Article
Banach Contraction Principle-Type Results for Some Enriched Mappings in Modular Function Spaces

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Abstract: The idea of enriched mappings in normed spaces is relatively a newer idea. In this paper, we initiate the study of enriched mappings in modular function spaces. We first introduce the concepts of enriched $\rho$-contractions and enriched $\rho$-Kannan mappings in modular function spaces. We then establish some Banach Contraction Principle type theorems for the existence of fixed points of such mappings in this setting. Our results for enriched $\rho$-contractions are generalizations of the corresponding results from Banach spaces to modular function spaces and those from contractions to enriched $\rho$-contractions. We make a first ever attempt to prove existence results for enriched $\rho$-Kannan mappings and deduce the result for $\rho$-Kannan mappings. Note that even $\rho$-Kannan mappings in modular function spaces have not been considered yet. We validate our main results by examples.

Keywords: fixed point; enriched $\rho$-contraction; enriched $\rho$-Kannan mapping; iterative process; modular function space

MSC: 46A80; 47H09; 47H10

1. Introduction

Fixed point theory has numerous applications in different fields like basic sciences, economics, engineering, game theory, computer science, image processing, and mathematics itself. That’s why a lot of research is being carried out in this flourishing area. Fixed point theory in modular function spaces provides modular equivalent concepts in norm and metric fixed point theory. Thus they enjoy a close relationship. Modular spaces are basically generalizations of Lebesgue and Orlicz spaces. Many a times, the conditions considered in this setting are naturally better and easier to verify than their corresponding metric counterparts, see for example, Khamsi and Kozlowski [1].

Banach Contraction Principle is deservedly one of the most celebrated theorems in Banach spaces and complete metric spaces. It is a fundamental existence theorem for fixed points. It finds its applications in nonlinear integral equations and differential equations among many others. A number of attempts have been made to find generalizations of Banach Contraction Principle in different directions. One such direction is to consider an abstract functional on a linear space. Such a functional is able to control the growth of the members of the underlying space. It is called modular and defines a modular space. The idea of modular spaces was coined by Nakano [2] while studying the theory of ordered spaces, see also [3]. Kozlowski [4–6] initiated the study of the class of modular function spaces which constitutes a subclass of modular spaces. Khamsi and Kozlowski have done a lot of important basic work, see for example [1,7]. In their book on modular function spaces, Khamsi and Kozlowski [1] write on their importance. Considering the Hammerstein nonlinear integral equation, which has an important role in the elasticity theory, the Hammerstein operator in this integral equation fails to operate in any of the
We can then use this approach to apply a pertinent modular function space fixed point theorem to find a solution of integral equations, for example, Urysohn integral equation. Modular function spaces are not only important in that they contain Banach spaces or F-spaces but also that they are armed with modular equivalents of the concepts of metric or normed spaces. More so, the tools like almost convergence everywhere and convergence in submeasure are available in these spaces.

Existence of fixed points in modular function spaces has been a problem of interest to many mathematicians. We refer the reader to Khamisi and Kozlowski [1] and the references already cited there. A work on quasicontractions in the frame of modular function spaces can be seen in [8]. For some other results in modular function spaces, the reader can see such as [9–14] and the references therein.

On the other hand, the idea of enriched mappings in a metric or a normed space is a relatively newer idea introduced by Berinde [15]. As much as we know, no work has so far been initiated for enriched mappings in modular function spaces. In this paper, we initiate research on this topic. We first introduce the concepts of enriched ρ-constractions and enriched ρ-Kannan mappings in modular function spaces. Basically we transform these ideas from normed spaces to modular function spaces. We then establish Banach Contraction Principle type theorems for the existence of fixed points of such mappings in this setting. Our results for enriched ρ-constractions are generalizations of the corresponding results of [7,15]. We make a first ever attempt to prove existence results even for ρ-Kannan mappings in modular function spaces though we establish results for enriched ρ-Kannan mappings and deduce the result for ρ-Kannan mappings. We also include examples to validate our results.

2. Prelude

This section, as the name indicates, discusses the tools necessary in proving our results in modular function spaces. The concepts have already been discussed in Khamisi and Kozlowski [1].

Suppose that Ω is a nonempty set and Σ is a nontrivial σ-algebra of subsets of Ω. We denote by ℙ a δ-ring of subsets of Ω with $E \cap A \in ℙ$ for any $E \in ℙ$ and $A \in Σ$. Take an increasing sequence of sets $K_n \in ℙ$ such that $Ω = \cup K_n$. Such a sequence does exist, for example, in a σ-finite measure space, we can take ℙ as the class of sets of finite measure. $1_A$ stands for the characteristic function of the set $A$ in Ω. Let $E$ be the linear space of all simple functions with supports from ℙ. The space of all extended measurable functions is symbolized as $M_\infty$. Mathematically,

$$M_\infty = \{ f : Ω \to [-\infty, \infty] : \exists \{g_n\} \subset E, |g_n| \leq |f| \land g_n(ω) \to f(ω) \forall ω \in Ω \}.$$  

**Definition 1.** Let $ρ : M_\infty \to [0, \infty]$ be a nontrivial, convex and even function. $ρ$ is termed as a regular convex function pseudomodular if

1. $ρ(0) = 0$;
2. $ρ$ is monotone, that is, $|f(ω)| \leq |g(ω)|$ for any $ω \in Ω$ implies $ρ(f) \leq ρ(g)$, where $f, g \in M_\infty$;
3. $ρ$ is orthogonally subadditive, that is, $ρ(f1_{A\cup B}) \leq ρ(f1_A) + ρ(f1_B)$ for any $A, B \in Σ$ such that $A \cap B \neq φ$, $f \in M_\infty$;
4. $ρ$ has Fatou property, that is, $|f_n(ω)| \uparrow |f(ω)|$ for all $ω \in Ω$ implies $ρ(f_n) \uparrow ρ(f)$, where $f \in M_\infty$;
5. $ρ$ is order continuous in $E$, that is, $g_n \in E$, and $|g_n(ω)| \downarrow 0$ implies $ρ(g_n) \downarrow 0$.  

$L^p$ spaces. And yet an Orlicz space can be found where the Hammerstein operator is well-defined and possesses properties allowing to apply some fixed point results for finding the solutions of the corresponding integral equation. They have further indicated that the equipment available in the modular function spaces can be used to construct a function modular and, in turn, we can find a space where this operator has desired properties. We can then use this approach to apply a pertinent modular function space fixed point theorem to find a solution of integral equations, for example, Urysohn integral equation. Modular function spaces are not only important in that they contain Banach spaces or F-spaces but also that they are armed with modular equivalents of the concepts of metric or normed spaces. More so, the tools like almost convergence everywhere and convergence in submeasure are available in these spaces.
A set \( A \in \Sigma \) is \( \rho \)-null if \( \rho(g1_A) = 0 \) for every \( g \in \mathcal{E} \). We say that the property \( p(\omega) \) holds \( \rho \)-almost everywhere, shortly \( \rho \)-a.e., provided that \( \{ \omega \in \Omega : p(\omega) \text{ does not hold} \} \) is \( \rho \)-null.

As per routine, a pair of measurable sets with symmetric difference \( \rho \)-null is identified as a pair of measurable functions differencing only on a \( \rho \)-null set. Let us consider \( f \in \mathcal{M}(\Omega, \Sigma, \mathcal{P}, \rho) \) an equivalence class of functions equal \( \rho \)-a.e. and not merely a function, and then define

\[
\mathcal{M}(\Omega, \Sigma, \mathcal{P}, \rho) = \{ f \in \mathcal{M}_\kappa : |f(\omega)| < \infty \rho \text{-a.e.} \}.
\]

We write just \( \mathcal{M} \) instead of \( \mathcal{M}(\Omega, \Sigma, \mathcal{P}, \rho) \) when we are clear about the notation. The following properties of \( \rho : \mathcal{M} \rightarrow [0, \infty] \) can be quickly verified.

**Properties**

1. \( \rho(f) = 0 \) iff \( f = 0 \).
2. \( \rho(\alpha f) = \rho(f) \) for every scalar \( \alpha \) with \( |\alpha| = 1 \) and \( f \in \mathcal{M} \).
3. \( \rho(\alpha f + \beta g) \leq \rho(f) + \rho(g) \) if \( \alpha + \beta = 1 \), \( \alpha, \beta \geq 0 \), and \( f, g \in \mathcal{M} \).

If on top, the following property also holds, we say that \( \rho \) is convex modular.

3’. \( \rho(\alpha f + \beta g) \leq \alpha \rho(f) + \beta \rho(g) \) if \( \alpha + \beta = 1 \), \( \alpha, \beta \geq 0 \), and \( f, g \in \mathcal{M} \).

**Definition 2.** A regular function pseudomodular \( \rho \) is called a regular convex function modular if \( \rho(f) = 0 \) implies \( f = 0 \) \( \rho \)-a.e.

We denote by \( \mathbb{R} \) the class of all nonzero regular convex function modulars defined on \( \Omega \). The modular function space \( L_\rho \) is defined as follows:

\[
L_\rho = \{ f \in \mathcal{M} : \rho(\lambda f) \to 0 \text{ as } \lambda \to 0 \}
\]

where \( \rho \) is the convex function modular. The modular \( \rho \) cannot be treated as a norm or a distance because it does not satisfy subadditivity in general. Even so, we can define an \( F \)-norm on \( L_\rho \) as under:

\[
\|f\|_\rho = \inf \{ \alpha > 0 : \rho \left( \frac{f}{\alpha} \right) \leq 1 \}.
\]

If \( \rho \) is convex modular, then the following also defines a norm on \( L_\rho \) and is termed as the Luxemburg norm:

\[
\|f\|_\rho = \inf \{ \alpha > 0 : \rho \left( \frac{f}{\alpha} \right) \leq 1 \}
\]

We next define the following:

\[
L^0_\rho = \{ f \in L_\rho : \rho(f, \cdot) \text{ is order continuous} \}.
\]

\[
E_\rho = \{ f \in L_\rho : \lambda f \in L^0_\rho \text{ for every } \lambda > 0 \}.
\]

Note that \( E_\rho \) is a linear space.

**Definition 3.** Let \( \{ D_k \} \) be a sequence decreasing to \( \phi \) such that \( \sup_{n \geq 1} \rho(f_n, D_k) \to 0 \) as \( k \to \infty \). Then \( \rho \in \mathbb{R} \) satisfies the \( \Delta_2 \)-condition provided \( \sup_{n \geq 1} \rho(2f_n, D_k) \to 0 \) as \( k \to \infty \).

In case, \( \rho \) is convex and satisfies the \( \Delta_2 \)-condition, then \( L_\rho = E_\rho \). Further, \( F \)-norm convergence is equivalent to modular convergence if and only if \( \rho \) satisfies the \( \Delta_2 \)-condition.

**Condition 1.** Let \( f_n \) and \( f \) belong to \( L^0_\rho \) and \( H \in \mathcal{P} \). Let \( \{ f_n \} \) converge uniformly to \( f \) on \( H \). Then the function modular \( \rho \) satisfies the \( * \)-condition if

\[
\rho(f, H) \leq \lim \sup \rho(f_n, H).
\]
Definition 4. Let $\rho \in \mathbb{R}$. The sequence $\{f_n\} \subset L_\rho$:

- $\rho$-converges to $f \in L_\rho$ if $\rho(f_n - f) \to 0$ as $n \to \infty$.
- is $\rho$-Cauchy, if $\rho(f_n - f_m) \to 0$ as $n$ and $m \to \infty$.

We know that $\rho$ does not satisfy the triangle inequality therefore $\rho$-convergence does not imply $\rho$-Cauchy, in general. However, it is true if and only if $\rho$ satisfies the $\Delta_2$-condition.

Definition 5. Let $\rho \in \mathbb{R}$.

- If the $\rho$-limit (a.e. limit) of a $\rho$-convergent sequence (a.e. convergent) of $D$ always belongs to $D$, then $D$ is called $\rho$-closed (a.e. closed).
- If every sequence in $D$ has a $\rho$-convergent subsequence in $D$, then $D$ is called $\rho$-compact.
- A similar definition for $\rho$-a.e. compact may be obtained from the above.
- If $\text{diam}_\rho(D) = \sup \{\rho(f - g) : f, g \in D\} < \infty$, then $D$ is called $\rho$-bounded.

3. Banach Contraction Principle Type Results for Certain Enriched Mappings

In this section, we initiate the study of enriched mappings in modular function spaces. In particular, we prove some Banach Contraction Principle type results in modular function spaces for some enriched mappings. We include examples to validate our results. For a mapping $T : L_\rho \to L_\rho$, we say that $f \in L_\rho$ is a fixed point of $T$ if $f = T f$.

$F_\rho(T)$ denotes the set of all fixed points of $T$.

Following is the definition of the well-known $\rho$-contraction generalized from contraction in metric or normed spaces.

Definition 6. Let $D \subset L_\rho$. A mapping $T : D \to D$ is called $\rho$-contraction if there exists $k \in [0, 1)$ such that

$$\rho(Tf - Tg) \leq k\rho(f - g)$$

for all $f, g \in D$.

On similar lines, we can define Kannan mappings [16,17] in modular function spaces as follows.

Definition 7. Let $D \subset L_\rho$. A mapping $T : D \to D$ is called $\rho$-Kannan mapping if there exists $c \in (0, \frac{1}{2})$ such that

$$\rho(Tf - Tg) \leq c[\rho(f - Tg) + \rho(g - Tf)]$$

for all $f, g \in D$.

We now reformulate the above definitions for the so-called enriched mappings. The original idea in metric spaces is due to Berinde and Pacurar [15].

We first define and consider enriched $\rho$-contractions in modular function spaces.

Definition 8. Let $D \subset L_\rho$. We say that a mapping $T : D \to D$ is an enriched $\rho$-contraction if there exist $a \in (0, 1]$ and $\lambda \in [0, \frac{1}{2})$ such that

$$\rho((1-a)(f - g) + a(Tf - Tg)) \leq \lambda a \rho(f - g)$$

for all $f, g \in D$. (1)

We consider the iteration process as defined below.

$$f_0 \in D,$$

$$f_{n+1} = (1-a)f_n + aTf_n, \ n \in \mathbb{N} \cup \{0\}$$

for any $a \in (0, 1]$.

Before we proceed further, we prove the following very important proposition.
Proposition 1. Let \( \{f_n\} \) be as in (2) for enriched \( \rho \)-contractions \( T \) defined by (1). Put
\[
T_a f = (1 - a) f + aT f. \tag{3}
\]
Then

(i) \( T_a \) is a \( \rho \)-contraction and (2) can be written as
\[
f_{n+1} = T_a f_n, \ n \in \mathbb{N} \cup \{0\}.
\]

(ii) The set of fixed points of \( T \) and \( T_a \) are same. That is, \( F(T) = F(T_a) \).

Proof. (i) Putting \( a = 1 \), we have by (1), \( \rho(Tf - Tg) \leq \lambda \rho(f - g) \) for all \( f, g \in D \) where \( \lambda \in (0, 1) \). It means that \( T \) is a \( \rho \)-contraction. Moreover, by (3),
\[
\rho(T_a f - T_a g) = \rho(T f - T g) \leq \lambda \rho(f - g) \text{ for all } f, g \in D.
\]

Hence \( T_a \) is a \( \rho \)-contraction.

When \( a \neq 1 \), put \( c = a \lambda \). Then by (3), we can re-write (1) as
\[
\rho(T_a f - T_a g) \leq c(\rho(f - g) \text{ for all } f, g \in D
\]
where \( c \in [0, 1) \) because \( \lambda \in [0, \frac{1}{2}) \) and \( a \in (0, 1) \). Hence \( T_a \) is a \( \rho \)-contraction.

Next by (3), (2) becomes
\[
f_{n+1} = T_a f_n, \ n \in \mathbb{N} \cup \{0\}
\]
or equivalently
\[
f_{n+1} = T_a^n f_0.
\]

(ii) Let \( f \in F(T) \). Then \( Tf = f \) implies that \( T_a f = (1 - a) f + af = f \). Conversely, if \( T_a f = f \) then \( (1 - a) f + af = f \) implies \( aTf = af \) and hence \( Tf = f \) because \( a \neq 0 \). \( \square \)

We now prove the following Banach Contraction Principle type result in modular function spaces.

Theorem 1. Let \( \rho \in \mathbb{R} \). Let \( D \) be \( \rho \)-closed subset of \( L_\rho \). Let \( T : D \to D \) be an enriched \( \rho \)-contraction as defined by (1) and \( T_a \) as defined by (3).

(i) Suppose that there exists \( f_0 \in D \) such that \( \sup \rho(2T_a^n f_0) < \infty \). Then \( T \) has a fixed point in \( D \).

(ii) Additionally if \( \rho(g - f) < \infty \), then such a fixed point is unique.

Proof. (i) First consider the case \( a \neq 1 \). In this case, \( \rho(T_a f - T_a g) \leq k(f - g) \) for all \( f, g \in D \) and \( k = a \lambda \in (0, 1) \) as shown before. Let \( f_0 \in D \) as in (2) such that \( s = \sup \rho(2T_a^n f_0) < \infty \).

For all \( n, m \in \mathbb{N} \),
\[
\rho(T_a^{n+m} f_0 - T_a^n f_0) \leq k\rho(T_a^{n+m-1} f_0 - T_a^{n-1} f_0) \leq k^2 \rho(T_a^{n+m-2} f_0 - T_a^{n-2} f_0) \\
\vdots \\
\leq k^n \rho(T_a^n f_0 - f_0) = k^n [\rho(\frac{1}{2} (2T_a^n f_0) + \frac{1}{2} (-2f_0))] \leq k^n [\rho(2T_a^n f_0) + \rho((-1)2f_0)] = k^n [\rho(2T_a^n f_0) + \rho(2f_0)] \leq k^n (s + s).
\]
Let $$T : D \to D$$ be an enriched $$\rho$$-contraction as defined by (1) and $$T_a$$ as defined by (3). Then $$T$$ has a unique fixed point in $$D$$.

We can also prove the following result on the lines similar to our Theorem 1. This constitutes a generalization of Theorem 2.9 of Khamsi et al. [7].

Theorem 3. Let $$\rho$$ satisfy (*) condition and $$D$$ be $$\rho$$-a.e. compact, $$\rho$$-bounded subset of $$L_\rho$$. Assume that $$g - f \in D$$ provided $$g, f \in D.$$ Let $$T : D \to D$$ be an enriched $$\rho$$-contraction as defined by (1) and $$T_a$$ as defined by (3). Then $$T$$ has a unique fixed point in $$D$$.

Proof. The case $$a \neq 1$$ can be proved using $$T_a$$ instead of $$T$$ in the Theorem 2.9 of Khamsi et al. [7]. We leave a smooth sailing for the reader. The case $$a = 1$$ is exactly their Theorem 2.9. \)

Now we give an example to validate our Theorem 1 as follows.

Example 1. Consider the modular space $$L_\rho = \mathbb{R}$$ equipped with the norm given by $$\rho(f) = |f|$$ and $$D = \{f \in L_\rho : 0 \leq f \leq 1\}.$$ Obviously $$D$$ is a nonempty $$\rho$$-closed subset of $$L_\rho.$$ Define $$T : D \to D$$ by

$$Tf = 1 - f \text{ for all } f \in D.$$
When \( a = 1 \), \( T \) is clearly enriched \( \rho \)-contraction.
When \( a \neq 1 \), we have
\[
\rho((1 - a)(f - g) + a(Tf - Tg)) \leq \lambda a \rho(f - g)
\]
iff
\[
\rho((1 - a)(f - g) + a(-f + g)) \leq \lambda a \rho(f - g)
\]
iff
\[
|1 - 2a| |f - g| \leq \lambda a |f - g|
\]
which is true for all \( f, g \in D \) if we choose \( \frac{2a - 1}{a} = \lambda \in [0, \frac{1}{a}) \) for any \( a \in \left[\frac{1}{2}, 1\right) \). Hence \( T \) is enriched \( \rho \)-contraction. See also [15].

Now take \( f_0 = 1 \in D \) and define
\[
T_a f_0 = (1 - a)f_0 + aTf.
\]

Since we have shown that (1) is true for any \( a \in \left[\frac{1}{2}, 1\right) \), we may choose \( a = \frac{1}{2} \) without any loss of generality. Thus
\[
T_a f_0 = (1 - a)f_0 + aTf_0
= (1 - a)f_0 + a(1 - f_0)
= 1 - a
= \frac{1}{2}
\]

Next,
\[
T_a^2 f_0 = T_a(T_a f_0)
= T_a(\frac{1}{2})
= (1 - \frac{1}{2}) \frac{1}{2} + \frac{1}{2} T(\frac{1}{2})
= (1 - \frac{1}{2}) \frac{1}{2} + \frac{1}{2} (1 - \frac{1}{2})
= \frac{1}{2} \frac{1}{2} + \frac{1}{2} \frac{1}{2}
= \frac{1}{2}
\]

Continuing in this way, we get \( T_a^n f_0 = \frac{1}{2} \) for all \( n = 1, 2, 3, ... \). Thus \( \rho(2T_a^n f_0) = \rho(1) = 1 \) for all \( n = 1, 2, 3, ... \) Hence \( \sup(\rho(2T_a^n f_0)) = 1 \) \( < \infty \) for \( f_0 = 1 \in D \) and the condition in (i) of Theorem 1 is satisfied.

The condition \( \rho(g - f) < \infty \) of Theorem 1 also holds because
\[
\rho(g - f) = |g - f|
\]
\[
\leq 1 \text{ for all } f, g \in D
\]
\[
< \infty.
\]

Thus all the conditions of Theorem 1 are satisfied so \( T \) must have a unique fixed point and it indeed does because the only fixed point of \( T \) is \( \frac{1}{2} \).

Next, we turn our attention to Kannan mappings [16,17]. The idea was extended to define enriched Kannan mappings on normed spaces [18]. To the best of our knowledge, Kannan mappings have not so far been considered in modular function spaces. We initiate this study by extending the idea of Kannan mappings in two ways. We first define \( \rho \)-Kannan mappings and then generalize them further to define enriched \( \rho \)-Kannan mappings in modular function spaces. So let us first define \( \rho \)-Kannan mappings as follows.
**Definition 9.** Let $D \subset L_{\rho}$. We say that a mapping $T : D \rightarrow D$ is $\rho$-Kannan if there exists a $c \in [0, \frac{1}{2})$ such that
\[
\rho(Tf - Tg) \leq c[\rho(f - Tf) + \rho(g - Tg)] \text{ for all } f, g \in D.
\] (5)

Now a further extended concept of enriched $\rho$-Kannan mappings is as follows.

**Definition 10.** Let $D \subset L_{\rho}$. We call a mapping $T : D \rightarrow D$ enriched $\rho$-Kannan if there exist $a \in (0, 1]$ and $\lambda \in [0, \frac{1}{2})$ such that
\[
\rho((1 - a)(f - g) + a(Tf - Tg)) \leq \lambda[\rho(a(f - Tf)) + \rho(a(g - Tg))] \quad (6)
\]
for all $f, g \in D$.

We first give our results on enriched $\rho$-Kannan mappings.

**Proposition 2.** Let $\{f_n\}$ be defined by (2) for enriched $\rho$-Kannan mappings $T$ as defined in (6) and $T_a$ as in (3). Then $T_a$ is a $\rho$-Kannan mapping (5) and (2) can be written as
\[
f_{n+1} = T_a f_n, \quad n \in \mathbb{N} \cup \{0\}.
\]

**Proof.** Putting $a = 1$ in (6), we have $\rho(Tf - Tg) \leq \lambda[\rho(f - Tf) + \rho(g - Tg)]$ for all $f, g \in D$. It means that $T$ is a $\rho$-Kannan mapping. Moreover, by (3),
\[
\rho(T_a f - T_a g) = \rho(Tf - Tg) \leq \lambda[\rho(f - Tf) + \rho(g - Tg)] \text{ for all } f, g \in D.
\]

Hence $T_a$ is a $\rho$-Kannan mapping.

When $a \neq 1$, the term in the brackets on the right hand side of (6) can be written $\rho(af - aTf) + \rho(ag - aTg) = \rho(f - (1 - a)f - aTf) + \rho(g - (1 - a)g - aTg)$. Then by (3), we can re-write (6) as
\[
\rho(T_a f - T_a g) \leq \lambda[\rho(f - T_a f) + \rho(g - T_a g)] \text{ for all } f, g \in D.
\]

Hence $T_a$ is a $\rho$-Kannan mapping.

Next by (3), (2) becomes
\[
f_{n+1} = T_a f_n, \quad n \in \mathbb{N} \cup \{0\}
\]
or equivalently
\[
f_{n+1} = T_a^n f_0.
\]

The following is our second Banach Contraction Principle type result involving enriched $\rho$-Kannan mappings.

**Theorem 4.** Let $\rho \in \mathbb{R}$. Let $D$ be $\rho$-closed subset of $L_{\rho}$. Let $T : D \rightarrow D$ be an enriched $\rho$-Kannan mapping as defined by (6) and $T_a$ as defined by (3). Suppose that there exists $f_0 \in D$ such that $\sup \rho(2T_a^n(f_0)) < \infty$. Then $T$ has a unique fixed point in $D$.

**Proof.** First consider the case $a \neq 1$. In this case, $\rho(T_a f - T_a g) \leq \lambda[\rho(f - T_a f) + \rho(g - T_a g)]$ for all $f, g \in D$ where $\lambda \in (0, \frac{1}{2})$ as shown before. Let $f_0 \in D$ as in (2). Put $s = \sup \rho(2T_a^n f_0) < \infty$. We follow the technique of our Theorem 1.
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For all \( n, m \in \mathbb{N} \),
\[
\rho(T_a^{n+m} f_0 - T_a^n f_0) = \rho(T_a T_a^{n+m-1} f_0 - T_a T_a^{n-1} f_0) \\
\leq \lambda \rho(T_a^{n+m-1} f_0 - T_a^{n+m-1} f_0) + \rho(T_a^{n-1} f_0 - T_a^n f_0) \\
\leq \lambda^2 \rho(T_a^{n+m-2} f_0 - T_a^{n+m-2} f_0) + \rho(T_a^{n-1} f_0 - T_a^n f_0) \\
\leq \lambda^3 \rho(T_a^{n+m-3} f_0 - T_a^{n+m-3} f_0) + \rho(T_a^{n-2} f_0 - T_a^n f_0) \\
\leq \lambda^n \rho(T_a^{n-1} f_0 - T_a^n f_0) + \rho(T_a^n f_0 - T_a^n f_0) \\
\leq \lambda^n \rho(T_a^{n-1} f_0 - T_a^n f_0) + \rho(T_a^n f_0 - T_a^n f_0).
\]

That is, \( \rho(T_a^{n+m} f_0 - T_a^n f_0) \leq (2\lambda)^n s. \) Since \( 2\lambda \in (0, 1) \) and \( s < \infty, \) \( \{T_a f_0\} \) is a Cauchy sequence in \( L_\rho. \) Completeness of \( L_\rho \) implies existence of an \( f \in L_\rho \) such that \( \lim_{n \to \infty} \rho(f - T_a^n f_0) = 0. \) For simplicity, we write
\[
\lim_{n \to \infty} \rho(f - f_{n+1}) = 0.
\] (7)

Since \( D \) is a \( \rho \)-closed subset of \( L_\rho, \) therefore \( f \in D. \) We now prove that this \( f \) is actually the fixed point of \( T \) as follows.
\[
\rho(f - T_a f) = \lim_{n \to \infty} \rho(f_{n+1} - T_a f) \\
= \lim_{n \to \infty} \rho(T_a f_n - T_a f) \\
\leq \lim_{n \to \infty} \lambda [\rho(f_n - T_a f_n) + \rho(f - T_a f)].
\]

This implies \( (1 - \lambda) \rho(f - T_a f) \leq \lim_{n \to \infty} \lambda \rho(f_n - T_a f_n) = 0 \) in view of (7). Thus \( f = T_a f \) and, in turn, \( f = T_f. \)

In case \( a = 1 \) we have \( T_a = T (= T_1) \) and the result follows from the above.

For uniqueness, suppose on contrary that there exist \( g, f \in D \) with \( T_a g = g \) and \( T_a f = f. \) Then \( \rho(g - f) = \rho(T_a g - T_a f) \leq \lambda [\rho(g - T_a g) + \rho(f - T_a f)] = 0 \) implies that \( g = f. \) \( \square \)

**Remark 3.** Note that for the uniqueness of the fixed point for enriched \( \rho \)-Kannan mappings, we do not need the condition \( \rho(g - f) < \infty \) on contrary to enriched \( \rho \)-contractions. Compare Theorem 1 with the above Theorem 4.

As mentioned earlier that as far as we know, there is no result on the existence of fixed points even for (non-enriched) \( \rho \)-Kannan mappings. Thus, although we obtain the following result from our above Theorem 4 yet it is new in itself.

**Theorem 5.** Let \( \rho \in \mathbb{R}. \) Let \( D \) be \( \rho \)-closed subset of \( L_\rho. \) Let \( T : D \to D \) be an \( \rho \)-Kannan mapping as defined by (6) and \( T_a \) as defined by (3). Suppose that there exists \( f_0 \in D \) such that \( \sup \rho(2T_a^n(f_0)) < \infty. \) Then \( T \) has a unique fixed point in \( D. \)
**Proof.** Consider the case \( a = 1 \) in the proof of the above theorem as already mentioned. 

Finally we give an example to validate our Theorem 4.

**Example 2.** Consider the modular space \( L_p = \mathbb{R} \) equipped with the norm given by \( \rho(f) = |f| \) and \( D = \{ f \in L_p : 0 \leq f \leq 2 \} \). Obviously \( D \) is a nonempty \( \rho \)-closed subset of \( L_p \). Define \( T : D \to D \) by

\[
Tf = 2 - f \quad \text{for all } f \in D.
\]

\( T \) is not \( \rho \)-Kannan because

\[
\rho(Tf - Tg) = |2 - f - 2 + g| = |f - g|
\]

and

\[
c[\rho(f - Tf) + \rho(g - Tg)] = c[|f - 2 + f + g - 2 + g|] = 2c|f - g|
\]

But \( |f - g| \leq 2c|f - g| \) iff \( c \geq \frac{1}{2} \) which contradicts the fact that \( c \in (0, \frac{1}{2}) \).

But \( T \) is enriched \( \rho \)-Kannan because (6) is true as follows. To prove that \( T \) is enriched \( \rho \)-Kannan, we need to find \( \lambda \in [0, \frac{1}{2}) \) and \( a \in (0, 1) \) such that (6) remains true. Now

\[
\rho(\lambda(1-a)(f-g) + a(Tf - Tg)) \leq \lambda \rho(a(f- Tf)) + \rho(a(g - Tg))
\]

iff \( |(1-a)(f-g) + a(Tf - Tg)| \leq \lambda |a(f - Tf)| + |a(g - Tg)|\)

iff \( |(1-a)(f-g) + a(2f - 2 + g)| \leq \lambda |a(2f - 2 + g)| + |a(g - 2 + g)|\)

iff \( |(1 - 2a)(f-g)| \leq a\lambda |2f - 2| + |2g - g|\)

Choose \( \lambda = \frac{2-a-1}{2a} \). Then \( \lambda \in \left[0, \frac{1}{2}\right) \) for all \( a \in \left[\frac{1}{2}, 1\right) \) and the above inequality is true for such choices. The rest of the steps follow the Example 1. Hence our Theorem 4 is validated. The unique fixed point in this case is \( T(1) = 1 \).

4. Conclusions

We initiated the study of enriched mappings in modular function spaces by first introducing the concepts of enriched \( \rho \)-contractions and enriched \( \rho \)-Kannan mappings in these spaces. We established results for the existence of fixed points of such mappings in this setting. We validated our main results by examples. Main feature of our results is that they are first ever attempt for not only enriched but also non-enriched mappings in modular function spaces. Our results also constitute generalizations of the corresponding results of [15] from Banach spaces to modular function spaces and those of [7] from contractions to enriched contractions.

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