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Determinantal Expressions, Identities, Concavity, Maclaurin Power Series Expansions for van der Pol Numbers, Bernoulli Numbers, and Cotangent

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Abstract: In this paper, basing on the generating function for the van der Pol numbers, utilizing the Maclaurin power series expansion and two power series expressions of a function involving the cotangent function, and by virtue of the Wronski formula and a derivative formula for the ratio of two differentiable functions, the authors derive four determinantal expressions for the van der Pol numbers, discover two identities for the Bernoulli numbers and the van der Pol numbers, prove the increasing property and concavity of a function involving the cotangent function, and establish two alternative Maclaurin power series expansions of a function involving the cotangent function. The coefficients of the Maclaurin power series expansions are expressed in terms of specific Hessenberg determinants whose elements contain the Bernoulli numbers and binomial coefficients.

Keywords: determinantal expression; identity; concavity; van der Pol number; Bernoulli number; Maclaurin power series expansion; cotangent; Wronski's formula; derivative formula; Hessenberg determinant; binomial coefficient

MSC: 11B68; 11B83; 15A15; 26A06; 26A09; 26A24; 26A51; 33B10; 41A58

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1. Motivations

In [1] (p. 235), see also Section 1 in [2], the sequence of rational numbers V_k for $k \ge 0$ was defined by means of

$$\begin{aligned}\frac{x^3}{6x(e^x+1)-12(e^x-1)} &= \sum_{k=0}^{\infty} V_k \frac{x^k}{k!}\\ &= 1-\frac{x}{2}+\frac{x^2}{10}-\frac{x^3}{120}-\frac{x^4}{8400}+\frac{x^5}{16{,}800}+\frac{x^6}{756{,}000}-\frac{x^7}{1{,}512{,}000}-\frac{37x^8}{2{,}328{,}480{,}000}-\cdots. \end{aligned}\tag{1}$$

The first few values of V_k for $0 \le k \le 8$ are

$$\begin{aligned} &V_0=1, && V_1=-\frac{1}{2}, && V_2=\frac{1}{5}, && V_3=-\frac{1}{20}, && V_4=-\frac{1}{350},\\ &V_5=\frac{1}{140}, && V_6=\frac{1}{1050}, && V_7=-\frac{1}{300}, && V_8=-\frac{37}{57{,}750}. \end{aligned}\tag{2}$$

Let $J_\nu(z)$ denote the Bessel function and

$$\sigma_{2k}(\nu)=\sum_{m=1}^{\infty}\frac{1}{j_{\nu,m}^{2k}}, \quad k\ge 1,$$

where $j_{\nu,m}$ are the zeros of $\frac{J_\nu(z)}{z^\nu}$. Then,

$$\sigma_{2k}\left(\frac{3}{2}\right) = (-1)^{k-1}\frac{3 \times 2^{2k-1}}{(2k)!}V_{2k}, \quad k > 1.$$

The van der Pol numbers V_k are also related to the Bernoulli numbers B_k for $k \geq 0$ via the relation

$$\sigma_{2k}\left(\frac{1}{2}\right) = (-1)^{k-1}\frac{2^{2k-1}}{(2k)!}B_{2k}, \quad k \geq 1,$$

where the Bernoulli numbers B_k for $k \geq 0$ are generated by

$$\frac{z}{e^z - 1} = \sum_{k=0}^{\infty} B_k \frac{z^k}{k!} = 1 - \frac{z}{2} + \sum_{k=1}^{\infty} B_{2k}\frac{z^{2k}}{(2k)!}, \quad |z| < 2\pi.$$

The Rayleigh function $\sigma_k(\nu)$ for $k > 1$ has an alternative notation

$$\zeta_\nu(k) = \sum_{m=1}^{\infty} \frac{1}{j_{\nu,m}^k}, \quad k > 1.$$

This is also called the Bessel zeta function in [3] and was originally introduced and studied in papers [4–7]. In paper [8], the authors derived closed-form expressions of the Bessel zeta function $\zeta_\nu(2k)$ for $k \geq 1$ in terms of specific Hessenberg determinants.

At the site https://math.stackexchange.com/q/4447783 (accessed on 27 May 2023), a question was asked: is there an explicit formula for the Maclaurin expansion of the function $\frac{x^2}{1 - x\cot x}$ around the origin $x = 0$? At the site https://math.stackexchange.com/a/4449466 (accessed on 5 June 2023), an answer to this question was given. We now quote this answer as follows.

Using Bessel functions, we find

$$\begin{aligned}
\frac{x^2}{1 - x\cot x} &= 3 - x\frac{J_{5/2}(x)}{J_{3/2}(x)} \\
&= \frac{3}{2} + x\frac{J_{3/2}'(x)}{J_{3/2}(x)} \\
&= 3 - 2x^2 \sum_{k=1}^{\infty} \frac{1}{1 - \left(x/j_{3/2,k}^2\right)^2} \\
&= 3 - 2x^2 \sum_{k=1}^{\infty}\sum_{n=0}^{\infty} \frac{1}{j_{3/2,k}^{2n}} x^{2n} \\
&= 3 - 2x^2 \sum_{n=0}^{\infty} \left(\sum_{k=1}^{\infty} \frac{1}{j_{3/2,k}^{2n}}\right) x^{2n} \\
&= 3 - 2x^2 \sum_{n=0}^{\infty} \sigma_{2n}\left(\frac{3}{2}\right) x^{2n},
\end{aligned}$$

where $j_{3/2,k}$ denotes the kth positive zero of $J_{3/2}$ and σ_n is the Rayleigh function of order n. If $n \geq 1$, we can write

$$\sigma_{2n}\left(\frac{3}{2}\right) = (-1)^{n-1} 3 \times 2^{2n-1}\frac{V_{2n}}{(2n)!},$$

where V_n is the nth van der Pol number. See paper [2] for the properties of these numbers, including recurrence relations. In particular, the generating function is Equation (d) in Section 1.

In [2] (Sections 1 and 6), we find the series expansion

$$\frac{z^2}{1 - z\cot z} = 3 - \frac{z^2}{5} + 3\sum_{k=2}^{\infty}(-1)^k 2^{2k} V_{2k} \frac{z^{2k}}{(2k)!} \tag{3}$$

by defining $\frac{z^2}{1-z\cot z}$ to be 3 at $z = 0$. According to the discussion in [2] (Section 6), the Maclaurin power series expansion (3) has a finite positive radius of convergence, and this radius of convergence, denoted by R, is located between $\frac{4\pi}{3}$ and $\frac{3\pi}{2}$. In fact, the number R is the first positive zero of the equation $\tan x - x = 0$.

Let

$$G(x) = \begin{cases} \dfrac{x^2}{1 - x\cot x}, & 0 < |x| < R; \\ 3, & x = 0. \end{cases} \tag{4}$$

It is clear that the function $G(x)$ is even on $(-R, R)$.

In [9] (p. 75, Entry 4.3.70), it was listed that

$$\cot z = \frac{1}{z} - \sum_{k=1}^{\infty} \frac{(-1)^{k-1} 2^{2k}}{(2k)!} B_{2k} z^{2k-1}, \quad |z| < \pi.$$

Hence, we have

$$G(x) = \frac{1}{\sum_{k=0}^{\infty} \frac{2^{2k+2}}{(2k+2)!} |B_{2k+2}| x^{2k}}, \quad 0 < |x| < R. \tag{5}$$

On the other hand,

$$G(x) = \frac{x^2 \sin x}{\sin x - x\cos x} = \frac{1}{2} \frac{\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k}}{\sum_{k=0}^{\infty} \frac{(-1)^k (k+1)}{(2k+3)!} x^{2k}}, \quad |x| < R. \tag{6}$$

In this paper, basing on the generating function (1) for the van der Pol numbers V_n, on the Maclaurin power series expansion (3) of the function $G(z)$, and on the expressions in (5) and (6) of the function $G(x)$, by virtue of Wronski's formula in [10] (p. 17, Theorem 1.3), and in view of a derivative formula for the ratio of two differentiable functions, we will derive four determinantal expressions for the van der Pol numbers V_n and V_{2k}, discover two identities for the Bernoulli numbers B_{2k} and the van der Pol numbers V_{2k}, prove the increasing property and concavity of the function $G(x)$, and establish two alternative Maclaurin power series expansions of the function $G(x)$, whose coefficients are expressed in terms of specific Hessenberg determinants in which the Bernoulli numbers B_{2k} and binomial coefficients are involved.

2. Lemmas

For smoothly solving the above two problems, we need the following lemmas.

Lemma 1 (Wronski's formula [10] (p. 17, Theorem 1.3))**.** *If $a_0 \neq 0$ and*

$$P(x) = a_0 + a_1 x + a_2 x^2 + \cdots$$

is a formal series, then the coefficients of the reciprocal series

$$\frac{1}{P(x)} = b_0 + b_1 x + b_2 x^2 + \cdots$$

are given by

$$b_r = \frac{(-1)^r}{a_0^{r+1}} \begin{vmatrix} a_1 & a_0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ a_2 & a_1 & a_0 & 0 & \cdots & 0 & 0 & 0 \\ a_3 & a_2 & a_1 & a_0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{r-2} & a_{r-3} & a_{r-4} & a_{r-5} & \cdots & a_1 & a_0 & 0 \\ a_{r-1} & a_{r-2} & a_{r-3} & a_{r-4} & \cdots & a_2 & a_1 & a_0 \\ a_r & a_{r-1} & a_{r-2} & a_{r-3} & \cdots & a_3 & a_2 & a_1 \end{vmatrix}, \quad r \in \mathbb{N}. \tag{7}$$

Lemma 2 ([11] (p. 40, Exercise 5))**.** *Let $u(x)$ and $v(x) \neq 0$ be two n-time differentiable functions on an interval I for a given integer $n \geq 0$. Then, the nth derivative of the ratio $\frac{u(x)}{v(x)}$ is*

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[\frac{u(x)}{v(x)}\right] = (-1)^n \frac{|W_{(n+1)\times(n+1)}(x)|}{v^{n+1}(x)}, \quad n \geq 0, \tag{8}$$

where the matrix

$$W_{(n+1)\times(n+1)}(x) = \begin{pmatrix} U_{(n+1)\times 1}(x) & V_{(n+1)\times n}(x) \end{pmatrix}_{(n+1)\times(n+1)},$$

the matrix $U_{(n+1)\times 1}(x)$ is an $(n+1)\times 1$ matrix whose elements satisfy $u_{k,1}(x) = u^{(k-1)}(x)$ for $1 \leq k \leq n+1$, the matrix $V_{(n+1)\times n}(x)$ is an $(n+1)\times n$ matrix whose elements are

$$v_{\ell,j}(x) = \begin{cases} \dbinom{\ell-1}{j-1} v^{(\ell-j)}(x), & \ell - j \geq 0 \\ 0, & \ell - j < 0 \end{cases}$$

for $1 \leq \ell \leq n+1$ and $1 \leq j \leq n$, and the notation $|W_{(n+1)\times(n+1)}(x)|$ denotes the determinant of the $(n+1)\times(n+1)$ matrix $W_{(n+1)\times(n+1)}(x)$.

3. A Relation and Two Identities for van der Pol and Bernoulli Numbers

In this section, by comparing the power series expansion (3) and two expressions (5) and (6), we derive a relation and two identities for van der Pol numbers V_{2k} and the Bernoulli numbers B_{2k}.

Theorem 1. *For $k \geq 2$, we have*

$$V_{2k} = (2k)!\left(\frac{3}{4}\right)^k \begin{vmatrix} a_1 & a_0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ a_2 & a_1 & a_0 & 0 & \cdots & 0 & 0 & 0 \\ a_3 & a_2 & a_1 & a_0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{k-2} & a_{k-3} & a_{k-4} & a_{k-5} & \cdots & a_1 & a_0 & 0 \\ a_{k-1} & a_{k-2} & a_{k-3} & a_{k-4} & \cdots & a_2 & a_1 & a_0 \\ a_k & a_{k-1} & a_{k-2} & a_{k-3} & \cdots & a_3 & a_2 & a_1 \end{vmatrix}, \tag{9}$$

where

$$a_k = \frac{2^{2k+2}}{(2k+2)!}|B_{2k+2}|, \qquad k \geq 0.$$

Proof. Comparing the series expansion (3) and Expression (5), we find that

$$\left[3 - \frac{1}{5}x^2 + 3\sum_{k=2}^{\infty}(-1)^k \frac{2^{2k}}{(2k)!}V_{2k}x^{2k}\right]\sum_{k=0}^{\infty}\frac{2^{2k+2}}{(2k+2)!}|B_{2k+2}|x^{2k} = 1.$$

Applying Formula (7) in Lemma 1 and rearranging yield Formula (9). □

Remark 1. *When $k = 4$, we have*

$$V_8 = 8!\left(\frac{3}{4}\right)^4 \begin{vmatrix} a_1 & a_0 & 0 & 0 \\ a_2 & a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 & a_0 \\ a_4 & a_3 & a_2 & a_1 \end{vmatrix} = 8!\left(\frac{3}{4}\right)^4 \begin{vmatrix} \frac{1}{45} & \frac{1}{3} & 0 & 0 \\ \frac{2}{945} & \frac{1}{45} & \frac{1}{3} & 0 \\ \frac{1}{4725} & \frac{2}{945} & \frac{1}{45} & \frac{1}{3} \\ \frac{2}{93555} & \frac{1}{4725} & \frac{2}{945} & \frac{1}{45} \end{vmatrix} = -\frac{37}{57750}.$$

When $k = 3$, we have

$$V_6 = 6!\left(\frac{3}{4}\right)^3 \begin{vmatrix} a_1 & a_0 & 0 \\ a_2 & a_1 & a_0 \\ a_3 & a_2 & a_1 \end{vmatrix} = 6!\left(\frac{3}{4}\right)^3 \begin{vmatrix} \frac{1}{45} & \frac{1}{3} & 0 \\ \frac{2}{945} & \frac{1}{45} & \frac{1}{3} \\ \frac{1}{4725} & \frac{2}{945} & \frac{1}{45} \end{vmatrix} = \frac{1}{1050}.$$

When $k = 2$, we have

$$V_4 = 4!\left(\frac{3}{4}\right)^2 \begin{vmatrix} a_1 & a_0 \\ a_2 & a_1 \end{vmatrix} = 4!\left(\frac{3}{4}\right)^2 \begin{vmatrix} \frac{1}{45} & \frac{1}{3} \\ \frac{2}{945} & \frac{1}{45} \end{vmatrix} = -\frac{1}{350}.$$

These three values are congruent to the corresponding ones in (2).

Theorem 2. *For $k \ge 0$, we have*

$$\sum_{j=0}^{k} (-1)^j 2^{2j} \binom{2k+3}{2j+2} |B_{2j+2}| = \frac{k+1}{2}. \tag{10}$$

Proof. From Expressions (5) and (6), it follows that

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{(-1)^k (k+1)}{(2k+3)!} x^{2k} &= \frac{1}{2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k} \sum_{k=0}^{\infty} \frac{2^{2k+2}}{(2k+2)!} |B_{2k+2}| x^{2k} \\ &= \frac{1}{2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+3)!} \left[\sum_{j=0}^{k} (-1)^j \binom{2k+3}{2j+2} 2^{2j+2} |B_{2j+2}| \right] x^{2k}. \end{aligned}$$

Accordingly, we obtain

$$\frac{(-1)^k (k+1)}{(2k+3)!} = \frac{1}{2} \frac{(-1)^k}{(2k+3)!} \left[\sum_{j=0}^{k} (-1)^j \binom{2k+3}{2j+2} 2^{2j+2} |B_{2j+2}| \right], \quad k \ge 0,$$

which can be simplified as the identity (10). □

Theorem 3. *For $k \ge 2$, we have*

$$\sum_{j=2}^{k} (k-j+1) 2^{2j} \binom{2k+3}{2j} V_{2j} = -\frac{4}{15} k(k^2-1). \tag{11}$$

Proof. Considering the series expansion (3) and Expression (6), we arrive at

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k} &= 2\left[3 - \frac{1}{5}x^2 + 3 \sum_{k=2}^{\infty} (-1)^k \frac{2^{2k}}{(2k)!} V_{2k} x^{2k} \right] \sum_{k=0}^{\infty} \frac{(-1)^k (k+1)}{(2k+3)!} x^{2k} \\ &= 2 \sum_{k=0}^{\infty} (-1)^k \left[\frac{3(k+1)}{(2k+3)!} + \frac{1}{5} \frac{k}{(2k+1)!} + 3 \sum_{j=2}^{k} \frac{k-j+1}{(2j)!(2k-2j+3)!} 2^{2j} V_{2j} \right] x^{2k}. \end{aligned}$$

This means that

$$\frac{1}{(2k+1)!}=2\left[\frac{3(k+1)}{(2k+3)!}+\frac{1}{5}\frac{k}{(2k+1)!}+3\sum_{j=2}^{k}\frac{k-j+1}{(2j)!(2k-2j+3)!}2^{2j}V_{2j}\right],$$

which can be rewritten as identity (11). The proof of Theorem 3 is complete. □

4. A Determinantal Expression of van der Pol Numbers

In [12] (Theorem 2.7), an explicit formula of the van der Pol numbers V_n for $n\ge 0$ was given by

$$\begin{aligned}V_n=&\sum_{k=0}^{n}\frac{k!}{(n+2k)!}\binom{n}{k}\sum_{q=0}^{n-k}\langle -3k\rangle_q(n-k-q)!\binom{n-k}{q}\binom{n+2k}{n-k-q}\\&\times\sum_{\ell=0}^{k}12^{\ell}\binom{k}{\ell}\sum_{p=0}^{n+2k}(n+2k-p)!\binom{n+2k}{p}\frac{S(p+\ell,\ell)}{\binom{p+\ell}{\ell}}\\&\times\sum_{s=0}^{k-\ell}\binom{k-\ell}{s}\frac{(-6)^s}{(n-p+2\ell+2s)!}\sum_{\beta=1}^{s}\binom{s}{\beta}\beta^{n-p+2\ell+2s},\end{aligned}\tag{12}$$

where $S(p,q)$ for $p\ge q\ge 0$ denotes the Stirling numbers of the second kind and

$$\langle z\rangle_n=\prod_{k=0}^{n-1}(z-k)=\begin{cases}z(z-1)\cdots(z-n+1), & n\in\mathbb{N}\\ 1, & n=0\end{cases}$$

is the nth falling factorial of $z\in\mathbb{C}$.

In this section, by virtue of Lemma 2, we deduce an alternative formula of the van der Pol numbers V_n in terms of specific Hessenberg determinants.

Theorem 4. *For $n\ge 1$, the van der Pol numbers V_n can be expressed by the determinant*

$$\begin{aligned}V_n&=(-1)^n\begin{vmatrix}\binom{1}{0}\frac{1}{2} & \binom{1}{1} & 0 & \cdots & 0\\ \binom{2}{0}\frac{3}{10} & \binom{2}{1}\frac{1}{2} & \binom{2}{2} & \cdots & 0\\ \binom{3}{0}\frac{1}{5} & \binom{3}{1}\frac{3}{10} & \binom{3}{2}\frac{1}{2} & \cdots & 0\\ \binom{4}{0}\frac{1}{7} & \binom{4}{1}\frac{1}{5} & \binom{4}{2}\frac{3}{10} & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \binom{n-2}{0}\frac{6}{n(n+1)} & \binom{n-2}{1}\frac{6}{(n-1)n} & \binom{n-2}{2}\frac{6}{(n-2)(n-1)} & \cdots & 0\\ \binom{n-1}{0}\frac{6}{(n+1)(n+2)} & \binom{n-1}{1}\frac{6}{n(n+1)} & \binom{n-1}{2}\frac{6}{(n-1)n} & \cdots & \binom{n-1}{n-1}\\ \binom{n}{0}\frac{6}{(n+2)(n+3)} & \binom{n}{1}\frac{6}{(n+1)(n+2)} & \binom{n}{2}\frac{6}{n(n+1)} & \cdots & \binom{n}{n-1}\frac{1}{2}\end{vmatrix}\\&=(-1)^n\left|v_{i,j}\right|_{n\times n},\end{aligned}\tag{13}$$

where

$$v_{i,j}=\begin{cases}0, & 3\le i+2\le j\le n;\\ \dbinom{i}{j-1}\dfrac{6}{(i-j+3)(i-j+4)}, & n\ge i+2>j\ge 1.\end{cases}$$

Proof. The generating function in (1) of the van der Pol numbers V_k can be written as

$$\frac{x^3}{6x(\mathrm{e}^x+1)-12(\mathrm{e}^x-1)}=\frac{1}{6}\frac{1}{\sum_{k=0}^{\infty}\frac{k+1}{(k+3)!}x^k}.$$

Therefore, by virtue of the derivative Formula (8), we obtain

$$
\begin{aligned}
V_n &= \lim_{x\to 0}\left[\frac{x^3}{6x(e^x+1)-12(e^x-1)}\right]^{(n)} \\
&= \lim_{x\to 0}\left[\frac{1}{\sum_{k=0}^{\infty}\frac{6(k+1)}{(k+3)!}x^k}\right]^{(n)} \\
&= (-1)^n \begin{vmatrix}
1 & 1 & 0 & 0 & \cdots & 0 \\
0 & \frac{1}{2} & \binom{1}{1} & 0 & \cdots & 0 \\
0 & \frac{3}{10} & \binom{2}{1}\frac{1}{2} & \binom{2}{2} & \cdots & 0 \\
0 & \frac{1}{5} & \binom{3}{1}\frac{3}{10} & \binom{3}{2}\frac{1}{2} & \cdots & 0 \\
0 & \frac{1}{7} & \binom{4}{1}\frac{1}{5} & \binom{4}{2}\frac{3}{10} & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \frac{6}{(n+1)(n+2)} & \binom{n-1}{1}\frac{6}{n(n+1)} & \binom{n-1}{2}\frac{6}{(n-1)n} & \cdots & \binom{n-1}{n-1} \\
0 & \frac{6}{(n+2)(n+3)} & \binom{n}{1}\frac{6}{(n+1)(n+2)} & \binom{n}{2}\frac{6}{n(n+1)} & \cdots & \binom{n}{n-1}\frac{1}{2}
\end{vmatrix} \\
&= (-1)^n \begin{vmatrix}
\binom{1}{0}\frac{1}{2} & \binom{1}{1} & 0 & \cdots & 0 \\
\binom{2}{0}\frac{3}{10} & \binom{2}{1}\frac{1}{2} & \binom{2}{2} & \cdots & 0 \\
\binom{3}{0}\frac{1}{5} & \binom{3}{1}\frac{3}{10} & \binom{3}{2}\frac{1}{2} & \cdots & 0 \\
\binom{4}{0}\frac{1}{7} & \binom{4}{1}\frac{1}{5} & \binom{4}{2}\frac{3}{10} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\binom{n-2}{0}\frac{6}{n(n+1)} & \binom{n-2}{1}\frac{6}{(n-1)n} & \binom{n-2}{2}\frac{6}{(n-2)(n-1)} & \cdots & 0 \\
\binom{n-1}{0}\frac{6}{(n+1)(n+2)} & \binom{n-1}{1}\frac{6}{n(n+1)} & \binom{n-1}{2}\frac{6}{(n-1)n} & \cdots & \binom{n-1}{n-1} \\
\binom{n}{0}\frac{6}{(n+2)(n+3)} & \binom{n}{1}\frac{6}{(n+1)(n+2)} & \binom{n}{2}\frac{6}{n(n+1)} & \cdots & \binom{n}{n-1}\frac{1}{2}
\end{vmatrix} \\
&= (-1)^n \left|v_{i,j}\right|_{n\times n}
\end{aligned}
$$

for $n \ge 1$. The proof of Theorem 4 is thus complete. □

Remark 2. *It is obvious that the determinantal expression (13) is more beautiful, symmetric, and computable than the explicit expression (12) of the van der Pol numbers V_n.*

Remark 3. *By the determinantal expression (13), we can compute*

$$
V_4 = (-1)^4 \begin{vmatrix}
\binom{1}{0}\frac{1}{2} & \binom{1}{1} & 0 & 0 \\
\binom{2}{0}\frac{3}{10} & \binom{2}{1}\frac{1}{2} & \binom{2}{2} & 0 \\
\binom{3}{0}\frac{1}{5} & \binom{3}{1}\frac{3}{10} & \binom{3}{2}\frac{1}{2} & \binom{3}{3} \\
\binom{4}{0}\frac{1}{7} & \binom{4}{1}\frac{1}{5} & \binom{4}{2}\frac{3}{10} & \binom{4}{3}\frac{1}{2}
\end{vmatrix} = -\frac{1}{350},
$$

$$
V_3 = (-1)^3 \begin{vmatrix}
\binom{1}{0}\frac{1}{2} & \binom{1}{1} & 0 \\
\binom{2}{0}\frac{3}{10} & \binom{2}{1}\frac{1}{2} & \binom{2}{2} \\
\binom{3}{0}\frac{1}{5} & \binom{3}{1}\frac{3}{10} & \binom{3}{2}\frac{1}{2}
\end{vmatrix} = -\frac{1}{20},
$$

$$
V_2 = (-1)^2 \begin{vmatrix}
\binom{1}{0}\frac{1}{2} & \binom{1}{1} \\
\binom{2}{0}\frac{3}{10} & \binom{2}{1}\frac{1}{2}
\end{vmatrix} = \frac{1}{5}, \quad V_1 = (-1)^1 \left|\binom{1}{0}\frac{1}{2}\right| = -\frac{1}{2}.
$$

These values of V_k for $1 \le k \le 4$ coincide with the corresponding ones in (2).

5. Decreasing Property and Concavity

In this section, we discuss the decreasing property and concavity of the even function $G(x)$ defined by (4).

Theorem 5. *The even function $G(x)$ defined by (4) is decreasing on $[0, R) \supset [0, \pi]$ and is concave on $[0, \pi] \subset [0, R)$.*

Proof. From Expression (5), it is easy to see that the even function $G(x)$ is decreasing on $(0, R)$.

Considering the first equality in (6) and directly differentiating give

$$G'(x) = \left(\frac{x^2 \sin x}{\sin x - x\cos x}\right)' = -\frac{x[x\sin(2x) + 2\cos(2x) + 2x^2 - 2]}{2(\sin x - x\cos x)^2}$$

and

$$G''(x) = -\frac{\sin(3x) - (4x^4 - 12x^2 + 3)\sin x - 8x^3 \cos x}{2(\sin x - x\cos x)^3}.$$

Let

$$h(x) = \sin(3x) - (4x^4 - 12x^2 + 3)\sin x - 8x^3 \cos x. \tag{14}$$

The function $h(x)$ can be expanded to

$$h(x) = x^9 \sum_{j=0}^{\infty} (-1)^j \frac{3^{2j+9} - (64j^4 + 896j^3 + 4640j^2 + 10{,}552j + 8931)}{(2j+9)!} x^{2j}.$$

Let

$$T_j = \frac{3^{2j+9} - (64j^4 + 896j^3 + 4640j^2 + 10{,}552j + 8931)}{(2j+9)!}, \quad j \ge 0.$$

Then

$$h(x) = x^9 \sum_{j=0}^{\infty} (T_{2j} - T_{2j+1}x^2)x^{4j} = x^9 \sum_{j=0}^{\infty} T_{2j+1}\left(\frac{T_{2j}}{T_{2j+1}} - x^2\right)x^{4j}.$$

By induction, we can verify that

$$3^{2j+9} > 64j^4 + 896j^3 + 4640j^2 + 10{,}552j + 8931, \quad j \ge 0.$$

This means that $T_j > 0$ for $j \ge 0$.

In order to prove that the inequality

$$\frac{T_{2j}}{T_{2j+1}} > \left(\frac{3\pi}{2}\right)^2 = 22.2\ldots > \pi^2 = 9.8\cdots \tag{15}$$

holds for $j \ge 0$, it suffices to show $T_{2j} > 25T_{2j+1}$ for $j \ge 0$, which is equivalent to

$$\begin{aligned} 3^{4j+9}(16j^2 + 84j - 115) > 16{,}384j^6 + 200{,}704j^5 + 986{,}112j^4 \\ + 2{,}454{,}784j^3 + 3{,}186{,}032j^2 + 1{,}932{,}844j + 355{,}335, \quad j \ge 0. \end{aligned}$$

For $j \ge 2$, it is sufficient to prove that

$$3^{4j+9} > \frac{16{,}384j^6 + 200{,}704j^5 + 9{,}861{,}12j^4 + 2{,}454{,}784j^3 + 3{,}186{,}032j^2 + 1{,}932{,}844j + 355{,}335}{16j^2 + 84j - 115},$$

which can be verified by induction on $j \ge 2$. In short, Inequality (15) holds for $j \ge 2$.

Let

$$\begin{aligned} Y(y) = \sum_{j=0}^{1} T_{2j+1}\left(\frac{T_{2j}}{T_{2j+1}} - y\right)y^{2j} = \sum_{j=0}^{3} (-1)^j T_j y^j \\ = -\frac{713y^3}{65{,}488{,}500} + \frac{y^2}{4050} - \frac{2y}{525} + \frac{4}{135}, \quad 0 \le y \le \pi^2. \end{aligned}$$

Differentiating gives

$$Y'(y) = -\frac{713y^2}{21{,}829{,}500} + \frac{y}{2025} - \frac{2}{525} \quad \text{and} \quad Y''(y) = \frac{1}{2025} - \frac{713y}{10{,}914{,}750}.$$

The second derivative $Y''(y)$ is decreasing and has a zero $\frac{10{,}914{,}750}{713\times 2025} = \frac{5390}{713}$. The first derivative $Y'(y)$ has a maximum

$$Y'\left(\frac{5390}{713}\right) = -\frac{19{,}637}{10{,}106{,}775}.$$

Thus, the first derivative $Y'(y)$ is negative and the function $Y(y)$ is decreasing. Since

$$Y(\pi^2) = \frac{4}{135} - \frac{2\pi^2}{525} + \frac{\pi^4}{4050} - \frac{713\pi^6}{65{,}488{,}500} = 0.0056\ldots,$$

the function $Y(y)$ is positive on $[0, \pi^2]$.

Combining the above arguments, we conclude that the function $h(x)$ is positive on $(0, \pi]$. Then, the second derivative $G''(x)$ is negative on $(0, \pi)$. Hence, the function $G(x)$ is concave on $[0, \pi]$. □

Remark 4. *We guess that the function $h(x)$ defined in (14) is positive, and even increasing, on the interval $\left(0, \frac{3\pi}{2}\right] \supset (0, R]$. If this guess were true, then the even function $G(x)$ would be concave on $[0, R) \subset \left(0, \frac{3\pi}{2}\right)$.*

Remark 5. *We note that a concave function must be a logarithmically concave function, but not conversely. However, a logarithmically convex function must be a convex function, but not conversely.*

6. Power Series Expansions

In this section, basing on (5) and (6) and utilizing the derivative Formula (8) in Lemma 2, we derive two Maclaurin power series expansions of the function $G(x)$ defined by (4) in terms of specific Hessenberg determinants. These two Maclaurin power series expansions are different from (3) in form.

Theorem 6. *For $m \ge 0$, let*

$$E_m = \frac{2^{2m+1}}{2m+1} \frac{|B_{2m+2}|}{m+1}$$

and

$$D_{2m} = \begin{vmatrix} 0 & \binom{1}{1}\frac{1}{3} & 0 & 0 & \cdots & 0 \\ \frac{2}{45} & 0 & \binom{2}{2}\frac{1}{3} & 0 & \cdots & 0 \\ 0 & \binom{3}{1}\frac{2}{45} & 0 & \binom{3}{3}\frac{1}{3} & \cdots & 0 \\ \frac{16}{315} & 0 & \binom{4}{2}\frac{2}{45} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ E_{m-1} & 0 & \binom{2m-2}{2}E_{m-2} & 0 & \cdots & 0 \\ 0 & \binom{2m-1}{1}E_{m-1} & 0 & \binom{2m-1}{3}E_{m-2} & \cdots & \binom{2m-1}{2m-1}\frac{1}{3} \\ E_m & 0 & \binom{2m}{2}E_{m-1} & 0 & \cdots & 0 \end{vmatrix}$$
$$= |e_{i,j}|_{(2m)\times(2m)}$$

with the convention $D_0 = 1$, where

$$e_{i,j} = \begin{cases} 0, & 2m \ge j > i+1 \ge 2; \\ 0, & 2m-2 \ge i-j = 2k \ge 0; \\ \dbinom{i}{j-1} E_{(i-j+1)/2}, & 2m-1 \ge i-j = 2k-1 \ge -1. \end{cases}$$

Then the function $G(x)$ defined in (4) can be expanded to

$$G(x) = \sum_{m=0}^{\infty} 3^{2m+1} D_{2m} \frac{x^{2m}}{(2m)!} = 3 - \frac{x^2}{5} - \frac{x^4}{175} - \frac{2x^6}{7875} - \frac{37x^8}{3{,}031{,}875} - \cdots. \tag{16}$$

Proof. Since the function $G(x)$ is even, its odd order derivative $G^{(2m+1)}(0) = 0$ for $m \ge 0$.

Let $u(x) = 1$ and

$$v(x) = \sum_{k=0}^{\infty} \frac{2^{2k+2}}{(2k+2)!} |B_{2k+2}| x^{2k},$$

which are both even functions. Then

$$v^{(n)}(0) = \lim_{x\to 0} \sum_{k=0}^{\infty} \frac{2^{2k+2}}{(2k+2)!} |B_{2k+2}| \langle 2k \rangle_n x^{2k-n} = \begin{cases} 0, & n = 2m+1 \\ \dfrac{2^{2m+1}}{2m+1} \dfrac{|B_{2m+2}|}{m+1}, & n = 2m \end{cases}$$

for $m \ge 0$. Applying the derivative Formula (8) to Expression (5) leads to

$$\begin{aligned} G^{(2m)}(0) &= \lim_{x\to 0} \left[\frac{1}{\sum_{k=0}^{\infty} \frac{2^{2k+2}}{(2k+2)!} |B_{2k+2}| x^{2k}} \right]^{(2m)} \\ &= \frac{1}{E_0^{2m+1}} \begin{vmatrix} 1 & \binom{0}{0}E_0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \binom{1}{1}E_0 & 0 & \cdots & 0 \\ 0 & \binom{2}{0}E_1 & 0 & \binom{2}{2}E_0 & \cdots & 0 \\ 0 & 0 & \binom{3}{1}E_1 & 0 & \cdots & 0 \\ 0 & \binom{4}{0}E_2 & 0 & \binom{4}{2}E_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \binom{2m-2}{0}E_{m-1} & 0 & \binom{2m-2}{2}E_{m-2} & \cdots & 0 \\ 0 & 0 & \binom{2m-1}{1}E_{m-1} & 0 & \cdots & \binom{2m-1}{2m-1}E_0 \\ 0 & \binom{2m}{0}E_m & 0 & \binom{2m}{2}E_{m-1} & \cdots & 0 \end{vmatrix} \\ &= 3^{2m+1} \begin{vmatrix} 0 & \binom{1}{1}\frac{1}{3} & 0 & 0 & \cdots & 0 \\ \frac{2}{45} & 0 & \binom{2}{2}\frac{1}{3} & 0 & \cdots & 0 \\ 0 & \binom{3}{1}\frac{2}{45} & 0 & \binom{3}{3}\frac{1}{3} & \cdots & 0 \\ \frac{16}{315} & 0 & \binom{4}{2}\frac{2}{45} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ E_{m-1} & 0 & \binom{2m-2}{2}E_{m-2} & 0 & \cdots & 0 \\ 0 & \binom{2m-1}{1}E_{m-1} & 0 & \binom{2m-1}{3}E_{m-2} & \cdots & \binom{2m-1}{2m-1}\frac{1}{3} \\ E_m & 0 & \binom{2m}{2}E_{m-1} & 0 & \cdots & 0 \end{vmatrix} \\ &= 3^{2m+1} D_{2m}. \end{aligned}$$

Consequently, the Maclaurin power series expansion is

$$G(x) = \sum_{n=0}^{\infty} G^{(n)}(0) \frac{x^n}{n!} = \sum_{m=0}^{\infty} G^{(2m)}(0) \frac{x^{2m}}{(2m)!} = \sum_{m=0}^{\infty} 3^{2m+1} D_{2m} \frac{x^{2m}}{(2m)!}.$$

The proof of Theorem 6 is complete. □

Remark 6. *When $m = 2$ and $m = 1$, we obtain*

$$D_4 = \begin{vmatrix} 0 & \binom{1}{1}E_0 & 0 & 0 \\ \binom{2}{0}E_1 & 0 & \binom{2}{2}E_0 & 0 \\ 0 & \binom{3}{1}E_1 & 0 & \binom{3}{3}E_0 \\ \binom{4}{0}E_2 & 0 & \binom{4}{2}E_1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & \binom{1}{1}\frac{1}{3} & 0 & 0 \\ \binom{2}{0}\frac{2}{45} & 0 & \binom{2}{2}\frac{1}{3} & 0 \\ 0 & \binom{3}{1}\frac{2}{45} & 0 & \binom{3}{3}\frac{1}{3} \\ \binom{4}{0}\frac{16}{315} & 0 & \binom{4}{2}\frac{2}{45} & 0 \end{vmatrix} = -\frac{8}{14{,}175}$$

and

$$D_2 = \begin{vmatrix} 0 & \binom{1}{1}E_0 \\ \binom{2}{0}E_1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & \binom{1}{1}\frac{1}{3} \\ \binom{2}{0}\frac{2}{45} & 0 \end{vmatrix} = -\frac{2}{135}.$$

Accordingly, we have

$$\begin{aligned} G(x) &= 3D_0 + 3^3 D_2 \frac{x^2}{2!} + 3^5 D_4 \frac{x^4}{4!} + \sum_{m=3}^{\infty} 3^{2m+1} D_{2m} \frac{x^{2m}}{(2m)!} \\ &= 3 - \frac{x^2}{5} - \frac{x^4}{175} + \sum_{m=3}^{\infty} 3^{2m+1} D_{2m} \frac{x^{2m}}{(2m)!}, \end{aligned}$$

which coincides with the first three terms in the series expansions (3) *and* (16)*.*

Remark 7. *Comparing the series expansions* (3) *and* (16) *reveals*

$$3\sum_{k=2}^{\infty} (-1)^k 2^{2k} V_{2k} \frac{x^{2k}}{(2k)!} = \sum_{m=2}^{\infty} 3^{2m+1} D_{2m} \frac{x^{2m}}{(2m)!}.$$

As a result, we deduce the relation

$$V_{2k} = (-1)^k \left(\frac{3}{2}\right)^{2k} D_{2k}, \quad k \ge 2, \tag{17}$$

which is different to the determinantal expression (9)*.*

Theorem 7. *The function $G(x)$ defined in* (4) *can be expanded into*

$$G(x) = \sum_{m=0}^{\infty} 3^{2m+1} \mathcal{D}_{2m+1} \frac{x^{2m}}{(2m)!} = 3 - \frac{x^2}{5} - \frac{x^4}{175} - \frac{2x^6}{7875} - \frac{37x^8}{3{,}031{,}875} - \cdots, \tag{18}$$

where the determinants

$$\mathcal{D}_{2m+1} = \left|d_{i,j}\right|_{(2m+1)\times(2m+1)}$$

and the elements of the determinants $\mathcal{D}_{2m+1}$ are

$$d_{i,j} = \begin{cases} \dfrac{(-1)^{(i-1)/2}[1+(-1)^{i-1}]}{2i}, & 1 \le i \le 2m+1, j = 1; \\ \dbinom{i-1}{j-2} \dfrac{(-1)^{(i-j+1)/2}[1+(-1)^{i-j+1}]}{2(i-j+2)(i-j+4)}, & 1 \le i \le 2m+1, 2 \le j \le 2m+1. \end{cases}$$

Proof. Let

$$u(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k} \quad \text{and} \quad v(x) = 2\sum_{k=0}^{\infty} \frac{(-1)^k (k+1)}{(2k+3)!} x^{2k}.$$

Differentiating results in

$$
\begin{gathered}
u^{(2m)}(0)=\lim_{x\to0}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\langle 2k\rangle_{2m}x^{2k-2m}=\frac{(-1)^m}{(2m+1)!}\langle 2m\rangle_{2m}=\frac{(-1)^m}{2m+1},\\
v^{(2m)}(0)=2\lim_{x\to0}\sum_{k=0}^{\infty}\frac{(-1)^k(k+1)}{(2k+3)!}\langle 2k\rangle_{2m}x^{2k-2m}=\frac{(-1)^m}{(2m+1)(2m+3)},\\
u^{(2m+1)}(0)=0,\quad v^{2m+1}(0)=0
\end{gathered}
$$

for $m\ge0$. Applying the derivative Formula (8) to Expression (6) leads to

$$
\frac{G^{(2m)}(0)}{3^{2m+1}}=\begin{vmatrix}
1 & \frac{1}{1\cdot3} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & \binom{1}{1}\frac{1}{1\cdot3} & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0\\
\frac{-1}{3} & \frac{-1}{3\cdot5} & 0 & \binom{2}{2}\frac{1}{1\cdot3} & \cdots & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & \binom{3}{1}\frac{-1}{3\cdot5} & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0\\
\frac{1}{5} & \frac{1}{5\cdot7} & 0 & \binom{4}{2}\frac{-1}{3\cdot5} & \cdots & 0 & 0 & 0 & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & & \\
0 & 0 & \binom{2m-1}{1}\frac{(-1)^{m-1}}{(2m-1)(2m+1)} & 0 & \cdots & \binom{2m-1}{2m-5}\frac{1}{5\cdot7} & 0 & \binom{2m-1}{2m-3}\frac{1}{3\cdot5} & 0 & \binom{2m-1}{2m-1}\frac{1}{1\cdot3} & \\
\frac{(-1)^m}{2m+1} & \frac{(-1)^m}{(2m+1)(2m+3)} & 0 & \binom{2m}{2}\frac{(-1)^{m-1}}{(2m-1)(2m+1)} & \cdots & 0 & \binom{2m}{2m-4}\frac{1}{5\cdot7} & 0 & \binom{2m}{2m-2}\frac{1}{3\cdot5} & 0 &
\end{vmatrix}
$$

$$
=\mathcal{D}_{2m+1}
$$

for $m\ge0$. Consequently, we have the series expansion

$$
G(x)=\sum_{m=0}^{\infty}G^{(2m)}\frac{x^{2m}}{(2m)!}=\sum_{m=0}^{\infty}3^{2m+1}\mathcal{D}_{2m+1}\frac{x^{2m}}{(2m)!}.
$$

The proof of Theorem 7 is complete. □

Remark 8. *Comparing the series expansions (16) and (18), we acquire the relation*

$$
D_{2m}=\mathcal{D}_{2m+1},\quad m\ge0. \tag{19}
$$

We note that the elements of D_{2m} contain products of the Bernoulli numbers B_{2k} and binomial coefficients, while the elements of $\mathcal{D}_{2m+1}$ just contain products of common fractions and binomial coefficients.

Can one verify relation (19) by a linear algebraic operation?

Remark 9. *The determinants D_{2m} and $\mathcal{D}_{2m+1}$ are both connected with the van der Pol numbers V_{2k} via the relation (17) and*

$$
V_{2k}=(-1)^k\left(\frac{3}{2}\right)^{2k}\mathcal{D}_{2k+1},\quad k\ge2. \tag{20}
$$

Remark 10. *When $m = 2$ and $m = 1$, we acquire*

$$\mathcal{D}_5 = \begin{vmatrix} 1 & \frac{1}{1\cdot 3} & 0 & 0 & 0 \\ 0 & 0 & \binom{1}{1}\frac{1}{1\cdot 3} & 0 & 0 \\ \frac{-1}{3} & \frac{-1}{3\cdot 5} & 0 & \binom{2}{2}\frac{1}{1\cdot 3} & 0 \\ 0 & 0 & \binom{3}{1}\frac{-1}{3\cdot 5} & 0 & \binom{2}{2}\frac{1}{1\cdot 3} \\ \frac{1}{5} & \frac{1}{5\cdot 7} & 0 & \binom{4}{2}\frac{-1}{3\cdot 5} & 0 \end{vmatrix} = -\frac{8}{14{,}175}$$

and

$$\mathcal{D}_3 = \begin{vmatrix} 1 & \frac{1}{1\cdot 3} & 0 \\ 0 & 0 & \binom{1}{1}\frac{1}{1\cdot 3} \\ \frac{-1}{3} & \frac{-1}{3\cdot 5} & 0 \end{vmatrix} = -\frac{2}{135}.$$

Therefore, the relation (19) is confirmed for $m = 1, 2$.

7. Conclusions

In this paper, we established the following results.

1. Three determinantal expressions (9), (17), and (20) for the van der Pol numbers V_{2k} with $k \geq 2$ were derived in Theorem 1 and Remarks 7 and 9.
2. An identity (10) for the Bernoulli numbers B_{2k} with $k \geq 1$ was deduced in Theorem 2.
3. An identity (11) for the van der Pol numbers V_{2k} with $k \geq 1$ was acquired in Theorem 3.
4. A determinantal expression (13) for the van der Pol numbers V_n with $n \geq 1$ was presented in Theorem 4.
5. The even function $G(x)$ defined by (4) was proven in Theorem 5 to be decreasing in $[0, R) \supset [0, \pi]$ and to be concave in $[0, \pi] \subset [0, R)$.
6. Two Maclaurin power series expansions ((16) and (18)) of the function $G(x)$, whose coefficients are expressed in terms of specific Hessenberg determinants with elements containing the Bernoulli numbers B_{2k}, were discovered in Theorems 6 and 7.

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References

1. van der Pol, B. Smoothing and "unsmoothing". In *Probability and Related Topics in Physical Sciences*; Kac, M., Ed.; Amer Mathematical Society: New York, NY, USA, 1957; pp. 223–235.
2. Howard, F.T. Properties of the van der Pol numbers and polynomials. *J. Reine Angew. Math.* **1973**, *260*, 35–46. [CrossRef]
3. Moll, V.H.; Vignat, C. On polynomials connected to powers of Bessel functions. *Int. J. Number Theory* **2014**, *10*, 1245–1257. [CrossRef]
4. Kishore, N. A structure of the Rayleigh polynomial. *Duke Math. J.* **1964**, *31*, 513–518. [CrossRef]
5. Kishore, N. Binary property of the Rayleigh polynomial. *Duke Math. J.* **1965**, *32*, 429–435. [CrossRef]
6. Kishore, N. Congruence properties of the Rayleigh functions and polynomials. *Duke Math. J.* **1968**, *35*, 557–562. [CrossRef]
7. Kishore, N. The Rayleigh polynomial. *Proc. Amer. Math. Soc.* **1964**, *15*, 911–917. [CrossRef]

8. Hong, Y.; Guo, B.-N.; Qi, F. Determinantal expressions and recursive relations for the Bessel zeta function and for a sequence originating from a series expansion of the power of modified Bessel function of the first kind. *CMES Comput. Model. Eng. Sci.* **2021**, *129*, 409–423. [CrossRef]
9. Abramowitz, M.; Stegun, I.A. (Eds.) *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*; National Bureau of Standards, Applied Mathematics Series 55, Reprint of the 1972 edition; Dover Publications, Inc.: New York, NY, USA, 1992.
10. Henrici, P. *Applied and Computational Complex Analysis*; Volume 1, Pure and Applied Mathematics; Wiley-Interscience [John Wiley & Sons]: New York, NY, USA; London, UK; Sydney, Australia, 1974.
11. Bourbaki, N. *Elements of Mathematics: Functions of a Real Variable: Elementary Theory*; Translated from the 1976 French original by Philip Spain; Elements of Mathematics (Berlin); Springer: Berlin/Heidelberg, Germany, 2004. [CrossRef]
12. Qi, F.; Niu, D.-W.; Lim, D.; Yao, Y.-H. Special values of the Bell polynomials of the second kind for some sequences and functions. *J. Math. Anal. Appl.* **2020**, *491*, 124382. [CrossRef]