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Article

An Equivalent Form Related to a Hilbert-Type Integral Inequality

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Abstract: In the present paper, we establish an equivalent form related to a Hilbert-type integral inequality with a non-homogeneous kernel and a best possible constant factor. We also consider the case of homogeneous kernel as well as certain operator expressions.

Keywords: Hilbert-type integral inequality; weight function; equivalent form; operator; norm

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1. Introduction

As is well-known, in 1925, Hardy [1] proved the following famous integral inequality: If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $f(x), g(y) \geq 0$,

$$0 < \int_0^\infty f^p(x)dx < \infty \text{ and } 0 < \int_0^\infty g^q(y)dy < \infty,$$

then it holds

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \frac{\pi}{\sin(\pi/p)} \left(\int_0^\infty f^p(x)dx \right)^{\frac{1}{p}} \left(\int_0^\infty g^q(y)dy \right)^{\frac{1}{q}}, \quad (1)$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible.

For $p = q = 2$, (1) reduces to the well-known Hilbert integral inequality. Hilbert's integral inequality and (1) are two very important inequalities, which are well-known for their applicability in various domains of analysis (cf. [2,3]).

In 1934, Hardy et al. presented the following extension of (1):

If $k_1(x, y)$ is a non-negative homogeneous function of degree -1,

$$k_p = \int_0^\infty k_1(u, 1) u^{\frac{-1}{p}} du \in \mathbf{R}_+ = (0, \infty),$$

then we have

$$\int_0^\infty \int_0^\infty k_1(x, y) f(x) g(y) dxdy < k_p \left(\int_0^\infty f^p(x)dx \right)^{\frac{1}{p}} \left(\int_0^\infty g^q(y)dy \right)^{\frac{1}{q}}, \quad (2)$$

where the constant factor k_p is the best possible (cf. [2], Theorem 319). Furthermore, the following Hilbert-type integral inequality with non-homogeneous kernel holds true:
If $h(u) > 0, \phi(\sigma) = \int_0^\infty h(u)u^{\sigma-1}du \in \mathbf{R}_+$, then

$$\int_0^\infty \int_0^\infty h(xy)f(x)g(y)dxdy < \phi\left(\frac{1}{p}\right)\left(\int_0^\infty x^{p-2}f^p(x)dx\right)^{\left(\frac{1}{p}\right)}\left(\int_0^\infty g^q(y)dy\right)^{\frac{1}{q}}, \quad (3)$$

where the constant factor $\phi(\frac{1}{p})$ is the best possible (cf. [2], Theorem 350).

In 1998, by introducing an independent parameter $\lambda > 0$, Yang established an extension of Hilbert's integral inequality with the kernel $\frac{1}{(x+y)^\lambda}$ (cf. [4,5]). In 2004, by introducing two pairs of conjugate exponents (p,q) and (r,s) with an independent parameter $\lambda > 0$, Yang [6] proved the following extension of (1):

If $p, r > 1, \frac{1}{p} + \frac{1}{q} = \frac{1}{r} + \frac{1}{s} = 1, f(x), g(y) \geq 0$, such that

$$0 < \int_0^\infty x^{p(1-\frac{\lambda}{r})-1}f^p(x)dx < \infty \text{ and } 0 < \int_0^\infty y^{q(1-\frac{\lambda}{s})-1}g^q(y)dy < \infty,$$

then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x^\lambda + y^\lambda}dxdy < \frac{\pi}{\lambda\sin(\pi/r)}\left[\int_0^\infty x^{p(1-\frac{\lambda}{r})-1}f^p(x)dx\right]^{\frac{1}{p}}\left[\int_0^\infty y^{q(1-\frac{\lambda}{s})-1}g^q(y)dy\right]^{\frac{1}{q}}, \quad (4)$$

where the constant factor $\frac{\pi}{\lambda\sin(\pi/r)}$ is the best possible.

For $\lambda = 1, r = q, s = p$, (4) reduces to (1). In 2005, the work [7] also provided an extension of (1) with the kernel $\frac{1}{(x+y)^\lambda}$ and two pairs of conjugate exponents. In papers [8–12], the authors proved some interesting extensions and particular cases of (1)–(3) with parameters.

In 2009, Yang presented the following extension of (2) and (5) (cf. [13,14]):
If $\lambda_1 + \lambda_2 = \lambda \in \mathbf{R} = (-\infty, \infty)$, $k_\lambda(x,y)$ is a non-negative homogeneous function of degree $-\lambda$, satisfying

$$k_\lambda(ux, uy) = u^{-\lambda}k_\lambda(x,y)(u,x,y > 0),$$

$$k(\lambda_1) = \int_0^\infty k_\lambda(u,1)u^{\lambda_1-1}du \in \mathbf{R}_+,$$

then we have

$$\int_0^\infty \int_0^\infty k_\lambda(x,y)f(x)g(y)dxdy < k(\lambda_1)\left[\int_0^\infty x^{p(1-\lambda_1)-1}f^p(x)dx\right]^{\frac{1}{p}}\left[\int_0^\infty y^{q(1-\lambda_2)-1}g^q(y)dy\right]^{\frac{1}{q}}, \quad (5)$$

where the constant factor $k(\lambda_1)$ is the best possible.

For $\lambda = 1, \lambda_1 = \frac{1}{q}, \lambda_2 = \frac{1}{p}$, (5) reduces to (2), whereas for $\lambda > 0$, $\lambda_1 = \frac{\lambda}{r}$, $\lambda_2 = \frac{\lambda}{s}$, $k_\lambda(x,y) = \frac{1}{x^\lambda + y^\lambda}$, (5) reduces to (4).

Additionally, the extension below of (3) has been established:

$$\int_0^\infty \int_0^\infty h(xy)f(x)g(y)dxdy < \phi(\sigma)\left[\int_0^\infty x^{p(1-\sigma)-1}f^p(x)dx\right]^{\frac{1}{p}}\left[\int_0^\infty y^{q(1-\sigma)-1}g^q(y)dy\right]^{\frac{1}{q}}, \quad (6)$$

where the constant factor $\phi(\sigma)$ is the best possible (cf. [15]).

Some equivalent inequalities of (5) and (6) were constructed in [14] . In 2013, Yang [15] also studied the equivalency of (5) and (6). In 2017, Hong [16] investigated an equivalent condition between (5) and a few parameters. Since 2018, in the papers [17–26], the authors proved some novel extensions of the above Hilbert-type inequalities.

In the present paper, we establish an equivalent form related to a Hilbert-type integral inequality with the non-homogeneous kernel

$$|\ln xy| \prod_{k=1}^{s} \frac{(\min\{xy, c_k\})^{\frac{\alpha}{s}}}{(\max\{xy, c_k\})^{\frac{\lambda+\alpha}{s}}}$$

and a best possible constant factor. We also consider the case of homogeneous kernel and operator expressions.

2. An Example and a Lemma

In the following, we assume that $s, s_0 \in \mathbf{N} = \{1, 2, \cdots\}$, $0 < c_1 \leq \cdots \leq c_s < \infty$, $s_0 \leq s, 0 = c_0 \leq c_{s_0} \leq 1 < c_{s_0+1} \leq c_{s+1} = \infty$, $\lambda_1, \lambda_2 > -\alpha$, $\lambda_1 + \lambda_2 = \lambda$.

Example 1. *We consider the following function:*

$$h(u) := |\ln u| \prod_{k=1}^{s} \frac{(\min\{u, c_k\})^{\frac{\alpha}{s}}}{(\max\{u, c_k\})^{\frac{\lambda+\alpha}{s}}} \ (u \in \mathbf{R}_+), \tag{7}$$

and define

$$k(\lambda_1) := \int_0^\infty h(u) u^{\lambda_1 - 1} du = \int_0^\infty |\ln u| \left[\prod_{k=1}^{s} \frac{(\min\{u, c_k\})^{\frac{\alpha}{s}}}{(\max\{u, c_k\})^{\frac{\lambda+\alpha}{s}}} \right] u^{\lambda_1 - 1} du. \tag{8}$$

Note. For $0 < a \leq b \leq 1, \eta \neq 0$, we have

$$\begin{aligned} & \int_a^b x^{\eta-1} |\ln x| dx = \frac{1}{\eta} \int_a^b (-\ln x) dx^\eta = \frac{1}{\eta} [(-\ln x) x^\eta |_a^b + \int_a^b x^{\eta-1} dx] \\ = \ & \frac{1}{\eta^2} [\eta(-\ln x) x^\eta |_a^b + (b^\eta - a^\eta)]. \end{aligned} \tag{9}$$

Since we have

$$\begin{aligned} & \int_a^b x^{-1} |\ln x| dx = \int_a^b (-\ln x) d \ln x \\ = \ & -\frac{1}{2} \ln^2 x |_a^b = \lim_{\eta \to 0^+} \frac{1}{\eta^2} [\eta(-\ln x) x^\eta |_a^b + (b^\eta - a^\eta)], \end{aligned}$$

we still denote this as (9) for $\eta = 0$.

For $0 < a \leq b$, we also use the above viewpoint in the following.

By the above Note, indicating

$$\prod_{k=1}^{0} c_k^{\frac{\alpha}{s}} = \prod_{k=s+1}^{s} c_k^{\frac{\lambda+\alpha}{s}} = 1$$

we obtain

$$
\begin{aligned}
k(\lambda_1) &= \int_{c_0}^{1}(-\ln u)\left[\prod_{k=1}^{s}\frac{(\min\{u,c_k\})^{\frac{\alpha}{s}}}{(\max\{u,c_k\})^{\frac{\lambda+\alpha}{s}}}\right]u^{\lambda_1-1}du \\
&\quad + \int_{1}^{c_{s+1}}\ln u\left[\prod_{k=1}^{s}\frac{(\min\{u,c_k\})^{\frac{\alpha}{s}}}{(\max\{u,c_k\})^{\frac{\lambda+\alpha}{s}}}\right]u^{\lambda_1-1}du \\
&= \sum_{i=0}^{s_0-1}\int_{c_i}^{c_{i+1}}(-\ln u)\left(\prod_{k=1}^{i}\frac{c_k^{\frac{\alpha}{s}}}{u^{\frac{\lambda+\alpha}{s}}}\right)\left(\prod_{k=i+1}^{s}\frac{u^{\frac{\alpha}{s}}}{c_k^{\frac{\lambda+\alpha}{s}}}\right)u^{\lambda_1-1}du \\
&\quad + \int_{c_{s_0}}^{1}(-\ln u)\left(\prod_{k=1}^{s_0}\frac{c_k^{\frac{\alpha}{s}}}{u^{\frac{\lambda+\alpha}{s}}}\right)\left(\prod_{k=s_0+1}^{s}\frac{u^{\frac{\alpha}{s}}}{c_k^{\frac{\lambda+\alpha}{s}}}\right)u^{\lambda_1-1}du \\
&\quad + \int_{1}^{c_{s_0+1}}\ln u\left(\prod_{k=1}^{s_0}\frac{c_k^{\frac{\alpha}{s}}}{u^{\frac{\lambda+\alpha}{s}}}\right)\left(\prod_{k=s_0+1}^{s}\frac{u^{\frac{\alpha}{s}}}{c_k^{\frac{\lambda+\alpha}{s}}}\right)u^{\lambda_1-1}du \\
&\quad + \sum_{i=s_0+1}^{s}\int_{c_i}^{c_{i+1}}\ln u\left(\prod_{k=1}^{i}\frac{c_k^{\frac{\alpha}{s}}}{u^{\frac{\lambda+\alpha}{s}}}\right)\left(\prod_{k=i+1}^{s}\frac{u^{\frac{\alpha}{s}}}{c_k^{\frac{\lambda+\alpha}{s}}}\right)u^{\lambda_1-1}du \\
&= \sum_{i=0}^{s_0-1}\left[\int_{c_i}^{c_{i+1}}(-\ln u)u^{\lambda_1+\alpha-\frac{\lambda+2\alpha}{s}i-1}du\right]\frac{\prod_{k=1}^{i}c_k^{\frac{\alpha}{s}}}{\prod_{k=i+1}^{s}c_k^{\frac{\lambda+\alpha}{s}}} \\
&\quad + \left[\int_{c_{s_0}}^{1}(-\ln u)u^{\lambda_1+\alpha-\frac{\lambda+2\alpha}{s}s_0-1}du\right]\frac{\prod_{k=1}^{s_0}c_k^{\frac{\alpha}{s}}}{\prod_{k=s_0+1}^{s}c_k^{\frac{\lambda+\alpha}{s}}} \\
&\quad + \left[\int_{1}^{c_{s_0+1}}(\ln u)u^{\lambda_1+\alpha-\frac{\lambda+2\alpha}{s}s_0-1}du\right]\frac{\prod_{k=1}^{s_0}c_k^{\frac{\alpha}{s}}}{\prod_{k=s_0+1}^{s}c_k^{\frac{\lambda+\alpha}{s}}} \\
&\quad + \sum_{i=s_0+1}^{s}\left[\int_{c_i}^{c_{i+1}}(\ln u)u^{\lambda_1+\alpha-\frac{\lambda+2\alpha}{s}i-1}du\right]\frac{\prod_{k=1}^{i}c_k^{\frac{\alpha}{s}}}{\prod_{k=i+1}^{s}c_k^{\frac{\lambda+\alpha}{s}}}.
\end{aligned}
$$

Hence, we find that

$$
\begin{aligned}
k(\lambda_1) &= \sum_{i=0}^{s_0-1}\frac{1}{(\lambda_1+\alpha-\frac{\lambda+2\alpha}{s}i)^2}\Big\{[1-(\lambda_1+\alpha-\frac{\lambda+2\alpha}{s}i)\ln c_{i+1}]c_{i+1}^{(\lambda_1+\alpha-\frac{\lambda+2\alpha}{s}i)} \\
&\quad -[1-(\lambda_1+\alpha-\frac{\lambda+2\alpha}{s}i)\ln c_i]c_i^{(\lambda_1+\alpha-\frac{\lambda+2\alpha}{s}i)}\Big\}\frac{\prod_{k=1}^{i}c_k^{\frac{\alpha}{s}}}{\prod_{k=i+1}^{s}c_k^{\frac{\lambda+\alpha}{s}}} \\
&\quad + \frac{1-[1-(\lambda_1+\alpha-\frac{\lambda+2\alpha}{s}s_0)\ln c_{s_0}]c_{s_0}^{(\lambda_1+\alpha-\frac{\lambda+2\alpha}{s}s_0)}}{(\lambda_1+\alpha-\frac{\lambda+2\alpha}{s}s_0)^2}\frac{\prod_{k=1}^{s_0}c_k^{\frac{\alpha}{s}}}{\prod_{k=s_0+1}^{s}c_k^{\frac{\lambda+\alpha}{s}}} \\
&\quad + \frac{[(\lambda_1+\alpha-\frac{\lambda+2\alpha}{s}s_0)\ln c_{s_0+1}-1]c_{s_0+1}^{(\lambda_1+\alpha-\frac{\lambda+2\alpha}{s}s_0)}+1}{(\lambda_1+\alpha-\frac{\lambda+2\alpha}{s}s_0)^2}\frac{\prod_{k=1}^{s_0}c_k^{\frac{\alpha}{s}}}{\prod_{k=s_0+1}^{s}c_k^{\frac{\lambda+\alpha}{s}}}
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{i=s_0+1}^{s}\frac{1}{(\lambda_1+\alpha-\frac{\lambda+2\alpha}{s}i)^2}\Big\{[(\lambda_1+\alpha-\frac{\lambda+2\alpha}{s}i)\ln c_{i+1}-1]c_{i+1}^{(\lambda_1+\alpha-\frac{\lambda+2\alpha}{s}i)} \\
&-[(\lambda_1+\alpha-\frac{\lambda+2\alpha}{s}i)\ln c_i-1]c_i^{(\lambda_1+\alpha-\frac{\lambda+2\alpha}{s}i)}\Big\}\frac{\prod_{k=1}^{i}c_k^{\frac{\alpha}{s}}}{\prod_{k=i+1}^{s}c_k^{\frac{\lambda+\alpha}{s}}}. \qquad (10)
\end{aligned}
$$

In particular:

(1) For $s_0 = s, 0 < c_1 \leq \cdots \leq c_s \leq 1$, we have

$$\begin{aligned} k(\lambda_1) &= \sum_{i=0}^{s-1} \frac{1}{(\lambda_1 + \alpha - \frac{\lambda+2\alpha}{s} i)^2} \\ &\times \left\{ [1 - (\lambda_1 + \alpha - \frac{\lambda + 2\alpha}{s} i) \ln c_{i+1}] c_{i+1}^{(\lambda_1 + \alpha - \frac{\lambda+2\alpha}{s} i)} \right. \\ &\left. - [1 - (\lambda_1 + \alpha - \frac{\lambda + 2\alpha}{s} i) \ln c_i] c_i^{(\lambda_1 + \alpha - \frac{\lambda+2\alpha}{s} i)} \right\} \frac{\prod_{k=1}^{i} c_k^{\frac{\alpha}{s}}}{\prod_{k=i+1}^{s} c_k^{\frac{\lambda+\alpha}{s}}} \\ &+ \frac{2 - [1 + (\lambda_2 + \alpha) \ln c_s] c_s^{-(\lambda_2 + \alpha)}}{(\lambda_2 + \alpha)^2} \prod_{k=1}^{s} c_k^{\frac{\alpha}{s}}; \end{aligned}$$

(2) For $s_0 = 0, 1 < c_1 \leq \cdots \leq c_s$, we have

$$\begin{aligned} k(\lambda_1) = & \frac{[(\lambda_1 + \alpha) \ln c_1 - 1] c_1^{(\lambda_1 + \alpha)} + 2}{(\lambda_1 + \alpha)^2} \frac{1}{\prod_{k=1}^{s} c_k^{\frac{\lambda+\alpha}{s}}} \\ & + \sum_{i=1}^{s} \frac{1}{(\lambda_1 + \alpha - \frac{\lambda+2\alpha}{s} i)^2} \left\{ [(\lambda_1 + \alpha - \frac{\lambda + 2\alpha}{s} i) \ln c_{i+1} - 1] c_{i+1}^{(\lambda_1 + \alpha - \frac{\lambda+2\alpha}{s} i)} \right. \\ & \left. - [(\lambda_1 + \alpha - \frac{\lambda + 2\alpha}{s} i) \ln c_i - 1] c_i^{(\lambda_1 + \alpha - \frac{\lambda+2\alpha}{s} i)} \right\} \frac{\prod_{k=1}^{i} c_k^{\frac{\alpha}{s}}}{\prod_{k=i+1}^{s} c_k^{\frac{\lambda+\alpha}{s}}}; \end{aligned}$$

(3) For $s = 1$ (or $c_s = \cdots = c_1$),

$$h(u) = \frac{|\ln u| (\min\{u, c_1\})^{\alpha}}{(\max\{u, c_1\})^{\lambda+\alpha}},$$

in view of (1) and (2), we deduce that

$$\begin{aligned} k(\lambda_1) &= \int_0^{\infty} |\ln u| \frac{(\min\{u, c_1\})^{\alpha} u^{\lambda_1 - 1}}{(\max\{u, c_1\})^{\lambda+\alpha}} du \\ &= \begin{cases} \left[\frac{1-(\lambda_1+\alpha)\ln c_1}{(\lambda_1+\alpha)^2} + \frac{2c_1^{(\lambda_2+\alpha)} - 1 - (\lambda_2+\alpha)\ln c_1}{(\lambda_2+\alpha)^2} \right] \frac{1}{c_1^{\lambda_2}}, c_1 \leq 1 \\ \left[\frac{(\lambda_1+\alpha)\ln c_1 - 1 + 2^{-(\lambda_1+\alpha)}}{(\lambda_1+\alpha)^2} + \frac{1+(\lambda_2+\alpha)\ln c_1}{(\lambda_2+\alpha)^2} \right] \frac{1}{c_1^{\lambda_2}}, c_1 > 1 \end{cases}; \end{aligned}$$

(4) For $\alpha = 0$,

$$h(u) = \frac{|\ln u|}{\prod_{k=1}^{s} (\max\{u, c_k\})^{\frac{\lambda}{s}}}, \quad \lambda_1, \lambda_2 > 0,$$

we get that

$$
\begin{aligned}
k(\lambda_1) \quad = \quad & \sum_{i=0}^{s_0-1} \frac{1}{(\lambda_1 - \frac{\lambda i}{s})^2} \bigg\{ [1 - (\lambda_1 - \frac{\lambda i}{s}) \ln c_{i+1}] c_{i+1}^{(\lambda_1 - \frac{\lambda i}{s})} \\
& - [1 - (\lambda_1 - \frac{\lambda i}{s}) \ln c_i] c_i^{(\lambda_1 - \frac{\lambda i}{s})} \bigg\} \frac{1}{\prod_{k=i+1}^{s} c_k^{\frac{\lambda}{s}}} \\
& + \frac{1 - [1 - (\lambda_1 - \frac{\lambda}{s} s_0) \ln c_{s_0}] c_{s_0}^{(\lambda_1 - \frac{\lambda}{s} s_0)}}{(\lambda_1 - \frac{\lambda}{s} s_0)^2} \frac{1}{\prod_{k=s_0+1}^{s} c_k^{\frac{\lambda}{s}}} \\
& + \frac{[(\lambda_1 - \frac{\lambda}{s} s_0) \ln c_{s_0+1} - 1] c_{s_0+1}^{(\lambda_1 - \frac{\lambda}{s} s_0)} + 1}{(\lambda_1 - \frac{\lambda}{s} s_0)^2} \frac{1}{\prod_{k=s_0+1}^{s} c_k^{\frac{\lambda}{s}}} \\
& + \sum_{i=s_0+1}^{s} \frac{1}{(\lambda_1 - \frac{\lambda i}{s})^2} \bigg\{ [(\lambda_1 - \frac{\lambda i}{s}) \ln c_{i+1} - 1] c_{i+1}^{(\lambda_1 - \frac{\lambda i}{s})} \\
& - [(\lambda_1 - \frac{\lambda i}{s}) \ln c_i - 1] c_i^{(\lambda_1 - \frac{\lambda i}{s})} \bigg\} \frac{1}{\prod_{k=i+1}^{s} c_k^{\frac{\lambda}{s}}};
\end{aligned}
$$

(5) For $\lambda = 0$,

$$
h(u) = |\ln u| \prod_{k=1}^{s} \left(\frac{\min\{u, c_k\}}{\max\{u, c_k\}} \right)^{\frac{\alpha}{s}}, \ |\lambda_1| < \alpha \ (\alpha > 0),
$$

we have

$$
\begin{aligned}
k(\lambda_1) \quad = \quad & \sum_{i=0}^{s_0-1} \frac{1}{(\lambda_1 + \alpha - \frac{2\alpha}{s} i)^2} \bigg\{ [1 - (\lambda_1 + \alpha - \frac{2\alpha}{s} i) \ln c_{i+1}] c_{i+1}^{(\lambda_1 + \alpha - \frac{2\alpha}{s} i)} \\
& - [1 - (\lambda_1 + \alpha - \frac{2\alpha}{s} i) \ln c_i] c_i^{(\lambda_1 + \alpha - \frac{2\alpha}{s} i)} \bigg\} \frac{\prod_{k=1}^{i} c_k^{\frac{\alpha}{s}}}{\prod_{k=i+1}^{s} c_k^{\frac{\alpha}{s}}} \\
& + \frac{1 - [1 - (\lambda_1 + \alpha - \frac{2\alpha}{s} s_0) \ln c_{s_0}] c_{s_0}^{(\lambda_1 + \alpha - \frac{2\alpha}{s} s_0)}}{(\lambda_1 + \alpha - \frac{2\alpha}{s} s_0)^2} \frac{\prod_{k=1}^{s_0} c_k^{\frac{\alpha}{s}}}{\prod_{k=s_0+1}^{s} c_k^{\frac{\alpha}{s}}} \\
& + \frac{[(\lambda_1 + \alpha - \frac{2\alpha}{s} s_0) \ln c_{s_0+1} - 1] c_{s_0+1}^{(\lambda_1 + \alpha - \frac{2\alpha}{s} s_0)} + 1}{(\lambda_1 + \alpha - \frac{2\alpha}{s} s_0)^2} \frac{\prod_{k=1}^{s_0} c_k^{\frac{\alpha}{s}}}{\prod_{k=s_0+1}^{s} c_k^{\frac{\alpha}{s}}} \\
& + \sum_{i=s_0+1}^{s} \frac{1}{(\lambda_1 + \alpha - \frac{2\alpha}{s} i)^2} \bigg\{ [(\lambda_1 + \alpha - \frac{2\alpha}{s} i) \ln c_{i+1} - 1] c_{i+1}^{(\lambda_1 + \alpha - \frac{2\alpha}{s} i)} \\
& - [(\lambda_1 + \alpha - \frac{2\alpha}{s} i) \ln c_i - 1] c_i^{(\lambda_1 + \alpha - \frac{2\alpha}{s} i)} \bigg\} \frac{\prod_{k=1}^{i} c_k^{\frac{\alpha}{s}}}{\prod_{k=i+1}^{s} c_k^{\frac{\alpha}{s}}};
\end{aligned}
$$

(6) For $\lambda = -\alpha \ (\alpha > 0)$,

$$
h(u) = |\ln u| \prod_{k=1}^{s} (\min\{u, c_k\})^{\frac{\alpha}{s}},
$$

we derive that

$$\begin{aligned} k(\lambda_1) &= \sum_{i=0}^{s_0-1}\frac{1}{(\lambda_1+\alpha-\frac{\alpha i}{s})^2}\Big\{[1-(\lambda_1+\alpha-\frac{\alpha i}{s})\ln c_{i+1}]c_{i+1}^{(\lambda_1+\alpha-\frac{\alpha i}{s})} \\ &-[1-(\lambda_1+\alpha-\frac{\alpha i}{s})\ln c_i]c_i^{(\lambda_1+\alpha-\frac{\alpha i}{s})}\Big\}\prod_{k=1}^{i}c_k^{\frac{\alpha}{s}} \\ &+\frac{1-[1-(\lambda_1+\alpha-\frac{\alpha}{s}s_0)\ln c_{s_0}]c_{s_0}^{(\lambda_1+\alpha-\frac{\alpha}{s}s_0)}}{(\lambda_1+\alpha-\frac{\alpha}{s}s_0)^2}\prod_{k=1}^{s_0}c_k^{\frac{\alpha}{s}} \\ &+\frac{[(\lambda_1+\alpha-\frac{\alpha}{s}s_0)\ln c_{s_0+1}-1]c_{s_0+1}^{(\lambda_1+\alpha-\frac{\alpha}{s}s_0)}+1}{(\lambda_1+\alpha-\frac{\alpha}{s}s_0)^2}\prod_{k=1}^{s_0}c_k^{\frac{\alpha}{s}} \\ &+\sum_{i=s_0+1}^{s}\frac{1}{(\lambda_1+\alpha-\frac{\alpha i}{s})^2}\Big\{[(\lambda_1+\alpha-\frac{\alpha i}{s})\ln c_{i+1}-1]c_{i+1}^{(\lambda_1+\alpha-\frac{\alpha i}{s})} \\ &-[(\lambda_1+\alpha-\frac{\alpha i}{s})\ln c_i-1]c_i^{(\lambda_1+\alpha-\frac{\alpha i}{s})}\Big\}\prod_{k=1}^{i}c_k^{\frac{\alpha}{s}}. \end{aligned}$$

For $n \in \mathbf{N}$, we consider the following two expressions:

$$I_1 := \int_1^\infty\left\{\int_0^1 |\ln xy|\left[\prod_{k=1}^{s}\frac{(\min\{xy,c_k\})^{\frac{\alpha}{s}}}{(\max\{xy,c_k\})^{\frac{\lambda+\alpha}{s}}}\right]x^{\lambda_1+\frac{1}{pn}-1}dx\right\}y^{\sigma_1-\frac{1}{qn}-1}dy, \quad (11)$$

$$I_2 := \int_0^1\left\{\int_1^\infty |\ln xy|\left[\prod_{k=1}^{s}\frac{(\min\{xy,c_k\})^{\frac{\alpha}{s}}}{(\max\{xy,c_k\})^{\frac{\lambda+\alpha}{s}}}\right]x^{\lambda_1-\frac{1}{pn}-1}dx\right\}y^{\sigma_1+\frac{1}{qn}-1}dy. \quad (12)$$

Setting $u = xy$ in (11) and (12), by Fubini's theorem (cf. [27]), we obtain

$$\begin{aligned} I_1 &= \int_1^\infty\left\{\int_0^y |\ln u|\left[\prod_{k=1}^{s}\frac{(\min\{u,c_k\})^{\frac{\alpha}{s}}}{(\max\{u,c_k\})^{\frac{\lambda+\alpha}{s}}}\right]\left(\frac{u}{y}\right)^{\lambda_1+\frac{1}{pn}-1}\frac{du}{y}\right\}y^{\sigma_1-\frac{1}{qn}-1}dy \\ &= \int_1^\infty y^{(\sigma_1-\lambda_1)-\frac{1}{n}-1}\left\{\int_0^y |\ln u|\left[\prod_{k=1}^{s}\frac{(\min\{u,c_k\})^{\frac{\alpha}{s}}}{(\max\{u,c_k\})^{\frac{\lambda+\alpha}{s}}}\right]u^{\lambda_1+\frac{1}{pn}-1}du\right\}dy \\ &= \int_1^\infty y^{(\sigma_1-\lambda_1)-\frac{1}{n}-1}dy\int_0^1(-\ln u)\left[\prod_{k=1}^{s}\frac{(\min\{u,c_k\})^{\frac{\alpha}{s}}}{(\max\{u,c_k\})^{\frac{\lambda+\alpha}{s}}}\right]u^{\lambda_1+\frac{1}{pn}-1}du \\ &+ \int_1^\infty y^{(\sigma_1-\lambda_1)-\frac{1}{n}-1}\int_1^y \ln u\left[\prod_{k=1}^{s}\frac{(\min\{u,c_k\})^{\frac{\alpha}{s}}}{(\max\{u,c_k\})^{\frac{\lambda+\alpha}{s}}}\right]u^{\lambda_1+\frac{1}{pn}-1}dudy \\ &= \int_1^\infty y^{(\sigma_1-\lambda_1)-\frac{1}{n}-1}dy\int_0^1(-\ln u)\left[\prod_{k=1}^{s}\frac{(\min\{u,c_k\})^{\frac{\alpha}{s}}}{(\max\{u,c_k\})^{\frac{\lambda+\alpha}{s}}}\right]u^{\lambda_1+\frac{1}{pn}-1}du \\ &+ \int_1^\infty[\int_u^\infty y^{(\sigma_1-\lambda_1)-\frac{1}{n}-1}dy]\ln u\left[\prod_{k=1}^{s}\frac{(\min\{u,c_k\})^{\frac{\alpha}{s}}}{(\max\{u,c_k\})^{\frac{\lambda+\alpha}{s}}}\right]u^{\lambda_1+\frac{1}{pn}-1}du, \end{aligned} \quad (13)$$

$$\begin{aligned}
I_2 &= \int_0^1 \left\{ \int_y^\infty |\ln u| \left[\prod_{k=1}^s \frac{(\min\{u,c_k\})^{\frac{\alpha}{s}}}{(\max\{u,c_k\})^{\frac{\lambda+\alpha}{s}}} \right] \left(\frac{u}{y}\right)^{\lambda_1 - \frac{1}{pn} - 1} \frac{du}{y} \right\} y^{\sigma_1 + \frac{1}{qn} - 1} dy \\
&= \int_0^1 y^{(\sigma_1 - \lambda_1) + \frac{1}{n} - 1} \left\{ \int_y^\infty |\ln u| \left[\prod_{k=1}^s \frac{(\min\{u,c_k\})^{\frac{\alpha}{s}}}{(\max\{u,c_k\})^{\frac{\lambda+\alpha}{s}}} \right] u^{\lambda_1 - \frac{1}{pn} - 1} du \right\} dy \\
&= \int_0^1 y^{(\sigma_1 - \lambda_1) + \frac{1}{n} - 1} dy \int_y^1 (-\ln u) \left[\prod_{k=1}^s \frac{(\min\{u,c_k\})^{\frac{\alpha}{s}}}{(\max\{u,c_k\})^{\frac{\lambda+\alpha}{s}}} \right] u^{\lambda_1 - \frac{1}{pn} - 1} du \\
&\quad + \int_0^1 y^{(\sigma_1 - \lambda_1) + \frac{1}{n} - 1} \int_1^\infty \ln u \left[\prod_{k=1}^s \frac{(\min\{u,c_k\})^{\frac{\alpha}{s}}}{(\max\{u,c_k\})^{\frac{\lambda+\alpha}{s}}} \right] u^{\lambda_1 - \frac{1}{pn} - 1} du dy \\
&= \int_0^1 \left[\int_0^u y^{(\sigma_1 - \lambda_1) + \frac{1}{n} - 1} dy \right] (-\ln u) \left[\prod_{k=1}^s \frac{(\min\{u,c_k\})^{\frac{\alpha}{s}}}{(\max\{u,c_k\})^{\frac{\lambda+\alpha}{s}}} \right] u^{\lambda_1 - \frac{1}{pn} - 1} du \\
&\quad + \int_0^1 y^{(\sigma_1 - \lambda_1) + \frac{1}{n} - 1} dy \int_1^\infty \ln u \left[\prod_{k=1}^s \frac{(\min\{u,c_k\})^{\frac{\alpha}{s}}}{(\max\{u,c_k\})^{\frac{\lambda+\alpha}{s}}} \right] u^{\lambda_1 - \frac{1}{pn} - 1} du. \qquad (14)
\end{aligned}$$

Lemma 1. *Suppose that $p > 1, \frac{1}{p} + \frac{1}{q} = 1, \sigma_1 \in \mathbf{R}$. If there exists a constant M, such that for any non-negative measurable functions $f(x)$ and $g(y)$ in $(0, \infty)$, the following inequality*

$$\begin{aligned}
I &:= \int_0^\infty \int_0^\infty |\ln xy| \left[\prod_{k=1}^s \frac{(\min\{xy,c_k\})^{\frac{\alpha}{s}}}{(\max\{xy,c_k\})^{\frac{\lambda+\alpha}{s}}} \right] f(x) g(y) dx dy \\
&\leq M \left[\int_0^\infty x^{p(1-\lambda_1)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\sigma_1)-1} g^q(y) dy \right]^{\frac{1}{q}} \qquad (15)
\end{aligned}$$

holds, then we have $\sigma_1 = \lambda_1$. When $\sigma_1 = \lambda_1$, we have $M \geq k(\lambda_1)$.

Proof. If $\sigma_1 < \lambda_1$, then for

$$n > \frac{1}{\lambda_1 - \sigma_1} \ (n \in \mathbf{N}),$$

we set the following two functions

$$f_n(x) := \begin{cases} 0, 0 < x < 1 \\ x^{\lambda_1 - \frac{1}{pn} - 1}, x \geq 1 \end{cases}, \quad g_n(y) := \begin{cases} y^{\sigma_1 + \frac{1}{qn} - 1}, 0 < y \leq 1 \\ 0, y > 1 \end{cases}.$$

Hence, we derive that

$$\begin{aligned}
J_2 &:= \left[\int_0^\infty x^{p(1-\lambda_1)-1} f_n^p(x) dx \right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\sigma_1)-1} g_n^q(y) dy \right]^{\frac{1}{q}} \\
&= \left(\int_1^\infty x^{-\frac{1}{n}-1} dx \right)^{\frac{1}{p}} \left(\int_0^1 y^{\frac{1}{n}-1} dy \right)^{\frac{1}{q}} = n.
\end{aligned}$$

By (14) and (15), we have

$$\begin{aligned}
&\int_0^1 \left[\int_0^u y^{(\sigma_1 - \sigma) + \frac{1}{n} - 1} dy \right] (-\ln u) \left[\prod_{k=1}^s \frac{(\min\{u,c_k\})^{\frac{\alpha}{s}}}{(\max\{u,c_k\})^{\frac{\lambda+\alpha}{s}}} \right] u^{\lambda_1 - \frac{1}{pn} - 1} du \\
\leq\ & I_2 = \int_0^\infty \int_0^\infty |\ln xy| \left[\prod_{k=1}^s \frac{(\min\{xy,c_k\})^{\frac{\alpha}{s}}}{(\max\{xy,c_k\})^{\frac{\lambda+\alpha}{s}}} \right] f_n(x) g_n(y) dx dy \\
\leq\ & M J_2 = Mn. \qquad (16)
\end{aligned}$$

Since

$$(\sigma_1 - \lambda_1) + \frac{1}{n} < 0,$$

it follows that for any $u \in (0,1)$,

$$\int_0^u y^{(\sigma_1-\lambda_1)+\frac{1}{n}-1} dy = \infty.$$

By (16), in view of

$$(-\ln u)\left[\prod_{k=1}^{s} \frac{(\min\{u, c_k\})^{\frac{\alpha}{s}}}{(\max\{u, c_k\})^{\frac{\lambda+\alpha}{s}}}\right] u^{\lambda_1 - \frac{1}{pn} - 1} > 0,\ u \in (0,1),$$

we obtain that $\infty \leq Mn < \infty$, which is a contradiction.

If $\sigma_1 > \lambda_1$, then for

$$n > \frac{1}{\sigma_1 - \lambda_1}\ (n \in \mathbf{N}),$$

we set

$$\widetilde{f}_n(x) := \begin{cases} x^{\lambda_1 + \frac{1}{pn} - 1}, 0 < x \leq 1 \\ 0,\ x > 1 \end{cases},\ \widetilde{g}_n(y) := \begin{cases} 0, 0 < y < 1 \\ y^{\sigma_1 - \frac{1}{qn} - 1}, y \geq 1 \end{cases},$$

and find that

$$\begin{aligned} \widetilde{J}_2 &:= \left[\int_0^\infty x^{p(1-\lambda_1)-1} \widetilde{f}_n^p(x) dx\right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\sigma_1)-1} \widetilde{g}_n^q(y) dy\right]^{\frac{1}{q}} \\ &= \left(\int_0^1 x^{\frac{1}{n}-1} dx\right)^{\frac{1}{p}} \left(\int_1^\infty y^{-\frac{1}{n}-1} dy\right)^{\frac{1}{q}} = n. \end{aligned}$$

By (13) and (15), we have

$$\begin{aligned} &\int_1^\infty y^{(\sigma_1-\lambda_1)-\frac{1}{n}-1} dy \int_0^1 (-\ln u)\left[\prod_{k=1}^{s} \frac{(\min\{u, c_k\})^{\frac{\alpha}{s}}}{(\max\{u, c_k\})^{\frac{\lambda+\alpha}{s}}}\right] u^{\lambda_1 + \frac{1}{pn} - 1} du \\ \leq\ & I_1 = \int_0^\infty \int_0^\infty |\ln xy| \left[\prod_{k=1}^{s} \frac{(\min\{xy, c_k\})^{\frac{\alpha}{s}}}{(\max\{xy, c_k\})^{\frac{\lambda+\alpha}{s}}}\right] \widetilde{f}_n(x) \widetilde{g}_n(y) dx dy \\ \leq\ & M\widetilde{J}_2 = Mn. \end{aligned} \tag{17}$$

Since $(\sigma_1 - \lambda_1) - \frac{1}{n} > 0$, it follows that

$$\int_1^\infty y^{(\sigma_1-\lambda_1)-\frac{1}{n}-1} dy = \infty.$$

By (17), in view of

$$\int_0^1 (-\ln u)\left[\prod_{k=1}^{s} \frac{(\min\{u, c_k\})^{\frac{\alpha}{s}}}{(\max\{u, c_k\})^{\frac{\lambda+\alpha}{s}}}\right] u^{\lambda_1 + \frac{1}{pn} - 1} du > 0,$$

we have $\infty \leq Mn < \infty$, which is a contradiction.

Hence, we conclude that $\sigma_1 = \lambda_1$.

For $\sigma_1 = \lambda_1$, we reduce (13) and then use (17) as follows:

$$\begin{aligned}\frac{1}{n}I_1 &= \frac{1}{n}\left\{\int_1^\infty y^{-\frac{1}{n}-1}dy\int_0^1(-\ln u)\left[\prod_{k=1}^s\frac{(\min\{u,c_k\})^{\frac{\alpha}{s}}}{(\max\{u,c_k\})^{\frac{\lambda+\alpha}{s}}}\right]u^{\lambda_1+\frac{1}{pn}-1}du\right.\\ &\quad\left.+\int_1^\infty\left(\int_u^\infty y^{-\frac{1}{n}-1}dy\right)\ln u\left[\prod_{k=1}^s\frac{(\min\{u,c_k\})^{\frac{\alpha}{s}}}{(\max\{u,c_k\})^{\frac{\lambda+\alpha}{s}}}\right]u^{\lambda_1+\frac{1}{pn}-1}du\right\}\\ &= \int_0^1(-\ln u)\left[\prod_{k=1}^s\frac{(\min\{u,c_k\})^{\frac{\alpha}{s}}}{(\max\{u,c_k\})^{\frac{\lambda+\alpha}{s}}}\right]u^{\lambda_1+\frac{1}{pn}-1}du\\ &\quad+\int_1^\infty\ln u\left[\prod_{k=1}^s\frac{(\min\{u,c_k\})^{\frac{\alpha}{s}}}{(\max\{u,c_k\})^{\frac{\lambda+\alpha}{s}}}\right]u^{\lambda_1-\frac{1}{qn}-1}du\le\frac{1}{n}M\widetilde{J}_2=M. \qquad (18)\end{aligned}$$

Since

$$(-\ln u)\left[\prod_{k=1}^s\frac{(\min\{u,c_k\})^{\frac{\alpha}{s}}}{(\max\{u,c_k\})^{\frac{\lambda+\alpha}{s}}}\right]u^{\lambda_1+\frac{1}{pn}-1}$$

$\left(\ln u\left[\prod_{k=1}^s\frac{(\min\{u,c_k\})^{\frac{\alpha}{s}}}{(\max\{u,c_k\})^{\frac{\lambda+\alpha}{s}}}\right]u^{\lambda_1-\frac{1}{qn}-1}\right)$ is nonnegative and increasing in $(0,1)$ $((1,\infty))$, by Levi's theorem (cf. [27]), we derive that

$$\begin{aligned}k(\lambda_1) &= \int_0^1\lim_{n\to\infty}(-\ln u)\left[\prod_{k=1}^s\frac{(\min\{u,c_k\})^{\frac{\alpha}{s}}}{(\max\{u,c_k\})^{\frac{\lambda+\alpha}{s}}}\right]u^{\lambda_1+\frac{1}{pn}-1}du\\ &\quad+\int_1^\infty\lim_{n\to\infty}\ln u\left[\prod_{k=1}^s\frac{(\min\{u,c_k\})^{\frac{\alpha}{s}}}{(\max\{u,c_k\})^{\frac{\lambda+\alpha}{s}}}\right]u^{\lambda_1-\frac{1}{qn}-1}du\\ &= \lim_{n\to\infty}\left\{\int_0^1(-\ln u)\left[\prod_{k=1}^s\frac{(\min\{u,c_k\})^{\frac{\alpha}{s}}}{(\max\{u,c_k\})^{\frac{\lambda+\alpha}{s}}}\right]u^{\lambda_1+\frac{1}{pn}-1}du\right.\\ &\quad\left.+\int_1^\infty\ln u\left[\prod_{k=1}^s\frac{(\min\{u,c_k\})^{\frac{\alpha}{s}}}{(\max\{u,c_k\})^{\frac{\lambda+\alpha}{s}}}\right]u^{\lambda_1-\frac{1}{qn}-1}du\right\}\le M<\infty. \qquad (19)\end{aligned}$$

This completes the proof of the lemma. □

3. Main Results and Operator Expressions

Theorem 1. *Suppose that* $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $\sigma_1\in\mathbf{R}$. *The following statements are equivalent:*

(i) *There exists a constant M such that for any* $f(x)\ge 0$, *with*

$$0<\int_0^\infty x^{p(1-\lambda_1)-1}f^p(x)dx<\infty,$$

the following inequality holds true:

$$\begin{aligned}J &:= \left\{\int_0^\infty y^{p\sigma_1-1}\left[\int_0^\infty|\ln xy|\left(\prod_{k=1}^s\frac{(\min\{xy,c_k\})^{\frac{\alpha}{s}}}{(\max\{xy,c_k\})^{\frac{\lambda+\alpha}{s}}}\right)f(x)dx\right]^p dy\right\}^{\frac{1}{p}}\\ &< M\left[\int_0^\infty x^{p(1-\lambda_1)-1}f^p(x)dx\right]^{\frac{1}{p}}; \qquad (20)\end{aligned}$$

(ii) *There exists a constant M, such that for any* $f(x)$, $g(y)\ge 0$, *with*

$$0<\int_0^\infty x^{p(1-\lambda_1)-1}f^p(x)dx<\infty$$

and

$$0<\int_0^\infty y^{q(1-\sigma_1)-1}g^q(y)dy<\infty,$$

the following inequality holds true:

$$\begin{aligned} I &= \int_0^\infty \int_0^\infty |\ln xy| \left[\prod_{k=1}^{s} \frac{(\min\{xy, c_k\})^{\frac{\alpha}{s}}}{(\max\{xy, c_k\})^{\frac{\lambda+\alpha}{s}}} \right] f(x) g(y) dx dy \\ &< M \left[\int_0^\infty x^{p(1-\lambda_1)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\sigma_1)-1} g^q(y) dy \right]^{\frac{1}{q}}; \end{aligned} \tag{21}$$

(iii) $\sigma_1 = \lambda_1$.

If Condition (iii) is satisfied, then $M \geq k(\lambda_1)$ *and the constant factor* $M = k(\lambda_1)$ *in* (20) *and* (21) *is the best possible.*

Proof. "$(i) \Rightarrow (ii)$". By Hölder's inequality (cf. [28]), we have

$$\begin{aligned} I &= \int_0^\infty \left[y^{\sigma_1 - \frac{1}{p}} \int_0^\infty |\ln xy| \left(\prod_{k=1}^{s} \frac{(\min\{xy, c_k\})^{\frac{\alpha}{s}}}{(\max\{xy, c_k\})^{\frac{\lambda+\alpha}{s}}} \right) f(x) dx \right] \left(y^{\frac{1}{p} - \sigma_1} g(y) \right) dy \\ &\leq J \left[\int_0^\infty y^{q(1-\sigma_1)-1} g^q(y) dy \right]^{\frac{1}{q}}. \end{aligned} \tag{22}$$

Then by (20), we deduce (21).

"$(ii) \Rightarrow (iii)$". By Lemma 1, we have $\sigma_1 = \lambda_1$.

"$(iii) \Rightarrow (i)$". Setting $u = xy$, we obtain the following weight function:

For $y > 0$,

$$\begin{aligned} \omega(\lambda_1, y) &:= y^{\lambda_1} \int_0^\infty |\ln xy| \left[\prod_{k=1}^{s} \frac{(\min\{xy, c_k\})^{\frac{\alpha}{s}}}{(\max\{xy, c_k\})^{\frac{\lambda+\alpha}{s}}} \right] x^{\lambda_1 - 1} dx \\ &= \int_0^\infty |\ln u| \left[\prod_{k=1}^{s} \frac{(\min\{u, c_k\})^{\frac{\alpha}{s}}}{(\max\{u, c_k\})^{\frac{\lambda+\alpha}{s}}} \right] u^{\lambda_1 - 1} du = k(\lambda_1). \end{aligned} \tag{23}$$

By Hölder's inequality with weight and (23), we obtain that

$$\begin{aligned} &\left\{ \int_0^\infty |\ln xy| \left[\prod_{k=1}^{s} \frac{(\min\{xy, c_k\})^{\frac{\alpha}{s}}}{(\max\{xy, c_k\})^{\frac{\lambda+\alpha}{s}}} \right] f(x) dx \right\}^p \\ &= \left\{ \int_0^\infty |\ln xy| \left[\prod_{k=1}^{s} \frac{(\min\{xy, c_k\})^{\frac{\alpha}{s}}}{(\max\{xy, c_k\})^{\frac{\lambda+\alpha}{s}}} \right] \frac{y^{(\lambda_1-1)/p} f(x)}{x^{(\lambda_1-1)/q}} \frac{x^{(\lambda_1-1)/q}}{y^{(\lambda_1-1)/p}} dx \right\}^p \\ &\leq \int_0^\infty |\ln xy| \left[\prod_{k=1}^{s} \frac{(\min\{xy, c_k\})^{\frac{\alpha}{s}}}{(\max\{xy, c_k\})^{\frac{\lambda+\alpha}{s}}} \right] \frac{y^{\lambda_1-1}}{x^{(\lambda_1-1)p/q}} f^p(x) dx \\ &\quad \times \left\{ \int_0^\infty |\ln xy| \left[\prod_{k=1}^{s} \frac{(\min\{xy, c_k\})^{\frac{\alpha}{s}}}{(\max\{xy, c_k\})^{\frac{\lambda+\alpha}{s}}} \right] \frac{x^{\lambda_1-1}}{y^{(\lambda_1-1)q/p}} dx \right\}^{p/q} \\ &= \left[\frac{\omega(\lambda_1, y)}{y^{q(\lambda_1-1)+1}} \right]^{p-1} \int_0^\infty |\ln xy| \left[\prod_{k=1}^{s} \frac{(\min\{xy, c_k\})^{\frac{\alpha}{s}}}{(\max\{xy, c_k\})^{\frac{\lambda+\alpha}{s}}} \right] \frac{y^{\lambda_1-1} f^p(x)}{x^{(\lambda_1-1)p/q}} dx \\ &= \frac{(k(\lambda_1))^{p-1}}{y^{p\lambda_1-1}} \int_0^\infty |\ln xy| \left[\prod_{k=1}^{s} \frac{(\min\{xy, c_k\})^{\frac{\alpha}{s}}}{(\max\{xy, c_k\})^{\frac{\lambda+\alpha}{s}}} \right] \frac{y^{\lambda_1-1} f^p(x)}{x^{(\lambda_1-1)p/q}} dx. \end{aligned} \tag{24}$$

If (24) assumes the form of equality for some $y \in (0, \infty)$, then (cf. [28]) there exist constants A and B, such that they are not all zero and

$$A \frac{y^{\lambda_1-1}}{x^{(\lambda_1-1)p/q}} f^p(x) = B \frac{x^{\lambda_1-1}}{y^{(\lambda_1-1)q/p}} \text{ a.e. in } \mathbf{R}_+.$$

We suppose that $A \neq 0$ (otherwise, $B = A = 0$). Then, it follows that

$$x^{p(1-\lambda_1)-1} f^p(x) = y^{q(1-\lambda_1)} \frac{B}{Ax} \text{ a.e. in } \mathbf{R}_+,$$

which contradicts the fact that

$$0 < \int_0^\infty x^{p(1-\lambda_1)-1} f^p(x) dx < \infty.$$

Hence, (24) assumes the form of strict inequality.

Therefore, for $\sigma_1 = \lambda_1$, by Fubini's theorem, we derive that

$$\begin{aligned} J &< (k(\lambda_1))^{\frac{1}{q}} \left\{ \int_0^\infty \int_0^\infty |\ln xy| \left[\prod_{k=1}^{s} \frac{(\min\{xy, c_k\})^{\frac{\alpha}{s}}}{(\max\{xy, c_k\})^{\frac{\lambda+\alpha}{s}}} \right] \frac{y^{\lambda_1 - 1}}{x^{(\lambda_1 - 1)p/q}} f^p(x) dx dy \right\}^{\frac{1}{p}} \\ &= (k(\lambda_1))^{\frac{1}{q}} \left\{ \int_0^\infty \left[\int_0^\infty |\ln xy| \left(\prod_{k=1}^{s} \frac{(\min\{xy, c_k\})^{\frac{\alpha}{s}}}{(\max\{xy, c_k\})^{\frac{\lambda+\alpha}{s}}} \right) \frac{y^{\lambda_1 - 1} dy}{x^{(\lambda_1 - 1)p/q}} \right] f^p(x) dx \right\}^{\frac{1}{p}} \\ &= (k(\lambda_1))^{\frac{1}{q}} \left[\int_0^\infty \omega(\lambda_1, x) x^{p(1-\lambda_1)-1} f^p(x) dx \right]^{\frac{1}{p}} \\ &= k(\lambda_1) \left[\int_0^\infty x^{p(1-\lambda_1)-1} f^p(x) dx \right]^{\frac{1}{p}}. \end{aligned}$$

Setting $M \geq k(\lambda_1)$, (20) follows.

Thus, the conditions (i), (ii) and (iii) are equivalent.

When Condition (iii) is satisfied, if there exists a constant $M \leq k(\lambda_1)$, such that (21) holds true, then by Lemma 1 we have that $M \geq k(\lambda_1)$. Then the constant factor $M = k(\lambda_1)$ in (21) is the best possible. The constant factor $M = k(\lambda_1)$ in (20) is still the best possible. Otherwise, by (22) (for $\sigma_1 = \lambda_1$), we would conclude that the constant factor $M = k(\lambda_1)$ in (21) is not the best possible.

This completes the proof of the theorem. □

Setting $y = \frac{1}{Y}, G(Y) = Y^{\lambda-2} g(\frac{1}{Y}), \sigma_2 = \lambda - \sigma_1$ in Theorem 1, then replacing Y ($G(Y)$) by y ($g(y)$), we derive the following corollary.

Corollary 1. *Suppose that $p > 1, \frac{1}{p} + \frac{1}{q} = 1, \sigma_2 \in \mathbf{R}$. The following conditions are equivalent:*

(i) There exists a constant M, such that for any $f(x) \geq 0$ satisfying

$$0 < \int_0^\infty x^{p(1-\lambda_1)-1} f^p(x) dx < \infty,$$

we have the following Hilbert-type inequality with the homogeneous kernel:

$$\begin{aligned} &\left\{ \int_0^\infty y^{p\sigma_2 - 1} \left[\int_0^\infty |\ln \frac{x}{y}| \left(\prod_{k=1}^{s} \frac{(\min\{x, c_k y\})^{\frac{\alpha}{s}}}{(\max\{x, c_k y\})^{\frac{\lambda+\alpha}{s}}} \right) f(x) dx \right]^p dy \right\}^{\frac{1}{p}} \\ < \; & M \left[\int_0^\infty x^{p(1-\lambda_1)-1} f^p(x) dx \right]^{\frac{1}{p}}; \end{aligned} \quad (25)$$

(ii) There exists a constant M, such that for any $f(x), g(y) \geq 0$, satisfying

$$0 < \int_0^\infty x^{p(1-\lambda_1)-1} f^p(x) dx < \infty,$$

and

$$0 < \int_0^\infty y^{q(1-\sigma_2)-1} g^q(y) dy < \infty,$$

we have the following inequality:

$$\begin{aligned} &\int_0^\infty \int_0^\infty |\ln \frac{x}{y}| \left[\prod_{k=1}^{s} \frac{(\min\{x, c_k y\})^{\frac{\alpha}{s}}}{(\max\{x, c_k y\})^{\frac{\lambda+\alpha}{s}}}\right] f(x)g(y)dxdy \\ < \; & M\left[\int_0^\infty x^{p(1-\lambda_1)-1} f^p(x)dx\right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\sigma_2)-1} g^q(y)dy\right]^{\frac{1}{q}}; \end{aligned} \quad (26)$$

(iii) $\sigma_2 = \lambda_2$.

If Condition (iii) is satisfied, then we have $M \geq k(\lambda_2)$, *and the constant* $M = k(\lambda_2)$ *in (25) and (26) is the best possible.*

Remark 1. *On the other hand, setting* $y = \frac{1}{Y}$, $G(Y) = Y^{\lambda-2} g(\frac{1}{Y}), \sigma_1 = \lambda - \sigma_2$, *in Corollary 1, then replacing* Y $(G(Y))$ *by* y $(g(y))$, *we deduce Theorem 1. Hence, Theorem 1 and Corollary 1 are equivalent.*

For $p > 1, \frac{1}{p} + \frac{1}{q} = 1$, we set the following functions:

$$\varphi(x) := x^{p(1-\lambda_1)-1}, \;\; \psi(y) := y^{q(1-\lambda_1)-1}, \;\; \phi(y) := y^{q(1-\lambda_2)-1},$$

wherefrom,

$$\psi^{1-p}(y) = y^{p\lambda_1 - 1}, \; \phi^{1-p}(y) = y^{p\lambda_2 - 1} \; (x, y \in \mathbf{R}_+).$$

Define the following real normed linear spaces:

$$\begin{aligned} L_{p,\varphi}(\mathbf{R}_+) &= \left\{ f : ||f||_{p,\varphi} := \left(\int_0^\infty \varphi(x)|f(x)|^p dx\right)^{\frac{1}{p}} < \infty \right\}, \\ L_{q,\psi}(\mathbf{R}_+) &= \left\{ g : ||g||_{q,\psi} := \left(\int_0^\infty \psi(y)|g(y)|^q dy\right)^{\frac{1}{q}} < \infty \right\}, \\ L_{q,\phi}(\mathbf{R}_+) &= \left\{ g : ||g||_{q,\phi} := \left(\int_0^\infty \phi(y)|g(y)|^q dy\right)^{\frac{1}{q}} < \infty \right\}, \\ L_{p,\psi^{1-p}}(\mathbf{R}_+) &= \left\{ h : ||h||_{p,\psi^{1-p}} = \left(\int_0^\infty \psi^{1-p}(y)|h(y)|^p dy\right)^{\frac{1}{p}} < \infty \right\}, \\ L_{q,\phi^{1-p}}(\mathbf{R}_+) &= \left\{ h : ||h||_{p,\phi^{1-p}} = \left(\int_0^\infty \phi^{1-p}(y)|h(y)|^p dy\right)^{\frac{1}{p}} < \infty \right\}. \end{aligned}$$

(a) In view of Theorem 1 (with $\sigma_1 = \lambda_1$), for $f \in L_{p,\varphi}(\mathbf{R}_+)$, setting

$$h_1(y) := \int_0^\infty |\ln xy| \left(\prod_{k=1}^{s} \frac{(\min\{xy, c_k\})^{\frac{\alpha}{s}}}{(\max\{xy, c_k\})^{\frac{\lambda+\alpha}{s}}}\right) f(x)dx \; (y \in \mathbf{R}_+),$$

by (20), we obtain that

$$||h_1||_{p,\psi^{1-p}} = \left(\int_0^\infty \psi^{1-p}(y) h_1^p(y)dy\right)^{\frac{1}{p}} < M||f||_{p,\varphi} < \infty. \quad (27)$$

Definition 1. *Define a Hilbert-type integral operator with the non-homogeneous kernel* $T^{(1)} : L_{p,\varphi}(\mathbf{R}_+) \to L_{p,\psi^{1-p}}(\mathbf{R}_+)$ *as follows:*

For any $f \in L_{p,\varphi}(\mathbf{R}_+)$, *there exists a unique representation* $T^{(1)} f = h_1 \in L_{p,\psi^{1-p}}(\mathbf{R}_+)$, *satisfying* $T^{(1)} f(y) = h_1(y)$, *for any* $y \in \mathbf{R}_+$.

In view of (27), it follows that

$$||T^{(1)}f||_{p,\psi^{1-p}} = ||h_1||_{p,\psi^{1-p}} \leq M||f||_{p,\varphi},$$

and then the operator $T^{(1)}$ is bounded satisfying

$$||T^{(1)}|| = \sup_{f(\neq\theta)\in L_{p,\varphi}(\mathbf{R}_+)} \frac{||T^{(1)}f||_{p,\psi^{1-p}}}{||f||_{p,\varphi}} \leq M.$$

If we define the formal inner product of $T^{(1)}f$ and g as follows:

$$(T^{(1)}f, g) := \int_0^\infty \left[\int_0^\infty |\ln xy| \left(\prod_{k=1}^s \frac{(\min\{xy, c_k\})^{\frac{\alpha}{s}}}{(\max\{xy, c_k\})^{\frac{\lambda+\alpha}{s}}}\right) f(x)dx\right] g(y)dy,$$

then we can rewrite Theorem 1 (for $\sigma_1 = \lambda_1$) as follows:

Theorem 2. *Suppose that $p > 1, \frac{1}{p} + \frac{1}{q} = 1$. The following conditions are equivalent:*

(i) *There exists a constant M, such that for any $f(x) \geq 0$, $f \in L_{p,\varphi}(\mathbf{R}_+)$, $||f||_{p,\varphi} > 0$, we have the following inequality:*

$$||T^{(1)}f||_{p,\psi^{1-p}} < M||f||_{p,\varphi}; \quad (28)$$

(ii) *There exists a constant M, such that for any $f(x), g(y) \geq 0$, $f \in L_{p,\varphi}(\mathbf{R}_+)$, $g \in L_{q,\psi}(\mathbf{R}_+)$, $||f||_{p,\varphi}, ||g||_{q,\psi} > 0$, we have the following inequality:*

$$(T^{(1)}f, g) < M||f||_{p,\varphi}||g||_{q,\psi}. \quad (29)$$

We still have $||T^{(1)}|| = k(\lambda_1) \leq M$.

(b) In view of Corollary 1 (with $\sigma_2 = \lambda_2$), for $f \in L_{p,\varphi}(\mathbf{R}_+)$, setting

$$h_2(y) := \int_0^\infty |\ln \frac{x}{y}| \left[\prod_{k=1}^s \frac{(\min\{x, c_k y\})^{\frac{\alpha}{s}}}{(\max\{x, c_k y\})^{\frac{\lambda+\alpha}{s}}}\right] f(x)dx \ (y \in \mathbf{R}_+),$$

by (27), we have

$$||h_2||_{p,\phi^{1-p}} = \left(\int_0^\infty \phi^{1-p}(y) h_2^p(y) dy\right)^{\frac{1}{p}} < M||f||_{p,\varphi} < \infty. \quad (30)$$

Definition 2. *Define a Hilbert-type integral operator with the homogeneous kernel $T^{(2)} : L_{p,\varphi}(\mathbf{R}_+) \to L_{p,\phi^{1-p}}(\mathbf{R}_+)$ as follows:*

For any $f \in L_{p,\varphi}(\mathbf{R})$, there exists a unique representation $T^{(2)}f = h_2 \in L_{p,\phi^{1-p}}(\mathbf{R}_+)$, satisfying $T^{(2)}f(y) = h_2(y)$, for any $y \in \mathbf{R}_+$.

In view of (30), it follows that

$$||T^{(2)}f||_{p,\phi^{1-p}} = ||h_2||_{p,\phi^{1-p}} \leq M||f||_{p,\varphi},$$

and then the operator $T^{(2)}$ is bounded satisfying

$$||T^{(2)}|| = \sup_{f(\neq\theta)\in L_{p,\varphi}(\mathbf{R}_+)} \frac{||T^{(2)}f||_{p,\phi^{1-p}}}{||f||_{p,\varphi}} \leq M.$$

If we define the formal inner product of $T^{(2)}f$ and g as follows:

$$(T^{(2)}f, g) := \int_0^\infty \left\{ \int_0^\infty |\ln \frac{x}{y}| \left[\prod_{k=1}^{s} \frac{(\min\{x, c_k y\})^{\frac{\alpha}{s}}}{(\max\{x, c_k y\})^{\frac{\lambda+\alpha}{s}}} \right] f(x)dx \right\} g(y)dy,$$

then we can rewrite Corollary 1 (for $\sigma_2 = \lambda_2$) as follows:

Corollary 2. *Suppose that* $p > 1, \frac{1}{p} + \frac{1}{q} = 1$. *The following conditions are equivalent:*

(i) *There exists a constant* M*, such that for any* $f(x) \geq 0, f \in L_{p,\varphi}(\mathbf{R}_+), ||f||_{p,\varphi} > 0$*, we have the following inequality:*

$$||T^{(2)}f||_{p,\phi^{1-p}} < M||f||_{p,\varphi}; \quad (31)$$

(ii) *There exists a constant* M*, such that for any* $f(x), g(y) \geq 0, f \in L_{p,\varphi}(\mathbf{R}_+), g \in L_{q,\phi}(\mathbf{R}_+)$, $||f||_{p,\varphi}, ||g||_{q,\phi} > 0$*, we have the following inequality:*

$$(T^{(2)}f, g) < M||f||_{p,\varphi}||g||_{q,\phi}. \quad (32)$$

We still have $||T^{(2)}|| = k(\lambda_1) \leq M$.

Remark 2. *Theorem 2 and Corollary 2 are equivalent.*

4. Conclusions

In this paper, by means of real analysis, an equivalent form related to a Hilbert-type integral inequality with the non-homogeneous kernel

$$|\ln xy| \prod_{k=1}^{s} \frac{(\min\{xy, c_k\})^{\frac{\alpha}{s}}}{(\max\{xy, c_k\})^{\frac{\lambda+\alpha}{s}}}$$

and a best possible constant factor is given in Theorem 1. We also consider the case of the homogeneous kernel and the operator expressions in Corollary 1, Corollary 2 and Theorem 2. The lemmas and theorems provide an extensive account of this type of inequalities.

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