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Existence, Approximation and Stability of Fixed Point for Cirić Contraction in Convex b-Metric Spaces

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Abstract: We establish a new fixed point theorem in the setting of convex b-metric spaces that ensures the existence of fixed point for Cirić contraction with the assumption $k < \frac{1}{s^2}$. Also, the fixed point is approximated by Krasnoselskij iterative procedure. Moreover, we discuss the stability of fixed point for the aforesaid contraction. As a consequence, we develop a common fixed point and coincidence point result. Finally, we provide a number of examples to illustrate the findings presented here and incorporate these findings to solve an initial value problem.

Keywords: b-metric spaces; convex metric spaces; approximation; stability; cirić contraction

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Citation: Gazal; Rathee, S.; Mahima; Kadyan, A.; Minakshi; Kumar, A.; Existence, Approximation and Stability of Fixed Point for Cirić Contraction in Convex b-Metric Spaces. *Axioms* **2023**, *12*, 646. https://doi.org/10.3390/axioms12070646

Academic Editor: Hsien-Chung Wu

Received: 31 May 2023
Revised: 22 June 2023
Accepted: 25 June 2023
Published: 28 June 2023

1. Introduction and Preliminaries

In the field of fixed point theory, the most useful and widely applied fixed point theorem was proved by Stefan Banach [1] in 1922, where he ensured the existence of fixed point for a contraction defined on a complete metric space. In the literature, this result is also known as the Banach contraction principle. Moreover, as this result played a pivotal role in solving various real-life problems of nonlinear analysis, it has been extended by the researchers either by weakening the contractive condition or by enlarging the structure of the ambient space. In 1974, pursuing the former course of action, Cirić [2] weakened the contractive condition of Banach [1] by defining the notion of quasi contraction (also called Cirić contraction) and succeeded in obtaining a generalization of not only the Banach contraction principle but also the Kannan fixed point theorem and Chatterjea fixed point theorem existing in the literature. On the other hand, in lieu of extending this contraction principle, the notion of b-metric spaces was introduced by Bakhtin [3] in 1989. One can refer to [4–8] and references therein to learn more about this space. In the recent years, Chen et al. [9] introduced the notion of convex b-metric spaces by utilizing the concept of convex structure of Takahashi [10] in b- metric space which is given as under:

Definition 1 ([9]). *Let $\Xi \neq \phi$ and $s \geq 1$ (a real number). A mapping $\rho_b : \Xi \times \Xi \to [0, \infty)$ is said to be a b-metric if the following holds for every $\sigma, \mu, \xi \in \Xi$*

1. $\rho_b(\sigma, \mu) = 0$ *iff* $\sigma = \mu$
2. $\rho_b(\sigma, \mu) = \rho_b(\mu, \sigma)$
3. $\rho_b(\sigma, \mu) \leq s[\rho_b(\sigma, \xi) + \rho_b(\xi, \mu)]$

Further, a function $\Delta : \Xi \times \Xi \times I \to \Xi$(where I=[0,1]) is said to have convex structure on Ξ if

$$\rho_b(\xi, \Delta(\sigma, \mu; \alpha) \leq \alpha\rho_b(\xi, \sigma) + (1-\alpha)\rho_b(\xi, \mu) \text{ for each } \xi, \sigma, \mu \in \Xi.$$

The triplet (Ξ, ρ_b, Δ) is called a convex b-metric space.

Additionally, they extended the Mann's iterative algorithm in convex b-metric space and employed it to establish the Banach contraction principle in the framework of this newly introduced space. In 2022, Rathee et al. [11] extended this result by establishing a fixed point theorem for Cirić contraction which is stated as under:

Theorem 1 ([11])**.** *Suppose $\Upsilon : (\Xi, \rho_b, \Delta) \to (\Xi, \rho_b, \Delta)$ is a quasi-contraction, that is, Υ satisfies*

$$\rho_b(\Upsilon\sigma, \Upsilon\mu) \leq k \max\{\rho_b(\sigma, \mu), \rho_b(\sigma, \Upsilon\sigma), \rho_b(\mu, \Upsilon\mu), \rho_b(\sigma, \Upsilon\mu), \rho_b(\mu, \Upsilon\sigma)\}, \tag{1}$$

for all $\sigma, \mu \in \Xi$ and some $k \in (0,1)$, where (Ξ, ρ_b, Δ) is a complete convex b-metric space with $s > 1$. Let $\sigma_n = \Delta(\sigma_{n-1}, \Upsilon\sigma_{n-1}; \alpha_{n-1})$ be a sequence defined by choosing an initial point $\sigma_0 \in \Xi$ with the property $\rho_b(\sigma_0, \Upsilon\sigma_0) < \infty$, where $0 \leq \alpha_{n-1} < 1$ for each $n \in \mathbb{N}$. If $k < \min\left\{\frac{1}{s^2(s+1)}, \frac{1}{s^4}\right\}$ and $0 \leq \alpha_{n-1} < \min\left\{\frac{1}{s^2} - (s+1)k, \frac{\frac{1}{s^4} - k}{\frac{1}{s^2} - k}\right\}$ for each $n \in \mathbb{N}$, then Υ has a fixed point in Ξ that is unique.

The main aim of this work is to improve the above theorem by stretching the domain of constant k from $\left[0, \frac{1}{s^4}\right)$ to $\left[0, \frac{1}{s^2}\right)$ by motivating with the idea of Djafari-Rouhani and Moradi [12]. Furthermore, the fixed point is approximated by means of Krasnoselskij iteration, and then, we discuss the stability of the obtained fixed point. Moreover, some examples are presented to clarify the universality of the proven results over Theorem 1 as well as over the similar results existing in the literature.The obtained results can be utilized in various branch of mathematics, such as the theory of differential equation and integral equation, in numerical methods and in the theory of fractal. For example, we applied the main result of this paper to the initial value problem (27) and ensured that there is a unique solution to the given initial value problem. Hence, it can be said that these results can be helpful for solving real-life problems of nonlinear analysis, which can be formulated in any of the above-mentioned classes. Besides, as a consequence of main result, we obtained some coincidence and common fixed point theorem and hence the obtained results play a crucial role in the further development of fixed point theory.

2. Main Results

We start this section with the following lemma that is required in the sequel to assure the existence, approximation and stability of fixed point.

Lemma 1. *Let $\Upsilon : \Xi \to \Xi$ be a self mapping defined on (Ξ, ρ_b), a complete b-metric space with parameter $s \geq 1$, such that for all $\sigma, \mu \in \Xi$ and some $\kappa \in [0,1)$, it satisfies*

$$\rho_b(\Upsilon\sigma, \Upsilon\mu) \leq \kappa \max\{\rho_b(\sigma, \mu), \rho_b(\sigma, \Upsilon\sigma), \rho_b(\mu, \Upsilon\mu), \rho_b(\sigma, \Upsilon\mu), \rho_b(\mu, \Upsilon\sigma)\}. \tag{2}$$

If $\kappa < \frac{1}{s^2}$, then the following statements are equivalent:

1. *Υ has a unique fixed point.*
2. *Υ has approximate fixed point property, i.e., $\inf\{\rho_b(\sigma, \Upsilon\sigma);\ \sigma \in \Xi\} = 0$.*

Proof. (1) $\Longrightarrow$ (2)

Firstly, presume that a unique fixed point of Υ, say σ, exists, i.e., $\Upsilon\sigma = \sigma$. Then,

$$\begin{aligned} \rho_b(\sigma, \Upsilon\sigma) &= 0, \\ \Longrightarrow \quad \inf\{\rho_b(\sigma, \Upsilon\sigma);\ \sigma \in \Xi\} &= 0. \end{aligned}$$

Thus, Υ exhibits approximate fixed point property.

(2) $\Longrightarrow$ (1)

Conversely, presume that Υ exhibits approximate fixed point property, i.e., $\inf\{\rho_b(\sigma, \Upsilon\sigma);\ \sigma \in \Xi\} = 0$. This indicates the existence of $\langle\sigma_n\rangle_{n\in\mathbb{N}}$, a sequence in Ξ satisfying $\lim_{n\to\infty} \rho_b(\sigma_n, \Upsilon\sigma_n) = 0$ and by using (2) and triangle inequality for all $m, n \in \Xi$, we have,

$$\begin{aligned}
\rho_b(\Upsilon\sigma_n, \Upsilon\sigma_m) &\leq \kappa \max\{\rho_b(\sigma_n, \sigma_m), \rho_b(\sigma_n, \Upsilon\sigma_n), \rho_b(\sigma_m, \Upsilon\sigma_m), \\
&\quad \rho_b(\sigma_n, \Upsilon\sigma_m), \rho_b(\sigma_m, \Upsilon\sigma_n)\} \\
&\leq \kappa \max\{s\rho_b(\sigma_n, \Upsilon\sigma_n) + s\rho_b(\Upsilon\sigma_n, \sigma_m), \rho_b(\sigma_n, \Upsilon\sigma_n), \\
&\quad \rho_b(\sigma_m, \Upsilon\sigma_m), s\rho_b(\sigma_n, \Upsilon\sigma_n) + s\rho_b(\Upsilon\sigma_n, \Upsilon\sigma_m), \\
&\quad s\rho_b(\sigma_m, \Upsilon\sigma_m) + s\rho_b(\Upsilon\sigma_m, \Upsilon\sigma_n)\} \\
&\leq \kappa \max\{s\rho_b(\sigma_n, \Upsilon\sigma_n) + s^2\rho_b(\Upsilon\sigma_n, \Upsilon\sigma_m) \\
&\quad + s^2\rho_b(\Upsilon\sigma_m, \sigma_m), s\rho_b(\sigma_n, \Upsilon\sigma_n) + s\rho_b(\Upsilon\sigma_n, \Upsilon\sigma_m), \\
&\quad s\rho_b(\sigma_m, \Upsilon\sigma_m) + s\rho_b(\Upsilon\sigma_m, \Upsilon\sigma_n)\} \\
&= \kappa[s\rho_b(\sigma_n, \Upsilon\sigma_n) + s^2\rho_b(\Upsilon\sigma_n, \Upsilon\sigma_m) + s^2\rho_b(\Upsilon\sigma_m, \sigma_m)] \\
\implies (1 - \kappa s^2)\rho_b(\Upsilon\sigma_n, \Upsilon\sigma_m) &\leq \kappa[s\rho_b(\sigma_n, \Upsilon\sigma_n) + s^2\rho_b(\Upsilon\sigma_m, \sigma_m)].
\end{aligned}$$

Now, since $\lim_{n\to\infty} \rho_b(\sigma_n, \Upsilon\sigma_n) = 0$ and $\kappa < \frac{1}{s^2}$, we are left with a Cauchy sequence $\langle\Upsilon\sigma_n\rangle_{n\in\mathbb{N}}$ as $n \to \infty$. Also, the space (Ξ, ρ_b), being complete, proposes the existence of an element $\sigma \in \Xi$ satisfying $\lim_{n\to\infty} \Upsilon\sigma_n = \sigma$.
Again using triangle inequality

$$\rho_b(\sigma_n, \sigma) \leq s[\rho_b(\sigma_n, \Upsilon\sigma_n) + \rho_b(\Upsilon\sigma_n, \sigma)].$$

Taking limit as $n \to \infty$, we get

$$\lim_{n\to\infty} \rho_b(\sigma_n, \sigma) = 0 \implies \sigma_n \to \sigma.$$

Also, consider

$$\begin{aligned}
\rho_b(\Upsilon\sigma_n, \Upsilon\sigma) &\leq \kappa \max\{\rho_b(\sigma_n, \sigma), \rho_b(\sigma_n, \Upsilon\sigma_n), \rho_b(\sigma, \Upsilon\sigma), \rho_b(\sigma_n, \Upsilon\sigma), \\
&\quad \rho_b(\sigma, \Upsilon\sigma_n)\}.
\end{aligned}$$

Now taking limit as $n \to \infty$, we get

$$\begin{aligned}
\frac{1}{s}\rho_b(\sigma, \Upsilon\sigma) &\leq \kappa \max\{0, 0, \rho_b(\sigma, \Upsilon\sigma), s\rho_b(\sigma, \Upsilon\sigma), 0\} \\
&= \kappa s\rho_b(\sigma, \Upsilon\sigma) \\
\implies \rho_b(\sigma, \Upsilon\sigma) &\leq \kappa s^2\rho_b(\sigma, \Upsilon\sigma) \\
(1 - \kappa s^2)\rho_b(\sigma, \Upsilon\sigma) &\leq 0.
\end{aligned}$$

Since $\kappa < \frac{1}{s^2}$, i.e., $1 - \kappa s^2 < 1$, we get

$$\rho_b(\sigma, \Upsilon\sigma) = 0 \implies \Upsilon\sigma = \sigma$$

and $\sigma \in \Xi$. Thus Υ has a fixed point in Ξ.

Now, if possible, let us consider two fixed points of Υ, say σ and μ, exist and thus $\rho_b(\sigma, \mu) \neq 0$. By using inequality (2), we get

$$\begin{aligned}
\rho_b(\sigma, \mu) &= \rho_b(\Upsilon\sigma, \Upsilon\mu) \\
&\leq \kappa \max\{\rho_b(\sigma, \mu), \rho_b(\sigma, \Upsilon\sigma), \rho_b(\mu, \Upsilon\mu), \rho_b(\sigma, \Upsilon\mu), \rho_b(\mu, \Upsilon\sigma)\} \\
&= \kappa \max\{\rho_b(\sigma, \mu), \rho_b(\sigma, \sigma), \rho_b(\mu, \mu), \rho_b(\sigma, \mu), \rho_b(\mu, \sigma)\} \\
&= \kappa\rho_b(\sigma, \mu)
\end{aligned}$$

which is a contradiction since $\kappa \in [0,1)$.
$\Longrightarrow$ $\rho_b(\sigma,\mu) = 0$ and thus $\sigma = \mu$, i.e., the fixed point is unique. □

Theorem 2. *Let $\Upsilon : \Xi \to \Xi$ be a self mapping defined on (Ξ, ρ_b, Δ), a complete convex b-metric space with parameter $s \geq 2$ such that for all $\sigma, \mu \in \Xi$ and some $\kappa \in [0,1)$, it satisfies*

$$\rho_b(\Upsilon\sigma,\Upsilon\mu) \leq \kappa \max\{\rho_b(\sigma,\mu),\rho_b(\sigma,\Upsilon\sigma),\rho_b(\mu,\Upsilon\mu),\rho_b(\sigma,\Upsilon\mu),\rho_b(\mu,\Upsilon\sigma)\} \tag{3}$$

If $\kappa < \frac{1}{s^2}$, then Υ has approximate fixed point property.

Proof. For every $\sigma \in \Xi$, we have

$$\begin{aligned}
\rho_b(\Upsilon^{n+1}\sigma,\Upsilon^n\sigma) &\leq \kappa \max\{\rho_b(\Upsilon^n\sigma,\Upsilon^{n-1}\sigma),\rho_b(\Upsilon^n\sigma,\Upsilon^{n+1}\sigma),\\
&\quad \rho_b(\Upsilon^{n-1}\sigma,\Upsilon^n\sigma),\rho_b(\Upsilon^n\sigma,\Upsilon^n\sigma),\rho_b(\Upsilon^{n-1}\sigma,\Upsilon^{n+1}\sigma)\}\\
&= \kappa \max\{\rho_b(\Upsilon^{n-1}\sigma,\Upsilon^n\sigma),\rho_b(\Upsilon^n\sigma,\Upsilon^{n+1}\sigma),\\
&\quad \rho_b(\Upsilon^{n-1}\sigma,\Upsilon^{n+1}\sigma)\}\\
&\leq \kappa \max\{\rho_b(\Upsilon^{n-1}\sigma,\Upsilon^n\sigma),\rho_b(\Upsilon^n\sigma,\Upsilon^{n+1}\sigma),\\
&\quad s\rho_b(\Upsilon^{n-1}\sigma,\Upsilon^n\sigma)+s\rho_b(\Upsilon^n\sigma,\Upsilon^{n+1}\sigma)\}\\
&= \kappa s[s\rho_b(\Upsilon^{n-1}\sigma,\Upsilon^n\sigma)+s\rho_b(\Upsilon^n\sigma,\Upsilon^{n+1}\sigma)]\\
\Longrightarrow (1-\kappa s)\rho_b(\Upsilon^{n+1}\sigma,\Upsilon^n\sigma) &\leq \kappa s\rho_b(\Upsilon^{n-1}\sigma,\Upsilon^n\sigma)\\
\rho_b(\Upsilon^{n+1}\sigma,\Upsilon^n\sigma) &\leq \frac{\kappa s}{1-\kappa s}\rho_b(\Upsilon^{n-1}\sigma,\Upsilon^n\sigma)\\
&< \rho_b(\Upsilon^{n-1}\sigma,\Upsilon^n\sigma),
\end{aligned}$$

since $\kappa < \frac{1}{s^2}$. Thus the sequence $\langle \rho_b(\Upsilon^{n+1}\sigma,\Upsilon^n\sigma)\rangle_{n\in\mathbb{N}}$ is non-increasing and for $\lambda \in \mathbb{N}$,

$$\rho_b(\Upsilon^{\lambda+1}\sigma,\Upsilon^\lambda\sigma) < \rho_b(\Upsilon^\lambda\sigma,\Upsilon^{\lambda-1}\sigma) < \cdots < \rho_b(\Upsilon^2\sigma,\Upsilon\sigma) < \rho_b(\Upsilon\sigma,\sigma).$$

Now, consider

$$\begin{aligned}
\rho_b(\Upsilon^\lambda\sigma,\Upsilon^{\lambda+2}\sigma) &\leq \kappa \max\{\rho_b(\Upsilon^{\lambda-1}\sigma,\Upsilon^{\lambda+1}\sigma),\rho_b(\Upsilon^{\lambda-1}\sigma,\Upsilon^\lambda\sigma),\rho_b(\Upsilon^{\lambda+1}\sigma,\Upsilon^{\lambda+2}\sigma),\\
&\quad \rho_b(\Upsilon^{\lambda-1}\sigma,\Upsilon^{\lambda+2}\sigma),\rho_b(\Upsilon^{\lambda+1}\sigma,\Upsilon^\lambda\sigma)\}\\
&< \kappa \max\{\rho_b(\Upsilon^{\lambda-1}\sigma,\Upsilon^{\lambda+1}\sigma),\rho_b(\sigma,\Upsilon\sigma),\rho_b(\Upsilon^{\lambda-1}\sigma,\Upsilon^{\lambda+2}\sigma)\}.
\end{aligned} \tag{4}$$

Then, the following cases exist:

Case 1. *If $\max\{\rho_b(\Upsilon^{\lambda-1}\sigma,\Upsilon^{\lambda+1}\sigma),\rho_b(\sigma,\Upsilon\sigma),\rho_b(\Upsilon^{\lambda-1}\sigma,\Upsilon^{\lambda+2}\sigma)\} = \rho_b(\Upsilon^{\lambda-1}\sigma,\Upsilon^{\lambda+1}\sigma)$, then by using inequality (5), we get*

$$\begin{aligned}
\rho_b(\Upsilon^\lambda\sigma,\Upsilon^{\lambda+2}\sigma) &\leq \kappa\rho_b(\Upsilon^{\lambda-1}\sigma,\Upsilon^{\lambda+1}\sigma)\\
&\leq \kappa^2\rho_b(\Upsilon^{\lambda-2}\sigma,\Upsilon^\lambda\sigma)\\
&\ \ \vdots\\
&\leq \kappa^\lambda\rho_b(\sigma,\Upsilon^2\sigma)\\
&\leq \kappa^\lambda s[\rho_b(\sigma,\Upsilon\sigma)+\rho_b(\Upsilon\sigma,\Upsilon^2\sigma)]\\
&< 2\kappa^\lambda s\rho_b(\sigma,\Upsilon\sigma).
\end{aligned} \tag{5}$$

Case 2. *If $\max\{\rho_b(\Upsilon^{\lambda-1}\sigma,\Upsilon^{\lambda+1}\sigma),\rho_b(\sigma,\Upsilon\sigma),\rho_b(\Upsilon^{\lambda-1}\sigma,\Upsilon^{\lambda+2}\sigma)\} = \rho_b(\sigma,\Upsilon\sigma)$, then by using inequality (5), we get*

$$\rho_b(\Upsilon^\lambda\sigma,\Upsilon^{\lambda+2}\sigma) \leq \kappa\rho_b(\sigma,\Upsilon\sigma). \tag{6}$$

Case 3. *If* $\max\{\rho_b(\Upsilon^{\lambda-1}\sigma, \Upsilon^{\lambda+1}\sigma), \rho_b(\sigma, \Upsilon\sigma), \rho_b(\Upsilon^{\lambda-1}\sigma, \Upsilon^{\lambda+2}\sigma)\} = \rho_b(\Upsilon^{\lambda-1}\sigma, \Upsilon^{\lambda+2}\sigma)$, *then by using inequality (5), we get*

$$\begin{aligned}
\rho_b(\Upsilon^{\lambda}\sigma, \Upsilon^{\lambda+2}\sigma) &\leq \kappa\rho_b(\Upsilon^{\lambda-1}\sigma, \Upsilon^{\lambda+2}\sigma) \\
&\leq \kappa s[\rho_b(\Upsilon^{\lambda-1}\sigma, \Upsilon^{\lambda}\sigma) + \rho_b(\Upsilon^{\lambda}\sigma, \Upsilon^{\lambda+2}\sigma)] \\
\implies (1-\kappa s)\rho_b(\Upsilon^{\lambda}\sigma, \Upsilon^{\lambda+2}\sigma) &\leq \kappa s\rho_b(\Upsilon^{\lambda-1}\sigma, \Upsilon^{\lambda}\sigma) \\
\rho_b(\Upsilon^{\lambda}\sigma, \Upsilon^{\lambda+2}\sigma) &\leq \frac{\kappa s}{1-\kappa s}\rho_b(\Upsilon^{\lambda-1}\sigma, \Upsilon^{\lambda}\sigma) \\
&\leq \frac{\kappa s}{1-\kappa s}\rho_b(\sigma, \Upsilon\sigma).
\end{aligned} \tag{7}$$

Therefore, using inequalities (6)–(8), we get

$$\rho_b(\Upsilon^{\lambda}\sigma, \Upsilon^{\lambda+2}\sigma) \leq \eta\rho_b(\sigma, \Upsilon\sigma) \tag{8}$$

where $\eta = 2\kappa s$ since

$$\max\{2\kappa^{\lambda}s, \kappa, \frac{\kappa s}{1-\kappa s}\} \leq \max\{2\kappa s, \kappa, \frac{\kappa s}{1-\kappa s}\} = \max\{2\kappa s, \frac{\kappa s}{1-\kappa s}\} = 2\kappa s = \rho, \ say,$$

for $\kappa < \frac{1}{2s}$ and $\kappa < \frac{1}{s^2} \leq \frac{1}{2s}$ for $s \geq 2$ here.
We let $\inf\{\rho_b(\sigma, \Upsilon\sigma);\ \sigma \in \Xi\} = \gamma$. We need to prove that this $\gamma = 0$. For this, let $\langle\sigma_n\rangle$ be a sequence such that $\lim_{n\to\infty}\rho_b(\sigma_n, \Upsilon\sigma_n) = \gamma$, i.e., by (8), we have, for every $n \in \mathbb{N}$ and some $\lambda(n) \in \mathbb{N}$,

$$\rho_b(\Upsilon^{\lambda(n)}\sigma_n, \Upsilon^{\lambda(n)+2}\sigma_n) \leq \eta\rho_b(\sigma_n, \Upsilon\sigma_n). \tag{9}$$

Now, (Ξ, ρ_b, Δ) being a complete convex b-metric space, defining $\zeta_n = \Delta(\Upsilon^{\lambda(n)+1}\sigma_n, \Upsilon^{\lambda(n)+2}\sigma_n, \alpha)$ leads to a well defined ζ_n belonging to Ξ, where $\alpha \in (0,1)$ and we have,

$$\begin{aligned}
\rho_b(\zeta_n, \Upsilon\zeta_n) &\leq \alpha\rho_b(\Upsilon^{\lambda(n)+1}\sigma_n, \Upsilon\zeta_n) + (1-\alpha)\rho_b(\Upsilon^{\lambda(n)+2}\sigma_n, \Upsilon\zeta_n) \\
&\leq \kappa\alpha\max\{\rho_b(\Upsilon^{\lambda(n)}\sigma_n, \zeta_n), \rho_b(\Upsilon^{\lambda(n)}\sigma_n, \Upsilon^{\lambda(n)+1}\sigma_n), \rho_b(\zeta_n, \Upsilon\zeta_n), \\
&\quad \rho_b(\Upsilon^{\lambda(n)}\sigma_n, \Upsilon\zeta_n), \rho_b(\zeta_n, \Upsilon^{\lambda(n)+1}\sigma_n)\} \\
&\quad +\kappa(1-\alpha)\max\{\rho_b(\Upsilon^{\lambda(n)+1}\sigma_n, \zeta_n), \rho_b(\Upsilon^{\lambda(n)+1}\sigma_n, \Upsilon^{\lambda(n)+2}\sigma_n), \\
&\quad \rho_b(\zeta_n, \Upsilon\zeta_n), \rho_b(\Upsilon^{\lambda(n)+1}\sigma_n, \Upsilon\zeta_n), \rho_b(\zeta_n, \Upsilon^{\lambda(n)+2}\sigma_n)\} \\
&\leq \kappa\alpha\max\{\rho_b(\Upsilon^{\lambda(n)}\sigma_n, \zeta_n), \rho_b(\sigma_n, \Upsilon\sigma_n), \rho_b(\zeta_n, \Upsilon\zeta_n), \\
&\quad s\rho_b(\Upsilon^{\lambda(n)}\sigma_n, \zeta_n) + s\rho_b(\zeta_n, \Upsilon\zeta_n), \rho_b(\zeta_n, \Upsilon^{\lambda(n)+1}\sigma_n)\} \\
&\quad +\kappa(1-\alpha)\max\{\rho_b(\Upsilon^{\lambda(n)+1}\sigma_n, \zeta_n), \rho_b(\sigma_n, \Upsilon\sigma_n), \rho_b(\zeta_n, \Upsilon\zeta_n), \\
&\quad s\rho_b(\Upsilon^{\lambda(n)+1}\sigma_n, \zeta_n) + s\rho_b(\zeta_n, \Upsilon\zeta_n), \rho_b(\zeta_n, \Upsilon^{\lambda(n)+2}\sigma_n)\} \\
&= \kappa\alpha\max\{\rho_b(\sigma_n, \Upsilon\sigma_n), s\rho_b(\Upsilon^{\lambda(n)}\sigma_n, \zeta_n) + s\rho_b(\zeta_n, \Upsilon\zeta_n), \\
&\quad \rho_b(\zeta_n, \Upsilon^{\lambda(n)+1}\sigma_n)\} + \kappa(1-\alpha)\max\{\rho_b(\sigma_n, \Upsilon\sigma_n), \\
&\quad s\rho_b(\Upsilon^{\lambda(n)+1}\sigma_n, \zeta_n) + s\rho_b(\zeta_n, \Upsilon\zeta_n), \rho_b(\zeta_n, \Upsilon^{\lambda(n)+2}\sigma_n)\} \\
&\leq \kappa\alpha\max\{\rho_b(\sigma_n, \Upsilon\sigma_n), s\alpha\rho_b(\Upsilon^{\lambda(n)}\sigma_n, \Upsilon^{\lambda(n)+1}\sigma_n) \\
&\quad +s(1-\alpha)\rho_b(\Upsilon^{\lambda(n)}\sigma_n, \Upsilon^{\lambda(n)+2}\sigma_n) + s\rho_b(\zeta_n, \Upsilon\zeta_n), \\
&\quad (1-\alpha)\rho_b(\Upsilon^{\lambda(n)+2}\sigma_n, \Upsilon^{\lambda(n)+1}\sigma_n)\} + \kappa(1-\alpha)\max\{\rho_b(\sigma_n, \Upsilon\sigma_n), \\
&\quad s(1-\alpha)\rho_b(\Upsilon^{\lambda(n)+1}\sigma_n, \Upsilon^{\lambda(n)+2}\sigma_n) + s\rho_b(\zeta_n, \Upsilon\zeta_n), \\
&\quad \alpha\rho_b(\Upsilon^{\lambda(n)+1}\sigma_n, \Upsilon^{\lambda(n)+2}\sigma_n)\} \\
&\leq \kappa\alpha\max\{\rho_b(\sigma_n, \Upsilon\sigma_n), s\alpha\rho_b(\sigma_n, \Upsilon\sigma_n) + s(1-\alpha)\eta\rho_b(\sigma_n, \Upsilon\sigma_n) \\
&\quad +s\rho_b(\zeta_n, \Upsilon\zeta_n), (1-\alpha)\rho_b(\sigma_n, \Upsilon\sigma_n)\} + \kappa(1-\alpha)\max\{\rho_b(\sigma_n, \Upsilon\sigma_n), \\
&\quad s(1-\alpha)\rho_b(\sigma_n, \Upsilon\sigma_n) + s\rho_b(\zeta_n, \Upsilon\zeta_n), \alpha\rho_b(\sigma_n, \Upsilon\sigma_n)\}
\end{aligned} \tag{10}$$

$$
\begin{aligned}
&= \kappa\alpha \max\{\rho_b(\sigma_n, \Upsilon\sigma_n), s\alpha\rho_b(\sigma_n, \Upsilon\sigma_n) + s(1-\alpha)\eta\rho_b(\sigma_n, \Upsilon\sigma_n) \\
&\quad + s\rho_b(\zeta_n, \Upsilon\zeta_n)\} + \kappa(1-\alpha)\max\{\rho_b(\sigma_n, \Upsilon\sigma_n), s(1-\alpha)\rho_b(\sigma_n, \Upsilon\sigma_n) \\
&\quad + s\rho_b(\zeta_n, \Upsilon\zeta_n)\}.
\end{aligned}
$$

Now suppose that $\lim_{n\to\infty} \rho_b(\zeta_n, \Upsilon\zeta_n) = \beta$ and using inequality (11), we find that β is finite. By definition of γ and β, the inequality $\gamma \leq \beta$ holds. We shall now prove that $\beta = 0$, which in turn, shall prove that $\gamma = 0$.

For this, take lim sup as $n \to \infty$ on both sides of inequality (11) and using the inequality $\gamma \leq \beta$, we must have,

$$
\begin{aligned}
\beta &\leq \kappa\alpha \max\{\beta, s\alpha\beta + s(1-\alpha)\rho\beta + s\beta\} + \kappa(1-\alpha)\max\{\beta, s(1-\alpha)\beta + s\beta\} \\
&= \kappa\alpha[s\alpha\beta + s(1-\alpha)\rho\beta + s\beta] + \kappa(1-\alpha)[2s\beta - s\alpha\beta] \\
&= \kappa[s\alpha^2 + s\rho\alpha - s\alpha^2\eta + s\alpha + 2s - s\alpha - 2s\alpha + s\alpha^2]\beta \\
&= \kappa[2s\alpha^2 + s\rho\alpha - s\alpha^2\eta + 2s - 2s\alpha]\beta.
\end{aligned} \tag{11}
$$

If possible, suppose that $\beta > 0$. Then, by inequality (12), we get

$$
\begin{aligned}
1 &\leq \kappa[2s\alpha^2 + s\rho\alpha - s\alpha^2\eta + 2s - 2s\alpha] \\
&\leq \kappa s[(\rho-2)\alpha(1-\alpha) + 2] \\
&< 2\kappa s \\
\implies \kappa &> \frac{1}{2s},
\end{aligned}
$$

which is a contradiction since $\kappa < \frac{1}{s^2} \leq \frac{1}{2s}$, $\eta = 2\kappa s < \frac{2}{s} \leq 1$ for $s \geq 2$ and $\alpha, (1-\alpha) \in (0,1)$ implying $(\rho-2)\alpha(1-\alpha) < 0$.

Thus our supposition is wrong, i.e., $\beta = 0$ and, in turn, $\gamma = \inf\{\rho_b(\sigma, \Upsilon\sigma);\ \sigma \in \Xi\} = 0$. Therefore, Υ has approximate fixed point property. □

Let $\Upsilon : \Xi \to \Xi$ be a self mapping defined on (Ξ, ρ_b, Δ), a convex b-metric space. We state the following Lemma to show the relation between the set of fixed points of the self mappings Υ and $\Upsilon_\alpha : \Xi \to \Xi$ defined by

$$\Upsilon_\alpha\sigma = \Delta(\sigma, \Upsilon\sigma; \alpha),\ \sigma \in \Xi.$$

Here, set of fixed points of the mappings Υ and Υ_α are denoted by $Fix(\Upsilon)$ and $Fix(\Upsilon_\alpha)$, respectively.

Lemma 2. *Let $\Upsilon : \Xi \to \Xi$ be a self mapping defined on (Ξ, ρ_b, Δ), a convex b-metric space with parameter $s \geq 1$. Define another self mapping $\Upsilon_\alpha : \Xi \to \Xi$ by*

$$\Upsilon_\alpha\sigma = \Delta(\sigma, \Upsilon\sigma; \alpha),\ \sigma \in \Xi.$$

Then, for any $\alpha \in [0,1)$,

$$Fix(\Upsilon) = Fix(\Upsilon_\alpha).$$

Proof. By definition,

$$\Upsilon_\alpha\sigma = \alpha\sigma + (1-\alpha)\Upsilon\sigma.$$

If $\alpha = 0$, then

$$
\begin{aligned}
\Upsilon_\alpha\sigma &= \Upsilon\sigma\ \ \forall\, \sigma \in \Xi \\
i.e.,\ \Upsilon_\alpha &= \Upsilon \\
\implies Fix(\Upsilon) &= Fix(\Upsilon_\alpha).
\end{aligned}
$$

Now assume that $\alpha \in (0,1)$ and let a fixed point of Υ, say σ^*, exists i.e., $\sigma^* = \Upsilon\sigma^*$ and therefore,

$$\begin{aligned} \rho_b(\sigma^*, \Upsilon_\alpha \sigma^*) &= \rho_b(\sigma^*, \Delta(\sigma^*, \Upsilon\sigma^*; \alpha)) \\ &\leq \alpha\rho_b(\sigma^*, \sigma^*) + (1-\alpha)\rho_b(\sigma^*, \Upsilon\sigma^*) = 0 \\ \implies \sigma^* &= \Upsilon_\alpha \sigma^* \end{aligned}$$

i.e., σ^* is a fixed point of Υ_α.

Conversely, suppose that σ^* is a fixed point of Υ_α, i.e., $\rho_b(\sigma^*, \Upsilon_\alpha\sigma^*) = 0$, then

$$\begin{aligned} \rho_b(\sigma^*, \Delta(\sigma^*, \Upsilon\sigma^*; \alpha)) &= 0 \\ \alpha\rho_b(\sigma^*, \sigma^*) + (1-\alpha)\rho_b(\sigma^*, \Upsilon\sigma^*) &= 0 \\ (1-\alpha)\rho_b(\sigma^*, \Upsilon\sigma^*) &= 0. \end{aligned}$$

Since $\alpha \neq 1$, this implies that $\rho_b(\sigma^*, \Upsilon\sigma^*) = 0$, i.e., $\Upsilon\sigma^* = \sigma^*$. Therefore σ^* is a fixed point of Υ. □

Lemma 1 and Theorem 2 imply the following result, extending the Cirić fixed point theorem in the case of Convex b-metric spaces by Rathee et al. [11].

Theorem 3. *Let $\Upsilon : \Xi \to \Xi$ be a self mapping defined on (Ξ, ρ_b, Δ), a complete convex b-metric space with parameter $s \geq 2$ such that for all $\sigma, \mu \in \Xi$ and some $\kappa \in [0,1)$, it satisfies*

$$\rho_b(\Upsilon\sigma, \Upsilon\mu) \leq \kappa \max\{\rho_b(\sigma, \mu), \rho_b(\sigma, \Upsilon\sigma), \rho_b(\mu, \Upsilon\mu), \rho_b(\sigma, \Upsilon\mu), \rho_b(\mu, \Upsilon\sigma)\} \tag{12}$$

If $\kappa < \frac{1}{s^2}$, then

1. *A fixed point of Υ, say σ, exists that is unique.*
2. *The sequence $\langle\sigma_n\rangle_{n\in\mathbb{N}}$ converges to σ for any $\sigma_0 \in \Xi$ that is obtained from the iterative procedure*
$$\sigma_{n+1} = \Delta(\sigma_n, \Upsilon\sigma_n; \alpha);\ n \geq 0.$$
3. *The error estimate*
$$\frac{1}{s}\rho_b(\sigma_{n+i-1}, \sigma) \leq \frac{\delta^i}{1-\delta}\rho_b(\sigma_n, \sigma_{n-1})$$
holds for $n = 1, 2, \cdots;\ i = 1, 2, \cdots$.

Proof. 1. With the given conditions, by Theorem 2, we arrive at the conclusion that Υ has approximate fixed point property. By Lemma 1, a fixed point of Υ, say σ, exists that is unique.

2. We observe that Krasnoselskij iterative procedure is nothing but the Picard iteration associated with Υ_α and defined by $\sigma_{n+1} = \Delta(\sigma_n, \Upsilon\sigma_n; \alpha)$, i.e.,
$$\sigma_{n+1} = \Upsilon_\alpha\sigma_n;\ n \geq 0.$$
Now, in inequality (12), taking $\sigma = \sigma_n$ and $\mu = \sigma_{n-1}$, we get
$$\begin{aligned} \rho_b(\sigma_{n+1}, \sigma_n) &\leq \kappa \max\{\rho_b(\sigma_n, \sigma_{n-1}), \rho_b(\sigma_n, \sigma_{n+1}), \rho_b(\sigma_{n-1}, \sigma_n), \\ &\quad \rho_b(\sigma_n, \sigma_n), \rho_b(\sigma_{n-1}, \sigma_{n+1})\} \\ &\leq \kappa \max\{\rho_b(\sigma_n, \sigma_{n-1}), \rho_b(\sigma_n, \sigma_{n+1}), s\rho_b(\sigma_{n-1}, \sigma_n) \\ &\quad + s\rho_b(\sigma_n, \sigma_{n+1})\} \\ &= \kappa s[\rho_b(\sigma_{n-1}, \sigma_n) + \rho_b(\sigma_n, \sigma_{n+1})]. \end{aligned}$$

This implies

$$\begin{aligned}
\rho_b(\sigma_{n+1},\sigma_n) &\leq \frac{\kappa s}{1-\kappa s}\rho_b(\sigma_n,\sigma_{n-1}) \\
&\leq \kappa s^2 \rho_b(\sigma_n,\sigma_{n-1}) \\
&= \delta\rho_b(\sigma_n,\sigma_{n-1}),\ say,\ \kappa s^2 = \delta \\
&\leq \delta(\delta\rho_b(\sigma_{n-1},\sigma_{n-2})) \\
&= \delta^2\rho_b(\sigma_{n-1},\sigma_{n-2}) \\
&\ \vdots \\
&\leq \delta^n\rho_b(\sigma_1,\sigma_0)
\end{aligned} \tag{13}$$

As $\delta \in [0,1)$, we have

$$\lim_{n\to\infty}\rho_b(\sigma_{n+1},\sigma_n) = 0. \tag{14}$$

We shall now verify that the sequence $\langle\sigma_n\rangle_{n\in\mathbb{N}}$ is Cauchy. For this, consider the points σ and μ as σ_{n+k} and σ_n, respectively, in inequality (12).

$$\begin{aligned}
\rho_b(\sigma_{n+k+1},\sigma_{n+1}) &\leq \kappa\max\{\rho_b(\sigma_{n+k},\sigma_n),\rho_b(\sigma_{n+k},\sigma_{n+k+1}), \\
&\quad \rho_b(\sigma_n,\sigma_{n+1}),\rho_b(\sigma_{n+k},\sigma_{n+1}),\rho_b(\sigma_n,\sigma_{n+k+1})\} \\
&\leq \kappa\max\{s\rho_b(\sigma_{n+k},\sigma_{n+k+1})+s\rho_b(\sigma_{n+k+1},\sigma_n), \\
&\quad \rho_b(\sigma_{n+k},\sigma_{n+k+1}),\rho_b(\sigma_n,\sigma_{n+1}),s\rho_b(\sigma_{n+k},\sigma_{n+k+1}) \\
&\quad +s\rho_b(\sigma_{n+k+1},\sigma_{n+1}),s\rho_b(\sigma_n,\sigma_{n+1})+s\rho_b(\sigma_{n+1},\sigma_{n+k+1})\} \\
&\leq \kappa\max\{s\rho_b(\sigma_{n+k},\sigma_{n+k+1})+s^2\rho_b(\sigma_{n+k+1},\sigma_{n+1}) \\
&\quad +s^2\rho_b(\sigma_{n+1},\sigma_n),s\rho_b(\sigma_{n+k},\sigma_{n+k+1})+s\rho_b(\sigma_{n+k+1},\sigma_{n+1}), \\
&\quad s\rho_b(\sigma_n,\sigma_{n+1})+s\rho_b(\sigma_{n+1},\sigma_{n+k+1})\} \\
&= \kappa[s\rho_b(\sigma_{n+k},\sigma_{n+k+1})+s^2\rho_b(\sigma_{n+k+1},\sigma_{n+1}) \\
&\quad +s^2\rho_b(\sigma_{n+1},\sigma_n)].
\end{aligned}$$

This implies

$$\begin{aligned}
(1-\kappa s^2)\rho_b(\sigma_{n+k+1},\sigma_{n+1}) &\leq \kappa s\rho_b(\sigma_{n+k},\sigma_{n+k+1})+\kappa s^2\rho_b(\sigma_{n+1},\sigma_n) \\
&< \kappa s^2\rho_b(\sigma_{n+k},\sigma_{n+k+1})+\kappa s^2\rho_b(\sigma_{n+1},\sigma_n) \\
\Longrightarrow (1-\delta)\rho_b(\sigma_{n+k+1},\sigma_{n+1}) &\leq \delta[\rho_b(\sigma_{n+k},\sigma_{n+k+1})+\rho_b(\sigma_{n+1},\sigma_n)] \\
and\ \rho_b(\sigma_{n+k+1},\sigma_{n+1}) &\leq \frac{\delta}{1-\delta}[\rho_b(\sigma_{n+k},\sigma_{n+k+1})+\rho_b(\sigma_{n+1},\sigma_n)].
\end{aligned} \tag{15}$$

In inequality (16), taking limit as $n\to\infty$ and using condition (14), we get,

$$\lim_{n\to\infty}\rho_b(\sigma_{n+k+1},\sigma_{n+k}) = 0.$$

This shows that the aforementioned sequence $\langle\sigma_n\rangle_{n\in\mathbb{N}}$ is Cauchy and owing to completeness of the space (Ξ,ρ_b,Δ), converges to some point, say μ. Now, consider the inequality (14),

$$\begin{aligned}
\rho_b(\sigma_{n+1},\sigma_n) &\leq \delta^n\rho_b(\sigma_1,\sigma_0) \\
\Longrightarrow \rho_b(\Upsilon_\alpha\sigma_n,\sigma_n) &\leq \delta^n\rho_b(\sigma_1,\sigma_0).
\end{aligned}$$

Now taking limit as $n\to\infty$, we get,

$$\begin{aligned}
\frac{1}{s}\rho_b(\Upsilon_\alpha\mu,\mu) &= 0 \\
\Longrightarrow \rho_b(\Upsilon_\alpha\mu,\mu) &= 0.
\end{aligned}$$

Thus, $\Upsilon_\alpha \mu = \mu$, and therefore μ is a fixed point of Υ_α. But by using Lemma 2, we must have

$$Fix(\Upsilon) = Fix(\Upsilon_\alpha),$$

and $Fix(\Upsilon) = \{\sigma\}$, i.e., Fixed point of Υ is σ, which is unique.
So, $\mu = \sigma$ and thus $\langle \sigma_n \rangle_{n\in\mathbb{N}}$ obtained from the above iteration converges to σ.

3. Using inequalities (16) and (14), we have

$$\begin{aligned}\rho_b(\sigma_{n+m}, \sigma_n) &\leq \frac{\delta}{1-\delta}[\rho_b(\sigma_{n+m-1}, \sigma_{n+m}) + \rho_b(\sigma_n, \sigma_{n-1})] \\ &\leq \frac{\delta}{1-\delta}[\delta^{n+m-1}\rho_b(\sigma_1, \sigma_0) + \delta^{n-1}\rho_b(\sigma_1, \sigma_0)] \\ &= \frac{\delta^n(\delta^m+1)}{1-\delta}\rho_b(\sigma_1, \sigma_0).\end{aligned}$$

Now letting $m \to \infty$, we get,

$$\frac{1}{s}\rho_b(\sigma, \sigma_n) \leq \frac{\delta^n}{1-\delta}\rho_b(\sigma_1, \sigma_0) \tag{16}$$

and

$$\begin{aligned}\rho_b(\sigma_{n+m}, \sigma_n) &\leq \frac{\delta}{1-\delta}[\rho_b(\sigma_{n+m-1}, \sigma_{n+m}) + \rho_b(\sigma_n, \sigma_{n-1})] \\ &\leq \frac{\delta}{1-\delta}[\delta^{m-1}\rho_b(\sigma_{n-1}, \sigma_n) + \rho_b(\sigma_n, \sigma_{n-1})] \\ &= \frac{\delta(\delta^{m-1}+1)}{1-\delta}\rho_b(\sigma_{n-1}, \sigma_n).\end{aligned}$$

Now letting $m \to \infty$, we get,

$$\frac{1}{s}\rho_b(\sigma, \sigma_n) \leq \frac{\delta}{1-\delta}\rho_b(\sigma_{n-1}, \sigma_n). \tag{17}$$

Thus, we get the following error estimate after merging inequalities (16) and (17),

$$\frac{1}{s}\rho_b(\sigma_{n+i-1}, \sigma) \leq \frac{\delta^i}{1-\delta}\rho_b(\sigma_n, \sigma_{n-1}).$$

□

The following example illustrates the importance of the above theorem.

Example 1. *Let the set of non-negative real numbers be $\Xi = \mathbb{R}_0^+$ and $\rho_b(\sigma, \mu) = (\sigma - \mu)^2$ for all $\sigma, \mu \in \Xi$. Here, we perceive that*

1. $\rho_b(\sigma, \mu) \geq 0$ *for all* $\sigma, \mu \in \Xi$;
2. $\rho_b(\sigma, \mu) = 0 \iff \sigma = \mu$;
3. $\rho_b(\sigma, \mu) = \rho_b(\mu, \sigma)$;
4. $\rho_b(\sigma, \mu) \leq 2[\rho_b(\sigma, \xi) + \rho_b(\xi, \mu)]$, $\xi \in \Xi$ *as*

$$\begin{aligned}\rho_b(\sigma, \mu) &= (\sigma - \mu)^2 \\ &= [(\sigma - \xi) + (\xi - \mu)]^2 \\ &\leq 2\Big[\Big((\sigma - \xi)^2 + (\xi - \mu)^2\Big)\Big] \\ &= 2[\rho_b(\sigma, \xi) + \rho_b(\xi, \mu)].\end{aligned}$$

We define the convex structure $\Delta : \Xi \times \Xi \times [0,1] \to \Xi$ as

$$\Delta(\sigma, \mu; \alpha) = \alpha\sigma + (1-\alpha)\mu,$$

for any $\sigma, \mu \in \Xi$ and $\alpha \in [0,1]$. As a consequence,

$$\begin{aligned}
\rho_b(\xi, \Delta(\sigma, \mu; \alpha)) &= (\xi - (\alpha\sigma + (1-\alpha)\mu))^2 \\
&\leq (\alpha|\xi - \sigma| + (1-\alpha)|\xi - \mu|)^2 \\
&\leq \alpha^2(\xi - \sigma)^2 + (1-\alpha)^2(\xi - \mu)^2 + 2\alpha(1-\alpha)|\xi - \sigma| \cdot |\xi - \mu| \\
&\leq \alpha^2(\xi - \sigma)^2 + (1-\alpha)^2(\xi - \mu)^2 + \alpha(1-\alpha)((\xi - \sigma)^2 + (\xi - \mu)^2) \\
&= \alpha(\xi - \sigma)^2 + (1-\alpha)(\xi - \mu)^2 \\
&= \alpha\rho_b(\xi, \sigma) + (1-\alpha)\rho_b(\xi, \mu).
\end{aligned}$$

Thus, for $s \geq 2$, (Ξ, ρ_b, Δ) is a convex b-metric space . However, the metric triangle inequality is not satisfied by ρ_b, for example,

$$\rho_b(1,5) = 16 > \rho_b(1,3) + \rho_b(3,5) = 8.$$

Therefore, (Ξ, ρ_b) is not a metric space and hence it is not a convex metric space.
Let the mapping $\Upsilon : \Xi \to \Xi$ be defined as

$$\Upsilon(\sigma) = \begin{cases} \frac{\sigma}{\sqrt{5}} \ , & \sigma \in \Lambda = [0,1) \\ \frac{\sigma}{\sqrt{7}} \ , & \sigma \in \Sigma = [1,\infty). \end{cases}$$

Thereafter, to prove that Υ satisfies inequality (2), the following four cases exist:

1. *If both $\sigma, \mu \in \Lambda$, then*

$$\begin{aligned}
\rho_b(\Upsilon\sigma, \Upsilon\mu) &= (\Upsilon\sigma - \Upsilon\mu)^2 \\
&= \left(\frac{\sigma}{\sqrt{5}} - \frac{\mu}{\sqrt{5}}\right)^2 \\
&= \frac{1}{5}(\sigma - \mu)^2 \\
&= \frac{1}{5}\rho_b(\sigma, \mu).
\end{aligned}$$

2. *If $\sigma \in \Lambda$ and $\mu \in \Sigma$, then*

$$\begin{aligned}
\rho_b(\Upsilon\sigma, \Upsilon\mu) &= (T\sigma - T\mu)^2 \\
&= \left(\frac{\sigma}{\sqrt{5}} - \frac{\mu}{\sqrt{7}}\right)^2 \\
&= \frac{1}{5}\left(\sigma - \sqrt{\frac{5}{7}}\mu\right)^2 \\
&\leq \frac{1}{5}\left(\sigma - \frac{1}{\sqrt{7}}\mu\right)^2 \\
&= \frac{1}{5}\rho_b(\sigma, \Upsilon\mu).
\end{aligned}$$

3. *If $\sigma \in \Sigma$ and $\mu \in \Lambda$, then as in the former case, we get*

$$\rho_b(\Upsilon\sigma, \Upsilon\mu) \leq \frac{1}{5}\rho_b(\mu, \Upsilon\sigma).$$

4. *If both* $\sigma, \mu \in \Sigma = [1, \infty)$

$$\begin{aligned} \rho_b(\Upsilon\sigma, \Upsilon\mu) &= (\Upsilon\sigma - \Upsilon\mu)^2 \\ &= \left(\frac{\sigma}{\sqrt{7}} - \frac{\mu}{\sqrt{7}}\right)^2 \\ &= \frac{1}{7}(\sigma - \mu)^2 \\ &< \frac{1}{5}(\sigma - \mu)^2 \\ &= \frac{1}{5}\rho_b(\sigma, \mu). \end{aligned}$$

which infers that for all $\sigma, \mu \in \Xi$

$$\rho_b(\Upsilon\sigma, \Upsilon\mu) \leq \frac{1}{5}\max\{\rho_b(\sigma, \mu), \rho_b(\sigma, \Upsilon\sigma), \rho_b(\mu, \Upsilon\mu), \rho_b(\sigma, \Upsilon\mu), \rho_b(\mu, \Upsilon\sigma)\}.$$

Therefore, for $k = \frac{1}{5} < \frac{1}{s^2}$, Υ *satisfies the inequality (6).*

Next, we choose σ_0 *as an initial point in* Ξ *and generate the sequence by Krasnoselskij's iteration* $\sigma_n = \Upsilon_\alpha\sigma_{n-1} = \Delta(\sigma_{n-1}, \Upsilon\sigma_{n-1}; \alpha)$ *with* $0 < \alpha = \frac{3}{4} < 1$. *There are two possibilities for* σ_0:

1. *If* $\sigma_0 < 1$, *then*

$$\begin{aligned} \Upsilon\sigma_0 &= \frac{\sigma_0}{\sqrt{5}} \\ \sigma_1 &= \Upsilon_\alpha\sigma_0 = \frac{3}{4}\sigma_0 + \frac{1}{4}\Upsilon\sigma_0 = \left(\frac{3}{4} + \frac{1}{4\sqrt{5}}\right)\sigma_0 \\ \sigma_2 &= \Upsilon_\alpha\sigma_1 = \frac{3}{4}\sigma_1 + \frac{1}{4}\sigma_1 = \left(\frac{3}{4} + \frac{1}{4\sqrt{5}}\right)^2\sigma_0 \\ &\vdots \\ \sigma_n &= \Upsilon_\alpha\sigma_{n-1} = \frac{3}{4}\sigma_{n-1} + \frac{1}{4}\Upsilon\sigma_{n-1} = \left(\frac{3}{4} + \frac{1}{4\sqrt{5}}\right)^n\sigma_0. \end{aligned}$$

Certainly, $\sigma_n \to 0$ *as* $n \to \infty$.

2. *If* $\sigma_0 \geq 1$, *then*

$$\begin{aligned} \Upsilon\sigma_0 &= \frac{\sigma_0}{\sqrt{7}} \\ \sigma_1 &= \Upsilon_\alpha\sigma_0 = \frac{3}{4}\sigma_0 + \frac{1}{4}T\sigma_0 = \left(\frac{3}{4} + \frac{1}{4\sqrt{7}}\right)\sigma_0. \end{aligned}$$

If $\sigma_1 \in \Lambda$, *as* $n \to \infty$, $\sigma_n \to 0$ *as in the former case. If* $\sigma_1 \in \Sigma = [1, \infty)$, *then* $\frac{\sigma_2}{\sigma_1} = \frac{3}{4} + \frac{1}{4} \cdot \frac{\Upsilon\sigma_1}{\sigma_1} = \frac{3}{4} + \frac{1}{4\sqrt{7}}$. *Continuing in comparable manner, we presume that* $\sigma_{n-1} \in \Sigma = [1, \infty)$ *yielding*

$$\frac{\sigma_n}{\sigma_{n-1}} = \frac{3}{4} + \frac{1}{4} \cdot \frac{\Upsilon\sigma_{n-1}}{\sigma_{n-1}} = \frac{3}{4} + \frac{1}{4\sqrt{7}},$$

and

$$\frac{\sigma_n}{\sigma_0} = \frac{\sigma_1}{\sigma_0} \cdot \frac{\sigma_2}{\sigma_1} \cdots \frac{\sigma_n}{\sigma_{n-1}} = \left(\frac{3}{4} + \frac{1}{4\sqrt{7}}\right)^n,$$

and hence $\lim_{n\to\infty} \sigma_n = 0$.

Now, if $\sigma_0 < 1$*, consider*

$$\begin{aligned}\rho_b(\sigma_n, \Upsilon\sigma_n) &= (\sigma_n - \Upsilon\sigma_n)^2 \\ &= \left[\left(\frac{3}{4} + \frac{1}{4\sqrt{5}}\right)^n \sigma_0 - \left(\frac{3}{4} + \frac{1}{4\sqrt{5}}\right)^{n+1} \sigma_0\right]^2 \\ &= \frac{1}{16}\left(\frac{3}{4} + \frac{1}{4\sqrt{5}}\right)^{2n}\left(1 - \frac{1}{\sqrt{5}}\right)^2 \sigma_0^2.\end{aligned}$$

Clearly,

$$\lim_{n\to\infty} \rho_b(\sigma_n, \Upsilon\sigma_n) = 0. \tag{18}$$

Also, if $\sigma_0 \geq 1$*, then*

$$\begin{aligned}\rho_b(\sigma_n, \Upsilon\sigma_n) &= (\sigma_n - \Upsilon\sigma_n)^2 \\ &= \left[\left(\frac{3}{4} + \frac{1}{4\sqrt{7}}\right)^n \sigma_0 - \left(\frac{3}{4} + \frac{1}{4\sqrt{7}}\right)^{n+1} \sigma_0\right]^2 \\ &= \frac{1}{16}\left(\frac{3}{4} + \frac{1}{4\sqrt{7}}\right)^{2n}\left(1 - \frac{1}{\sqrt{7}}\right)^2 \sigma_0^2.\end{aligned}$$

Clearly,

$$\lim_{n\to\infty} \rho_b(\sigma_n, \Upsilon\sigma_n) = 0. \tag{19}$$

Thus, from (18) and (19), we get

$$\inf\{\rho_b(\sigma, \Upsilon\sigma);\ \sigma \in \Xi\} = 0.$$

Thus, Υ *has approximate fixed point property and, hence, a unique fixed point exists which is equal to the limit of sequence obtained by applying Mann's iteration, i.e.,* 0.

Remark 1. *If we take* $\sigma = 0$ *and* $\mu = \frac{1}{2}$*, then* $\Upsilon\sigma = 0$ *and* $\Upsilon\mu = \frac{1}{5\sqrt{2}}$ *which yields*

$$\begin{aligned}\rho_b(\Upsilon\sigma, \Upsilon\mu) &\leq \kappa \max\{\rho_b(\sigma,\mu), \rho_b(\sigma,\Upsilon\sigma), \rho_b(\mu,\Upsilon\mu), \rho_b(\sigma,\Upsilon\mu, \rho_b(\mu,\Upsilon\sigma))\} \\ \frac{1}{20} &= \kappa \max\left\{\frac{1}{4}, 0, \frac{6-2\sqrt{5}}{20}, \frac{1}{20}, \frac{1}{4}\right\} \\ \Longrightarrow \frac{1}{20} &\leq \frac{\kappa}{4}\end{aligned} \tag{20}$$

which is true for all $\kappa \geq \frac{1}{5} < \frac{1}{s^2}$ *and* $\kappa \geq \frac{1}{5} > \frac{1}{s^4}$ *and therefore, Theorem 2 of Rathee et al. [11] does not guarantee the existence and uniqueness of a fixed point in this scenario. Thus, results provided by Theorem 3 extend the Cirić fixed point theorem proved by Rathee et al. [11].*

Theorem 4. *Let* $\Pi : \Xi \to \Xi$ *be a self mapping defined on* (Ξ, ρ_b, Δ)*, a complete convex b-metric space with parameter* $s \geq 2$ *such that a natural cardinal* N *exists for all* $\sigma, \mu \in \Xi$ *and some* $\kappa \in [0, 1)$*, it satisfies*

$$\begin{aligned}\rho_b(\Pi^N\sigma, \Pi^N\mu) \leq\ & \kappa \max\{\rho_b(\sigma,\mu), \rho_b(\sigma,\Pi^N\sigma), \rho_b(\mu,\Pi^N\mu), \rho_b(\sigma,\Pi^N\mu), \\ & \rho_b(\mu,\Pi^N\sigma)\}\end{aligned} \tag{21}$$

If $\kappa < \frac{1}{s^2}$*, then*

1. *A unique fixed point of* Π*, say* σ *exists that is unique.*
2. *The sequence* $\langle\sigma_n\rangle_{n\in\mathbb{N}}$ *obtained from the iterative procedure*

$$\sigma_{n+1} = \Delta(\sigma_n, \Pi^N\sigma_n; \alpha);\ n \geq 0$$

converges to σ for any $\sigma_0 \in \Xi$.

Proof. 1. Applying Theorem 3 for the mapping $\Upsilon = \Pi^N$, we obtain that Π^N has a unique fixed point, say σ. Also, we have

$$\Pi^N(\Pi(\sigma)) = \Pi^{N+1}(\sigma) = \Pi(\Pi^N(\sigma)) = \Pi(\sigma).$$

This shows that $\Pi(\sigma)$ is a fixed point of Π^N. However, there is a unique fixed point of Π^N, σ. This implies that $\Pi(\sigma) = \sigma$ and thus, Π has a unique fixed point, σ.

2. Applying Theorem 3,we observe that the sequence $\langle \sigma_n \rangle_{n\in\mathbb{N}}$ obtained from the iterative procedure

$$\sigma_{n+1} = \Delta(\sigma_n, \Pi^N\sigma_n; \alpha);\ n \geq 0$$

converges to σ for any $\sigma_0 \in \Xi$.

□

As far as approximation of fixed points is concerned, we prove that the convergence of every orbit of self mapping Υ is to its unique fixed point that too for $\kappa < \frac{1}{s^2}$, even in the case of any complete b-metric space.

Theorem 5. *Let $\Upsilon : \Xi \to \Xi$ be a self mapping defined on a complete b-metric space (Ξ, ρ_b) with parameter $s \geq 2$ such that for all $\sigma, \mu \in \Xi$ and some $\kappa \in [0,1)$, it satisfies*

$$\rho_b(\Upsilon\sigma, \Upsilon\mu) \leq \kappa \max\{\rho_b(\sigma,\mu), \rho_b(\sigma,\Upsilon\sigma), \rho_b(\mu,\Upsilon\mu), \rho_b(\sigma,\Upsilon\mu), \rho_b(\mu,\Upsilon\sigma)\} \tag{22}$$

Then, if $\kappa < \frac{1}{s^2}$, a fixed point of Υ exists that is unique. Besides, the sequence $\langle \Upsilon^n\sigma_0 \rangle_{n\in\mathbb{N}}$ of Picard iterates, for each $\sigma_0 \in \Xi$, converges to this fixed point.

Proof. Let the sequence $\langle \sigma_n \rangle_{n\in\mathbb{N}}$ be defined by

$$\sigma_{n+1} = \Upsilon\sigma_n = \Upsilon^n\sigma_0,$$

where σ_0 is arbitrary in Ξ.

Preserving generality, assume that, for every $n \in \mathbb{N}$, $\sigma_{n+1} \neq \sigma_n$, as the result holds trivially if $\sigma_{n+1} = \sigma_n$.

Now, we have

$$\begin{aligned}
\rho_b(\sigma_{n+1}, \sigma_n) &\leq \kappa \max\{\rho_b(\sigma_n, \sigma_{n-1}), \rho_b(\sigma_n, \sigma_{n+1}), \rho_b(\sigma_{n-1}, \sigma_n), \\
&\quad \rho_b(\sigma_n, \sigma_n), \rho_b(\sigma_{n-1}, \sigma_{n+1})\} \\
&\leq \kappa \max\{\rho_b(\sigma_n, \sigma_{n-1}), \rho_b(\sigma_n, \sigma_{n+1}), s\rho_b(\sigma_{n-1}, \sigma_n) \\
&\quad + s\rho_b(\sigma_n, \sigma_{n+1})\} \\
&= \kappa s[\rho_b(\sigma_{n-1}, \sigma_n) + \rho_b(\sigma_n, \sigma_{n+1})].
\end{aligned}$$

This implies

$$\begin{aligned}
\rho_b(\sigma_{n+1}, \sigma_n) &\leq \frac{\kappa s}{1-\kappa s}\rho_b(\sigma_n, \sigma_{n-1}) \\
&\leq \kappa s^2 \rho_b(\sigma_n, \sigma_{n-1}) \\
&= \delta\rho_b(\sigma_n, \sigma_{n-1}),\ say,\ \kappa s^2 = \delta \\
&\leq \delta^2 \rho_b(\sigma_{n-1}, \sigma_{n-2}) \\
&\ \vdots \\
&\leq \delta^n \rho_b(\sigma_1, \sigma_0).
\end{aligned}$$

This shows that the aforementioned sequence $\langle \sigma_n \rangle_{n\in\mathbb{N}}$ is Cauchy and owing to completeness of the space, is convergent too. Let σ be its limit.

Consider now

$$\rho_b(\sigma_{n+1}, \Upsilon\sigma) \leq \kappa \max\{\rho_b(\sigma_n, \sigma), \rho_b(\sigma_n, \sigma_{n+1}), \rho_b(\sigma, \Upsilon\sigma), \rho_b(\sigma_n, \Upsilon\sigma), \rho_b(\sigma, \sigma_{n+1})\}$$

Taking limit as $n \to \infty$, we get

$$\begin{aligned} \frac{1}{s}\rho_b(\sigma, \Upsilon\sigma) &\leq \kappa \max\{0, 0, \rho_b(\sigma, \Upsilon\sigma), s\rho_b(\sigma, \Upsilon\sigma), 0\} \\ &= \kappa s \rho_b(\sigma, \Upsilon\sigma) \\ \implies \rho_b(\sigma, \Upsilon\sigma) &\leq \kappa s^2 \rho_b(\sigma, \Upsilon\sigma) \\ &< \rho_b(\sigma, \Upsilon\sigma) \end{aligned}$$

since $\kappa < \frac{1}{s^2}$.

Thus, $\rho_b(\sigma, \Upsilon\sigma) = 0$, i.e., $\Upsilon\sigma = \sigma$ and this proves that Υ has a fixed point σ.

For uniqueness, let us suppose that Υ has two distinct fixed points, say σ and μ, such that $\rho_b(\sigma, \mu) \neq 0$, then

$$\begin{aligned} \rho_b(\sigma, \mu) &= \rho_b(\Upsilon\sigma, \Upsilon\mu) \\ &\leq \kappa \max\{\rho_b(\sigma, \mu), \rho_b(\sigma, \Upsilon\sigma), \rho_b(\mu, \Upsilon\mu), \rho_b(\sigma, \Upsilon\mu), \rho_b(\mu, \Upsilon\sigma)\} \\ &= \kappa \max\{\rho_b(\sigma, \mu), 0, 0, \rho_b(\sigma, \mu), \rho_b(\sigma, \mu)\} \\ &= \kappa \rho_b(\sigma, \mu) \end{aligned} \tag{23}$$

which is a contradiction since $\kappa \in [0, 1)$. Thus, the fixed point so obtained is unique. □

Example 2. *The pair* (Ξ, ρ_b), *in Example 1, makes a complete b-metric space. If we take the sequence of Picard iterates , then*

1. *for* $\sigma_0 < 1$, *we have*

$$\begin{aligned} \sigma_1 &= \Upsilon\sigma_0 = \left(\frac{1}{\sqrt{5}}\right)\sigma_0 \\ \sigma_2 &= \Upsilon^2\sigma_0 = \left(\frac{1}{\sqrt{5}}\right)^2\sigma_0 \\ &\vdots \\ \sigma_n &= \Upsilon^n\sigma_0 = \left(\frac{1}{\sqrt{5}}\right)^n\sigma_0, \end{aligned}$$

2. *and for* $\sigma_0 \geq 1$, *we have*

$$\sigma_1 = \Upsilon\sigma_0 = \left(\frac{1}{\sqrt{7}}\right)\sigma_0$$

If $\sigma_1 \in [0, 1)$, *then the sequence can be evaluated as in the above case. If* $\sigma_1 \in \Sigma = [1, \infty)$, *then* $\frac{\sigma_2}{\sigma_1} = \left(\frac{1}{\sqrt{7}}\right)$. *Continuing in a comparable manner, presume that* $\sigma_{n-1} \in \Sigma = [1, \infty)$, *yielding*

$$\frac{\sigma_n}{\sigma_{n-1}} = \left(\frac{1}{\sqrt{7}}\right),$$

and

$$\frac{\sigma_n}{\sigma_0} = \frac{\sigma_1}{\sigma_0} \cdot \frac{\sigma_2}{\sigma_1} \cdots \frac{\sigma_n}{\sigma_{n-1}} = \left(\frac{1}{\sqrt{7}}\right)^n.$$

Hence $\lim_{n\to\infty} \sigma_n = 0$ *for both the cases. Therefore, the sequence of Picard iterates converge to fixed point* 0.

3. Stability of Fixed Point

This section is concerned with the stability results for fixed points of mappings satisfying the Cirić contraction.

Definition 2. *Let $\langle \Upsilon_n \rangle_{n\in\mathbb{N}}$ be a sequence of self mapping defined on a convex b-metric space. Then stability is nothing but a relation between the convergence of the sequence $\langle \Upsilon_n \rangle_{n\in\mathbb{N}}$ and their fixed points.*

Theorem 6. *Let (Ξ, ρ_b) be a complete b-metric space with parameters ≥ 1 and suppose $\langle \Upsilon_n \rangle_{n\in\mathbb{N}}$ be a sequence of self mappings $\Upsilon_n : \Xi \to \Xi$ such that for all $\sigma, \mu \in \Xi$; $n \in \mathbb{N}$ and $\kappa_n \in [0,1)$, it satisfies*

$$\rho_b(\Upsilon_n\sigma, \Upsilon_n\mu) \leq \kappa_n \max\{\rho_b(\sigma,\mu), \rho_b(\sigma,\Upsilon_n\sigma), \rho_b(\mu,\Upsilon_n\mu), \rho_b(\sigma,\Upsilon_n\mu), \rho_b(\mu,\Upsilon_n\sigma)\}$$

Also, let $\Upsilon : \Xi \to \Xi$ be a self mapping satisfying

$$\rho_b(\Upsilon\sigma, \Upsilon\mu) \leq \kappa \max\{\rho_b(\sigma,\mu), \rho_b(\sigma,\Upsilon\sigma), \rho_b(\mu,\Upsilon\mu), \rho_b(\sigma,\Upsilon\mu), \rho_b(\mu,\Upsilon\sigma)\}$$

for all $n \in \mathbb{N}$ and $\sigma, \mu \in \Xi$, where $\kappa \in [0,1)$. Let Υ has a fixed point ζ and for every n, ζ_n be the fixed points of Υ_n. Presuming $\Upsilon_n \to \Upsilon$ pointwise and $\kappa_n \to \kappa$, then $\zeta_n \to \zeta$ if $\kappa, \kappa_n < \frac{1}{s^2}$.

Proof. For every $n \in \mathbb{N}$, we have

$$\begin{aligned}
\rho_b(\zeta,\zeta_n) &\leq s[\rho_b(\zeta,\Upsilon_n\zeta) + \rho_b(\Upsilon_n\zeta,\zeta_n)] \\
&\leq s[\rho_b(\zeta,\Upsilon_n\zeta) + \rho_b(\Upsilon_n\zeta,\Upsilon_n\zeta_n)] \\
&\leq s[\rho_b(\zeta,\Upsilon_n\zeta) + \kappa_n \max\{\rho_b(\zeta,\zeta_n), \rho_b(\zeta,\Upsilon_n\zeta), \rho_b(\zeta_n,\Upsilon_n\zeta_n), \\
&\quad \rho_b(\zeta,\Upsilon_n\zeta_n), \rho_b(\zeta_n,\Upsilon_n\zeta)\}] \\
&= s[\rho_b(\zeta,\Upsilon_n\zeta) + \kappa_n \max\{\rho_b(\zeta,\zeta_n), \rho_b(\zeta,\Upsilon_n\zeta), \rho_b(\zeta,\zeta_n), \\
&\quad \rho_b(\zeta_n,\Upsilon_n\zeta)\}] \\
&= s[\rho_b(\zeta,\Upsilon_n\zeta) + \kappa_n \max\{\rho_b(\zeta,\zeta_n), \rho_b(\zeta,\Upsilon_n\zeta), \rho_b(\zeta_n,\Upsilon_n\zeta)\}] \\
&\leq s[\rho_b(\zeta,\Upsilon_n\zeta) + \kappa_n \max\{\rho_b(\zeta,\zeta_n), \rho_b(\zeta,\Upsilon_n\zeta), s\rho_b(\zeta_n,\zeta) \\
&\quad +s\rho_b(\zeta,\Upsilon_n\zeta)\}] \\
&= s[\rho_b(\zeta,\Upsilon_n\zeta) + \kappa_n s[\rho_b(\zeta_n,\zeta) + \rho_b(\zeta,\Upsilon_n\zeta)]] \\
&= \kappa_n s^2 \rho_b(\zeta_n,\zeta) + (\kappa_n s^2 + s)\rho_b(\zeta,\Upsilon_n\zeta).
\end{aligned}$$

This implies

$$(1-\kappa_n s^2)\rho_b(\zeta_n,\zeta) \leq (\kappa_n s^2 + s)\rho_b(\zeta,\Upsilon_n\zeta).$$

Taking limit as $n \to \infty$, we get

$$\begin{aligned}
\lim_{n\to\infty}(1-\kappa_n s^2)\rho_b(\zeta_n,\zeta) &\leq \lim_{n\to\infty} s(\kappa_n s + 1)\rho_b(\zeta,\Upsilon_n\zeta) \\
(1-\kappa s^2)\lim_{n\to\infty}\rho_b(\zeta_n,\zeta) &\leq s^2(\kappa s + 1)\rho_b(\zeta,\Upsilon\zeta) \;=\; 0 \\
\Longrightarrow \lim_{n\to\infty}\rho_b(\zeta_n,\zeta) &= 0 \text{ as } \kappa < \frac{1}{s^2} \\
\text{i.e., } \zeta_n &\to \zeta.
\end{aligned}$$

□

Theorem 6 can also be restated as

Theorem 7. *Let a sequence $\langle \Upsilon_n \rangle_{n\in\mathbb{N}}$ of self mappings $\Upsilon_n : \Xi \to \Xi$ be defined on a complete b-metric space (Ξ, ρ_b) with $s \geq 1$ having fixed points ζ_n and for all $\sigma, \mu \in \Xi$; $n \in \mathbb{N}$, satisfying*

$$\rho_b(\Upsilon_n\sigma, \Upsilon_n\mu) \leq \kappa_n \max\{\rho_b(\sigma,\mu), \rho_b(\sigma,\Upsilon_n\sigma), \rho_b(\mu,\Upsilon_n\mu), \rho_b(\sigma,\Upsilon_n\mu), \rho_b(\mu,\Upsilon_n\sigma)\},$$

where $\kappa_n < \frac{1}{s^2}$ and $\kappa_n \in [0,1)$. Also for all $\sigma, \mu \in \Xi$, presume a self mapping $\Upsilon : \Xi \to \Xi$ satisfying

$$\rho_b(\Upsilon\sigma, \Upsilon\mu) \leq \kappa \max\{\rho_b(\sigma,\mu), \rho_b(\sigma,\Upsilon\sigma), \rho_b(\mu,\Upsilon\mu), \rho_b(\sigma,\Upsilon\mu), \rho_b(\mu,\Upsilon\sigma)\}$$

where $\kappa < \frac{1}{s^2}$ and $\kappa \in [0,1)$. If $\Upsilon_n \to \Upsilon$ pointwise and $\kappa_n \to \kappa$, then $\langle \zeta_n \rangle_{n\in\mathbb{N}}$ is convergent if Υ has a fixed point ζ and in that case, $\zeta_n \to \zeta$ as $n \to \infty$.

Proof. We presume that a fixed point ζ of map Υ exists i.e., $\zeta = \Upsilon\zeta$.

Now as proved in Theorem 6, $\rho_b(\zeta_n, \zeta) \to 0$ as $n \to \infty$, i.e., $\langle \zeta_n \rangle_{n\in\mathbb{N}}$ is convergent sequence and $\zeta_n \to \zeta$ as $n \to \infty$. □

Theorem 8. *Let a sequence $\langle \Upsilon_n \rangle_{n\in\mathbb{N}}$ of self mappings $\Upsilon_n : \Xi \to \Xi$ be defined on a complete b-metric space (Ξ, ρ_b) with $s \geq 1$ having fixed points ζ_n and for all $\sigma, \mu \in \Xi$; $n \in \mathbb{N}$, satisfying*

$$\rho_b(\Upsilon_n\sigma, \Upsilon_n\mu) \leq \kappa_n \max\{\rho_b(\sigma,\mu), \rho_b(\sigma,\Upsilon_n\sigma), \rho_b(\mu,\Upsilon_n\mu), \rho_b(\sigma,\Upsilon_n\mu), \rho_b(\mu,\Upsilon_n\sigma)\}$$

where $\kappa_n < \frac{1}{s^2}$ and $\kappa_n \in [0,1)$. Also, for all $\sigma, \mu \in \Xi$, presume a self mapping $\Upsilon : \Xi \to \Xi$ satisfying

$$\rho_b(\Upsilon\sigma, \Upsilon\mu) \leq \kappa \max\{\rho_b(\sigma,\mu), \rho_b(\sigma,\Upsilon\sigma), \rho_b(\mu,\Upsilon\mu), \rho_b(\sigma,\Upsilon\mu), \rho_b(\mu,\Upsilon\sigma)\}$$

where $\kappa < \frac{1}{s^2}$ and $\kappa \in [0,1)$. If $\Upsilon_n \to \Upsilon$ pointwise and $\kappa_n \to \kappa$, then a fixed point of Υ, say ζ, exists if $\langle \zeta_n \rangle_{n\in\mathbb{N}}$ is convergent and in that case, $\zeta_n \to \zeta$ as $n \to \infty$.

Proof. Presume that $\langle \zeta_n \rangle_{n\in\mathbb{N}}$ is convergent and $\zeta_n \to \zeta$ as $n \to \infty$. Then,

$$\begin{aligned}
\rho_b(\Upsilon_n\zeta, \zeta_n) &= \rho_b(\Upsilon_n\zeta, \Upsilon_n\zeta_n) \\
&\leq \kappa_n \max\{\rho_b(\zeta,\zeta_n), \rho_b(\zeta,\Upsilon_n\zeta), \rho_b(\zeta_n,\Upsilon_n\zeta_n), \rho_b(\zeta,\Upsilon_n\zeta_n), \rho_b(\zeta_n,\Upsilon_n\zeta)\} \\
&= \kappa_n \max\{\rho_b(\zeta,\zeta_n), \rho_b(\zeta,\Upsilon_n\zeta), \rho_b(\zeta_n,\zeta_n), \rho_b(\zeta,\zeta_n), \rho_b(\zeta_n,\Upsilon_n\zeta)\} \\
&= \kappa_n \max\{\rho_b(\zeta,\zeta_n), \rho_b(\zeta,\Upsilon_n\zeta), \rho_b(\zeta_n,\Upsilon_n\zeta)\}.
\end{aligned}$$

Now taking limit as $n \to \infty$

$$\begin{aligned}
\frac{1}{s}\rho_b(\Upsilon\zeta, \zeta) &\leq \kappa \max\{0, s\rho_b(\zeta,\Upsilon\zeta), s\rho_b(\zeta,\Upsilon\zeta)\} \\
\implies \rho_b(\zeta, \Upsilon\zeta) &\leq \kappa s^2 s\rho_b(\zeta,\Upsilon\zeta) \\
(1-\kappa s^2)\rho_b(\zeta, \Upsilon\zeta) &\leq 0 \\
\implies \rho_b(\zeta, \Upsilon\zeta) &= 0 \textit{ as } \kappa < \frac{1}{s^2}. \\
\textit{Thus, } \Upsilon\zeta &= \zeta.
\end{aligned}$$

Therefore, a fixed point ζ of map Υ exists. □

Theorem 7 and 8 can be combined to get the subsequent outcome:

Theorem 9. *Let a sequence $\langle \Upsilon_n \rangle_{n\in\mathbb{N}}$ of self mappings $\Upsilon_n : \Xi \to \Xi$ be defined on a complete b-metric space (Ξ, ρ_b) with $s \geq 1$ having fixed points ζ_n and for all $\sigma, \mu \in \Xi$; $n \in \mathbb{N}$, satisfying*

$$\rho_b(\Upsilon_n\sigma, \Upsilon_n\mu) \leq \kappa_n \max\{\rho_b(\sigma,\mu), \rho_b(\sigma,\Upsilon_n\sigma), \rho_b(\mu,\Upsilon_n\mu), \rho_b(\sigma,\Upsilon_n\mu), \rho_b(\mu,\Upsilon_n\sigma)\}$$

where $\kappa_n < \frac{1}{s^2}$ and $\kappa_n \in [0,1)$. Also, let $\Upsilon : \Xi \to \Xi$ be a self mapping and for all $\sigma, \mu \in \Xi$, satisfying

$$\rho_b(\Upsilon\sigma, \Upsilon\mu) \leq \kappa \max\{\rho_b(\sigma,\mu), \rho_b(\sigma,\Upsilon\sigma), \rho_b(\mu,\Upsilon\mu), \rho_b(\sigma,\Upsilon\mu), \rho_b(\mu,\Upsilon\sigma)\}$$

where $\kappa < \frac{1}{s^2}$ and $\kappa \in [0,1)$. If $\Upsilon_n \to \Upsilon$ pointwise and $\kappa_n \to \kappa$, then a fixed point of Υ, say ζ, exists if and only if $\langle\zeta_n\rangle_{n\in\mathbb{N}}$ is convergent and in that case, $\zeta_n \to \zeta$ as $n \to \infty$.

By virtue of fixed points of sequence of self-mappings $\langle\Upsilon_n\rangle_{n\in\mathbb{N}}$; $\Upsilon_n : \Xi \to \Xi$ defined on a complete metric space (Ξ, ρ_b) and satisfying Círic contractive condition, we provide an approximation result for fixed points of self-mapping $\Upsilon : \Xi \to \Xi$ satisfying Círic contractive condition where $\Upsilon_n \to \Upsilon$ pointwise.

Corollary 1. *Let a self mapping $\Upsilon : \Xi \to \Xi$ be defined on (Ξ, ρ_b, Δ), a complete convex b-metric space with $s \geq 1$, such that for all $\sigma, \mu \in \Xi$ and $\kappa \in [0,1)$, it satisfies*

$$\rho_b(\Upsilon\sigma, \Upsilon\mu) \leq \kappa \max\{\rho_b(\sigma,\mu), \rho_b(\sigma,\Upsilon\sigma), \rho_b(\mu,\Upsilon\mu), \rho_b(\sigma,\Upsilon\mu), \rho_b(\mu,\Upsilon\sigma)\}, \tag{24}$$

where $\kappa < \frac{1}{s^2}$. Also suppose that a sequence $\langle\Upsilon_n\rangle_{n\in\mathbb{N}}$ of self mappings $\Upsilon_n : \Xi \to \Xi$ exists and for all $\sigma, \mu \in \Xi$, it satisfies

$$\rho_b(\Upsilon_n\sigma, \Upsilon_n\mu) \leq \kappa_n \max\{\rho_b(\sigma,\mu), \rho_b(\sigma,\Upsilon_n\sigma), \rho_b(\mu,\Upsilon_n\mu), \rho_b(\sigma,\Upsilon_n\mu), \rho_b(\mu,\Upsilon_n\sigma)\},$$

where $\langle\kappa_n\rangle_{n\in\mathbb{N}}$ is a sequence such that $\forall\, n \in \mathbb{N}$, $\kappa_n < \frac{1}{s^2} \in [0,1)$. Presume $\Upsilon_n \to \Upsilon$ pointwise and $\kappa_n \to \kappa$. Then, the sequence of fixed points $\langle\zeta_n\rangle_{n\in\mathbb{N}}$ of mappings Υ_n is convergent and its limit is the fixed point ζ of Υ.

Proof. By Theorem 3, Υ has a fixed point ζ which is unique.
By Theorem 5, Υ_n have fixed points ζ_n which are unique for all $n \in \mathbb{N}$.
Finally, by Theorem 9, $\zeta_n \to \zeta$. □

Example 3. *In Example 1, we consider the sequence $\langle\Upsilon_n\rangle_{n\in\mathbb{N}}$ of self mappings $\Upsilon_n : \Xi \to \Xi$ such that*

$$\Upsilon_n(\sigma) = \begin{cases} \frac{1}{25n} + \frac{\sigma}{\sqrt{5}}, & \sigma \in \Lambda = [0,1) \\ \frac{1}{26n} + \frac{\sigma}{\sqrt{7}}, & \sigma \in \Sigma = [1,\infty). \end{cases}$$

Thereafter, to prove that Υ_n satisfies the inequality (24), the following four cases exist:

1. *If both $\sigma, \mu \in \Lambda$, then*

$$\begin{aligned} \rho_b(\Upsilon_n\sigma, \Upsilon_n\mu) &= (\Upsilon_n\sigma - \Upsilon_n\mu)^2 \\ &= \left(\frac{1}{25n} + \frac{\sigma}{\sqrt{5}} - \frac{1}{25n} - \frac{\mu}{\sqrt{5}}\right)^2 \\ &= \frac{1}{5}(\sigma-\mu)^2 \\ &= \frac{1}{5}\rho_b(\sigma,\mu) \\ &< \left(\frac{1}{5} + \frac{1}{25n}\right)\rho_b(\sigma,\mu). \end{aligned}$$

2. *If $\sigma \in \Lambda$ and $\mu \in \Sigma$, then*

$$\begin{aligned} \rho_b(\Upsilon_n\sigma, \Upsilon_n\mu) &= (\Upsilon_n\sigma - \Upsilon_n\mu)^2 \\ &= \left(\frac{1}{25n} + \frac{\sigma}{\sqrt{5}} - \frac{1}{26n} - \frac{\mu}{\sqrt{7}}\right)^2 \end{aligned}$$

$$\begin{aligned}
&< \left(\frac{1}{25n}+\frac{\sigma}{\sqrt{5}}-\frac{\mu}{\sqrt{7}}\right)^2 \\
&= \left(\frac{\mu}{\sqrt{7}}-\frac{1}{25n}-\frac{\sigma}{\sqrt{5}}\right)^2 \\
&= \frac{1}{7}\left(\mu-\frac{\sqrt{7}}{25n}-\sqrt{\frac{7}{5}}\sigma\right)^2 \\
&\leq \frac{1}{7}\left(\mu-\frac{1}{25n}-\frac{\sigma}{\sqrt{5}}\right)^2 \\
&= \frac{1}{7}(\mu-\Upsilon_n\sigma)^2 \\
&= \frac{1}{7}\rho_b(\mu,\Upsilon_n\sigma) \\
&< \left(\frac{1}{5}+\frac{1}{25n}\right)\rho_b(\mu,\Upsilon_n\sigma)
\end{aligned}$$

3. *If $\sigma\in\Sigma$ and $\mu\in\Lambda$, then as in the former case, we get*

$$\rho_b(\Upsilon_n\sigma,\Upsilon_n\mu) \leq \left(\frac{1}{5}+\frac{1}{25n}\right)\rho_b(\sigma,\Upsilon_n\mu).$$

4. *If both $\sigma,\mu\in\Sigma$*

$$\begin{aligned}
\rho_b(\Upsilon_n\sigma,\Upsilon_n\mu) &= (\Upsilon_n\sigma-\Upsilon_n\mu)^2 \\
&= \left(\frac{1}{26n}\frac{\sigma}{\sqrt{7}}-\frac{1}{26n}-\frac{\mu}{\sqrt{7}}\right)^2 \\
&= \frac{1}{7}(\sigma-\mu)^2 \\
&< \frac{1}{5}(\sigma-\mu)^2 \\
&< \left(\frac{1}{5}+\frac{1}{25n}\right)\rho_b(\sigma,\mu).
\end{aligned}$$

which infers that for all $\sigma,\mu\in\Xi$,

$$\begin{aligned}
\rho_b(\Upsilon_n\sigma,\Upsilon_n\mu) \leq{}& \left(\frac{1}{5}+\frac{1}{25n}\right)\max\{\rho_b(\sigma,\mu),\rho_b(\sigma,\Upsilon_n\sigma),\rho_b(\mu,\Upsilon_n\mu),\rho_b(\sigma,\Upsilon_n\mu), \\
&\rho_b(\mu,\Upsilon_n\sigma)\}.
\end{aligned}$$

Therefore, for $k=\frac{1}{5}+\frac{1}{25n}<\frac{1}{s^2}$, Υ_n satisfies the inequality (24).

We observe that $\Upsilon_n\to\Upsilon$ pointwise and $\kappa_n\to\kappa$. Also, the sequence of fixed points of Υ_n given by $\langle\frac{1}{5\sqrt{5}(\sqrt{5}-1)n}\rangle_{n\in\mathbb{N}}$ is convergent and this sequence converges to 0 which is the fixed point of self mapping Υ.

4. Consequence

Presume self mappings Υ *and* $I:\Xi\to\Xi$ defined on a non-empty set Ξ. For the mappings Υ and I, a point $\sigma\in\Xi$ for which $\Upsilon\sigma=I\sigma$ ($\sigma=\Upsilon\sigma=I\sigma$) is termed a coincidence point (common fixed point). Moreover, if the mappings Υ and I commute at every coincidence point, then the mappings Υ and I are termed weakly compatible.

Lemma 3. *Let Ξ be a nonempty set and $\Upsilon:\Xi\to\Xi$ be a self mapping defined on it. Then a subset of Ξ, say Θ, exists such that the mapping $\Upsilon:\Theta\to\Xi$ is one-to-one and $\Upsilon(\Theta)=\Upsilon(\Xi)$.*

Subsequently, a common fixed point theorem is obtained in continuation of the primary results established in the former section.

Theorem 10. *Let* Υ *and* $I : \Xi \to \Xi$ *be self mappings defined on a convex b-metric space* (Ξ, ρ_b, Δ) *with parameter* $s \geq 2$ *such that for all* $\sigma, \mu \in \Xi$ *and* $\kappa < 1$*, they satisfy*

$$b_m(\Upsilon\sigma, \Upsilon\mu) \leq \kappa \max\{\rho_b(I\sigma, I\mu), \rho_b(I\sigma, \Upsilon\sigma), \rho_b(I\mu, \Upsilon\mu), \rho_b(I\sigma, \Upsilon\mu), \rho_b(I\mu, \Upsilon\sigma)\}. \tag{25}$$

If $\kappa < \frac{1}{s^2}$*,* $I(\Xi)$ *is complete and* $\Upsilon(\Xi) \subseteq I(\Xi)$ *then a unique coincidence point of mappings* Υ *and* I *exists . Besides, if* Υ *and* I *are weakly compatible mappings, then a common fixed point of these mappings exists that is unique.*

Proof. By lemma 3, a subset of Ξ, say Θ, exists such that the mapping $I : \Theta \to \Xi$ is one-to-one and $I(\Theta) = I(\Xi)$. Further, let $\pi : I(\Theta) \to I(\Theta)$ be another self mapping defined by $\pi(I\sigma) = \Upsilon\sigma$. Then, since the mapping I is one-to-one, π is clearly well defined. Thus, for all $I\sigma, I\mu \in I(\Theta) = I(\Xi)$, we arrive that

$$\begin{aligned}
\rho_b(\pi(I\sigma), \pi(I\mu)) &= \rho_b(\Upsilon\sigma, \Upsilon\mu) \\
&\leq \kappa \max\{\rho_b(I\sigma, I\mu), \rho_b(I\sigma, \Upsilon\sigma), \rho_b(I\mu, \Upsilon\mu), \rho_b(I\sigma, \Upsilon\mu), \rho_b(I\mu, \Upsilon\sigma)\} \\
&= \kappa \max\{\rho_b(I\sigma, I\mu), \rho_b(I\sigma, \pi(I\sigma)), \rho_b(I\mu, \pi(I\mu)), \rho_b(I\sigma, \pi(I\mu)), \rho_b(I\mu, \pi(I\sigma))\}.
\end{aligned}$$

As $\kappa < 1$ with $\kappa < \frac{1}{s^2}$, then π is a Cirić contraction on $I(\Xi)$. Besides, a unique point $\sigma^* \in \Pi \subseteq \Xi$ exists on account of Theorem 3 since $I(\Xi)$ is complete yielding $p(I\sigma^*) = I\sigma^*$ implying $I\sigma^* = \Upsilon\sigma^*$. Thus, a coincidence point σ^* of mappings Υ and I exists that is unique.

Let $\zeta = I\sigma^* = \Upsilon\sigma^*$. Furthermore, let Υ and I be weakly compatible mappings following $\Upsilon\zeta = \Upsilon I\sigma^* = I\Upsilon\sigma^* = I\zeta$. As a result,

$$\begin{aligned}
\rho_b(\Upsilon\zeta, \zeta) &= \rho_b(\Upsilon\zeta, \Upsilon\sigma^*) \\
&\leq \kappa \max\{\rho_b(I\zeta, I\sigma^*), \rho_b(I\zeta, \Upsilon\zeta), \rho_b(I\sigma^*, \Upsilon\sigma^*), \rho_b(I\zeta, \Upsilon\sigma^*), \rho_b(I\sigma^*, \Upsilon\zeta)\} \\
&= \kappa\rho_b(\Upsilon\zeta, \zeta),
\end{aligned} \tag{26}$$

which is true for $\kappa \in [0, 1)$ if $\zeta = \Upsilon\zeta = I\zeta$ and hence a common fixed point ζ of mappings Υ and I exists that is unique. □

5. Application to Initial Value Problem

In this section, the existence of unique solution to an Initial Value Problem containing a differential equation of second order with two initial conditions is discussed.

$$\left.\begin{aligned}
&\frac{d^2\mu(\tau)}{d\tau^2} + \phi(\tau)\frac{d\mu(\tau)}{d\tau} + \psi(\tau)\mu(\tau) = \theta(\tau), \\
&\textit{with } \mu(\beta) = c_1 \\
&\textit{and } \frac{d\mu(\beta)}{d\tau} = c_2,
\end{aligned}\right\} \tag{27}$$

where $\phi(\tau), \psi(\tau), \theta(\tau)$ and $\mu(\tau)$ are continuous functions in $[\beta, \delta]$ and $\phi(\tau)$ is differentiable in $[\beta, \delta]$.

First we shall convert this Initial Value Problem (27) into Voltera Integral Equation of the second kind.

Lemma 4. *The Initial Value Problem (27) is equivalent to Voltera Integral Equation of the second kind*

$$\mu(\tau) = \Phi(\tau) + \int_\beta^\tau \Psi(\tau, \gamma, \mu(\gamma))\, d\gamma,$$

where $\Phi(\tau)$ *is continuous and* $\Psi : [\beta, \delta] \times [\beta, \delta] \times \mathbb{R} \to \mathbb{R}$.

Proof. Integrating first equation of (27) from β to τ and using remaining two initial conditions, we have,

$$\frac{d\mu(\tau)}{d\tau} - \frac{d\mu(\beta)}{d\tau} + |\phi(\gamma)\mu(\gamma)|_\beta^\tau - \int_\beta^\tau \frac{d\phi(\gamma)}{d\gamma}\mu(\gamma)\,d\gamma + \int_\beta^\tau \psi(\gamma)\mu(\gamma)\,d\gamma = \int_\beta^\tau \theta(\gamma)\,d\gamma$$
$$\frac{d\mu(\tau)}{d\tau} - c_2 + \phi(\tau)\mu(\tau) - \phi(\beta)\mu(\beta) + \int_\beta^\tau \left\{\psi(\gamma) - \frac{d\phi(\gamma)}{d\gamma}\right\}\mu(\gamma)\,d\gamma = \int_\beta^\tau \theta(\gamma)\,d\gamma.$$

Integrating again from β to τ,

$$\begin{aligned} \mu(\tau) \quad - \quad & \mu(\beta) - [c_2 + c_1\phi(\beta)](\tau - \beta) + \int_\beta^\tau \phi(\gamma)\mu(\gamma)\,d\gamma \\ & + \int_\beta^\tau \int_\beta^\gamma \left\{\psi(\gamma) - \frac{d\phi(\gamma)}{d\gamma}\right\} d\gamma_1\,d\gamma = \int_\beta^\tau \int_\beta^\gamma \theta(\gamma_1)\,d\gamma_1\,d\gamma \\ \mu(\tau) \quad = \quad & c_1 + [c_2 + c_1\phi(\beta)](\tau - \beta) + \int_\beta^\tau (\tau - \gamma)\theta(\gamma)\,d\gamma \\ & + \int_\beta^\tau \left\{(\tau - \gamma)\left(\frac{d\phi(\gamma)}{d\gamma} - \psi(\gamma)\right) + \phi(\gamma)\right\}\mu(\gamma)\,d\gamma. \end{aligned}$$

Thus, we have integral equation of the kind

$$\begin{aligned} \mu(\tau) &= \Phi(\tau) + \int_\beta^\tau \Psi(\tau, \gamma, \mu(\gamma))\,d\gamma, \qquad (28) \\ \textit{where } \Phi(\tau) &= c_1 + [c_2 + c_1\phi(\beta)](\tau - \beta) + \int_\beta^\tau (\tau - \gamma)\theta(\gamma)\,d\gamma \\ \textit{and } \Psi(\tau, \gamma, \mu(\gamma)) &= \left\{(\tau - \gamma)\left(\frac{d\phi(\gamma)}{d\gamma} - \psi(\gamma)\right) + \phi(\gamma)\right\}\mu(\gamma) \end{aligned}$$

□

Thus, the Initial Value Problem (27) is equivalent to the Voltera Integral Equation of the second kind (28)

$$\mu(\tau) = \Phi(\tau) + \int_\beta^\tau \Psi(\tau, \gamma, \mu(\gamma))\,d\gamma \;\; \forall\, \tau, \gamma \in [\beta, \delta],$$

where $\Phi(\tau) \in C[\beta, \delta] = \{\sigma(\tau);\, \sigma : [\beta, \delta] \to \mathbb{R}\} = \Xi$, say, and $\Psi : [\beta, \delta] \times [\beta, \delta] \times \mathbb{R} \to \mathbb{R}$. Define $\rho_b : \Xi \times \Xi \to [0, \infty)$ by

$$\rho_b(\sigma, \mu) = \sup_{\tau \in [\beta, \delta]} (\sigma(\tau) - \mu(\tau))^2 \;\; \forall\, \sigma, \mu \in \Xi$$

Define $\Delta : \Xi \times \Xi \times [0, 1] \to \Xi$ as

$$\Delta(\sigma, \mu; \alpha) = \alpha\sigma + (1 - \alpha)\mu.$$

Additionally, presume a self mapping $\Upsilon : \Xi \to \Xi$ defined as

$$\Upsilon\mu(\tau) = \Phi(\tau) + \int_\beta^\tau \Psi(\tau, \gamma, \mu(\gamma))\,d\gamma \;\; \forall\, \tau, \gamma \in [\beta, \delta].$$

Then, existence of unique fixed point of map Υ implies the existence and uniqueness of solution of Voltera integral Equation of the second kind (28) and hence, the Initial Value Problem (27).

Lemma 5. *Suppose* $\Xi = C[\beta,\delta] = \{\sigma(\tau);\ \sigma : [\beta,\delta] \to \mathbb{R}\}$ *and define* $\rho_b : \Xi \times \Xi \to [0,\infty)$ *by*

$$\rho_b(\sigma,\mu) = \sup_{\tau\in[\beta,\delta]} (\sigma(\tau) - \mu(\tau))^2 \ \ \forall\, \sigma,\mu \in \Xi$$

Define the convex structure $\Delta : \Xi \times \Xi \times [0,1] \to \Xi$ *as*

$$\Delta(\sigma,\mu;\alpha) = \alpha\sigma + (1-\alpha)\mu \ \ \forall\, \sigma,\mu \in \Xi.$$

Then, (Ξ,ρ_b,Δ) *is a convex b-metric space with parameter* $s \geq 2$*.*

Proof. We perceive that

1. $\rho_b(\sigma,\mu) \geq 0 \ \ \forall\, \sigma,\mu \in \Xi$.
2. $\rho_b(\sigma,\mu) = 0 \iff \sigma = \mu$.
3. $\rho_b(\sigma,\mu) = \rho_b(\mu,\sigma)$.
4. $\rho_b(\sigma,\mu) \leq 2[\rho_b(\sigma,\xi) + \rho_b(\xi,\mu)]$ as

$$\begin{aligned}
\rho_b(\sigma,\mu) &= \sup_{\tau\in[\beta,\delta]} (\sigma(\tau) - \mu(\tau))^2 \\
&= \sup_{\tau\in[\beta,\delta]} (\sigma(\tau) - \xi(\tau) + \xi(\tau) - \mu(\tau))^2 \\
&\leq 2\left\{ \sup_{\tau\in[\beta,\delta]} (\sigma(\tau) - \xi(\tau))^2 + \sup_{\tau\in[\beta,\delta]} (\xi(\tau) - \mu(\tau))^2 \right\} \\
&= 2[\rho_b(\sigma,\xi) + \rho_b(\xi,\mu)].
\end{aligned}$$

Also, for $\Delta(\sigma,\mu;\alpha) = \alpha\sigma + (1-\alpha)\mu \ \ \forall\, \sigma,\mu \in \Xi$, we have

$$\begin{aligned}
\rho_b(\xi,\Delta(\sigma,\mu;\alpha)) &= \sup_{\tau\in[\beta,\delta]} (\xi(\tau) - \Delta(\sigma(\tau),\mu(\tau);\alpha))^2 \\
&= \sup_{\tau\in[\beta,\delta]} (\xi(\tau) - \{\alpha\sigma(\tau) + (1-\alpha)\mu(\tau)\})^2 \\
&\leq \sup_{\tau\in[\beta,\delta]} (\alpha|\xi(\tau) - \sigma(\tau)| + (1-\alpha)|\xi(\tau) - \mu(\tau)|)^2 \\
&= \sup_{\tau\in[\beta,\delta]} [\alpha^2(\xi(\tau) - \sigma(\tau))^2 + (1-\alpha)^2(\xi(\tau) - \mu(\tau))^2 \\
&\quad +2\alpha(1-\alpha)|\xi(\tau) - \sigma(\tau)||\xi(\tau) - \mu(\tau)|] \\
&\leq \sup_{\tau\in[\beta,\delta]} [\alpha^2(\xi(\tau) - \sigma(\tau))^2 + (1-\alpha)^2(\xi(\tau) - \mu(\tau))^2 \\
&\quad +\alpha(1-\alpha)\{(\xi(\tau) - \sigma(\tau))^2 + (\xi(\tau) - \mu(\tau))^2\}] \\
&\leq \alpha \sup_{\tau\in[\beta,\delta]} (\xi(\tau) - \sigma(\tau))^2 + (1-\alpha) \sup_{\tau\in[\beta,\delta]} (\xi(\tau) - \mu(\tau))^2 \\
&= \alpha\rho_b(\xi,\sigma) + (1-\alpha)\rho_b(\xi,\mu)
\end{aligned}$$

Thus, for $s \geq 2$, (Ξ,ρ_b,Δ) is convex b-metric space. □

Theorem 11. *Suppose that*

$$|\Psi(\tau,\gamma,\sigma(\gamma)) - \Psi(\tau,\gamma,\mu(\gamma))| \leq [\kappa M(\sigma,\mu)]^{\frac{1}{2}}$$

for all $\tau,\gamma \in [\beta,\delta]$*;* $\sigma,\mu \in \Xi$ *and some* $\kappa < \frac{1}{(\delta-\beta)^2} \leq \frac{1}{s^2}$ *where* $s \geq 1$ *and*

$$M(\sigma,\mu) = \max\{\rho_b(\sigma,\mu), \rho_b(\sigma,\Upsilon\sigma), \rho_b(\mu,\Upsilon\mu), \rho_b(\sigma,\Upsilon\mu), \rho_b(\mu,\Upsilon\sigma)\}.$$

Then, a unique solution exists for Voltera Integral Equation of the second kind (28).

Proof. Consider

$$
\begin{aligned}
(\Upsilon\sigma(\tau) - \Upsilon\mu(\tau))^2 &\leq \left(\int_\beta^\tau |\Psi(\tau,\gamma,\sigma(\gamma)) - \Psi(\tau,\gamma,\mu(\gamma))|\, d\gamma \right)^2 \\
&\leq \left(\int_\beta^\tau [\kappa M(\sigma,\mu)]^{\frac{1}{2}}\, d\gamma \right)^2 \\
&\leq \kappa M(\sigma,\mu) \left(\int_\beta^\tau d\gamma \right)^2 \\
&= \kappa M(\sigma,\mu)(\tau-\beta)^2 \\
&\leq \kappa M(\sigma,\mu)(\delta-\beta)^2,
\end{aligned}
$$

and thus all the hypothesis of Theorem 3 are satisfied for $\kappa < \frac{1}{(\delta-\beta)^2} \leq \frac{1}{s^2}$ implying Voltera Integral Equation of the second kind (28) and hence, the Initial Value Problem (27) has a solution that is unique. □

6. Discussion, Conclusions and Open Problems

In the framework of convex b-metric spaces, we established a fixed point theorem as an extension of the main result of Rathee et al. [11] that guarantees the availability of fixed point for Cirić contraction. Additionally, the Krasnoselskij iterative process is used for approximating the fixed point. Furthermore, we discussed about the fixed point's stability for the prior mentioned contraction. We constructed a common fixed point and coincidence point result as a consequence. Finally, we provided several examples to highlight the conclusions drawn here and use these conclusions to solve an initial value problem. Following open problems may be worked upon in future:

1. Rathee et al. [11] ensured the existence of fixed point for Cirić contraction for the constant $\kappa \in \left[0, \frac{1}{s^4}\right)$. In addition, we extended the same for $\kappa \in \left[\frac{1}{s^4}, \frac{1}{s^2}\right)$. Is it viable to further relax the condition for $\kappa \in \left[\frac{1}{s^2}, 1\right)$?
2. Besides, we proved that the fixed points so obtained are stable for $\kappa \in \left[0, \frac{1}{s^2}\right)$. Can the hypothesis be eased?

Author Contributions: Investigation, S.R., A.K. (Anshuka Kadyan), M. (Minakshi) and A.K. (Anil Kumar); Methodology, G., S.R., M. (Mahima), A.K. (Anshuka Kadyan), M. (Minakshi) and A.K. (Anil Kumar); Writing—original draft, S.R., A.K. (Anshuka Kadyan), M. (Minakshi) and A.K. (Anil Kumar); Writing—review & editing, G. and M. (Mahima); Software, G. and M. (Mahima); Writing and typesetting using LaTeX, G. and M. (Mahima). All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Data Availability Statement: Data sharing not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

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