{n^*-1,m^*} \) or the statement \( A_{n^*,m^*-1} \) must not hold true. For these statements, the sum of the indices is equal to \( n^* + m^* - 1 \), which cannot be due to the choice of the statement \( A_{n^*,m^*} \). So, all statements \( A_{n,m} \) hold true. \( \square \)

**Lemma 2.** Let \( k_1, k_2 \in \mathbb{N}_0, k_3 \in \mathbb{N} \) and \( k = k_1 + k_2 + k_3 \). Then, the function
\[
h_i^{k_3}(x, \zeta)h_4^{k_2}(x, \zeta)\mathcal{E}_{2m}^*(x, \zeta)
\]
is \( k \)-harmonic in \( x \in S \) for \( \zeta \in \bar{S} \).
Proof. 1°. Let us prove that the function

\[ F_{k_2,k_3}(x, \xi) = h_{k_2}^k(x, \xi) E_{2k_3}^s(x, \xi) \]

for \( k_2 \in \mathbb{N}_0, k_3 \in \mathbb{N} \) is \((k_2 + k_3)\)-harmonic in \( x \in S \). Let first \( n \in \mathbb{N}_{k_2+k_3} \). It is easy to see that in this case, \( n \in \mathbb{N}_{k_3} \), and hence, from (7), we get

\[ F_{k_2,k_3}(x, \xi) = C_{k_3} h_{k_2}^{k_3-n/2+k}(x, \xi) = \frac{C_{k_3}}{C_{k_2+k_3}} E_{2k_2+2k_3}^s(x, \xi), \]

where \( C_{k_3} \) is the numerical coefficient of \(|x-\xi|^{2k_3-n}\) in \( E_{2k_3} \) when \( n \in \mathbb{N}_{k_3} \). Therefore, by Lemma 1, the function \( F_{k_2,k_3}(x, \xi) \) is \((k_2 + k_3)\)-harmonic in \( x \in S \). If \( n \in \mathbb{N}_{k_2+k_3} \) but \( n \in \mathbb{N}_{k_3} \) then there is a natural \( s \) such that \( k_3 + 1 \leq s \leq k_2 + k_3 \) and \( 2s = n \). In this case,

\[ F_{k_2,k_3}(x, \xi) = C_{k_3} h_{k_2}^{k_3+k_2-s}(x, \xi). \]

Next, it is easy to calculate that

\[ \Delta |x|/|x| - |x|/|\xi| = a(a + n - 2)|\xi|^2 |x| - |x|/|\xi|^{a-2} \]

and hence, because \( 1 \leq k_3 + k_2 - s + 1 \leq k_2 \), then the polynomial \( F_{k_2,k_3}(x, \xi) \) is \((k_2 + 1)\)-harmonic in \( x \in S \). Because \( k_2 + 1 \leq k_2 + k_3 \), the statement in 1° holds true in this case.

Let now \( n \in \mathbb{N}_{k_3} \). Then, there is a natural number \( s \leq k_3 \) such that \( 2s = n \), and therefore, the polynomial \( h_{k_2}^{k_3+s}(x, \xi) \), similarly to the previous case, is \((k_3 - s + 1)\)-harmonic. From the equality (7), we find

\[ E_{2k_3}^s(x, \xi) = C_{k_3} h_{k_2}^{k_3-s}(x, \xi) \ln |x|/|x| - |x|/|\xi| - C_{k_3} L_{k_3} h_{k_2}^{k_3-s}(x, \xi), \]

where \( C_{k_3} \) is the numerical factor at \(|x-\xi|^{2k_3-n}\), and \( L_{k_3} \) is the numerical term at the logarithm in \( E_{2k_3} \), when \( n \in \mathbb{N}_{k_3} \). That is why

\[ F_{k_2,k_3}(x, \xi) = C_{k_3} h_{k_2}^{k_3+k_2-s}(x, \xi) \ln |x|/|x| - |x|/|\xi| - C_{k_3} L_{k_3} h_{k_2}^{k_3+k_2-s}(x, \xi). \]

Because \( n \in \mathbb{N}_{k_2+k_3} \), (7) implies

\[ h_{k_2}^{k_3+k_2-s}(x, \xi) \ln |x|/|x| - |x|/|\xi| = C_{k_2+k_3}^{-1} E_{2k_2+2k_3}^s(x, \xi) + L_{k_2+k_3} h_{k_2}^{k_2+k_3-s}(x, \xi) \]

and therefore,

\[ F_{k_2,k_3}(x, \xi) = C_{k_3} C_{k_2+k_3}^{-1} E_{2k_2+2k_3}^s(x, \xi) + C_{k_3} (L_{k_2+k_3} - L_{k_3}) h_{k_2}^{k_2+k_3-s}(x, \xi). \]

The first term in the resulting equality, by Lemma 1, is a \((k_2 + k_3)\)-harmonic function in \( x \in S \), and the second term is a \((k_2 + k_3 - s + 1)\)-harmonic polynomial. Because \( s \geq 1 \), 1° is then completely proved.

2°. Let us now prove that the function \( G_{k_1,k_2}(x, \xi) = h_{k_1}(x, \xi) F_{k_2}(x, \xi) \) is an \((k_2' + k_3')\)-harmonic function in \( x \in S \) and \( k_1 \in \mathbb{N}_0, k_2' \in \mathbb{N} \), is \((k_1 + k_2')\)-harmonic in \( x \in S \). The proof will be carried out by the induction (°) with respect to two indices \( k_1 \in \mathbb{N}_0 \) and \( k_2' \in \mathbb{N} \).

(1) If \( k_1 = 0 \), then \( G_{0,k_2}(x, \xi) = F_{k_2}(x, \xi) \), and therefore, by assumption, the function \( G_{0,k_2}(x, \xi) \) is \((k_2' + k_3')\)-harmonic. If \( k_2' = 1 \), then \( G_{k_1,1}(x, \xi) = h_{k_1}(x, \xi) F_1(x, \xi) \), and because \( h(x, \xi) = |x-\xi|^2 \), then, according to Almansi’s formula, the function \( G_{k_1,1}(x, \xi) \) is \((k_1 + 1)\)-harmonic.
(2). Suppose that functions of the form $h^{k_1-1}(x, \xi)F_{k_2}(x, \xi)$ and $h^{k_1}(x, \xi)F_{k_2-1}(x, \xi)$ are $(k_1 + k_2 - 1)$-harmonic in $x \in S$. Because, according to (8), we have

$$\frac{\partial}{\partial x_i} h^{k_1}(x, \xi) = 2k_1(x_i - \xi_i)h^{k_1-1}(x, \xi),$$

then, using the operator $\Lambda$ from (10), we get

$$\Delta_x G_{k_1,k_2}(x, \xi) = F_{k_2}(x, \xi)\Delta_x h^{k_1}(x, \xi) + 4k_1h^{k_1-1}(x, \xi) \sum_{i=1}^n (x_i - \xi_i) \frac{\partial F_{k_2}(x, \xi)}{\partial x_i}$$

$$+ h^{k_1}(x, \xi)\Delta_x F_{k_2}(x, \xi) = 2k_1(2k_1 + n - 2)h^{k_1-1}(x, \xi)F_{k_2}(x, \xi)$$

$$+ 4k_1h^{k_1-1}(x, \xi)(\Delta_x F_{k_2}(x, \xi) - \xi \cdot \nabla_x F_{k_2}(x, \xi)) + h^{k_1}(x, \xi)\Delta_x F_{k_2}(x, \xi). \quad (12)$$

In the last equality, similar to (11), the following identity is used:

$$\Delta_x h^{k_1}(x, \xi) = 2k_1(2k_1 + n - 2)h^{k_1-1}(x, \xi).$$

According to the induction hypothesis and by the property of the operator $\Lambda$, each term on the right-hand side of (12) is a $(k_1 + k_2 - 1)$-harmonic function in $x \in S$. Therefore, the function $G_{k_1,k_2}(x, \xi)$ is $(k_1 + k_2)$-harmonic in $x \in S$. Assertion 1$^0$. is proved.

3$^0$. Now, choose $F_{k_2}(x, \xi) = F_{k_2,k_3}(x, \xi)$, where $k_2' = k_2 + k_3$. This is possible because, firstly, by Lemma 1, the function $F_1(x, \xi) = F_{0,1}(x, \xi) = E_2^*(x, \xi)$ is harmonic in $x \in S$, and secondly, by Assertion 1$^0$. of the lemma, the function $F_{k_2,k_3}(x, \xi)$ is $k_2'$-harmonic in $x \in S$ when $\xi \in S$. Therefore, according to 2$^0$. the assertion of the lemma holds true. The lemma is completely proved. \( \square \)

3. Green’s Function

Now, we can present the Green’s function representation to the Dirichlet boundary value problem (5)–(6).

**Theorem 1.** The Green’s function $G_{2m}(x, \xi)$ of the Dirichlet problem (5)–(6) for $n \geq 2$ and $m \in \mathbb{N}$ can be written as

$$G_{2m}(x, \xi) = E_{2m}(x, \xi) - \sum_{k=0}^{m-1} (|x|^2 - 1)^k (|\xi|^2 - 1)^k \frac{k!}{(2m-2)^k (2,2)_k} E_{2m-2k}(\frac{x}{|x|}, |x|^2). \quad (13)$$

The symmetric function $G_{2m}(x, \xi)$ is $m$-harmonic in $x \in S$, $x \neq \xi \in \bar{S}$ and satisfies the equalities

$$G_{2m}(x, \xi)|_{x \in \partial S} = 0, \quad \frac{\partial G_{2m}(x, \xi)}{\partial v}|_{x \in \partial S} = 0, \ldots, \frac{\partial^{m-1} G_{2m}(x, \xi)}{\partial v^{m-1}}|_{x \in \partial S} = 0, \xi \in S. \quad (14)$$

**Proof.** 1. First, let us check whether the function $G_{2m}(x, \xi)$ defined in (13) corresponds to the functions (2) and (4) found earlier. For $m = 1$ from (13), taking into account the notation (8), the condition $(a, b)_0 = 1$ and the equality $E(x, \xi) = E_2^*(x, \xi)$, we obtain the well-known Green’s function for the Laplace equation. For $m = 2$, from the equality (13), we get

$$G_4(x, \xi) = E_4(x, \xi) - E_4^*(x, \xi) - \frac{(|x|^2 - 1)(|\xi|^2 - 1)}{2 \cdot 2} E_2^*(x, \xi),$$
which coincides with (2) for \( n \geq 3 \), because in this case, according to Remark 1, \( \mathcal{E}_4(x, \xi) = E_4(x, \xi) \). If \( m = 3 \) and \( n \geq 3 \), \( n \neq 4 \), then, again by Remark 1, we get \( \mathcal{E}_6(x, \xi) = E_6(x, \xi) \) and hence, taking into account the previous equalities, (13) gives the equality

\[
G_6(x, \xi) = \mathcal{E}_6(x, \xi)
\]

which matches (4).

2. Let us check the \( m \)-harmonicity in \( x \in S \) of the function \( G_{2m}(x, \xi) \) from (13) for \( x \neq \xi \in \hat{S} \). To achieve this, note that, in accordance with the notation (8), the equalities

\[
h_s(x, \xi) - h(x, \xi) = 1 - 2x \cdot \xi + |x|^2|\xi|^2 - |x|^2 + 2x \cdot \xi - |\xi|^2 = (|x|^2 - 1)(|\xi|^2 - 1),
\]

hold true, and therefore the equality (13) can be rewritten as

\[
G_{2m}(x, \xi) = \mathcal{E}_{2m}(x, \xi) - \sum_{k=0}^{m-1} \frac{(h_s(x, \xi) - h(x, \xi))^k}{(2m - 2k - 2)k!2^k} \mathcal{E}_k(x, \xi).
\]  

(15)

By Lemma 1, it is sufficient to check the \( m \)-harmonicity of functions under the summation sign in (15). However, this immediately follows from Lemma 2, because the common term of the sum in (15) is a \((k + m - k)\)-harmonic function in \( x \in S \) as \( \xi \in \hat{S} \).

3. Now, let us check the fulfillment of the boundary conditions (14). The outer unit normal derivative to the sphere of radius \( |x| < 1 \) of the function \( G_{2m}(x, \xi) \) is equal to

\[
\frac{\partial G_{2m}(x, \xi)}{\partial \nu_x} = \sum_{i=1}^{n} \frac{x_i}{|x|} \frac{\partial G_{2m}(x, \xi)}{\partial x_i} = \frac{1}{|x|} \Lambda_s G_{2m}(x, \xi).
\]

Thus,

\[
\frac{\partial^2 G_{2m}(x, \xi)}{\partial \nu_x^2} = \left( \frac{1}{|x|} \Lambda_s \frac{1}{|x|} \Lambda_s \right) G_{2m}(x, \xi) = \frac{1}{|x|^2} \Lambda_s (-1 + \Lambda_s) G_{2m}(x, \xi)
\]

and hence, using the notation \( \Lambda^{[k]} = \Lambda(\Lambda - 1) \ldots (\Lambda - k + 1) \), one can write

\[
\frac{\partial^k G_{2m}(x, \xi)}{\partial \nu_x^k} = \left( \frac{1}{|x|} \Lambda_s \right)^k G_{2m}(x, \xi)
\]

\[
= \frac{1}{|x|^k} \Lambda_s (\Lambda_s - 1) \ldots (\Lambda_s - k + 1) G_{2m}(x, \xi) = \frac{1}{|x|^k} \Lambda^{[k]} G_{2m}(x, \xi).
\]  

(16)

Thus, to satisfy the boundary conditions (14), we must prove that the equalities

\[
G_{2m}(x, \xi)|_{x \in \partial S} = 0, \quad \Lambda_s G_{2m}(x, \xi)|_{x \in \partial S} = 0, \ldots, \quad \Lambda_s^{m-1} G_{2m}(x, \xi)|_{x \in \partial S} = 0.
\]

It is easy to see that these equalities are equivalent to the following equalities

\[
G_{2m}(x, \xi)|_{x \in \partial S} = 0, \quad \Lambda_s G_{2m}(x, \xi)|_{x \in \partial S} = 0, \ldots, \quad \Lambda_s^{m-1} G_{2m}(x, \xi)|_{x \in \partial S} = 0.
\]  

(17)

Let us prove that they hold true, which means that the boundary conditions (14) are also satisfied. If we put \( x \in \partial S \) in (15), i.e., \( |x| = 1 \), then for \( \xi \in S \), taking into account that \( \mathcal{E}_{2m}(x, \xi) = \mathcal{E}_{2m}^1(x, \xi) \) and \( h(x, \xi) = h_s(x, \xi) \) on \( \partial S \), we get

\[
G_{2m}(x, \xi)|_{x \in \partial S} = \mathcal{E}_{2m}(x, \xi)|_{x \in \partial S} - \frac{\mathcal{E}_{2m}(x, \xi)}{(2m - 2, -2)0(2, 2)0} = 0
\]

for any \( m \geq 1 \).
Let us consider the next boundary condition from (17). It is easy to verify the equality

\[ \Lambda_x G_{2m}(x, \xi) = \Lambda_x e_{2m}(x, \xi) - \sum_{k=0}^{m-1} \frac{(h_s(x, \xi) - h(x, \xi))^k}{(2m - 2, -2)^k(2, 2)^k} \Lambda_x e_{2m-2k}(x, \xi) \]

\[ \quad - \sum_{k=1}^{m-1} k \Lambda_x (h_s(x, \xi) - h(x, \xi)) \frac{(h_s(x, \xi) - h(x, \xi))^{k-1}}{(2m - 2, -2)^k(2, 2)^k} e_{2m-2k}(x, \xi). \]  

(19)

Consider the last term in the first sum on the right in (19), which has the form

\[ \epsilon_{m-2}(x) = \frac{(h_s(x, \xi) - h(x, \xi))^{m-1}}{(2m - 2, -2)^m-1(2, 2)^m-1} \Lambda_x e_{2}^*(x, \xi). \]

By Lemma 2, this function is \( m \)-harmonic in \( S \). Because \( h_s(x, \xi) = h(x, \xi) \) for \( x \in \partial S \), then, after applying the operators \( \Lambda_x^k \) for \( k = 0, \ldots, m - 2 \) to the function \( \epsilon_{m-2}(x) \), the limits at \( x \to \partial S \) of all the functions obtained in this case turn to 0. The subscript of \( \epsilon_{m-2}(x) \) indicates the maximum degree of the operator \( \Lambda_x \) under which this property is satisfied. Therefore, in what follows, all functions that have this property will be denoted by a single symbol, \( \epsilon_{m-2}(x) \). These functions do not affect the fulfillment of boundary conditions (17).

Let \( n \in \mathbb{N}_m \). Then, in accordance with (7), the equalities

\[ \Lambda_x e_{2m}(x, \xi) = \Lambda_x (\frac{(-1)^m |x - \xi|^{2m-n}}{(2-n, 2)_{(2, 2)}^{-m-n}}) = (\frac{(-1)^m}{(2-n, 2)_{(2, 2)}^{-m-n}}) \Lambda_x h^{m-n/2}(x, \xi) \]

\[ = \frac{(m-n/2)}{(2m-2)} \Lambda_x h(x, \xi) = -\frac{\Lambda_x h(x, \xi)}{2(2m-2)} e_{2m-2}(x, \xi). \]  

(20)

hold true. Let us study the case \( n \in \mathbb{N}_m \). Let first \( n = 2m \), and hence \( n \in \mathbb{N}_{m-1} \). Then, according to (7), we have

\[ \Lambda_x e_{2m}(x, \xi) = \frac{(-1)^m \ln |x - \xi|^{2m-n}}{(2-n, 2)_{(2, 2)}^{-m-n}} = \frac{(m-n/2)}{(2m-2)} \Lambda_x h(x, \xi) \]

\[ = -\frac{\Lambda_x h(x, \xi)}{2(2m-2)} e_{2m-2}(x, \xi). \]

Let \( n \in \mathbb{N}_m \) but \( n \neq 2m \). Then, again according to (7), we have

\[ \Lambda_x e_{2m}(x, \xi) = \frac{(-1)^m |x - \xi|^{2m-n}}{(2-n, 2)_{(2, 2)}^{-m-n}} \left( \ln |x - \xi| - \frac{m-n/2}{2k} \sum_{k=1}^{m-n/2} \frac{1}{2k} \right) \]

\[ = -\frac{(m-n/2)}{(2m-2)} \frac{(-1)^{m-1} |x - \xi|^{2m-2-n} \Lambda_x h(x, \xi)}{(2m-2)(2-n, 2)_{(2, 2)}^{-m-2}} \left( \ln |x - \xi| - \frac{m-n/2}{2k} \sum_{k=1}^{m-n/2} \frac{1}{2k} \right) \]

\[ + \frac{(2m-2)(2-n, 2)_{(2, 2)}^{-m-2}}{2m-2} \frac{1}{2(2m-2)} \Lambda_x h(x, \xi) \]

\[ = -\frac{\Lambda_x h(x, \xi)}{2(2m-2)} (\frac{-1}{2m-2} - \frac{1}{2m-n}) \left( \ln |x - \xi| - \frac{m-n/2}{2k} \sum_{k=1}^{m-n/2} \frac{1}{2k} + \frac{1}{2m-n} \right) \]

\[ = -\frac{\Lambda_x h(x, \xi)}{2(2m-2)} e_{2m-2}(x, \xi). \]

Thus, the equality (20) is valid for all \( n \geq 2 \). Similarly to the above calculations, it is easy to obtain for \( n \geq 2 \) the equality

\[ \Lambda_x e_{2m}^*(x, \xi) = -\frac{\Lambda_x h(x, \xi)}{2(2m-2)} e_{2m-2}(x, \xi). \]
In accordance with (20), we rewrite the equality (19) as

$$\Lambda_x G_{2m}(x, \xi) = -\frac{\Lambda_x h(x, \xi)}{2(2m - 2)} E_{2m-2}(x, \xi)$$

$$+ \sum_{k=0}^{m-2} \left( \frac{h_s(x, \xi) - h(x, \xi)}{2(2m - 2k - 2)} \right)^k \frac{\Lambda_x h_s(x, \xi)}{2(2m - 2k - 2)} E_{2m-2k-2}(x, \xi) + \varepsilon_{m-2}(x)$$

$$- \sum_{k=0}^{m-2} (k + 1) \Lambda_x \left( h_s(x, \xi) - h(x, \xi) \right) \left( \frac{h_s(x, \xi) - h(x, \xi)}{2(2m - 2, -2)_k(2, 2)_k} \right) E_{2m-2k-2}(x, \xi). \quad (21)$$

Note that the change of the summation index \(k \rightarrow k + 1\) is made in the last sum. Combine the last two sums, which have the same upper and lower summation limits, into one sum

$$\sum_{k=0}^{m-2} \left( \frac{h_s(x, \xi) - h(x, \xi)}{2(2m - 2, -2)_k(2, 2)_k} \right)^k \Lambda_x h_s(x, \xi) \left( \frac{h_s(x, \xi) - h(x, \xi)}{2(2m - 2k - 2)(2k + 2)} \right)$$

$$= \sum_{k=0}^{m-2} \left( h_s(x, \xi) - h(x, \xi) \right)^k \Lambda_x h(x, \xi) \left( \frac{h_s(x, \xi) - h(x, \xi)}{2(2m - 2, -2)_k(2, 2)_k} \right) E_{2m-2k-2}(x, \xi),$$

and then, using the equality

$$(2m, -2)_k = (2m - 2, 2)_k (2m - 2) = (2m - 2)(2m - 4, -2)_k,$$

we get

$$\Lambda_x h(x, \xi) \sum_{k=0}^{m-2} \left( \frac{h_s(x, \xi) - h(x, \xi)}{2(2m - 4, -2)_k(2, 2)_k} \right) E_{2m-2k-2}(x, \xi).$$

Thus, (21) is converted to the form

$$\Lambda_x G_{2m}(x, \xi) = -\frac{\Lambda_x h(x, \xi)}{2(2m - 2)} E_{2m-2}(x, \xi)$$

$$+ \frac{\Lambda_x h(x, \xi)}{2(2m - 2)} \sum_{k=0}^{m-1} \left( \frac{h_s(x, \xi) - h(x, \xi)}{2(2m - 1, -2)_k(2, 2)_k} \right) E_{2m-1-2k}(x, \xi) + \varepsilon_{m-2}(x),$$

from which we get

$$\Lambda_x G_{2m}(x, \xi) = -\frac{\Lambda_x h(x, \xi)}{4(m - 1)} G_{2m-2}(x, \xi) + \varepsilon_{m-2}(x). \quad (22)$$

From here, it immediately follows that

$$\Lambda_x^2 G_{2m}(x, \xi) = -\frac{\Lambda_x^2 h(x, \xi)}{4(m - 1)} G_{2m-2}(x, \xi) - \frac{\Lambda_x h(x, \xi)}{4(m - 1)} \Lambda_x G_{2m-2}(x, \xi) + \Lambda_x \varepsilon_{m-2}(x).$$

Due to the equality (22), considered for \(m - 1\) instead of \(m\), and taking into account the equalities

$$\Lambda_x \varepsilon_{m-2}(x) = \varepsilon_{m-3}(x), \quad -\frac{\Lambda_x h(x, \xi)}{4(m - 1)} \varepsilon_{m-3}(x) = \varepsilon_{m-3}(x),$$
and also because $\epsilon_{m-3}(x) + \epsilon_{m-3}(x) = \epsilon_{m-3}(x)$, we can write

$$
\Lambda^2_k G_{2m}(x, \xi) = -\frac{\Lambda^2_k h(x, \xi)}{4(m-1)} G_{2m-2}(x, \xi) + \frac{(\Lambda_x h(x, \xi))^2}{4^2(m-1)(m-2)} G_{2m-4}(x, \xi) + \epsilon_{m-3}(x). 
$$  (23)

Let us use induction on $k$. Based on the equality (23) and taking into account (22), where $k = 1$, suppose that for some $1 < k < m - 1$, the equality

$$
\Lambda^{k-1}_x G_{2m}(x, \xi) = \sum_{i=1}^{k-1} s^{(k-1)}_{2i}(x, \xi) G_{2m-2i}(x, \xi) + \epsilon_{m-k}(x)
$$

holds true, where $s^{(k-1)}_{2i}(x, \xi)$ is some polynomial of degree $2i$ in $x$. For example, from (22) and (23), where $k = 1$ and $k = 2$, respectively, we find

$$
s^{(1)}_2(x, \xi) = -\frac{\Lambda_x h(x, \xi)}{4(m-1)} s^{(2)}_2(x, \xi) = -\frac{\Lambda^2_k h(x, \xi)}{4(m-1)}, \quad s^{(2)}_4(x, \xi) = \frac{(\Lambda_x h(x, \xi))^2}{16(m-1)(m-2)}.
$$

Using (24) and (22) for $m - 1$ instead of $m$, we write

$$
\Lambda^k G_{2m}(x, \xi) = \sum_{i=1}^{k-1} \Lambda_x s^{(k-1)}_{2i}(x, \xi) G_{2m-2i}(x, \xi) - \sum_{i=1}^{k-1} s^{(k-1)}_{2i}(x, \xi) \frac{\Lambda_x h(x, \xi)}{4(m-i-1)}
$$

$$
\times G_{2m-2i-2}(x, \xi) + \sum_{i=1}^{k-1} s^{(k-1)}_{2i}(x, \xi) \epsilon_{m-i-2}(x) + \Lambda_x \epsilon_{m-k}(x).
$$

Hence, because

$$
\sum_{i=1}^{k-1} s^{(k-1)}_{2i}(x, \xi) \frac{\Lambda_x h(x, \xi)}{4(m-i-1)} G_{2m-2i-2}(x, \xi) = \sum_{i=2}^{k} s^{(k-1)}_{2i-2}(x, \xi) \frac{\Lambda_x h(x, \xi)}{4(m-i)} G_{2m-2i}(x, \xi),
$$

we obtain the equality

$$
\Lambda^k G_{2m}(x, \xi) = \sum_{i=1}^{k} \left( \Lambda_x s^{(k-1)}_{2i}(x, \xi) - s^{(k-1)}_{2i-2}(x, \xi) \frac{\Lambda_x h(x, \xi)}{4(m-i)} \right) G_{2m-2i}(x, \xi) + \epsilon_{m-k-1}(x),
$$

(25)

where it should be taken into account that $s^{(k-1)}_{2i} = 0$ for $i = 0$ or $i > k - 1$. Moreover, we also took into account that

$$
\sum_{i=1}^{k-1} s^{(k-1)}_{2i}(x, \xi) \epsilon_{m-i-2}(x) + \Lambda_x \epsilon_{m-k}(x) = \epsilon_{m-k-1}(x),
$$

because the smallest index of the functions $\epsilon_{m-i-2}(x)$ under the sum sign is equal to $m - k - 1$. If we denote

$$
s^{(k)}_{2i}(x, \xi) = \Lambda_x s^{(k-1)}_{2i}(x, \xi) - s^{(k-1)}_{2i-2}(x, \xi) \frac{\Lambda_x h(x, \xi)}{4(m-i)},
$$
Theorem 2. The solution to the homogeneous Dirichlet boundary value problem
\(u(x) = \frac{(-1)^m}{\omega_n} \int_S G_{2m}(x, \xi) f(\xi) \, d\xi,
\)
where \(\omega_n\) is the area of the unit sphere in \(\mathbb{R}^n\).

Proof. Let us calculate the result of applying the operator \(\Delta^{m-1}\) to the function \(u(x)\) given by the equality (27). To achieve this, note that the potential type integrals \(\int_\Lambda \frac{\rho(\xi)}{|x-\xi|^{n-2}} \, d\xi\) are functions from the class \(\mathcal{C}^p(\mathbb{R}^n)\) for a bounded and integrable function \(\rho(x)\) and, moreover,
the differentiation of order $p \in \mathbb{N}_0$ with respect to $x$ can be brought under the integral sign for any $p$ such that $\alpha + p < n$ [33]. In our case, for the singular term of the function $G_{2m}(x, \zeta)$ from (13), we have $\alpha = n - 2m + 1$, and hence, for the integral

$$u_1(x) = \frac{(-1)^m}{\omega_n} \int_{S} \mathcal{E}_{2m}(x, \zeta) f(\zeta) \, d\zeta$$

we have $p = 2m - 2$, and therefore $u_1 \in C^{2m-2}(\mathbb{R}^n)$, which means that the operator $\Delta^{m-1}$ can be brought under the integral sign.

Let $n \in \mathbb{N}_m$; then, from (9), it follows that $\mathcal{E}_{2m}(x, \zeta) = E_{2m}(x, \zeta)$ and, therefore, because of the assertion $\Delta E_{2m}(x, \zeta) = -E_{2m-2}(x, \zeta)$ (see Lemma 2 [17]) for $x \neq \zeta$, we get $\Delta \mathcal{E}_{2m}(x, \zeta) = -\mathcal{E}_{2m-2}(x, \zeta)$ for $x \neq \zeta$. That is why

$$\Delta^{m-1} \mathcal{E}_{2m}(x, \zeta) = (-1)^{m-1} \mathcal{E}_2(x, \zeta) = (-1)^{m-1} E_2(x, \zeta),$$

where $E_2(x, \zeta)$ is the same as $E(x, \zeta)$ from [28]. If $n \in \mathbb{N}_m$ and $n \neq 2$, then, given the equality $\Delta |x - \zeta|^\alpha = \alpha(n + 2)|x - \zeta|^{n-2}$ and according to (9), we get

$$\Delta^{m-1} \mathcal{E}_{2m}(x, \zeta) = (-1)^{m-1} (E_2(x, \zeta) - M),$$

where $M = \sum_{k=1}^{m-1} \frac{1}{k!}$. Hence, for $n > 2$, we have

$$\Delta^{m-1} u_1(x) = \frac{(-1)^m}{\omega_n} \int_{S} \Delta^{m-1} \mathcal{E}_{2m}(x, \zeta) f(\zeta) \, d\zeta = -\frac{1}{\omega_n} \int_{S} \mathcal{E}_2(x, \zeta) f(\zeta) \, d\zeta,$$

and for $n = 2$ we get, respectively,

$$\Delta^{m-1} u_1(x) = -\frac{1}{\omega_n} \int_{S} (E_2(x, \zeta) - M) f(\zeta) \, d\zeta.$$

As a result, for $n > 2$, by the property of the volume potential, we can write

$$\Delta^m u_1(x) = \Delta \left( -\frac{1}{\omega_n} \int_{S} E_2(x, \zeta) f(\zeta) \, d\zeta \right) = f(x), \quad x \in S,$$

and for $n = 2$, we obtain the same equality

$$\Delta^m u_1(x) = \Delta \left( -\frac{1}{\omega_n} \int_{S} E_2(x, \zeta) f(\zeta) \, d\zeta + \frac{M}{\omega_n} \int_{S} f(\zeta) \, d\zeta \right) = f(x), \quad x \in S.$$

The condition $f \in C^1(\mathbb{S})$ is sufficient for the equality $\Delta (\Delta^{m-1} u_1)(x) = f(x)$ [28]. In Lemma 2, taking into account the notation $(|x|^2 - 1)(|\zeta|^2 - 1) = h_+(x, \zeta) - h(x, \zeta)$ from Theorem 1, it is proved that a function of the form

$$\sum_{k=0}^{m-1} \frac{(|x|^2 - 1)(|\zeta|^2 - 1)^k}{(2m - 2, -2)_k} \mathcal{E}_{2m-2k}(x, \zeta) = \sum_{k=0}^{m-1} \frac{(h_+(x, \zeta) - h(x, \zeta))^k}{(2m - 2, -2)_k} \mathcal{E}_{2m-2k}(x, \zeta)$$
is $m$-harmonic with respect to $x$ in $S$ for any $\xi \in S$ and can be differentiated with respect to $x$ under the integral sign any number of times. Let us denote by $u_2(x)$ the integral over $\xi \in S$ of this function multiplied by $(-1)^{m}/\omega_m f(\xi)$. Then, we get

$$\Delta^m u_2(x) = \frac{(-1)^m}{\omega_n} \sum_{k=0}^{m-1} \int_S \Delta_x^k (|x|^2 - 1)^k (|\xi|^2 - 1)^k \xi_{2m-2k}(x, \xi) f(\xi) \, d\xi = 0.$$ 

Therefore, the function $u(x)$ from (27), bearing in mind the equality (13), satisfies Equation (5):

$$\Delta^m u(x) = \Delta^m u_1(x) - \Delta^m u_2(x) = f(x), \quad x \in S.$$ 

Furthermore, due to the fact that, for the function $u(x)$ from (27), we finally have the inclusion $u \in C^{2m-2}(S)$, the passage to the limit $x \to \partial S$ for the functions $\Lambda^k_x u(x)$, $k = 0, \ldots, m - 1$ can then be brought under the integral sign. Using (16) and (17), we find

$$\frac{\partial^k u}{\partial x^k} |_{x \in \partial S} = \frac{(-1)^m}{\omega_n} \int_S \Lambda^k_x G_{2m}(x, \xi) \big|_{x \in \partial S} f(\xi) \, d\xi = 0, \quad k = 0, \ldots, m - 1,$$

and hence, the function $u(x)$ from (27) satisfies all boundary conditions (6). The theorem is proved. $\square$

5. Polynomial Right-Hand Side

Consider the simplest particular case of the polynomial right-hand side of the $m$-harmonic Equation (5).

**Theorem 3.** Let $f(x) = |x|^{2l} H_k(x)$ in Equation (5), where $H_k(x)$ is a homogeneous harmonic polynomial of degree $k \in \mathbb{N}_0$, $l \in \mathbb{N}$ and $n \geq 2$. Then, the solution to the homogeneous Dirichlet boundary value problem (5)–(6) can be written in the form

$$u(x) = \frac{(-1)^m}{\omega_n} \int_{|\xi| < 1} G_{2m}(x, \xi) ||\xi||^2 H_k(\xi) \, d\xi = \frac{|x|^{2l+2m - m-1} \left( l + m \choose l \right) \left( |x|^2 - 1 \right)^l H_k(x)}{(2l + 2, 2)_m (2l + 2k + n, 2)_m}, \quad (28)$$

where $(a, b)_m$ is the generalized Pochhammer symbol defined in (7).

**Proof.** Let first $m = 3$. In Corollary 2 [12], it is established that the solution of the Dirichlet problem to the 3-harmonic equation for $f(x) = |x|^{2l} H_3(x)$ can be written as

$$u(x) = \frac{|x|^{2l+6 - 1 - (l+3)(|x|^2 - 1) - \frac{1}{2}(l+2)(l+3)(|x|^2 - 1)^2}}{(2l + 2, 2)_3 (2l + 2k + n, 2)_3} H_k(x).$$

Due to the uniqueness of the solution to the Dirichlet problem [30] and the equality (27), we easily obtain Formula (28) for $m = 3$. The equality (28) generalizes this formula to arbitrary $m \in \mathbb{N}$. Let us verify that the function $u(x)$ given by the right-hand side of (28) is a solution to the homogeneous Dirichlet problem with $f(x) = |x|^{2l} H_k(x)$. It is easy to see that, because

$$\Delta |x|^{2l} H_k(x) = 2l(2l + 2k + n - 2)|x|^{2l-2} H_k(x),$$

then, from (28), taking into account that the functions of the form $|x|^{2k} H_k(x)$ for $k \leq m - 1$ are $m$-harmonic polynomials, we find

$$\Delta^m u(x) = \frac{(2l + 2m) \ldots (2l + 2) \cdot (2l + 2m + 2k + n - 2) \ldots (2l + 2k + n)}{(2l + 2, 2)_m (2l + 2k + n, 2)_m} |x|^{2l} H_k(x) = |x|^{2l} H_k(x),$$

where $k = 0, \ldots, m - 1$. The proof is complete.
i.e., \( u(x) \) satisfies Equation (5).

Now, let us check the boundary conditions (6). First, we extract a constant factor from the function \( u(x) \), i.e., we represent this function as \( u(x) = (2l + 2m)(2l + 2k + n, 2)_m u^*(x) \). It is clear that, if \( u^*(x) \) satisfies the conditions (17), then the function \( u(x) \) also satisfies them, and hence it also satisfies the conditions (6). Next, we represent the function \( u^*(x) \) as a product of two polynomials \( u^*(x) = G_m(x) H_l(x) \), where

\[
G_m(x) = |x|^{2l+2m} - \sum_{i=0}^{m-1} \left( \frac{l + m - i}{i} \right) (|x|^2 - 1)^i.
\]

It is easy to see that \( G_m(x)|_{\partial S} = 0 \) for \( m \geq 1 \), and hence \( u^*(x)|_{\partial S} = 0 \). Computing \( \Lambda u^*(x) \), we have

\[
\Lambda u^*(x) = H_l(x) \Lambda G_m(x) + G_m(x) \Lambda H_l(x)
\]

\[
= (2l + 2m)|x|^{2l+2m} - \sum_{i=1}^{m-1} \left( \frac{l + m - i}{i} \right) 2i |x|^2 (|x|^2 - 1)^{i-1} H_l(x) + k G_m(x) H_l(x)
\]

\[
= (2l + 2m)|x|^{2l+2m-2} - \sum_{i=1}^{m-1} \left( \frac{l + m - i}{i - 1} \right) |x|^2 (|x|^2 - 1)^{i-1} |x|^2 H_l(x) + k G_m(x) H_l(x)
\]

\[
= (2l + 2m) G_{m-1}(x) H_{k+2i}(x) + k G_m(x) H_l(x),
\]

whence it follows that \( \Lambda u^*(x)|_{\partial S} = 0 \) for \( m \geq 2 \). Here, \( H_{k+2i}(x) = |x|^{2i} H_l(x) \) is denoted. In what follows, we assume that \( H_{k+2i+2}(x) = |x|^{2i} H_{k+2i+2}(x) \) for \( i \in \mathbb{N} \). It is easy to see that, similarly to what is carried out above, we have the equality

\[
\Lambda (G_{m-i}(x) H_{k+2i+2}(x)) = (2l + 2m - 2i) G_{m-i-1}(x) H_{k+2i+2}(x) + (k + 2i) G_{m-i}(x) H_{k+2i+2}(x),
\]

where \( m - i - 1 \geq 1 \). If we now continue applying the operator \( \Lambda \) to \( \Lambda u^*(x) \), then we can see that the polynomial \( \Lambda^s u^*(x) \) is written as a sum of polynomials of the form \( G_{m-i}(x) H_{k+2i+2}(x) \) with some coefficients, where \( i = 0, \ldots, s \). Furthermore, selecting from the polynomial \( \Lambda^s u^*(x) \) the term having the smallest subscript \( i \) of the polynomial \( G_i(x) \), and denoting the sum of the remaining terms with a higher subscript \( i \) of the polynomial \( G_i(x) \) as \( O_i(x) \), we can write the equality

\[
\Lambda^s u^*(x) = (2l + 2m) \ldots (2l + 2m - 2s + 2) G_{m-s}(x) H_{k+2s}(x) + O_{m-s}(x),
\]

where \( H_{k+2s}(x) = |x|^{2s} H_l(x) \) and \( m - 1 \geq s \). Because \( G_i(x)|_{\partial S} = 0 \) for \( i \geq 1 \), and hence \( O_{m-s}(x)|_{\partial S} = 0 \), for \( s \leq m - 1 \), then it follows from the obtained equality that \( \Lambda^s u^*(x)|_{\partial S} = 0 \), for \( s = 0, \ldots, m - 1 \). Thus, the boundary conditions (6) for the polynomial \( u^*(x) \), and hence for the polynomial \( u(x) \) from (28), are satisfied. The uniqueness of the solution of the Dirichlet problem [30] and the equality (27) imply the validity of (28). The theorem is proved.

6. Conclusions

The representation (13) of the Green’s function to the Dirichlet boundary value problem (5)–(6) obtained in this paper for the \( m \)-harmonic equation in the unit ball is uniform for different values of \( n \) and \( m \), which was not achieved earlier. The Green’s functions were found depending on the dimensionality \( n \) of the space and the order of the polyharmonicity \( m \) of the equation. Note that, in (13), the singularity of the Green’s function is contained only in the first term. Of course, finding the explicit value of the integral from (27) for a specific function \( f(x) \) is not an easy task. However, for the case of the simplest polynomial values of \( f(x) \), one can use the transparent Formula (28).
The solution of the complete Dirichlet problem in the case \( f(x) = 0 \) is given in ref. [9] and is not related to the obtained Green’s function. It might be worth thinking about the possibility of representing the solution of the complete Dirichlet problem in terms of the Green’s function, as it is known, for example, in the case of the Poisson equation \( m = 1 \) and in the case of the Riquier–Neumann [17] and Navier [15] problems.

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